

TI 2026-081/VI
Tinbergen Institute Discussion Paper

Heterogeneity in Expectations: Empirical and Experimental Evidence

*Xiaoming Cai*¹

*Pieter Gautier*²

¹ Peking University HSBC Business School

² Vrije Universiteit Amsterdam

Sorting in a Market for Lemons*

Xiaoming Cai[†] Pieter Gautier[‡]

August 28, 2026

Abstract

Search frictions normally raise the bar for sorting: both random and directed search require complementarities strictly stronger than the supermodularity that suffices in Becker's (1973) frictionless model. We introduce private information into this setting: firms post wages without observing worker productivity, while better workers have better outside options, creating a market for lemons (Akerlof 1970). We show that this information friction pushes sorting the other way: attracting high-productivity workers without attracting low-productivity ones requires a higher wage paired with a lower job-finding probability, so vacancies fill faster. This strict ranking of vacancy-filling rates sorts the market positively whenever output is complementary in worker and firm quality. Becker's condition therefore suffices once more, and private information makes positive sorting easier to sustain than complete information does.

JEL classification: C78, D82, D83, J64.

Keywords: competitive search, adverse selection, screening, sorting, positive assortative matching.

*We thank Philipp Kircher for helpful comments.

[†]Peking University HSBC Business School. email: xmingcai@gmail.com.

[‡]Vrije Universiteit Amsterdam and Tinbergen Institute. email: p.a.gautier@vu.nl.

1 Introduction

This paper studies a competitive-search labor market with heterogeneity on both sides, where agents' types are private information. Higher-type workers are more productive, but they also have a higher value of remaining unmatched (a higher value of home production or of being self-employed). So the market is one for lemons (Akerlof, 1970). Firms post wages, workers direct their search based on the posted wages, and meetings are bilateral. The question is how this information friction changes the sorting implications of competitive search. As a benchmark, Becker (1973) studies sorting in a frictionless Walrasian market and shows that supermodularity of output delivers positive assortative matching (PAM). Once search frictions are added, that condition is generally no longer enough: across both random and directed search, PAM requires complementarities strictly stronger than supermodularity (Shimer and Smith, 2000; Shi, 2001; Shimer, 2005; Eeckhout and Kircher, 2010, 2018; Lentz, 2010; Cai et al., 2025; Borovičková and Shimer, 2025; Chade et al., 2017). Our main result is that in this private-information environment, Becker's condition again suffices for PAM.

The result is counterintuitive. In a frictionless market, supermodularity compares whether a trader would rather match with one type or another on the other side, taking trade itself for granted. Search frictions add a second margin: whether trade occurs at all. Under complete information, the standard assumption that the value of trade rises with worker type implies that a higher type loses more from remaining unmatched and is therefore more eager to secure a quick match (Chade and Kircher, 2023). This matching-probability margin therefore works against sorting. High-type agents are attractive trading partners and can convert that advantage into high matching probabilities by targeting cheaper, lower-type counterparties, even if doing so requires conceding some payoff upon matching; symmetrically, lower-type agents may target higher-type counterparties and accept lower matching probabilities in exchange for a larger payoff when trade occurs. These matching-probability incentives push toward cross-matching; production complementarities must be strong enough to overcome them. This is why Eeckhout and Kircher (2010) require the value of trade to be root-supermodular, a strict strengthening of Becker's condition, for PAM in complete-information competitive search.

The role of this type-dependent cost of delay is underscored by Atakan (2006). In

his random-search model with a common fixed search cost and no discounting, the equilibrium Bellman equation implies a constant-surplus condition: for every type, the expected gain from continued search exactly offsets the same search cost. The cost of waiting is therefore independent of type, so all types respond symmetrically to search frictions and supermodularity once again suffices for positive sorting. Thus, whereas discounting makes the cost of delay rise with type, Atakan's common fixed search cost makes it constant and neutralizes the force toward cross-matching.

Our route is different. Private information reverses, rather than merely neutralizes, the relevant matching-probability force. To see why, note that a higher worker type raises both gross match output and the worker's *retention value*. Their difference is the *value of trade*. Under complete information, firms can condition their offers on worker type, so only this difference matters: the retention value does not act on its own. Under private information, however, firms cannot condition their offers on worker type: a posted wage is open to all applicants. The retention value therefore matters on its own, governing the worker's trade-off between pay and speed. A worker with more to fall back on is in less of a hurry and needs less extra pay to accept a lower chance of getting the job, so a better worker is the more valuable partner but also the more reluctant one. Under complete information this reluctance has no bite, because a low-type firm can act on the value-of-trade motive directly: to deliver the high-type worker's equilibrium utility at least cost to itself, it offers a high job-finding probability financed by a wage below what a high-type firm would pay. Private information removes this instrument: the same contract, aimed at poaching a high-type worker, is, if anything, even more attractive to the low-type workers it must screen out. Separation must therefore run the other way, placing high-type workers in submarkets with lower worker-matching probabilities and higher wages: precisely the bundle their reluctance makes them willing to accept. A low worker-matching probability is a high firm-matching probability, so the very distortion that screens workers also makes high-type workers attractive to high-type firms. The margin that complete-information firms use to cross-match is gone, and Becker's condition again suffices.

Existence of an equilibrium is not immediate. An equilibrium solves a system of two ordinary differential equations, one for the assignment and one for tightness, that must satisfy boundary conditions at both ends of the market simultaneously, with participation on each side itself endogenous. Our proof starts from alternative

descriptions of the bottom of the active market and traces the implied allocation upward. Some candidates use firm mass too slowly and leave part of the firm distribution unused; others use it too quickly and exhaust firms before reaching the highest worker type. Because the candidate allocation varies continuously between these cases, an intermediate starting point satisfies market clearing and the appropriate participation conditions. This then yields a differentiable PAM equilibrium.

The paper contributes to the literature on sorting with search frictions, where the recurring finding is that Becker’s condition is not enough. Under random search, [Shimer and Smith \(2000\)](#) show that PAM requires strictly more than log-supermodularity of output. [Borovičková and Shimer \(2025\)](#) weaken this requirement by adding match-specific shocks: a pair matches only after a sufficiently high realization, so selection in acceptance does the sorting work and log-supermodularity of output suffices. With bilateral meetings, [Shi \(2001\)](#) and [Eeckhout and Kircher \(2010\)](#) show that directed search is less demanding because posted terms steer workers toward particular submarkets. Yet, as noted above, the matching-probability margin still cuts against sorting, so for common meeting technologies the threshold is root-supermodularity (supermodularity of the square root of output). [Cai et al. \(2025\)](#) show that many-on-one meetings with ex post screening push the threshold back from square-root toward log-supermodularity as firms interview more applicants; the corner case in which firms can screen all applicants corresponds to the results of [Shimer \(2005\)](#).¹ These conditions are nested, with Becker’s supermodularity at the bottom and log-supermodularity as the limiting case, while private information returns the threshold to the bottom rung.

Our result also matters for applied work. A related quantitative literature measures the output losses from frictional mismatch ([Gautier and Teulings, 2006, 2015](#); [Lise et al., 2016](#)), while a large empirical literature uses matched employer–employee data on wages and transitions to identify worker and firm types, measure sorting, and recover production complementarities ([Eeckhout and Kircher, 2011](#); [Hagedorn et al., 2017](#); [de Melo, 2018](#); [Bagger and Lentz, 2019](#); [Bartolucci et al., 2018](#); [Borovičková and Shimer, 2020](#)). Related evidence links changes in the allocation of workers across firms to the evolution of wage inequality ([Card et al., 2013](#); [Song et al., 2018](#)). Our result implies that the mapping from these outcomes to the production function depends critically on what firms know when matches form. Under complete information, PAM

¹See [Chade et al. \(2017\)](#) for an overview of the literature on sorting with search frictions.

requires technological complementarities strong enough to offset the search-induced force toward cross-matching; under private information, screening itself contributes to PAM. A complete-information specification may therefore attribute screening-induced sorting to technology, potentially overstating complementarity and affecting quantitative assessments of mismatch losses and wage dispersion. This concern is empirically relevant. Employer-learning studies show that firms initially observe worker ability only imperfectly, while field evidence shows that credible information about workers' skills changes both search and hiring decisions (Farber and Gibbons, 1996; Altonji and Pierret, 2001; Carranza et al., 2022).

The paper also contributes to the literature on adverse selection in competitive search. In this literature, however, the uninformed side is homogeneous, so that the question of sorting does not arise. Guerrieri et al. (2010) and Chang (2018) show that the uninformed side can use the trading probability to screen types, so that equilibrium is separating, and Williams (2021) studies the choice among such screening instruments. Albrecht et al. (2025) show that separation survives when the informed side meets several counterparties, with auction reserve prices as the signal, whereas Auster and Gottardi (2019) show that pooling can arise when it is the uninformed side that meets several counterparties. We add heterogeneity on the uninformed side, and with it the assignment question.

The intersection of sorting and private information has also been studied outside search environments. Hendel et al. (2005) consider a dynamic durable-goods market in which quality is privately known but the number of times a good has changed hands signals it. With suitable rental contracts, each consumer self-selects into exactly the vintage a planner would assign her, so asymmetric information need not distort the allocation at all. Our environment differs in that the screening distortion is not eliminated: tightness in every submarket above the lowest lies strictly below its pointwise optimum. Our point is that this surviving distortion is aligned with sorting rather than opposed to it.

