TI 2026-067/VIII
Tinbergen Institute Discussion Paper

Claims on what remains

Erik Ansink [1]
Hans-Peter Weikard [2]

1 Vrije Universiteit Amsterdam
2 Wageningen University

Claims on What Remains*

Erik Ansink[†] Hans-Peter Weikard[‡]

Abstract

We study ordered claims problems in which availability of a resource changes across positions according to a stationary linear transition structure. Our main motivating example is the allocation of river water with conveyance loss. Allocations assigned to agents at different positions have different implications for their feasibility and for resource requirements. We characterise a class of geometric allocation rules in which claims are weighted by a constant multiplicative factor along the ordering and then scaled to the feasibility boundary. We apply these rules to two settings that capture opposing transition effects. In river water allocation the geometric weighting factor balances proportional sharing with efficiency considerations that would favour upstream agents. In a setting of river pollution with pollution decay, a geometric rule balances equal access to pollution rights with a reduction of pollution damage when favouring upstream agents.

Keywords: ordered claims problem, geometric rule, axiomatic characterisation, river sharing, pollution allocation, rights allocation.

JEL classification: C71, D63, K11, Q25, Q53.

1 Introduction

Proportionality is the natural benchmark in standard sharing problems with claims. When aggregate claims exceed the resource endowment, awards should follow claims: each claimant receives the same fraction of her claim. This idea is often represented or formalised by an anonymity axiom. Claims matter regardless of whose claim it is. This benchmark is central

*We thank Rosario Gomez and seminar participants at universities in Graz, Austria, and Amsterdam (UvA), The Netherlands, for inspiring discussions. We thank Jens Gudmundsson for thoughtful comments on a previous version of the paper.

[†]School of Business and Economics, Vrije Universiteit Amsterdam and Tinbergen Institute, the Netherlands.

[‡]Department of Social Sciences, Economics Section, Wageningen University, the Netherlands.

in the rights-arbitration and bankruptcy literature pioneered by O'Neill (1982) and surveyed by Thomson (2003). There is, however, a large class of allocation problems where agents are tied to locations. In such problems, anonymity is less convincing because location affects what it takes to satisfy a claim. This matters especially when resource availability changes across locations. Then, the same awards provided at different locations may require different amounts of the resource. In this setting proportionality is no longer innocent.

River water allocation with conveyance loss is our main application. Suppose water is supplied upstream and a fraction of the water is lost between neighbouring users because of seepage, evaporation, or other conveyance losses (e.g., Chakravorty et al., 1995). A simple two-user example makes the point.

Example 1. Consider a water endowment of 5 units and half of the water is lost in transit from upstream to downstream. If the upstream user has a claim to 2 units and the downstream user has a claim to 4 units, the endowment is insufficient to satisfy both claims. Satisfying claims would require $2 + 4/0.5 = 10$ units at the source. A proportional allocation gives each user $5/(2 + 4/0.5) = 1/2$ of her claim. Under this allocation, the upstream user receives 1 unit of water and the downstream user receives 2 units, while 2 units are lost in transit. This loss can be reduced if the downstream user receives less than the proportional allocation. $\circ$

More generally, with conveyance loss one unit delivered to a user requires more source water the further downstream the user's location. Hence, water sharing involves a basic trade-off. It is costly to be fair. Serving downstream users requires more source water because more is lost in transit; conversely favouring upstream users relaxes feasibility but departs from proportionality of water received.

The same transmission logic applies beyond water allocation. In energy transmission, line losses mean that delivery further along the grid requires more injection upstream (Bergantinos et al., 2017). In production and inventory systems, yield loss, spoilage, or leakage change what reaches later stages (Assari et al., 2023). In air pollution control, one unit emitted upwind counts differently at different downwind locations depending on distance, because dispersion, chemical transformation, and deposition alter along the transport route (Muller and Mendelsohn, 2007). Also note the similarity to intertemporal allocation. Due to discounting, one unit assigned to a later date has a smaller present value than one unit assigned earlier. These are different instances of the same general problem: claims are ordered, adjacent positions are linked by a stationary transition structure, and movement along the line changes the amount of the resource that remains available, even in the absence of extraction.

For such sharing problems with a stationary transition structure, we propose geometric rules. Claims are weighted by a constant multiplicative factor along the line, and the resulting weighted allocation profile is then scaled to the feasibility boundary. This connects neatly to the structure of the problem: when the same local transition effect applies when going from any location to the next, it is natural that the adjacent trade-off stays the same. This generates a constant multiplicative change in weights of claims. If the weight factor equals one, the rule is proportional. Lower values tilt the rule toward upstream claims, and higher values tilt it toward downstream claims. In river water allocation, where water is lost in transition, a geometric weight factor less than one reduces water loss, improving efficiency while departing from proportionality in delivered water.

Our paper makes two contributions. First, we formalise this setting as a general model of ordered claims problems with linear transitions. On this domain we characterise a class of geometric rules using four axioms. *No-Loss Proportionality* isolates claims differences by considering problems without transition losses: adjacent awards must then be proportional to adjacent claims. *Equal Treatment of Equal Neighbours* isolates transition losses by considering adjacent agents with equal claims: the ratio of neighbours' awards should be the same for any pair of neighbours. *Loss-Claim Separability* combines these two cases by requiring the adjacent award ratio in the general case, where both claims and losses differ, to be composed of the no-loss claims ratio and the equal-claims loss ratio. *Scale Efficiency* requires that the selected allocation lies on the feasibility boundary. We explore two applications in which transitions have opposing implications for allocation: river water allocation with conveyance loss and the allocation of a pollution budget with decay.

Second, the paper identifies physical transition as a distinct reason why proportionality may not be appropriate in ordered allocation problems. The ordering should matter only insofar as agents' positions generate different implications for feasibility through the resource transition. Existing departures from proportionality have been justified on several grounds, including considerations based on welfare trade-offs, or alternative ideas about the aggregation of claims. By contrast, our framework shows that departures from proportionality can naturally arise from a linear transition structure in a spatial setting: moving the resource across ordered locations changes how much of it remains available. A geometric parameter γ captures the direction and strength of the resulting location-based priority. Considering river water, this parameter receives a direct physical interpretation. When $\gamma = 1$, the rule distributes *delivered water* proportional to claims. When γ equals the transition factor, the rule distributes *source water* proportionally to claims, implying that downstream agents bear the cost associated with their disadvantaged location.