A complementary recent paper at the intersection of sorting and private information is Shimer and Wu (2026). In their model, types are private on both sides, and agents care directly about the type of partner they meet. Competitive, freely entering platforms post fees and participant distributions to coordinate this two-sided participation problem and induce self-selection through fees or rationing. Supermodularity makes pooling vulnerable to cream-skimming and therefore guarantees separation,

but it does not by itself determine how the separated types are matched. In their leading CES example, the interior condition for PAM is the same as under observable types; private information instead adds a boundary condition for the lowest type. When that condition fails, *anchor matching* pairs low types with better partners and may give them more utility than under observable types.

Our mechanism is different. A worker does not care directly about the firm's type; she evaluates a submarket through its posted wage and job-finding probability. The relevant screening problem is therefore one-sided: firms screen worker types, firm-type observability is irrelevant for workers' choices, and no platform is needed to coordinate participation on both sides. Shimer and Wu study how bilateral screening separates types and can reshape their assignment, whereas we show that one-sided adverse selection changes the classic directed-search condition for PAM, making Becker's supermodularity sufficient.

The paper is organized as follows. Section 2 presents the model. Section 3 develops the main analysis. Section 3.1 shows that supermodular output delivers PAM. Section 3.2 derives the differential conditions satisfied by a PAM equilibrium, decomposes the local sorting condition into technological complementarity and a positive screening term, characterizes equilibrium as a boundary-value problem, and establishes a converse result: if output is submodular somewhere, there are endowments for which no differentiable equilibrium exhibits PAM. Section 3.3 establishes existence. Section 4 concludes. All proofs are in the appendix, except for the existence proof, which is in the online appendix.

2 Model

Agents. We consider a static model with a continuum of heterogeneous, risk-neutral workers and firms. Each firm demands and each worker supplies one unit of indivisible labor. Firms are indexed by a type $y \in \mathcal{Y} = [\underline{y}, \bar{y}] \subseteq \mathbb{R}_+$, with cumulative distribution function (cdf) $G_f(y)$ and total measure N ; workers are indexed by a type $x \in \mathcal{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{R}_+$, with cdf $G_\ell(x)$ and total measure normalized to one. Both $G_f(y)$ and $G_\ell(x)$ are twice continuously differentiable on their supports, with strictly positive densities g_f and g_ℓ .²

²The second derivative is needed only for the existence argument in Section 3.3. It makes the densities locally Lipschitz, so that standard results deliver a solution to the differential system.

Both worker and firm types are private information. Higher worker types have a higher value of remaining unmatched, generating the “lemons” problem. We normalize the payoff of an unmatched worker of type x to x ; one interpretation is that a worker’s outside option is self-employment with productivity equal to x . An unmatched firm of type y receives $R(y)$. As shown below, firm-type observability is irrelevant and $R(y)$ can be absorbed into the production function, so we normalize $R(y) = 0$.

A match between a worker of type x and a firm of type y produces gross output $F(x, y)$, assumed twice continuously differentiable and strictly increasing in both arguments. The trade surplus is therefore $f(x, y) \equiv F(x, y) - x$, which is assumed to be strictly positive for any $(x, y) \in \mathcal{X} \times \mathcal{Y}$.³ We say that F is supermodular if $F_{xy}(x, y) \geq 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$; equivalently, for any $x_H > x_L$ and $y_H > y_L$, $F(x_H, y_H) + F(x_L, y_L) \geq F(x_H, y_L) + F(x_L, y_H)$. Because subtracting x does not affect the cross-partial, F is supermodular if and only if f is supermodular.

Search. Each firm posts and commits to a wage that cannot condition on privately observed worker type. After observing all posted wages, each worker chooses a firm to visit. We refer to the set of firms posting a given wage, together with the workers who choose to visit them, as a *submarket*. Within each submarket, meetings are bilateral: each worker meets at most one firm, and each firm meets at most one worker.

Let n_f and n_ℓ denote the measures of firms and workers in a submarket, and let total meetings be given by a constant-returns-to-scale matching function $M(n_f, n_\ell)$. Define market tightness as $\theta \equiv n_f/n_\ell$. Then $m(\theta) \equiv M(\theta, 1)$ is the probability that a worker meets a firm, and $q(\theta) \equiv \frac{m(\theta)}{\theta}$ is the probability that a firm meets a worker. We assume that m is twice continuously differentiable, $m(0) = 0$, $\lim_{\theta \rightarrow \infty} m(\theta) = 1$, $m'(\theta) > 0$, and $m''(\theta) < 0$. Hence $q(\theta)$ is strictly decreasing.

Payoffs. Let $\Theta(w)$ denote market tightness in submarket w , and let $\Gamma_w(x)$ denote the cdf of worker types among its applicants. Both objects are endogenously determined in equilibrium and defined for all wages, including those not posted. Let $\mathcal{W}^*$ denote the set of wages posted in equilibrium.

³Under complete information only $f(x, y)$ enters the analysis, and the literature accordingly assumes that it is strictly increasing in x , i.e., $F_x > 1$. We do not need this strengthening: our arguments use only $F_x > 0$.

A type- x worker applying to a submarket with wage w and tightness θ obtains a *net* payoff of

$$V(w, \theta, x) = m(\theta)(w - x). \quad (1)$$

A worker of type x visits the submarket that maximizes her expected payoff,

$$V(x) = \max_{w \in \mathcal{W}^*} V(w, \Theta(w), x), \quad (2)$$

where $V(x)$ is the equilibrium net payoff of a type- x worker.

Similarly, a type- y firm posts a wage to maximize expected profit:

$$\pi(y) = \max_w q(\Theta(w)) \left(\int_{\underline{x}}^{\bar{x}} F(x, y) d\Gamma_w(x) - w \right), \quad (3)$$

where $\pi(y)$ is the equilibrium payoff and the integral is expected output conditional on matching. Note that the choice is over all wages, not only those in $\mathcal{W}^*$: Θ and Γ_w are defined off path as well, and it is against those beliefs that a deviation is evaluated.

Equilibrium definition. Let $H_y(w)$ denote the cdf of wages posted by firms of type y in equilibrium. The aggregate cdf of posted wages is $\mathcal{H}_f(w) = \int_{\underline{y}}^{\bar{y}} H_y(w) g_f(y) dy$, with support denoted by $\mathcal{W}^*$. Let $H_x(w)$ denote the probability that a worker of type x applies to a job offering a wage weakly below w . The probability that an average worker applies to a job with wage weakly below w is then $\mathcal{H}_\ell(w) = \int_{\underline{x}}^{\bar{x}} H_x(w) g_\ell(x) dx$.

We can now formally define a competitive search equilibrium.

Definition 1. A competitive search equilibrium is a tuple $(H_x(w), H_y(w), \Theta(w), \Gamma_w(x), \pi(y), V(x))$ satisfying the following conditions:

1. *Workers' optimal search.* For each worker type x , any wage w in the support of $H_x(\cdot)$ solves the worker problem in (2).
2. *Profit maximization.* For each firm type y , any wage w in the support of $H_y(\cdot)$ solves the firm problem in (3).
3. *Beliefs.* For any wage $w \in \mathcal{W}^*$, market tightness is given by $\Theta(w) = N d\mathcal{H}_f(w)/d\mathcal{H}_\ell(w)$. Thus, $\Theta(w)$ is the ratio of the measure of firms post-

ing wage w to the measure of workers applying to that wage.⁴ Applicant composition $\Gamma_w(x)$ is determined from worker application strategies by Bayes' rule: for every $x \in \mathcal{X}$ and every wage w ,

$$\int_{\underline{x}}^x H_{\tilde{x}}(w) g_{\ell}(\tilde{x}) d\tilde{x} = \int_0^w \Gamma_{\tilde{w}}(x) d\mathcal{H}_{\ell}(\tilde{w}). \quad (4)$$

Both sides equal the measure of workers with type weakly below x who apply to wages weakly below w : the left-hand side integrates over worker types, whereas the right-hand side integrates over submarkets.

For any off-path wage $w \notin \mathcal{W}^*$, beliefs $(\Theta(w), \Gamma_w)$ satisfy

$$V(x) \geq m(\Theta(w))(w - x) \quad \text{for all } x,$$

with equality for every x in the support of $\Gamma_w(x)$.

Our equilibrium definition is standard except for its restriction on off-path beliefs, which we discipline in the spirit of subgame perfection, following [Guerrieri et al. \(2010\)](#) and [Chang \(2018\)](#). Because meetings are bilateral, a worker's payoff from applying to wage w depends only on $\Theta(w)$, not on applicant composition (see (2)). For each type x , there is a tightness threshold above which applying to w yields more than the equilibrium payoff.⁵ Subgame perfection selects the lowest such threshold: it sets $m(\Theta(w)) = \min_{x < w} V(x)/(w - x)$ and assigns all applicant mass to the worker types that attain the minimum. If $\Theta(w)$ were higher, some worker types would strictly prefer to deviate to w ; if it were lower, no worker would apply, implying $\Theta(w) = \infty$, a contradiction.

Although the definition above allows for mixed strategies and nondegenerate applicant composition, the equilibrium characterized below is much simpler: each active submarket attracts a single worker type, and higher types apply to strictly higher wages. The applicant distribution Γ_w is therefore degenerate, so the probability-measure formulation can be replaced by a deterministic assignment schedule. Section 3 establishes this and works with that schedule throughout.

⁴Formally, $\Theta(w)$ is the Radon–Nikodym derivative of $N\mathcal{H}_f$ with respect to $\mathcal{H}_{\ell}$. When both distributions have densities, it is simply their ratio.

⁵If $V(x) \geq w - x$, then type x obtains a payoff weakly below $V(x)$ from applying, regardless of $\Theta(w)$, and we may set the corresponding threshold equal to $+\infty$.

On the observability of firm types. A worker evaluating wage w cares only about the submarket tightness $\Theta(w)$, not about the type of the posting firm, so whether firm types are private or public is irrelevant for application decisions. A firm's payoff in turn depends on its own type, on $\Theta(w)$, and on the worker types that self-select into the submarket, but not on the types of the other firms posting there, so observability does not affect wage posting either. Hence equilibrium is the same whether firm types are observable or not.

Normalization of the production function. This irrelevance allows us to normalize firm outside options to zero. A firm of type y receives $F(x, y) - w$ when matched and $R(y)$ when unmatched. Its decisions therefore depend only on the gain from matching, $F(x, y) - w - R(y)$. We can thus subtract $R(y)$ from the firm's payoff in both states without changing its choices. Formally, define $\tilde{F}(x, y) \equiv F(x, y) - R(y)$. Under this normalization, an unmatched firm receives zero and a matched firm receives $\tilde{F}(x, y) - w$. Worker payoffs and the equilibrium allocation are unchanged. We therefore set $R(y) \equiv 0$ throughout and interpret F as output net of the firm's outside option.⁶

Worker outside options cannot be normalized away in the same fashion under private information. A type- x worker who applies to a submarket offering wage w and tightness θ obtains a net payoff of $m(\theta)(w - x)$. Thus, the outside option x affects not only the worker's net payoff level, but also how the worker trades off a higher wage against a higher probability of matching (at a given wage, the utility gain is larger for low-type workers, so a high matching probability matters more to them). Since firms cannot condition posted wages on worker type, they must combine wages with different matching probabilities to induce different types to select different submarkets. The outside option x is therefore part of the incentive constraints and cannot simply be subtracted from production.

Under complete information, firms can instead offer type-specific wages. Suppose a firm has already decided to target workers of type x . The worker receives w when matched and x when unmatched. We can subtract x from the worker's payoff in both states without changing the worker's choices. Thus only the net match surplus $f(x, y) = F(x, y) - x$ matters. The separate role of x in our model therefore comes

⁶Subtracting $R(y)$ preserves supermodularity because it depends only on y and therefore does not change the cross-partial with respect to x and y .

from private information and the restriction that posted wages cannot depend on worker type.

Positive assortative matching (PAM). Suppose that all workers of a given type match with firms of a single type, as we show below holds in equilibrium. Denote the assignment by $y = \psi(x)$. The equilibrium exhibits positive assortative matching (PAM) if ψ is strictly increasing; that is, higher-type firms match with higher-type workers.

3 The equilibrium analysis

The analysis proceeds in three subsections. Section 3.1 shows that supermodular production implies PAM. Section 3.2 takes PAM as given, derives the differential conditions that any such equilibrium must satisfy, and shows that these conditions, together with the appropriate boundary conditions, characterize equilibrium. Section 3.3 shows that a solution exists.

3.1 The PAM result

We establish positive assortative matching in three steps. We first characterize worker sorting across submarkets, then use firm optimality to rule out pooling, and finally show that the resulting firm payoff is strictly supermodular.

Worker optimality. We first show that workers' preferences satisfy a single-crossing condition: a contract with a higher wage and a lower worker job-finding probability is relatively more attractive to a higher worker type. A lower θ means a more congested queue for workers and a higher vacancy-filling probability for firms, and it is this congestion margin that screens worker types.

Lemma 1. *Consider two submarkets (w^a, θ^a) and (w^b, θ^b) with $\theta^a < \theta^b$ (and consequently $w^a > w^b$). If type x is indifferent between them, then every type $x' > x$ strictly prefers submarket a , whereas every type $x' < x$ strictly prefers submarket b .*

Proof. See Appendix A.1. □

Next, assume that an equilibrium exists and turn to the workers' incentive compatibility constraint (ICC). Suppose that in equilibrium workers of type x visit a submarket with wage $w(x)$ and tightness $\theta(x)$. Since low-type workers can always mimic high-type workers, the equilibrium net payoff function $V(x)$ is strictly decreasing. Furthermore, as in standard screening and mechanism-design models, ICC implies that a worker's expected payoff can be written as an integral of the matching probability, as the next lemma records.

Lemma 2. *Workers' ICC is satisfied if and only if $\theta(x)$ is decreasing and their equilibrium payoff function is given by*

$$V(x) = V(\underline{x}) - \int_{\underline{x}}^x m(\theta(\tilde{x})) d\tilde{x}. \quad (5)$$

Proof. See Appendix A.2. □

The wage schedule then follows from (1): since $V(x) = m(\theta(x))(w(x) - x)$,

$$w(x) = x + \frac{V(x)}{m(\theta(x))}. \quad (6)$$

Firm optimality. We now rule out pooling at a posted wage, starting from the weak monotonicity established above. Suppose that $\theta(x)$, and hence $w(x)$, were constant over an interval, so that a whole interval of worker types applied to the same wage. A firm could then raise its wage slightly and draw only workers from the upper end of that pool: the gain in applicant quality is of first order while the extra wage is arbitrarily small, and the vacancy-filling probability is weakly higher. Such a deviation is strictly profitable, so pooling cannot survive and tightness must fall strictly.

Lemma 3. *The equilibrium $\theta(x)$ is strictly decreasing.*

Proof. See Appendix A.3. □

We can thus index each active submarket by the worker type it attracts.

Positive assortative matching. Fix any worker-incentive-compatible separating schedule $(w(x), \theta(x))$. The payoff to a firm of type y from targeting worker type x is

$$\begin{aligned}\Omega(x, y) &= q(\theta(x)) [F(x, y) - w(x)] \\ &= q(\theta(x)) f(x, y) - \frac{V(x)}{\theta(x)},\end{aligned}\tag{7}$$

where, in the second equality, we have substituted $w(x)$ out using equation (6) and used $q(\theta) = m(\theta)/\theta$. This expression has a simple surplus interpretation: the first term is the expected trade surplus generated per firm and the second is the workers' expected gain per firm, so firm profit is expected trade surplus net of the utility delivered to workers.

Differentiating $\Omega(x, y)$ in equation (7) first with respect to y and then with respect to x gives

$$\Omega_{xy}(x, y) = \frac{\partial}{\partial x} [q(\theta(x)) F_y(x, y)] = q(\theta(x)) F_{xy}(x, y) + q'(\theta(x)) \theta'(x) F_y(x, y),\tag{8}$$

where we used the fact $f_y(x, y) = F_y(x, y)$.⁷ The first term on the right-hand side is the standard technological complementarity between worker and firm types. The second term captures the variation in the vacancy-filling probability across submarkets and is strictly positive because $\theta'(x) < 0$, $q'(\theta) < 0$, and $F_y(x, y) > 0$. Hence, when F is supermodular, $\Omega(x, y)$ is strictly supermodular.

Proposition 1. *Suppose $F(x, y)$ is supermodular. Then $\Omega(x, y)$ is strictly supermodular; hence the equilibrium exhibits PAM.*

Proof. See Appendix A.4. □

Equation (8) identifies two distinct sorting forces, both of which make submarkets with higher worker types relatively more attractive to higher-type firms. Technological complementarity makes the output gain from a higher firm type larger at a higher worker type. In addition, worker incentive compatibility separates higher worker types into submarkets with lower job-finding probabilities, which deliver higher vacancy-filling probabilities to the firms assigned there. The information friction therefore reinforces, rather than opposes, production complementarity.

⁷Here we assume that $\theta(x)$ is differentiable; being strictly decreasing, it is differentiable almost everywhere. The proof of Proposition 1 below uses a discrete-version argument and hence does not rely on differentiability of $\theta(x)$.

3.2 Equilibrium characterization

Assuming that the equilibrium exhibits PAM, we characterize it by a system of differential equations together with boundary conditions, and show conversely that under supermodularity of F any allocation and values satisfying them form an equilibrium. The boundary conditions depend on which types are active.

Active types. We allow some types to be inactive. Because lower-type workers can mimic higher-type workers (and gain more from any given submarket), active workers form a lower interval $[\underline{x}, x_A]$, with $x_A = \bar{x}$ when all workers are active. Because output is strictly increasing in y , if firms stay out, they must be the lowest types: active firms are bounded below by a threshold y_A and form an interval $[y_A, \bar{y}]$. Since $f(x, y) > 0$ for every (x, y) , workers and firms cannot be idle simultaneously, i.e., at most one of x_A and y_A is interior. The following lemma records these observations.