This paper connects to several literatures. Existing papers show that ordered location matters because of externalities, inflows, rights structure, or hierarchy. In the river-sharing literature, location matters because upstream use affects downstream availability and because rights must be assigned along a river. The seminal paper by Ambec and Sprumont (2002) addresses the welfare-based benchmark. Ambec and Ehlers (2008) extend the analysis to satiable-agents, Ansink and Weikard (2012, 2015) to claims-based river sharing, van den Brink et al. (2012, 2014); Martinez and Moreno-Ternero (2025) to related rights-based river allocation problems, Öztürk (2020) to a fairness-based social-ordering approach, and Gudmundsson et al. (2019) to decentralised river-sharing mechanisms; see Beal et al. (2013) for a survey. Related issues arise in pollution allocation, where upstream emissions affect a larger part of the river or a downstream environmental target (Ni and Wang, 2007; Alcalde-Unzu et al., 2015; van den Brink et al., 2018; Yang et al., 2025; Martinez and Moreno-Ternero, 2026b). More generally, the axiomatic literature shows that multiplicative or geometric profiles are natural in ordered or hierarchical environments (Hougaard et al., 2017), and serial allocation rules provide another benchmark for how consistency constrains ordered allocations (Hougaard and Østerdal, 2009). There is also a connection to claim-strength problems, where claims and feasible resources are expressed in different units (Van De Putte and Wintein, 2026). Our contribution is to isolate a different mechanism. Movement across ordered positions changes resource availability through loss or decay. Therefore, claims may receive location-dependent weights even before welfare or damage considerations enter. In addition, resource loss along the line generates an equity-efficiency trade-off.

The remainder of the paper proceeds as follows. Section 2 introduces the general model and the geometric class of rules. Section 3 states the axioms and presents the characterisation result. Section 4 applies the result to water allocation with conveyance loss and the allocation of a pollution budget with decay. Section 5 concludes. Proofs are collected in the Appendix.

2 Model

Consider a set of agents $N = \{1, \ldots, n\}$ with $n \geq 2$. Agents are ordered such that agent i comes before agent $i+1$. The agents are sharing a divisible resource $E > 0$. This resource is initially available at agent 1 and then moves down the order of agents. At each step, the resource suffers from a transition loss that we represent by a stationary loss rate $\lambda \in [0, 1)$. The sharing of the resource may not only be affected by transition losses but also by agents' claims to the resource. Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_{++}$ be the vector of agents' claims, where claims are exogenous. A resource allocation is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ that assigns a share of the resource to each agent. Since allocating x_i to agent i requires $\frac{x_i}{(1-\lambda)^{i-1}}$ of the initial endowment, an allocation is subject to the following feasibility constraint:

$$\sum_{i=1}^{n} \frac{x_i}{(1-\lambda)^{i-1}} \leq E. \tag{1}$$

Feasibility can also be represented recursively by denoting the remaining resource at agent i by $E_i = (1-\lambda)(E_{i-1} - x_{i-1})$. Then feasibility dictates $x_1 \leq E = E_1$ and $x_i \leq E_i,\ \forall i > 1$.

An ordered claims problem with transition losses, or simply a problem, is a triple $P = (E, c, \lambda)$. Let $\mathcal{P}$ denote the class of all ordered claims problems with transitions. For each problem $P \in \mathcal{P}$, let $\mathcal{X}(P) \subseteq \mathbb{R}^n_+$ denote the set of feasible allocation vectors satisfying Eq. (1). A solution is a function φ that maps each problem $P \in \mathcal{P}$ on an allocation $\varphi(P) \in \mathcal{X}(P)$.

Here we propose a geometric solution that assigns a weight function to the claims of the agents. Because agents are ordered, it is natural to consider rules that treat adjacent agents in a structured way. The relative award assigned to claims changes by a constant factor at each step downstream. This suggests a geometric pattern: at each step downstream the weight a claim receives is multiplied by the same factor γ. In this way, we obtain a vector of weighted claims $(c_1, \gamma c_2, \gamma^2 c_3, \ldots, \gamma^{n-1} c_n)$. These weighted claims are then scaled up as far as feasibility allows.

Definition 1 (Geometric rules)**.** A rule is geometric if, for each problem $P = (E, c, \lambda)$, awards are proportional to the geometrically weighted claims of the agents $c_1, \gamma c_2, \gamma^2 c_3, \ldots, \gamma^{n-1} c_n$, with $\gamma > 0$, and scaled to the feasibility boundary. For each agent $i \in N$ a geometric rule awards

$$\begin{aligned} \varphi_i^{\gamma}(P) &= \gamma^{i-1} c_i\ \sigma_{\gamma} \\ &= \gamma^{i-1} c_i\ \frac{E}{\sum_{j \in N} c_j \left(\frac{\gamma}{1-\lambda}\right)^{j-1}}. \end{aligned} \tag{2}$$

Moreover, for no-loss problems ($\lambda = 0$), the geometric parameter is fixed at $\gamma = 1$.

For no-loss problems ($\lambda = 0$), we constrain the geometric parameter. We do this since without transition losses location has no effect on feasibility and hence there is no reason for claims to receive location-dependent weights. The last term in Equation (2) is the maximal feasible scaling factor. We denote it by σ_γ. It can be derived when the feasibility constraint (1) holds with equality by substituting $x_i = \sigma_\gamma \gamma^{i-1} c_i$.

Geometric rules separate two issues. The parameter γ determines how the award assigned *per unit of claim* changes along the line. The maximal feasible scaling factor σ_γ then determines how far this profile can be scaled given the available resource and the transition losses. The class of geometric rules contains two rules as polar cases. If $\gamma = 1$ we have proportional sharing of *delivered water* $\varphi_i^1 = c_i \sigma_1$. In this case each unit of claim is awarded the same delivered water independently of the agent's position. If γ approaches zero: downstream claims receive vanishing weight, so awards are concentrated upstream. In the limit we obtain a priority rule that allocates all resources to the most upstream agent. Note that this rule will usually violate *Claims Boundedness*, a condition that precludes that any agent receives more than her claim.