Lemma 4. *In equilibrium, if some workers are idle, i.e., $x_A < \bar{x}$, then*

$$V(x_A) = 0, \quad \theta(x_A) = 0, \quad w(x_A) = x_A, \quad y_A = \underline{y}. \quad (9)$$

If some firms are idle, i.e., $y_A > \underline{y}$, then

$$\pi(y_A) = 0, \quad \theta(\underline{x}) = \infty, \quad w(\underline{x}) = F(\underline{x}, y_A), \quad x_A = \bar{x}. \quad (10)$$

Proof. See Appendix A.5. □

If some workers are inactive, workers are the long side of the market. The firm–worker ratio therefore falls to zero at the highest active worker type. To see why it cannot remain strictly positive, suppose instead that $\theta(x_A) > 0$. The top firm could then offer a wage slightly above x_A . Inactive workers just above x_A , who obtain zero equilibrium payoff, would apply to this wage, creating a submarket with tightness zero. The firm’s matching probability would rise from $q(\theta(x_A))$ to $q(0)$, and because $f(x_A, \bar{y}) > 0$, this gain would outweigh the arbitrarily small wage increase. The deviation would be profitable, so necessarily $\theta(x_A) = 0$. If instead some firms are inactive, the firm–worker ratio therefore becomes arbitrarily large at the lowest active worker type, so $\theta(\underline{x}) = \infty$.

Feasibility. Workers with types in $[x, x + dx]$ have measure $g_\ell(x) dx$. Their assigned firms have types in $[\psi(x), \psi(x) + \psi'(x)dx]$ and have measure $Ng_f(\psi(x))\psi'(x) dx$. Since $\theta(x)$ is the ratio of firms to workers in the submarket, local feasibility requires $Ng_f(\psi(x))\psi'(x) = \theta(x)g_\ell(x)$, that is,

$$\psi'(x) = \frac{\theta(x)g_\ell(x)}{Ng_f(\psi(x))}. \quad (11)$$

This condition determines how rapidly the assigned firm type rises with worker type.

Firms' first-order conditions. Because each active submarket is indexed by the worker type it attracts, a firm's choice of submarket is a choice of worker type. Recall from (7) that the expected profit of a type- y firm targeting workers of type x is $\Omega(x, y)$, in which x enters both directly through output and indirectly through the tightness and wage of the submarket serving type x . Differentiating with respect to x at fixed y yields

$$\begin{aligned} \Omega_x(x, y) &= q'(\theta(x))\theta'(x)f(x, y) + q(\theta(x))[F_x(x, y) - 1] - \frac{V'(x)}{\theta(x)} + \frac{V(x)\theta'(x)}{\theta(x)^2} \\ &= \frac{\theta'(x)}{\theta(x)} \left[\theta(x)q'(\theta(x))f(x, y) + \frac{V(x)}{\theta(x)} \right] + q(\theta(x))F_x(x, y). \end{aligned} \quad (12)$$

The second equality uses the envelope condition $V'(x) = -m(\theta(x))$ together with $m(\theta)/\theta = q(\theta)$.

Evaluating (12) along the equilibrium assignment and writing $D(x)$ for the bracketed term,

$$D(x) \equiv \theta(x)q'(\theta(x))f(x, \psi(x)) + \frac{V(x)}{\theta(x)}. \quad (13)$$

The firm's first-order condition $\Omega_x(x, \psi(x)) = 0$ can therefore be written as

$$\theta'(x)D(x) + m(\theta(x))F_x(x, \psi(x)) = 0.$$

Worker incentive compatibility implies $\theta'(x) \leq 0$. At every interior active worker type x , the above equation rules out $\theta'(x) = 0$, because $F_x > 0$. Thus $\theta'(x) < 0$, which

implies $D(x) > 0$ for every $x \in (x, x_A)$. Solving it for the slope of tightness gives

$$\theta'(x) = -\frac{m(\theta(x)) F_x(x, \psi(x))}{D(x)}. \quad (14)$$

Equation (14) shows how the direct productivity gain and the marginal congestion term jointly determine the slope of the separating tightness schedule.

No distortion at the bottom and strict distortion above it. The term $D(x)$ has a pointwise-optimality interpretation. Fix a worker–firm pair (x, y) , hold the worker’s promised utility $V(x)$ fixed, and let the firm vary tightness while adjusting the wage to preserve that utility. Its profit is

$$\Pi(\theta, x, y) = q(\theta)f(x, y) - \frac{V(x)}{\theta}, \quad (15)$$

which, by the definition of D in equation (13), implies that

$$D(x) = \theta(x)\Pi_\theta(\theta(x), x, \psi(x)), \quad (16)$$

where Π_θ is the partial derivative with respect to θ . Moreover, Π is strictly single-peaked in θ : the sign of Π_θ is that of $V(x) - [m(\theta) - \theta m'(\theta)]f(x, y)$, and $m(\theta) - \theta m'(\theta)$ is strictly increasing, its derivative being $-\theta m''(\theta) > 0$.

Hence $D(x)$ measures how far equilibrium tightness sits from the level that maximizes profit in the submarket serving type x . Under complete information, firms can choose tightness and worker types separately, so the pointwise optimum satisfies $D(x) = 0$. Since Π is single-peaked in θ , $D(x) > 0$ means that equilibrium tightness lies strictly below its pointwise optimum, and $D(x) = 0$ that the submarket is undistorted.⁸

⁸This pointwise comparison does not establish constrained efficiency. Following [Guerrieri et al. \(2010\)](#), the relevant comparison is with feasible allocations that preserve bilateral matching, self-selection, submarket-level break-even, and market clearing, while allowing menu restrictions. In their homogeneous-principal case, pooling can Pareto dominate separation when low-type workers are sufficiently scarce, because the gain from reducing the high-type distortion exceeds the required cross-subsidy. [Davoodalhosseini \(2019\)](#) studies a broader planner that permits budget-balanced cross-subsidies across submarkets to relax incentive constraints and raise welfare. With heterogeneous firms, the planner must also choose assignment. Under separation, worker incentive compatibility still implies that tightness falls with type, so Proposition 1 yields PAM under supermodular production, whereas pooling may induce bunching. We leave the full welfare analysis for future work.

The next lemma states that the bottom submarket is undistorted, while every active submarket above it has tightness strictly below its pointwise optimum.

Lemma 5. *In equilibrium, $D(\underline{x}) = 0$, whereas $D(x) > 0$ for every $x > \underline{x}$ with $\theta(x) > 0$.*

Proof. See Appendix A.6. □

Reduction to two schedules. By Lemma 2, the function $\theta(x)$ determines V up to the constant $V(\underline{x})$, which is pinned down by $D(\underline{x}) = 0$, or

$$V(\underline{x}) = -\theta_0^2 q'(\theta_0) f(\underline{x}, y_A) = [m(\theta_0) - \theta_0 m'(\theta_0)] f(\underline{x}, y_A), \quad \theta_0 \equiv \theta(\underline{x}). \quad (17)$$

Because the bottom submarket is undistorted, the bottom worker captures the fraction $m(\theta_0) - \theta_0 m'(\theta_0) \in (0, 1)$ of the surplus of the bottom match, consistent with the Hosios condition. Given V , the wage schedule follows from (6) and firm profits from $\pi(\psi(x)) = \Omega(x, \psi(x))$, so the pair (ψ, θ) determines every remaining equilibrium object.

Local sorting condition. The firm's first-order condition provides a direct way to characterize local sorting. Differentiating $\Omega_x(x, \psi(x)) = 0$ with respect to x yields

$$\Omega_{xx}(x, \psi(x)) + \psi'(x) \Omega_{xy}(x, \psi(x)) = 0. \quad (18)$$

At an interior optimum, $\Omega_{xx}(x, \psi(x)) < 0$ by the second-order condition, so $\psi'(x)$ has the same sign as $\Omega_{xy}(x, \psi(x))$.

Lemma 6. *At any interior point x , $\psi'(x) > 0$ if and only if*

$$\left. \frac{\partial}{\partial x} [q(\theta(x)) F_y(x, y)] \right|_{y=\psi(x)} > 0. \quad (19)$$

Proof. The result follows from the first equality in equation (8). □

To compare with the sorting condition under complete information derived by [Eeckhout and Kircher \(2010\)](#), rewrite (19) by taking logs and dividing by the positive

term $\partial_x \log F(x, y)$:

$$\left. \frac{\partial \log F_y(x, y)}{\partial \log F(x, y)} \right|_{y=\psi(x)} + \left. \frac{\partial \log q(\theta(x))}{\partial \log F(x, y)} \right|_{y=\psi(x)} > 0. \quad (20)$$

The first term equals $F(x, y)F_{xy}(x, y)/[F_x(x, y)F_y(x, y)]$, which is the elasticity of complementarity of the production function. The second term is the screening term. It is strictly positive because $\theta(x)$ is strictly decreasing and $q(\theta)$ is strictly decreasing in θ , so $q(\theta(x))$ is strictly increasing in x . Higher worker types therefore select submarkets with lower tightness, which raises the vacancy-filling probability and thereby makes those submarkets relatively more attractive to higher firm types. Thus, the screening term reinforces technological complementarity. Under complete information, by contrast, the corresponding matching-probability term works against positive sorting. [Eeckhout and Kircher \(2010\)](#) therefore require technological complementarity to be strong enough to offset this opposing force, yielding their root-supermodularity condition.⁹

Increasing trade surplus and worker participation. We now show that if trade surplus $f(x, y) = F(x, y) - x$ is weakly increasing in the worker's type (a condition standard in complete-information models), then all worker types are active in equilibrium. This observation will be useful in Proposition 2, where we construct a counterexample to PAM without relying on worker exit.