A focal rule is $\varphi^{1-\lambda}$. It sits between these two extremes. When the geometric parameter matches the physical transmission rate of the resource, $\gamma = 1 - \lambda$, we obtain a weighted claims vector $(c_1, (1-\lambda)c_2, (1-\lambda)^2 c_3, \ldots, (1-\lambda)^{n-1} c_n)$. The focal rule allocates $\varphi_i^{1-\lambda} = (1-\lambda)^{i-1} c_i \frac{E}{\sum_{j \in N} c_j}$. to each agent $i \in N$. Under this rule, *source water* is shared proportional to claims and each agent bears the losses associated with her position. By comparison, the proportional rule fully compensates downstream agents for their disadvantaged position.

Choosing $\gamma > 1 - \lambda$ gives relatively more weight to downstream claims than the focal rule. This moves the allocation toward proportional sharing, but since it requires more initial resource to deliver resources downstream, it lowers resource-use efficiency. Choosing $\gamma < 1 - \lambda$ gives even more weight to upstream claims. This moves the allocation toward priority, reduces transition losses, but creates larger departures from proportionality.

The next step is to ask which axioms characterise the class of geometric sharing rules.

3 Axiomatic Characterisation

This section presents four axioms for ordered claims problems with transition losses. Our main result characterises the class of geometric rules on this domain. Throughout this characterization, we restrict attention to positive-award rules. This ensures that the adjacent award ratios used in the axioms are well defined.

We first present two axioms that consider two different dimensions of the problem, each in isolation. One dimension concerns differences in claims between agents; the other concerns transition losses. *No-Loss Proportionality* isolates claim differences by removing losses, while *Equal Treatment of Equal Neighbours* isolates losses by removing claim differences. Each axiom then states what happens to the ratio of awards to any pair of adjacent agents.

Under *No-Loss Proportionality*, adjacent agents receive awards proportional to their claims. Since there are no transition losses, there is no reason to deviate from proportionality. *No-Loss Proportionality* has the spirit of an anonymity or symmetry axiom: if location is irrelevant for the feasibility of an allocation, then it should not matter for the awards. A fair allocation should then be purely driven by the strengths of the claims (Wintein and

Heilmann, 2024).

No-Loss Proportionality (NLP). For every no-loss problem $P^0 = (E, c, 0)$ and every adjacent pair $i, i+1$,

$$\frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} = \frac{c_{i+1}}{c_i}.$$

Under *Equal Treatment of Equal Neighbours*, adjacent agents with equal claims receive awards in a common ratio.

Equal Treatment of Equal Neighbours (ETEN). Consider any problem $P = (E, c, \lambda)$ and the derived problem $P' = (E, c', \lambda)$ with $c'_i = c_{i+1}$ and $c'_j = c_j$ for all $j \neq i$, then for every from P derived problem P'

$$\frac{\varphi_{i+1}(P')}{\varphi_i(P')} = \alpha > 0.$$

Our next axiom, *Loss-Claim Separability*, connects the two dimensions. It requires the adjacent award ratio in some problem to be multiplicatively separable into a claims component, obtained when losses are removed, and a loss component, obtained when the adjacent claims are equal. Thus the impacts of claims differences and losses on the allocation are separable. Together with *Equal Treatment of Equal Neighbours*, this makes the allocation rule stationary in losses: the effect of a given transition loss is stable across positions. This is common in ordered allocation and serial cost-sharing settings (Hougaard and Østerdal, 2009; Hougaard et al., 2017). The rationale is that claim differences and transition losses represent distinct features of the allocation problem. *Loss-Claim Separability* requires the effect of each feature on relative awards to be independent of the level of the other: changing claims affects the adjacent award ratio in the same proportional way regardless of the loss rate, while changing the loss environment affects that ratio independently of the agents' relative claims.

Loss-Claim Separability (LCS). For every problem $P = (E, c, \lambda)$, every adjacent pair $i, i+1$, every related no-loss problem $P^0 = (E, c, 0)$, and every related problem $P' = (E, c', \lambda)$ with $c'_i = c_{i+1}$ and $c'_j = c_j$ for all $j \neq i$,

$$\frac{\varphi_{i+1}(P)}{\varphi_i(P)} = \frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} \frac{\varphi_{i+1}(P')}{\varphi_i(P')}.$$

Our last axiom, *Scale Efficiency*, says that the rule φ selects an allocation that cannot be scaled up. This is a no-slack condition for any given pattern of awards, akin to efficiency or full-use requirements that are standard in river sharing (Ambec and Sprumont, 2002) and cost sharing (Moulin and Shenker, 1994). Note that *Scale Efficiency* allows for resource losses along the river as Example 1 in the Introduction shows. It is conceptually distinct from resource-use efficiency.

Scale Efficiency (SE). If $\varphi(P) = x$ and $x \in \mathcal{X}(P)$, then there is no $\sigma > 1$ such that $\sigma x \in \mathcal{X}(P)$.

The logic of the characterisation is as follows. *No-Loss Proportionality* fixes ordinary proportionality to claims when there are no transition losses. *Equal Treatment of Equal Neighbours* fixes the case when adjacent claims are equal. Then their award ratio is determined solely by a common factor associated with the transition loss. *Loss-Claim Separability* connects these two dimensions by requiring that the award ratio for any pair of adjacent agents is (separably) composed of the ratios due to claims differences and transition losses. *Scale Efficiency* imposes slack avoidance. Together these axioms imply that each adjacent award ratio is equal to the corresponding claims ratio multiplied by a location factor that we denote by α. Iterating these adjacent ratios gives a geometric path, and scale efficiency pushes this path to the feasibility boundary. Conversely, geometric rules satisfy exactly these requirements on the domain considered here. The theorem below therefore characterises the class of geometric rules for ordered claims problems with transition losses.

Theorem 1 (Characterisation of the class of geometric rules)**.** A solution to a claims problem with transition losses satisfies *No-Loss Proportionality*, *Equal Treatment of Equal Neighbours*, *Loss-Claim Separability*, and *Scale Efficiency* if and only if it is geometric.

That is, for every problem $P = (E, c, \lambda)$, there is a parameter $\gamma > 0$ such that

$$\varphi_i(P) = \gamma^{i-1} c_i \; \sigma_\gamma(P), \qquad i = 1, \ldots, n.$$

The proof is in the Appendix.

The theorem establishes a geometric form of the weights on claims but does not select a particular value of the parameter γ. The axioms therefore characterise the class of geometric rules, not yet a single member of that class. Particular members can be selected by fixing the parameter α in *Equal Treatment of Equal Neighbours*. Setting $\alpha = 1$ selects the proportional rule. Setting $\alpha = 1 - \lambda$ selects the focal rule, in which the geometric adjustment in awards mirrors the physical transition rate of the resource.

4 Applications

We now apply our model to two environmental allocation problems in which transitions have opposing implications for allocation: conveyance loss in river water allocation and decay in the allocation of pollution rights.

4.1 Water Allocation with Conveyance Loss

The river-water interpretation is the baseline case behind the model. If all river water is supplied upstream and λ is the conveyance-loss rate, then the feasibility constraint says that one unit delivered to agent i requires $1/(1-\lambda)^{i-1}$ units at the source. Section 3 characterises a class of geometric rules but does not select a particular value of the geometric factor $\gamma > 0$. We now consider two additional requirements for river-water allocation. Each selects one member of the geometric class.

Impartiality (IM). For every problem $P = (E, c, \lambda)$ and agents i, j:

$$c_i = c_j \implies \varphi_i = \varphi_j.$$

Impartiality requires equal claims to receive equal amounts of delivered water, independently of location. Within the geometric class, equal claims of agents i and j imply awards in the ratio γ^{j-i}. Thus *Impartiality* fixes the factor in *ETEN* at $\alpha = 1$. The axiom can be defended by a Rawlsian argument. If location is a circumstance beyond agents' control, a decision maker behind a veil of ignorance may wish to prevent it from affecting delivered-water awards.

Proposition 1 (Delivered-water proportional rule)**.** Within the class of geometric rules, *Impartiality* holds if and only if $\gamma = 1$, and is given by $\varphi_i^1 = c_i \sigma_1$.

Proof. For equal claims at agents i and j, a geometric rule gives

$$\frac{\varphi_j(P)}{\varphi_i(P)} = \gamma^{j-i}.$$

IM requires this ratio to equal one. Since $\gamma > 0$, it follows that $\gamma = 1$. □

The resulting rule allocates delivered water proportionally to claims. Each unit of claim receives the same delivered-water award, independently of the agent's position. The loss rate affects the common scaling factor through feasibility, but it does not affect relative awards. Thus the rule fully compensates downstream agents for the additional source water required to serve their claims.

An alternative view rejects compensation for downstream disadvantage, but instead requires equal access to water at the source.

Equal Access (EA). For every problem $P = (E, c, \lambda)$ and agents i, j such that $i < j$:

$$c_i = c_j \implies \frac{\varphi_i(P)}{(1-\lambda)^{i-1}} = \frac{\varphi_j(P)}{(1-\lambda)^{j-1}}.$$

Equal Access requires equal claims to receive equal access to water at source. Within the geometric class, this is equivalent to source-water awards being proportional to claims. Thus, *Equal Access* fixes the factor in *ETEN* at $\alpha = 1 - \lambda$.

Proposition 2 (Source-water proportional rule)**.** Within the class of geometric rules, *Equal Access* holds if and only if $\gamma = 1 - \lambda$, and is given by

$$\varphi_i^{1-\lambda} = (1-\lambda)^{i-1} c_i \frac{E}{\sum_{j \in N} c_j}.$$

Proof. For equal claims at agents i and j, a geometric rule gives

$$\frac{\varphi_j(P)}{\varphi_i(P)} = \gamma^{j-i}.$$

EA requires this ratio to equal $(1-\lambda)^{j-i}$. Hence $\gamma = 1 - \lambda$. □

Under this rule, source water is allocated proportionally to claims. Per unit of claim the delivered awards decline downstream with the physical loss factor: each agent bears the conveyance losses associated with her location. The rule therefore does not compensate downstream agents for their disadvantaged location but respects differences in claims.

The two rules allocate the burden of conveyance loss in different ways. The delivered-water proportional rule compensates downstream agents in delivered-water units, while the source-water proportional rule leaves each agent to bear the losses associated with her location. Figure 1 illustrates this difference. Returning to Example 1, let the endowment and claims remain fixed at $E = 5$ and $c = (2, 4)$, and vary the loss rate. For the geometric rule, the feasibility constraint (1) gives

$$x_1 = \frac{5(1-\lambda)}{1-\lambda+2\gamma}, \qquad x_2 = \frac{10\gamma(1-\lambda)}{1-\lambda+2\gamma}.$$

Under the delivered-water proportional rule ($\gamma = 1$), both allocations fall as losses rise because the ratio of awards remains constant. As the water constraint becomes tighter, both agents suffer. When source water is allocated proportionally to claims ($\gamma = 1 - \lambda$), the upstream allocation is unaffected by losses while the downstream allocation absorbs the loss. At $\lambda = 0.5$, the delivered-water proportional rule allocates $(1, 2)$, as in Example 1.

This formulation of the water-allocation problem differs from existing river-sharing models. In Ambec and Sprumont (2002), Ambec and Ehlers (2008), Ansink and Weikard (2012, 2015), van den Brink et al. (2012, 2014), and Martinez and Moreno-Ternero (2025), location matters because upstream use affects downstream availability and because rights must be