Lemma 7. *Suppose that $F_x(x, y) \geq 1$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Then all worker types are active in equilibrium: $x_A = \bar{x}$.*

Proof. See Appendix A.7. □

Suppose instead that the active worker support ended at some $x_A < \bar{x}$. As x approaches x_A , tightness approaches zero and the wage approaches the worker's outside option x_A . A firm targeting workers near x_A can therefore obtain a vacancy-filling probability arbitrarily close to its maximum $q(0)$ while delivering almost no rent to

⁹[Eeckhout and Kircher \(2010\)](#) state their condition in terms of trade surplus f rather than gross output F . Dividing by $\partial_x \log f(x, y)$ instead of $\partial_x \log F(x, y)$ gives their formulation. This requires the additional assumption $f_x > 0$; the numerators are unchanged because $F_y = f_y$, $F_{xy} = f_{xy}$, and the q -term is unaffected.

the worker. Furthermore, moving from a lower worker type toward x_A weakly increases trade surplus. A firm currently serving a lower worker type would therefore prefer to target workers near x_A , a contradiction.

Necessity of supermodularity for PAM. We now show that if production is submodular somewhere, endowments can be chosen so that the screening term is too small to overturn the technological force, and no differentiable equilibrium exhibits PAM.¹⁰ The assumption $F_x \geq 1$ serves only to ensure that all worker types are active in equilibrium.

Proposition 2. *Suppose that $F_x(x, y) \geq 1$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. If $F(x, y)$ is not supermodular, i.e., $F_{xy}(x_0, y_0) < 0$ at some point (x_0, y_0) , then we can find an endowment of workers and firms such that no differentiable equilibrium exhibits PAM.*

Proof. See Appendix A.8. □

The idea is to make the screening term arbitrarily weak by taking the measure of firms sufficiently small. By (20), this term operates through differences in vacancy-filling probabilities across submarkets. When firms are scarce enough, tightness in the two submarkets being compared is close to zero, so both vacancy-filling probabilities are close to $q(0) = m'(0)$ and their ratio is close to one. Screening is then too weak to offset strict submodularity.

Sufficiency of the firms' FOC. The conditions derived above are necessary. We now show that under supermodularity, they are also sufficient, so equilibrium is fully characterized by a boundary-value problem for the two schedules ψ and θ .

Proposition 3. *Suppose that $F(x, y)$ is supermodular. Then the schedules ψ and θ constitute an equilibrium if and only if they satisfy (11) and (14), where V is given by (5) with initial value (17), together with the applicable boundary conditions in (9) and (10).*

Proof. See Appendix A.9. □

¹⁰A *differentiable equilibrium* is one whose assignment and tightness schedules are continuous on the closures of the active type sets and continuously differentiable in their interiors; tightness at an endpoint may be zero or infinite. Proposition 4 below shows that one exists under supermodularity.

Note that there is no additional worker-side condition to verify: by Lemma 2, a decreasing tightness schedule together with the worker envelope condition is necessary and sufficient for worker incentive compatibility. It remains only to verify that firms have no profitable deviations. Only sufficiency requires further argument. The firms' first-order condition rules out local deviations, but equilibrium must also rule out distant ones. A firm may first target another on-path submarket. By Proposition 1, the targeting payoff Ω is strictly supermodular, so the first-order condition identifies the firm's global optimum.

Alternatively, a firm may post a wage outside the equilibrium wage range. Such a wage can attract only the lowest or highest active worker type, and at either endpoint tightness can move in only one direction. Lemma 5 gives $D(\underline{x}) = 0$ and $D > 0$ at any positive-tightness upper endpoint, which then rule out profitable movements in the feasible direction.

3.3 Equilibrium existence

Proposition 3 characterizes equilibrium. We now show that the differential system has a solution.

When all types are active, a candidate solution is determined by the bottom tightness $\theta_0 = \theta(\underline{x})$. This fixes $V(\underline{x})$ through (17). Starting from $\psi(\underline{x}) = \underline{y}$, we can then solve (11)–(14) upward. The correct value of θ_0 is the one for which workers and firms are exhausted at the same time: $\psi(\bar{x}) = \bar{y}$.

Equilibria need not have full participation. If some workers are inactive, the solution ends at an interior cutoff $x_A < \bar{x}$, where $\theta(x_A) = 0$. If some firms are inactive, the bottom match instead involves an interior firm cutoff $y_A > \underline{y}$, where $\theta(\underline{x}) = \infty$. The participation cutoffs must therefore be determined together with the solution, making existence a free-boundary problem.¹¹

Proposition 4. *Suppose that $F(x, y)$ is supermodular. Then there exists a differentiable PAM equilibrium.*

¹¹Under stronger restrictions, uniqueness can also be established. For example, suppose that $F(x, y) = ax + v(y)$, where $a \geq 1$ and v is concave, and that worker and firm types are uniformly distributed. In this case, it can be verified that the tightness paths generated by the shooting argument vary monotonically with the initial condition, which yields uniqueness. Beyond such special cases, however, establishing uniqueness appears considerably more difficult. In the complete-information model, Eeckhout and Kircher (2010) similarly prove equilibrium existence by analyzing the associated differential system, but do not pursue a general uniqueness result.

Proof. See Online Appendix B. □

The online appendix proves the proposition by a one-dimensional shooting argument. To see the basic idea, suppose first that all workers and firms are active. The path then starts from the bottom match $(\underline{x}, \underline{y})$, and the only initial value to guess is $\theta_0 \equiv \theta(\underline{x})$. Once θ_0 is chosen, the condition $D(\underline{x}) = 0$ fixes the remaining initial values, and the differential equations generate one path (ψ, θ) .

If θ_0 is too low, the path uses firm mass too slowly. It reaches $x = \bar{x}$ while $\psi(x) < \bar{y}$, so all worker types have been used but some firm types remain. If θ_0 is too high, the path uses firm mass too quickly. It reaches $\psi(x) = \bar{y}$ at some $x < \bar{x}$, so all firm types have been used while some worker types remain. The correct value lies between these two misses and makes the path reach $(\bar{x}, \bar{y})$ exactly. Under PAM, this is the required endpoint when all types participate: the highest worker type must be matched with the highest firm type.¹² Because the path varies continuously with θ_0 , it cannot move from one type of miss to the other without passing through such a value. This is the shooting argument illustrated in Figure 1.

This simple argument assumes that varying a finite θ_0 produces both kinds of miss. That need not be true. Even when θ_0 becomes arbitrarily large, the path may still reach $\bar{x}$ with some firms left over. In that case, we must start from an interior cutoff $y_A > \underline{y}$, leaving lower firm types inactive. Alternatively, every path that reaches $\bar{y}$ may do so while some workers are still left. The balancing path may then reach $\theta = 0$ at an interior cutoff $x_A < \bar{x}$, leaving higher worker types inactive. The online appendix allows for both participation boundaries and connects these cases to the full-participation case in one continuous family.¹³

¹²One may wonder why we do not instead shoot from the top. Lemma 5 shows $D(\underline{x}) = 0$, which supplies the missing equation linking θ_0 and $V(\underline{x})$ at the bottom, whereas $D(\bar{x}) > 0$ strictly, so no analogous relationship pins down $\theta(\bar{x})$ and $V(\bar{x})$ at the top; one would therefore have to guess both.

¹³Making this rigorous is more involved than the sketch above suggests. When the path starts from an interior firm cutoff, tightness at the bottom is infinite, so the starting point is described only in a limit rather than by an ordinary initial condition. Existence, uniqueness, and continuity in the cutoff must then be established separately, through a contraction-mapping argument rather than the usual Picard–Lindelöf theorem. Every path must also be shown not to explode before reaching a participation boundary. Only once this machinery is in place does the final shooting step, bracketing the two kinds of miss described above, deliver existence.

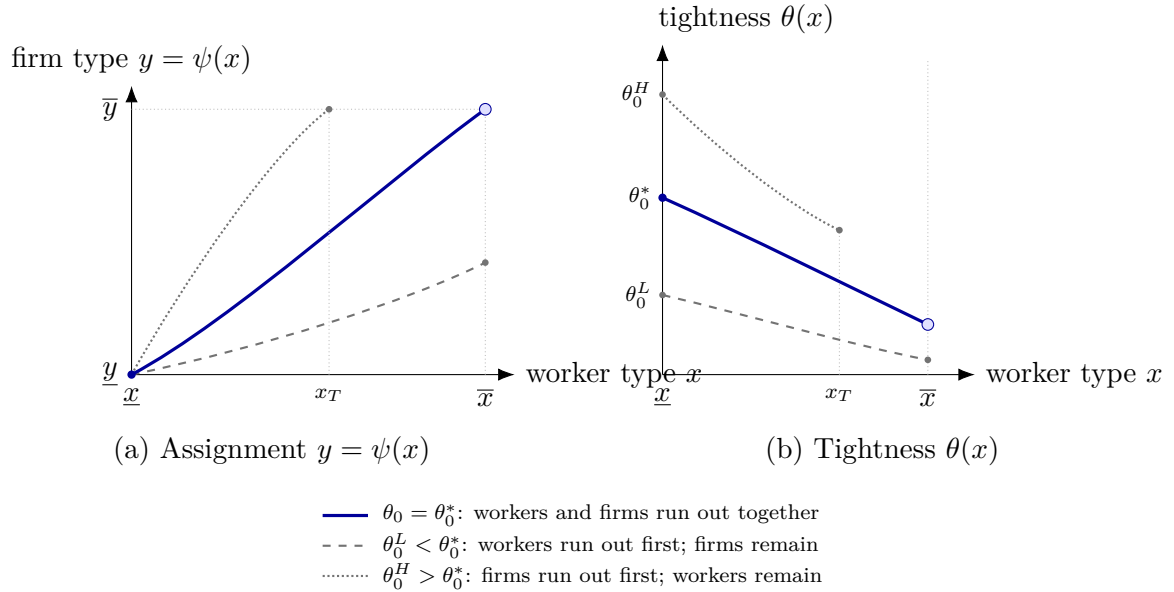


Figure 1: Shooting in the full-participation case. All three shots start from $(\underline{x}, y)$ and differ only in the initial tightness θ_0 . A low θ_0 uses firm mass too slowly and reaches $\bar{x}$ before $\bar{y}$; a high θ_0 uses firm mass too quickly and reaches $\bar{y}$ before $\bar{x}$. The correct value θ_0^* makes the assignment reach $(\bar{x}, \bar{y})$. Panel (b) shows the corresponding decreasing tightness schedules.

4 Final remarks

Private information changes the sorting force generated by competitive search. Worker incentive compatibility requires higher-productivity workers to enter submarkets with lower job-finding probabilities and therefore higher vacancy-filling probabilities and higher wages. Under supermodular production, this screening-induced ranking reinforces technological complementarity and gives firms strict increasing differences in their targets. Positive assortative matching follows.

The wage ranking above is easy to take to the data; the unemployment ranking is not. Because incentive compatibility pushes high-productivity workers into lower-tightness submarkets, they are the ones who take longer to match, whereas in the data unemployment is concentrated among low-productivity workers. Reconciling this ranking is a natural next step, and three extensions seem promising. Endogenous labor supply would work through a participation margin, independent of the screening wedge studied here, if low-productivity workers are also the first to exit the labor force when job finding is slow. Partial observability of skill, with education the

natural proxy, would let firms condition directly on the observable component and confine screening to the residual, weakening or reversing the tightness ordering established above. Most simply, low-productivity matches may be less stable: steady-state unemployment is governed jointly by job finding and job destruction, so it need not inherit the ranking of job-finding probabilities at all. We leave these extensions for future work.¹⁴

Another interesting extension would be to study optimal taxation in our environment. [Rothschild and Scheuer \(2013\)](#) study a Roy model of sectoral self-selection and show that, when sectoral inputs are complements, the resulting endogenous wage distribution makes optimal income taxes less progressive than in a standard single-sector model. Our environment differs: workers are screened not by a planner’s tax schedule but by firms competing over wages and job-finding probabilities, and the resulting self-selection across submarkets is sustained jointly by both margins. We conjecture that a similar conclusion follows for a different reason. Because a planner restricted to taxing wages cannot directly control the job-finding-probability margin, progressive taxation compresses the net-wage gap available for separating worker types. Firms may then need to offer larger gross-wage differences to sustain a given difference in job-finding probabilities, making separation more costly.

Appendix

A.1 Proof of Lemma 1

Indifference of type x means $m(\theta^a)(w^a - x) = m(\theta^b)(w^b - x)$. For any x' , subtracting the payoff in submarket b from the payoff in submarket a and using the indifference condition gives $m(\theta^a)(w^a - x') - m(\theta^b)(w^b - x') = [m(\theta^b) - m(\theta^a)](x' - x)$. The expression is positive for $x' > x$ and negative for $x' < x$ because m is strictly increasing in θ . $\square$

A.2 Proof of Lemma 2

The argument is the standard envelope-monotonicity characterization of one-dimensional screening problems with quasi-linear payoffs; see, for example, [Krishna](#)

¹⁴Note that [Heathcote et al. \(2010, 2023\)](#) show that wage dispersion, not employment dispersion, is the dominant source of U.S. earnings inequality.

(2010, Section 5.1.2).

Consider first necessity. By (1), a type x who visits the submarket intended for type r obtains $m(\theta(r))(w(r) - x) = V(r) + m(\theta(r))(r - x)$, so incentive compatibility is equivalent to

$$V(x) = \max_{r \in \mathcal{X}} \{m(\theta(r))w(r) - m(\theta(r))x\}. \quad (21)$$

Each element in braces is affine in x with slope $-m(\theta(r)) < 0$, so V is a pointwise maximum of affine functions and is therefore convex and strictly decreasing. The maximum is attained at $r = x$, hence $V(r) \geq V(x) - m(\theta(x))(r - x)$ for all x, r ; that is, $-m(\theta(x))$ is the slope of a line supporting V at x . Since the supporting slopes of a convex function are increasing (becoming less negative), $m(\theta(x))$ is thus decreasing and hence $\theta(x)$ is decreasing. Being convex and Lipschitz (as $0 < m \leq 1$), V is absolutely continuous and thus the integral of its derivative, and $V'(x) = -m(\theta(x))$ wherever V is differentiable. Integrating from $\underline{x}$ delivers (5).

Next, consider sufficiency. Suppose θ is decreasing and V satisfies (5). For any types x and r , $V(x) - [V(r) + m(\theta(r))(r - x)] = \int_r^x [m(\theta(r)) - m(\theta(s))] ds \geq 0$, because $m(\theta(x))$ is decreasing: the integrand is nonnegative when $x > r$ and nonpositive when $x < r$, in which case the orientation of the integral reverses the sign. Thus no type gains by visiting the submarket intended for another type, and the schedule is globally incentive compatible. $\square$

A.3 Proof of Lemma 3

By Lemma 2, $\theta(x)$ is weakly decreasing. Suppose that $\theta(x)$ is not strictly decreasing. Then there exists an interval $I = [a, b]$ of types on which $\theta(x)$ is constant. Let $\theta_0 = \theta(x)$ and $w_0 = w(x)$ for $x \in [a, b]$. Furthermore, let the interval be maximal, i.e., $\theta(x) > \theta_0$ for $x < a$ and $\theta(x) < \theta_0$ for $x > b$. The expected profit of a firm y posting w_0 is thus $\pi_I(y) = q(\theta_0)[\bar{F}_I(y) - w_0]$, where $\bar{F}_I(y) \equiv (G_\ell(b) - G_\ell(a))^{-1} \int_a^b F(x, y) dG_\ell(x)$, the average output of a firm with type y .

Now consider a deviation to the slightly higher wage $w_\varepsilon = w_0 + \varepsilon$, where $0 < \varepsilon < F(b, y) - \bar{F}_I(y)$. Let $\theta_\varepsilon \equiv \Theta(w_\varepsilon)$. Since $w_\varepsilon > w_0$, we have $\theta_\varepsilon < \theta_0$. The deviation has lower tightness than the pooled submarket, and type b weakly prefers the pooled submarket. Lemma 1 therefore implies that every type $x < b$ strictly prefers the pooled submarket to the deviation. Hence, no type below b applies to the deviating submarket. Every applicant to w_ε therefore has type at least b , which implies that the

deviating firm's expected output conditional on filling its vacancy is at least $F(b, y)$. Its deviation payoff is therefore bounded below by $q(\theta_\varepsilon)[F(b, y) - w_0 - \varepsilon]$. Since

$$q(\theta_\varepsilon)[F(b, y) - w_0 - \varepsilon] \geq q(\theta_0)[F(b, y) - w_0 - \varepsilon] > q(\theta_0)[\bar{F}_I(y) - w_0].$$

Thus, posting w_ε is a strictly profitable deviation, contradicting firm optimality. Therefore, $\theta(x)$ cannot be constant on any nondegenerate interval; it must be strictly decreasing. $\square$

A.4 Proof of Proposition 1

Take $x_H > x_L$. We have $q_H \equiv q(\theta(x_H)) > q_L \equiv q(\theta(x_L))$, since $q(\theta)$ is strictly decreasing and, by Lemma 3, $\theta(x_H) < \theta(x_L)$. Since $f_y(x, y) = F_y(x, y)$, $\Omega_y(x, y) = q(\theta(x))F_y(x, y)$ by equation (7). Thus, $\Omega_y(x_H, y) = q_H F_y(x_H, y) > q_L F_y(x_H, y) \geq q_L F_y(x_L, y) = \Omega_y(x_L, y)$. Hence, Ω is strictly supermodular, which implies that the equilibrium must exhibit PAM. $\square$

A.5 Proof of Lemma 4

Suppose first that $x_A < \bar{x}$. Continuity implies $V(x_A) = 0$. Moreover, worker incentive compatibility gives, for every active $x < x_A$, $0 = V(x_A) \geq m(\theta(x))(w(x) - x_A)$, so every equilibrium wage is at most x_A . Let $\theta_A = \lim_{x \uparrow x_A} \theta(x)$. If $\theta_A > 0$, then $w(x_A) = x_A$, and the top firm earns $q(\theta_A)f(x_A, \bar{y})$. Instead, it can post the off-path wage $x_A + \varepsilon$. Since all types in $[x_A, x_A + \varepsilon)$ have zero equilibrium payoff, the belief refinement assigns this wage tightness zero. The deviation therefore yields at least $q(0)[f(x_A, \bar{y}) - \varepsilon]$. Because $q(0) > q(\theta_A)$ and $f(x_A, \bar{y}) > 0$, this is profitable for ε small, a contradiction. Hence $\theta(x_A) = 0$. If also $y_A > \underline{y}$, an inactive firm $y < y_A$ can post the same wage $x_A + \varepsilon$ and earn at least $q(0)[f(x_A, y) - \varepsilon] > 0$ for ε small, again a contradiction. Thus $y_A = \underline{y}$.