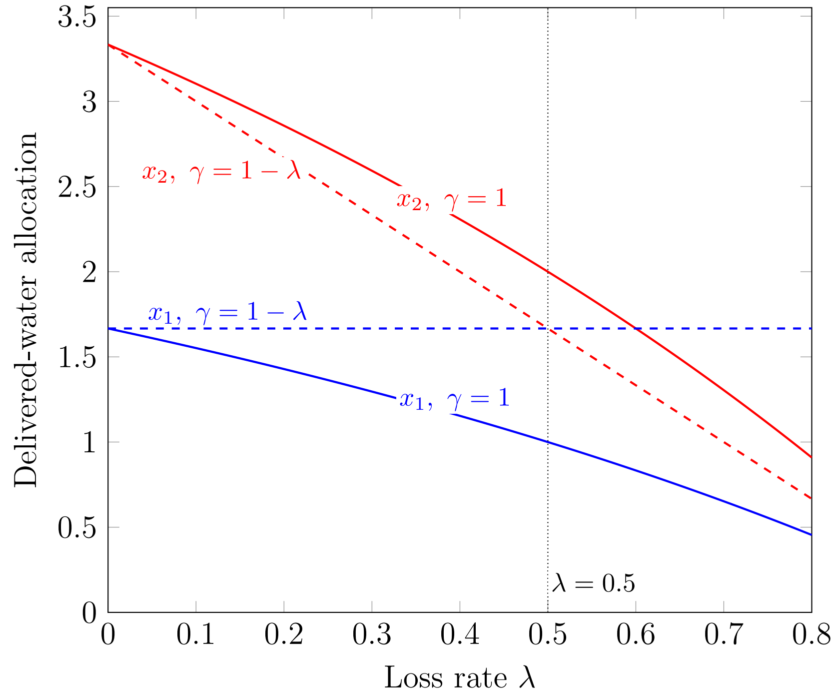

Figure 1: Water allocations as functions of the loss rate λ for $E = 5$ and $c = (2, 4)$. Solid lines show proportional sharing of delivered water ($\gamma = 1$); dashed lines show proportional sharing of source water ($\gamma = 1 - \lambda$).

assigned along a river. However, losses have not been considered. Here, due to losses, one delivered unit downstream requires more source water than one delivered unit upstream. This asymmetry turns the feasibility requirement into a weighted budget and gives direct economic meaning to the geometric parameter. A related but distinct use of geometric rules appears in Martinez and Moreno-Ternero (2026a), who study international river rights and derive geometric allocations from downstream transfers of inflows rather than from conveyance losses in a claims problem.

We end this section with two remarks, on possible violations of *Claims Boundedness* and an extension to non-stationary transitions.

Remark 1 (Claims boundedness)**.** For low values of the geometric parameter, a geometric rule may violate *Claims Boundedness*: an agent can receive more delivered water than her claim. Formally, Claims Boundedness requires $\varphi_i(P) \leq c_i$ for all $i \in N$. For a geometric rule, this condition is equivalent to $\sigma_\gamma(P) \leq 1$, because $\varphi_i(P) = \sigma_\gamma(P)\gamma^{i-1}c_i$ and $\gamma \leq 1$. Thus, when lowering γ a violation first occurs at the most upstream agent.

Using the binding feasibility constraint, define $\gamma_{\min}(P)$ implicitly by

$$E = \sum_{j \in N} \left(\frac{\gamma_{\min}(P)}{1-\lambda} \right)^{j-1} c_j.$$

For the non-trivial case in which this solution lies in $(0,1]$, *Claims Boundedness* holds if and only if $\gamma \geq \gamma_{\min}(P)$. For lower values, the stronger upstream weighting gives agent 1 more than her claim. Under the usual scarcity condition $E \leq \sum_{j \in N} c_j$, we have $\gamma_{\min}(P) \leq 1 - \lambda$. Hence, both the delivered-water proportional rule and the source-water proportional rule satisfy *Claims Boundedness.* ◦

Remark 2 (Non-stationary transitions)**.** Both the characterisation and the figure assume a stationary conveyance-loss rate. This assumption is restrictive but useful because it gives the water rule its simple geometric form. In actual rivers and canals, conveyance losses need not be the same between all neighbouring users, for three reasons. The first reason, or set of reasons, is given by biophysical conditions. Evaporation may differ across segments because of local temperature, wind, shade, or humidity. Infiltration may differ because the river bed crosses different soils and groundwater conditions. River morphology makes some segments more leaky than others. A second reason is distance. Even if the loss rate per kilometre is constant, adjacent agents may be located at unequal distances from each other, so the cumulative loss between neighbouring agents will differ. Finally, losses may be partly endogenous. Canal segments can be lined, covered, shaded, or otherwise adapted to reduce seepage or evaporation.

The resulting case is more general. It has the same logic, but no longer gives a one-parameter geometric rule. Let λ_j be the loss rate between agents j and $j+1$. Then the fraction of source water that reaches agent i is the product of all transmission factors on the path from the source to agent i:

$$s_i = \prod_{j<i} (1 - \lambda_j),$$

with $s_1 = 1$. Here, the product is taken over all links upstream of agent i. One unit delivered to agent i therefore requires $1/s_i$ units at the source. If the focal source-sharing logic is applied, such that source water is allocated proportionally to claims, the delivered award to agent i is proportional to $s_i c_i$. With a constant loss rate, $s_i = (1-\lambda)^{i-1}$, and these weights are geometric. With non-stationary losses, the weights follow the cumulative loss profile of the river. Thus stationarity is the assumption that turns cumulative transition losses into a geometric sequence. ◦

4.2 Allocation of a Pollution Budget with Decay

The requirements used in the water application are specific to the interpretation of claims as claims to delivered water. We now apply the same geometric class to a different allocation problem, in which claims concern local emission rights. Unlike in the water application, the resource E is not an upstream stock. It is a downstream-impact budget: the maximum total pollution impact that may reach a downstream control point. Not all pollution reaches this point because of pollution decay: the gradual reduction in the concentration of a pollutant over distance due to processes such as dilution or bio-chemical transformation.

Suppose agent i receives emission right $x_i \geq 0$. Let $\rho \in [0, 1)$ be the stationary decay rate between neighbouring locations, so that the fraction $1 - \rho$ of any emission survives one step downstream. The control point is downstream of all agents, so one unit emitted at location i contributes $(1 - \rho)^{n-i}$ units to the downstream impact. The relevant feasibility constraint is

$$\sum_{i=1}^{n} (1 - \rho)^{n-i} x_i \leq E. \tag{3}$$

Since

$$\sum_{i \in N} (1 - \rho)^{n-i} x_i = (1 - \rho)^{n-1} \sum_{i \in N} \frac{x_i}{(1 - \rho)^{i-1}},$$

constraint (3) is equivalent to

$$\sum_{i \in N} \frac{x_i}{(1 - \rho)^{i-1}} \leq \frac{E}{(1 - \rho)^{n-1}}.$$

Thus, after rescaling the endowment, the pollution problem has the same feasible-set form as constraint (1), with $\lambda = \rho$. This is a formal equivalence only. In water, transition losses make downstream deliveries more costly in source-water terms. In pollution, decay has the opposite implication: it allows larger upstream emission rights because part of their impact disappears before reaching the control point.

Claims are claims to local emission rights, and awards are measured in the same units. Decay enters through feasibility: one unit of local emissions at different locations uses different amounts of the downstream-impact budget. This creates a distinction between proportionality in local emission rights and proportionality in downstream impact.

Consider now a geometric rule on this pollution domain. The natural calibration in this application is to set the geometric weight equal to the survival factor, i.e., $\gamma = 1 - \rho$. This gives the focal pollution rule

$$\varphi_i^{1-\rho}(P) = \left(\frac{c_i}{\sum_{j\in N} c_j}\right)\frac{E}{(1-\rho)^{n-i}}. \tag{4}$$

The focal rule allocates the budget of downstream impact proportionally to local emission claims. Stronger decay allows larger upstream emission rights because a smaller share of upstream emissions reaches the downstream control point.

Compared with the source-water proportional rule assessed in Section 4.1, the pollution focal rule scales awards in the opposite direction. In water allocation, awards are scaled down by $(1-\lambda)^{i-1}$, because each awarded unit requires more source water the further downstream it is delivered. In the pollution application, awards are emission rights, while feasibility is measured in downstream-impact units. Upstream emission rights are therefore scaled up by $1/(1-\rho)^{n-i}$, because only the fraction $(1-\rho)^{n-i}$ reaches the downstream control point.

The application to pollution reveals a tension in the interpretation of claims. Claims and awards are measured in local emission-right units, so local claims boundedness would require $\varphi_i^{1-\rho}(P) \le c_i$. It turns out that the focal rule does not generally satisfy this condition. When decay is strong, an upstream agent can receive more raw emission rights than her local claim, because each unit of upstream emissions counts only partly against the downstream-impact cap. Thus the rule is proportional in downstream-impact units, but not necessarily bounded by local emission claims.

The next example illustrates this tension by plotting how the focal rule behaves as the decay rate varies.

Example 2 (Two-agent pollution problem)**.** Let $n = 2$, claims $c = (1, 2)$, and downstream cap $E = 2.5$. By the focal pollution rule in (4),

$$\varphi_1^{1-\rho} = \frac{2.5}{3(1-\rho)},$$
$$\varphi_2^{1-\rho} = \frac{5}{3}.$$

Figure 2 displays these awards over a range of decay rates. The figure shows that $\varphi_2^{1-\rho}$ is constant, while $\varphi_1^{1-\rho}$ rises nonlinearly with the decay rate. The reason is that the focal rule fixes downstream-impact shares. Since only the fraction $1-\rho$ of agent 1's emissions survives and reaches the control point, maintaining the same impact contribution requires division by a shrinking survival factor. For this reason, $\varphi_1^{1-\rho}$ exceeds the local claim $c_1 = 1$ whenever $2.5/[3(1-\rho)] > 1$, which is equivalent to $\rho > 1/6$. ◦

In existing polluted-river models, location usually matters through damage or responsibil-

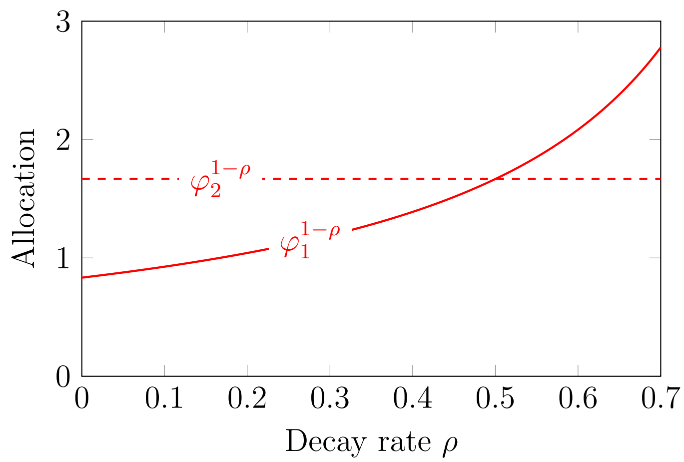

Figure 2: Pollution allocations as functions of the decay rate ρ for the focal rule where $\gamma = 1 - \rho$.

ity: upstream emissions affect a longer stretch of the river (Yang et al., 2025), or upstream agents are responsible for downstream cleaning costs (Ni and Wang, 2007; Alcalde-Unzu et al., 2015; van den Brink et al., 2018). Recent papers also obtain geometric rules in river-pollution settings, but from different mechanisms: Martinez and Moreno-Ternero (2026b) derive them from downstream transfers of residual claims, while Álvarez-Mozos et al. (2026) derive them in a cost-sharing setting based on responsibility and collusion neutrality. Here, location matters because physical decay changes how strongly emissions at different locations count against a downstream cap.

4.3 Combining Water Quantity and Quality

The two applications above treat water quantity and pollution separately. In many river settings they interact. Water withdrawals reduce downstream flow and therefore reduce dilution capacity, while emissions increase the pollution load that must be absorbed by the remaining flow. A simple way to represent this is to let

$$f_{i+1} = (1 - \lambda)(f_i - x_i)$$

describe flow and

$$p_{i+1} = (1 - \rho)(p_i + e_i)$$

describe pollutant load. If downstream water quality is constrained by a concentration standard, then feasibility requires

$$p_n \leq \bar{q} f_n,$$

where $\bar{q}$ is the maximum admissible concentration. This couples the two allocation dimensions: a larger water withdrawal tightens the feasible pollution budget downstream, while stronger decay relaxes it.

This combined problem illustrates the scope and the limits of the present paper. The transition logic remains the same: physical movement along the river changes the units in which claims bind feasibility. But the allocation problem is no longer a one-dimensional claims problem. It involves claims to water withdrawals and claims to emission rights, linked by a common downstream constraint. Characterising rules for such multidimensional claims would require a weighing of the relative status of water claims and emission claims. We therefore leave the combined quantity-quality problem for future work.

5 Conclusion

We study ordered claims problems in which movement from one location to the next changes what remains available. The main result is that *No-Loss Proportionality*, *Equal Treatment of Equal Neighbours*, *Loss-Claim Separability*, and *Scale Efficiency* characterise the class of geometric rules: awards take a geometric form and are then scaled to the feasibility boundary.