Now suppose that $y_A > \underline{y}$. A slightly lower firm can copy the marginal firm's wage, so continuity implies $\pi(y_A) = 0$. The implication established above, $x_A < \bar{x} \Rightarrow y_A = \underline{y}$, then yields $x_A = \bar{x}$ by contraposition. It remains to show that $\theta(\underline{x}) = \infty$. Suppose instead that $\theta_0 \equiv \theta(\underline{x}) < \infty$, and let w_0 be the wage posted by the marginal firm y_A . Since $\pi(y_A) = 0$, $w_0 = F(\underline{x}, y_A)$, while the bottom worker obtains $V(\underline{x}) = m(\theta_0)f(\underline{x}, y_A)$. Because $\theta_0 < \infty$, $m(\theta_0) < 1$. Hence, for $\varepsilon > 0$ small, the wage

$\hat{w} = w_0 - \varepsilon$ still gives the bottom worker more than $V(\underline{x})$ at sufficiently high, but finite, tightness. The off-path refinement therefore assigns $\hat{w}$ a finite tightness. Since every applicant has type at least $\underline{x}$, firm y_A earns at least $q(\Theta(\hat{w})) [F(\underline{x}, y_A) - \hat{w}] = q(\Theta(\hat{w}))\varepsilon > 0$, contradicting $\pi(y_A) = 0$. Therefore $\theta(\underline{x}) = \infty$. $\square$

A.6 Proof of Lemma 5

At every interior active point, worker incentive compatibility gives $\theta'(x) < 0$, and (14) therefore gives $D(x) > 0$ for all $x \in (\underline{x}, x_A)$. Continuity then implies $D(\underline{x}) \geq 0$.

Let $\theta_0 = \theta(\underline{x})$ and $y_A = \psi(\underline{x})$. Substituting $V(x)/\theta(x) = q(\theta(x))f(x, \psi(x)) - \pi(\psi(x))$, which follows from (7), into (13) and using $m'(\theta) = q(\theta) + \theta q'(\theta)$ yields the identity $D(x) = m'(\theta(x))f(x, \psi(x)) - \pi(\psi(x))$. If $\lim_{x \rightarrow \underline{x}} \theta(x) = \infty$, then $q(\theta(x)), m'(\theta(x)) \rightarrow 0$. As $x \rightarrow \underline{x}$, $0 \leq \pi = q(\theta)[F - w] \leq q(\theta)f \rightarrow 0$, so $\pi(y_A) = 0$. Hence this identity gives $D(\underline{x}) = 0$.

Suppose instead that $\theta_0 < \infty$. Lemma 4 then implies $y_A = \underline{y}$. For any $\hat{\theta} > \theta_0$, choose $\hat{w}$ so that $m(\hat{\theta})(\hat{w} - \underline{x}) = V(\underline{x})$. The bottom worker is indifferent, while Lemma 1 implies that every higher worker strictly prefers its equilibrium submarket. The deviation therefore attracts only the bottom type, and optimality of firm y_A requires $\Pi(\hat{\theta}, \underline{x}, y_A) \leq \Pi(\theta_0, \underline{x}, y_A)$. Thus $\Pi_\theta(\theta_0, \underline{x}, y_A) \leq 0$. Since $D(\underline{x}) = \theta_0 \Pi_\theta(\theta_0, \underline{x}, y_A)$, we obtain $D(\underline{x}) \leq 0$, and hence $D(\underline{x}) = 0$.

Next, consider the upper active endpoint x_A with $\theta(x_A) > 0$. The mirror deviation—a lower tightness and a higher wage—is feasible for the top firm and yields only $D(x_A) \geq 0$, which continuity already delivers. Below we use a different argument. Define $\theta^*(x)$ as the solution of $\Pi_\theta(\theta, x, \psi(x)) = 0$, i.e., $0 = V(x) - [m(\theta) - \theta m'(\theta)]f(x, \psi(x))$. The implicit function theorem gives that $\theta^*(x)$ is continuously differentiable and therefore moves at a bounded rate. Since $\Pi(\theta, x, \psi(x))$ is single-peaked in θ , $D(x) > 0$ if and only if $\theta(x) < \theta^*(x)$. Suppose now that $D(x_A) = 0$, so $\theta(x_A) = \theta^*(x_A)$. By (14),

$$\theta'(x) = -\frac{m(\theta(x))F_x(x, \psi(x))}{D(x)} \longrightarrow -\infty \quad \text{as } x \uparrow x_A.$$

Hence $\theta - \theta^*$ vanishes at x_A while its derivative diverges to $-\infty$, so $\theta(x) > \theta^*(x)$ and therefore $D(x) < 0$ just below x_A , a contradiction. Thus $D(x_A) > 0$.

Finally, if $x_A < \bar{x}$, Lemma 4 gives $V(x_A) = 0$ and $\theta(x_A) = 0$. Worker incentive

compatibility implies $x \leq w(x) \leq x_A$ for $x < x_A$, so $w(x) \rightarrow x_A$. Since $\psi(x) \rightarrow \bar{y}$, $\pi(\bar{y}) = q(0)f(x_A, \bar{y}) = m'(0)f(x_A, \bar{y})$, and therefore $D = m'(\theta)f - \pi$ gives $D(x_A) = 0$. $\square$

A.7 Proof of Lemma 7

Suppose, toward a contradiction, that $x_A < \bar{x}$. Lemma 4 gives $\theta(x_A) = 0$ and $w(x_A) = x_A$. Fix an active $x < x_A$ and let $y = \psi(x)$. Since $V(x) = m(\theta(x))[w(x) - x] \geq 0$, we have $w(x) \geq x$, so $\pi(y) = q(\theta(x))[F(x, y) - w(x)] \leq q(\theta(x))f(x, y) < q(0)f(x_A, y)$, where the strict inequality uses $\theta(x) > 0$, q strictly decreasing, and $f(x, y) > 0$, and the last step uses $f_x \geq 0$. If instead firm y copies the on-path wage $w(x')$ of an active type $x' < x_A$, it earns $q(\theta(x'))[F(x', y) - w(x')]$, which converges to $q(0)f(x_A, y)$ as $x' \uparrow x_A$ by continuity. The deviation is therefore strictly profitable for x' close enough to x_A , contradicting firm optimality. Hence $x_A = \bar{x}$. $\square$

A.8 Proof of Proposition 2

By continuity, choose a rectangle $R \equiv [a, b] \times [c, d]$ containing (x_0, y_0) with $F_{xy} < 0$ throughout, and set $\gamma \equiv \min_R \{-F_{xy}/F_y\}$, which is positive because R is compact and $F_y > 0$. Let the worker types be distributed on $[a, b]$ with total mass one and the firm types on $[c, d]$ with total mass N ; the shapes of the two distributions are immaterial. Let x_m denote the median worker type, so that $G_\ell(x_m) = 1/2$. We can choose $N > 0$ small enough that

$$\log \frac{q(0)}{q(2N)} < \gamma(b - x_m). \quad (22)$$

Suppose, toward a contradiction, that a differentiable equilibrium exhibits PAM. By Lemma 7, all workers are active. By (19), along the equilibrium path

$$\frac{d}{dx} \log q(\theta(x)) > -\frac{F_{xy}(x, \psi(x))}{F_y(x, \psi(x))} \geq \gamma,$$

where $\psi(x)$ is the type of the firm matched with worker type x . Integrating over (x_m, b) ,

$$\log \frac{q(\theta(b))}{q(\theta(x_m))} > \gamma(b - x_m). \quad (23)$$

We now obtain an upper bound on the same expression. Since θ is decreasing by worker incentive compatibility, feasibility gives, $N \geq \int_a^{x_m} \theta dG_\ell \geq \theta(x_m)G_\ell(x_m) = \theta(x_m)/2$, that is, $\theta(x_m) \leq 2N$. Since q is decreasing, $q(\theta(b)) \leq q(0)$ and $q(\theta(x_m)) \geq q(2N)$, so $\log \frac{q(\theta(b))}{q(\theta(x_m))} \leq \log \frac{q(0)}{q(2N)} < \gamma(b - x_m)$ by (22), contradicting (23). $\square$

A.9 Proof of Proposition 3

Necessity is already established; we prove sufficiency. Since $\theta(x)$ is decreasing and V satisfies (5), Lemma 2 makes the active schedule incentive compatible. If $x_A < \bar{x}$, then $V(x_A) = 0$ and no type above x_A wishes to enter an active submarket; if $x_A = \bar{x}$, all worker types are active.