We highlight physical transition as a distinct reason why simple proportionality may fail in ordered allocation problems and use that idea to unify two environmental applications within one framework. In the general model, Theorem 1 identifies the functional form. In the applications, physical parameters determine how that form should be read. In water allocation with conveyance loss, the calibration $\gamma = 1 - \lambda$ yields proportionality in source water. In the allocation of a pollution budget with decay, the calibration $\gamma = 1 - \rho$ yields proportionality in downstream impact, although it may violate local claims boundedness in raw emission-right units. In general, parameter γ reflects a normative stance on the equity-efficiency trade-off in resource use that emerges due to conveyance loss. More broadly, the paper places these applications next to ordered-allocation, serial-allocation, and claims-strength problems in which the units of claims and the units that bind feasibility do not coincide, and it applies equally to loss transitions and attenuating transitions.

Taken together, the results show that geometric allocation rules can arise from the way physical transitions transform claims across ordered positions.

A Proof of Theorem 1

Proof. The proof follows the structure outlined in the text, in three steps. First, we derive the common adjacent award ratio for an arbitrary problem. Second, we identify the maximal feasible scale of the resulting geometric allocation. Third, we verify the converse. For completeness, in a fourth step we show logical independence of the axioms.

Step 1 (Adjacent award ratio).

Consider any $P = (E, c, \lambda) \in \mathcal{P}$ and an adjacent pair of agents $i, i+1$. By (NLP), its related no-loss problem $P^0 = (E, c, 0)$ satisfies

$$\frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} = \frac{c_{i+1}}{c_i}.$$

Now consider the from P derived problem $P' = (E, c', \lambda)$, with $c'_i = c_{i+1}$ and $c'_j = c_j$ for all $j \neq i$. In P' the adjacent pair $i, i+1$ has equal claims. By (ETEN),

$$\frac{\varphi_{i+1}(P')}{\varphi_i(P')} = \alpha,$$

where $\alpha > 0$ is common to all problems derived from P. Set $\gamma = \alpha$. Then, by (LCS),

$$\frac{\varphi_{i+1}(P)}{\varphi_i(P)} = \frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} \frac{\varphi_{i+1}(P')}{\varphi_i(P')} = \gamma \frac{c_{i+1}}{c_i}.$$

Since the adjacent pair was arbitrary, this relation holds for every $i = 1, \ldots, n-1$. When $\lambda = 0$, (NLP) requires an award ratio of one for any equal-claims pair. Hence, in this case, $\gamma = 1$ and the rule is geometric. The following steps consider $\lambda > 0$.

Step 2 (Geometric representation and scale). Iterating the adjacent award ratios gives, for every $i = 1, \ldots, n$,

$$\frac{\varphi_i(P)}{\varphi_1(P)} = \prod_{k=1}^{i-1} \frac{\varphi_{k+1}(P)}{\varphi_k(P)} = \prod_{k=1}^{i-1} \gamma \frac{c_{k+1}}{c_k} = \gamma^{i-1} \frac{c_i}{c_1}.$$

Set

$$\sigma(P) = \frac{\varphi_1(P)}{c_1}.$$

Then

$$\varphi_i(P) = \gamma^{i-1} c_i \ \sigma(P), \quad i = 1, \dots, n.$$

Substituting this allocation into the feasibility constraint yields

$$\sigma(P) \sum_{j \in N} c_j \left(\frac{\gamma}{1-\lambda} \right)^{j-1} \leq E.$$

Suppose that this inequality is strict. Then the allocation can be multiplied by a scalar greater than one while remaining feasible, which contradicts (SE). The feasibility constraint therefore binds, so

$$\sigma(P) = \frac{E}{\sum_{j \in N} c_j \left(\frac{\gamma}{1-\lambda} \right)^{j-1}} = \sigma_\gamma(P).$$

Thus,

$$\varphi_i(P) = \gamma^{i-1} c_i \ \sigma_\gamma(P), \quad i = 1, \dots, n.$$

Since P was arbitrary, the solution has the stated geometric form for every problem.

Step 3 (Converse). Suppose that φ is geometric. We verify the four axioms.

For a no-loss problem $P^0 = (E, c, 0)$, the definition of a geometric rule requires $\gamma = 1$. Hence,

$$\frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} = \frac{c_{i+1}}{c_i},$$

so (NLP) holds.

Consider a from $P = (E, c, \lambda)$ derived problem $P' = (E, c', \lambda)$ with $c'_i = c_{i+1}$ and $c'_j = c_j$ for $j \neq i$. We have, for all $i < n$,

$$\begin{aligned} \varphi_i(P') &= \sigma_\gamma c_{i+1} \gamma^{i-1}, \\ \varphi_{i+1}(P') &= \sigma_\gamma c_{i+1} \gamma^{i}. \end{aligned}$$

Thus for every adjacent pair $i, i+1$ with $c'_i = c'_{i+1}$, the ratio is

$$\frac{\varphi_{i+1}(P')}{\varphi_i(P')} = \gamma.$$

The same ratio applies to every problem derived from P, so (ETEN) holds with $\alpha = \gamma$.

For any problem $P = (E, c, \lambda)$, its related no-loss problem P^0, and its related equal-claims problem P', as specified above, we have

$$\frac{\varphi_{i+1}(P)}{\varphi_i(P)} = \gamma \frac{c_{i+1}}{c_i},$$

$$\frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} = \frac{c_{i+1}}{c_i},$$

$$\frac{\varphi_{i+1}(P')}{\varphi_i(P')} = \gamma.$$

Therefore,

$$\frac{\varphi_{i+1}(P)}{\varphi_i(P)} = \frac{\varphi_{i+1}(P^0)}{\varphi_i(P^0)} \frac{\varphi_{i+1}(P')}{\varphi_i(P')},$$

so (LCS) holds.

Finally, a geometric rule is scaled to the feasibility boundary. Therefore, there is no scalar greater than one by which its allocation can be expanded while remaining feasible. Hence, (SE) holds.

Step 4 (Logical independence). The four axioms are logically independent. To construct counterexamples, we first define rules with prescribed adjacent award ratios. Let $\eta_i(P) > 0$ be the desired ratio of the award to agent $i+1$ to the award to agent i. Starting with $w_1(P) = 1$, define the weights recursively by

$$w_1(P) = 1,$$
$$w_i(P) = \prod_{k=1}^{i-1} \eta_k(P), \quad i = 2, \ldots, n.$$

Thus, $w_{i+1}(P)/w_i(P) = \eta_i(P)$. We obtain a feasible rule by scaling these weights to the feasibility boundary:

$$\psi_i(P) = w_i(P) \frac{E}{\sum_{j \in N} \frac{w_j(P)}{(1-\lambda)^{j-1}}}.$$

This rule is positive and satisfies (SE) by construction.

Without (NLP), take

$$\eta_i(P) = (1-\lambda)\left(\frac{c_{i+1}}{c_i}\right)^2.$$

This rule satisfies (ETEN) and (LCS), but violates (NLP) when $\lambda = 0$.

Without (ETEN), take

$$\eta_i(P) = (1-\lambda)^i \frac{c_{i+1}}{c_i}.$$

This rule satisfies (NLP) and (LCS), but the equal-claims award ratio depends on the location of the adjacent pair and therefore violates (ETEN).

Without (LCS), take

$$\eta_i(P) = (1-\lambda)\left(\frac{c_{i+1}}{c_i}\right)^{1+\lambda}.$$

This rule satisfies (NLP) and (ETEN), but violates (LCS) whenever $\lambda > 0$ and $c_i \neq c_{i+1}$.

Finally, let $\varphi^{1-\lambda}$ be the source-water proportional rule and define

$$\bar{\varphi}(P) = \frac{1}{2}\varphi^{1-\lambda}(P).$$

This rule satisfies (NLP), (ETEN), and (LCS), because these axioms concern award ratios, but it violates (SE).

$\square$

References

Alcalde-Unzu, J., Gómez-Rúa, M., and Molis, E. (2015). Sharing the costs of cleaning a river: The upstream responsibility rule. *Games and Economic Behavior*, 90:134–150.

Álvarez-Mozos, M., van den Brink, R., and Jané-Ballarín, M. (2026). Collusion in the polluted river problem. Discussion Paper TI 2026-021/II, Tinbergen Institute.

Ambec, S. and Ehlers, L. (2008). Sharing a river among satiable agents. *Games and Economic Behavior*, 64(1):35–50.

Ambec, S. and Sprumont, Y. (2002). Sharing a river. *Journal of Economic Theory*, 107(2):453–462.

Ansink, E. and Weikard, H.-P. (2012). Sequential sharing rules for river sharing problems. *Social Choice and Welfare*, 38(2):187–200.

Ansink, E. and Weikard, H.-P. (2015). Composition properties in the river claims problem. *Social Choice and Welfare*, 44(4):807–831.

Assari, M., Eruguz, A. S., Dullaert, W., and Heijungs, R. (2023). Incorporating product decay during transportation and storage into a sustainable inventory model. *Computers & Industrial Engineering*, 185:109653.

Beal, S., Ghintran, A., Remila, E., and Solal, P. (2013). The river sharing problem: A survey. *International Game Theory Review*, 15(3):1–19.

Bergantinos, G., Gonzalez-Diaz, J., Gonzalez-Rueda, A. M., and de Córdoba, M. P. F. (2017). Loss allocation in energy transmission networks. *Games and Economic Behavior*, 102:69–97.

Chakravorty, U., Hochman, E., and Zilberman, D. (1995). A spatial model of optimal water conveyance. *Journal of Environmental Economics and Management*, 29(1):25–41.

Gudmundsson, J., Hougaard, J. L., and Ko, C. Y. (2019). Decentralized mechanisms for river sharing. *Journal of Environmental Economics and Management*, 94:67–81.

Hougaard, J. L., Moreno-Ternero, J. D., Tvede, M., and Østerdal, L. P. (2017). Sharing the proceeds from a hierarchical venture. *Games and Economic Behavior*, 102:98–110.

Hougaard, J. L. and Østerdal, L. P. (2009). Decreasing serial cost sharing: An axiomatic characterization. *International Journal of Game Theory*, 38(4):469–479.

Martinez, R. and Moreno-Ternero, J. D. (2025). The allocation of riparian water rights. *Environmental and Resource Economics*, 88:3841–3872.

Martinez, R. and Moreno-Ternero, J. D. (2026a). The geometric adjudication of water rights in international rivers. Technical Report 2601.04150, arXiv.

Martinez, R. and Moreno-Ternero, J. D. (2026b). On the fair abatement of riparian pollution. Technical Report 2603.04345, arXiv.

Moulin, H. and Shenker, S. (1994). Average cost pricing versus serial cost sharing: An axiomatic comparison. *Journal of Economic Theory*, 64(1):178–201.

Muller, N. Z. and Mendelsohn, R. (2007). Measuring the damages of air pollution in the United States. *Journal of Environmental Economics and Management*, 54(1):1–14.

Ni, D. and Wang, Y. (2007). Sharing a polluted river. *Games and Economic Behavior*, 60:176–186.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 2(4):345–371.

Öztürk, Z. E. (2020). Fair social orderings for the sharing of international rivers: A leximin based approach. *Journal of Environmental Economics and Management*, 101:102302.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: A survey. *Mathematical Social Sciences*, 45:249–297.

Van De Putte, B. and Wintein, S. (2026). Claim-strength problems and mixtures: An axiomatic study. *Mathematical Social Sciences*, 141:103999.

van den Brink, J. R., Estévez-Fernández, A., van der Laan, G., and Moes, N. (2014). Independence of downstream and upstream benefits in river water allocation problems. *Social Choice and Welfare*, 43(1):173–194.

van den Brink, J. R., He, S., and Huang, J. P. (2018). Polluted river problems and games with a permission structure. *Games and Economic Behavior*, 108:182–205.

van den Brink, J. R., van der Laan, G., and Moes, N. (2012). Fair agreements for sharing international rivers with multiple springs and externalities. *Journal of Environmental Economics and Management*, 63(3):388–403.

Wintein, S. and Heilmann, C. (2024). How to be absolutely fair, part I: The fairness formula. *Economics and Philosophy*, 40:626–649.

Yang, Y., Ansink, E., and Gudmundsson, J. (2025). How to pollute a river if you must. *Journal of Environmental Economics and Management*, 130:103105.