Next, consider firm optimality. By (14), $\Omega_x(x, \psi(x)) = 0$ at every interior x . Fix an arbitrary assigned pair $(\tilde{x}, \tilde{y})$ with $\tilde{y} = \psi(\tilde{x})$; these are generic here and unrelated to the point (x_0, y_0) of Proposition 2. Take $x < \tilde{x}$, so that $\psi(x) < \tilde{y}$. Strict supermodularity of Ω and $\Omega_x(x, \psi(x)) = 0$ then give $\Omega_x(x, \tilde{y}) > 0$, and symmetrically $\Omega_x(x, \tilde{y}) < 0$ for $x > \tilde{x}$. Hence $\Omega(\cdot, \tilde{y})$ rises up to $\tilde{x}$ and falls thereafter, so $\tilde{x}$ is firm $\tilde{y}$'s unique best on-path choice; the same comparison applied one-sidedly covers the endpoint firms.

It remains to consider a firm posting $\hat{w}$ below $w(\underline{x})$ or above $w(x_A)$, and we first show that such a wage attracts only $\underline{x}$ or x_A . Let $p = m(\Theta(\hat{w}))$. The belief refinement requires the affine map $x \mapsto p(\hat{w} - x)$ to lie weakly below V , with equality at every applicant type. Since $V'(x) = -m(\theta(x))$, tangency at an interior applicant would force $p = m(\theta(x))$ and hence $\hat{w} = w(x)$, an on-path wage. Contact therefore occurs only at $\underline{x}$ or x_A , where the condition is one-sided and requires, respectively, $\Theta(\hat{w}) \geq \theta_0$ and $\Theta(\hat{w}) \leq \theta_1 \equiv \theta(x_A)$. The applicant obtains exactly its equilibrium utility, so firm y earns $\Pi(\Theta(\hat{w}), \underline{x}, y)$ or $\Pi(\Theta(\hat{w}), x_A, y)$.

Such a deviation moves tightness in the direction the firm does not want. If $\theta_0 < \infty$, then for every y ,

$$\theta_0 \Pi_\theta(\theta_0, \underline{x}, y) = D(\underline{x}) + \theta_0 q'(\theta_0) [f(\underline{x}, y) - f(\underline{x}, \underline{y})] \leq 0.$$

If $\theta_0 = \infty$, the only supporting endpoint contact has infinite tightness, where every firm earns zero. If $\theta_1 > 0$, then also

$$\theta_1 \Pi_\theta(\theta_1, x_A, y) = D(x_A) + \theta_1 q'(\theta_1) [f(x_A, y) - f(x_A, \bar{y})] \geq 0.$$

At a finite bottom, the first inequality uses $D(\underline{x}) = 0$, and the second uses $D(x_A) > 0$ when $\theta_1 > 0$; both use $q' < 0$ and the fact that f is increasing in y . Because Π is single-peaked in θ , at a finite bottom, $\Pi(\cdot, \underline{x}, y)$ is decreasing above θ_0 and, when $\theta_1 > 0$, $\Pi(\cdot, x_A, y)$ is increasing below θ_1 . If $\theta_1 = 0$, no strictly lower tightness is feasible; a wage strictly above the endpoint wage leaves the worker's zero payoff unchanged but lowers the firm's profit relative to the endpoint limit. Hence no such deviation beats the corresponding endpoint submarket, which firm y already weakly likes less than its own target. No firm therefore has a profitable deviation. $\square$

References

- Akerlof, G. A. (1970). The market for "lemons": Quality uncertainty and the market mechanism. *Quarterly Journal of Economics*, 84:488–500.
- Albrecht, J., Cai, X., Gautier, P., and Vroman, S. (2025). Competitive search with private information: Can price signal quality? *Mimeo Vrije Universiteit Amsterdam*.
- Altonji, J. G. and Pierret, C. R. (2001). Employer learning and statistical discrimination. *The Quarterly Journal of Economics*, 116(1):313–350.
- Atakan, A. E. (2006). Assortative matching with explicit search costs. *Econometrica*, 74:667–680.
- Auster, S. and Gottardi, P. (2019). Competing mechanisms in markets for lemons. *Theoretical Economics*, 14(3):927–970.
- Bagger, J. and Lentz, R. (2019). An empirical model of wage dispersion with sorting. *The Review of Economic Studies*, 86(1):153–190.
- Bartolucci, C., Devicienti, F., and Monzón, I. (2018). Identifying sorting in practice. *American Economic Journal: Applied Economics*, 10(4):408–438.
- Becker, G. S. (1973). A theory of marriage: Part I. *Journal of Political Economy*, 81(4):813–846.
- Borovičková, K. and Shimer, R. (2020). High wage workers work for high wage firms. Working paper.
- Borovičková, K. and Shimer, R. (2025). Assortative matching and wages: The role of selection. Working paper.
- Cai, X., Gautier, P., and Wolthoff, R. (2025). Spatial search. *Journal of Economic Theory*, 224:105976.

- Card, D., Heining, J., and Kline, P. (2013). Workplace heterogeneity and the rise of west german wage inequality. *The Quarterly Journal of Economics*, 128(3):967–1015.
- Carranza, E., Garlick, R., Orkin, K., and Rankin, N. (2022). Job search and hiring with limited information about workseekers’ skills. *American Economic Review*, 112(11):3547–3583.
- Chade, H., Eeckhout, J., and Smith, L. (2017). Sorting through search and matching models in economics. *Journal of Economic Literature*, 55(2):493–544.
- Chade, H. and Kircher, P. (2023). Matching with search frictions. In Echenique, F., Immorlica, N., and Vazirani, V. V., editors, *Online and Matching-Based Market Design*. Cambridge University Press.
- Chang, B. (2018). Adverse selection and liquidity distortion. *The Review of Economic Studies*, 85(1):275–306.
- Davoodalhosseini, S. M. (2019). Constrained efficiency with adverse selection and directed search. *Journal of Economic Theory*, 183:568–593.
- de Melo, R. L. (2018). Firm wage differentials and labor market sorting: Reconciling theory and evidence. *Journal of Political Economy*, 126(1):313–346.
- Eeckhout, J. and Kircher, P. (2010). Sorting and decentralized price competition. *Econometrica*, 78:539–574.
- Eeckhout, J. and Kircher, P. (2011). Identifying sorting - in theory. *Review of Economic Studies*, 78(3):872–906.
- Eeckhout, J. and Kircher, P. (2018). Assortative matching with large firms. *Econometrica*, 86(1):85–132.
- Farber, H. S. and Gibbons, R. (1996). Learning and wage dynamics. *The Quarterly Journal of Economics*, 111(4):1007–1047.
- Gautier, P. A. and Teulings, C. N. (2006). How large are search frictions? *Journal of the European Economic Association*, 4(6):1193–1225.
- Gautier, P. A. and Teulings, C. N. (2015). Sorting and the output loss due to search frictions. *Journal of the European Economic Association*, 13(6):1136–1166.
- Guerrieri, V., Shimer, R., and Wright, R. (2010). Adverse selection in competitive search equilibrium. *Econometrica*, 78(6):1823–1862.
- Hagedorn, M., Law, T. H., and Manovskii, I. (2017). Identifying equilibrium models of labor market sorting. *Econometrica*, 85(1):29–65.

- Heathcote, J., Perri, F., and Violante, G. L. (2010). Unequal we stand: An empirical analysis of economic inequality in the united states, 1967–2006. *Review of Economic Dynamics*, 13(1):15–51.
- Heathcote, J., Perri, F., Violante, G. L., and Zhang, L. (2023). More unequal we stand? Inequality dynamics in the united states, 1967–2021. *Review of Economic Dynamics*, 50:235–266.
- Hendel, I., Lizzeri, A., and Siniscalchi, M. (2005). Efficient sorting in a dynamic adverse-selection model. *Review of Economic Studies*, 72(2):467–497.
- Krishna, V. (2010). *Auction Theory*. Academic Press, San Diego, CA, 2nd edition.
- Lentz, R. (2010). Sorting by search intensity. *Journal of Economic Theory*, 145(4):1436 – 1452. Search Theory and Applications.
- Lise, J., Meghir, C., and Robin, J.-M. (2016). Matching, sorting and wages. *Review of Economic Dynamics*, 19:63 – 87. Special Issue in Honor of Dale Mortensen.
- Rothschild, C. and Scheuer, F. (2013). Redistributive taxation in the Roy model. *Quarterly Journal of Economics*, 128(2):623–668.
- Shi, S. (2001). Frictional assignment I: Efficiency. *Journal of Economic Theory*, 98:232–260.
- Shimer, R. (2005). The assignment of workers to jobs in an economy with coordination frictions. *Journal of Political Economy*, 113(5):996–1025.
- Shimer, R. and Smith, L. (2000). Assortative matching and search. *Econometrica*, 68(2):343–369.
- Shimer, R. and Wu, L. (2026). Competitive sorting with bilateral private information. Working paper; previously circulated as “Assortative Matching with Private Information”.
- Song, J., Price, D., Guvenen, F., Bloom, N., and von Wachter, T. (2018). Firming up inequality. Working paper.
- Williams, B. (2021). Search, liquidity, and retention: Screening multidimensional private information. *Journal of Political Economy*, 129(5):1487–1507.