TI 2026-021/II
Tinbergen Institute Discussion Paper

Collusion in the polluted river problem

Mikel Alvarez-Mozos[1]
Rene van den Brink[2]
Marti Jane-Ballarin[3]

1 Department of Mathematical Economics, Finance and Actuarial Sciences, and Barcelona Economics Analysis Team (BEAT), University of Barcelona

2 Vrije Universiteit Amsterdam, Tinbergen Institute

3 Department of Mathematical Economics, Finance and Actuarial Sciences, and Barcelona Economics Analysis Team (BEAT), University of Barcelona

Tinbergen Institute is the graduate school and research institute in economics of Erasmus University Rotterdam, the University of Amsterdam and Vrije Universiteit Amsterdam.

Contact: discussionpapers@tinbergen.nl

More TI discussion papers can be downloaded at https://www.tinbergen.nl

Tinbergen Institute has two locations:

Tinbergen Institute Amsterdam
Gustav Mahlerplein 117
1082 MS Amsterdam
The Netherlands
Tel.: +31(0)20 598 4580

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900

Collusion in the polluted river problem

Mikel Álvarez-Mozos[a], René van den Brink[b], Martí Jané-Ballarín[a,1]

[a]*Department of Mathematical Economics, Finance and Actuarial Sciences, and Barcelona Economics Analysis Team (BEAT), University of Barcelona, Avinguda Diagonal 690-696, Barcelona, 08034, Spain*
[b]*Department of Economics and Tinbergen Institute, VU University, De Boelelaan 1105, 1081 HV, Amsterdam, The Netherlands*

Abstract

We study the effects of collusion in the polluted river problem. In this model, agents along a river network have to share the costs of cleaning the entire river. We define a collusion neutrality property which requires that, if all agents upstream of a given point agree to collude and act as one, then the cost to be paid by this new entity equals the sum of the costs originally allocated to the colluding agents. This property is related to the Unlimited Territorial Integrity doctrine used to settle international disputes over water streams flowing through several territories. We introduce a new cost sharing method, called the Upstream Recursive Equal Sharing (URES) method, and axiomatically characterize it using our new collusion neutrality axiom. We also provide a new characterization for the well-known Local Responsibility Sharing (LRS) method that uses this axiom and which differs from the URES method only in one property that explains how costs are shared at the source of the river. Moreover, we derive and characterize a family of collusion neutral methods where the different methods are distinguished by a parametrized clean source property that balances between Unlimited Territorial Integrity and Absolute Territorial Sovereignty and that contains the LRS method as an extreme and the URES method as an intermediate method. Finally, we introduce and characterize the other extreme method in this class, which we call the limit recursive sharing method. All members of the family can be computed by means of a recursive algorithm.

Keywords: Polluted river, Cost sharing, Collusion neutrality, Axiomatization

JEL: C70, K32, K33, Q25

[1]Corresponding author: marti.jane@ub.edu

1. Introduction

Fresh water is an increasingly important natural resource, of which rivers are the most common source. According to Barrett (1994), at least 200 of the world's rivers flow through different countries: 148 rivers flow through two countries, 30 through three, 9 through four, and 13 through five or more. Many more flow through different provinces, regions or municipalities of the same country. Moreover, around 47% of the world's land area is located in an international river basin, a percentage that rises to 60% in Africa, Asia and South America (Kilgour and Dinar, 1995). Having access to the shared resource is a double-edged sword for countries in international basins: it confers them the right to use water from the river while, at the same time, they have the responsibility to keep the water clean and accessible for other countries.

Due to the increasing scarcity of water and its importance in developing economic activity, disputes over sharing international rivers are common, and can contribute to regional tensions and conflicts; see, for instance, Rieu-Clarke et al. (2012). These can be either directly tied to a reduction of the water level downstream due to excessive use or abuse by countries located upstream, or because of activities that pollute the river by upstream countries, resulting in a decline in water quality downstream. In either case, the flow of the river itself imposes a sense of directionality in the effects of mismanagement of the resource, from the upstream countries to those downstream.

Historically, states involved in conflicts regarding the use of river water have cited two legal principles in order to resolve their differences and avoid escalation: *Absolute Territorial Sovereignty*, also known as the Harmon doctrine, and *Unlimited Territorial Integrity* (Rieu-Clarke et al., 2012, p. 101). Absolute Territorial Integrity (ATS) states that a country has the right to use all the water that flows through its territory at its own will. Unlimited Territorial Integrity (UTI), on the other hand, requires that agents should not modify the natural conditions in their territory if it harms a downstream country. In terms of property rights, UTI contends that each country has a rightful claim to any water upstream of it; however, it is clear that not all such claims can be met at the same time.

In a seminal paper, Ambec and Sprumont (2002) use ATS and UTI to derive lower and upper bounds, respectively, to each country's welfare from water consumption. A unique welfare distribution is identified satisfying these bounds, thus providing a compromise between the two doctrines.[2] After their paper, the allocation of water resources has been addressed from different angles. Ansink and Weikard (2012) used bankruptcy rules to propose new solutions, while Öztürk (2020) maximizes a social welfare function to identify fair water allocations. With a focus on allocating the water resource itself instead of welfare, Gudmundsson et al. (2019) use a bargaining approach, and Martínez and Moreno-Tenero (2025) study two solutions aligned with the principle of Territorial Integration of Basin States (TIBS), which states that each basin state is entitled to a reasonable and equitable share of water.

Although initially proposed to settle disputes over rights to water use, the aforementioned legal principles have been adapted to assess each country's responsibility for pollution in the river. Taking pollution as a proxy for economic activity, van der Laan and Moes (2016) invoke ATS, UTI, and TIBS to determine fair monetary transfers between countries to be paid in compensation for polluting the river. Other models deal with the allocation of pollution permits, where a certain aggregate pollution level is tolerated. Hung and Shaw (2005) allow for the trading of these permits, in a model that distinguishes between pollutants by their transfer properties, that is, their tendency to either travel downstream with the river or accumulate at their origin. Finally, Yang et al. (2025) use a claims problem similar to that of Ansink and Weikard (2012) to allocate pollution permits over the agents (countries, regions, etc.) located along the river. Their key observation is that polluters upstream contribute to a higher aggregate disutility due to pollution, since they affect a longer section of the river, thus being reminiscent of the UTI principle.

Our paper looks at a cost sharing problem associated with cleaning up a polluted river. This problem was first modeled by Ni and Wang (2007), who used a weighted (directed) line graph to represent the course of a river with the nodes representing the agents, the edges being the segments of the river and the weights being the nonnegative costs needed to clean the polluted river at the corresponding river segment. The main question is how to allocate the cleaning costs among the agents. Ni and Wang (2007) propose two cost sharing

[2] Later, Ambec and Ehlers (2008) extended these results to a model with satiated agents, which was further generalized to rivers with multiple springs by van den Brink et al. (2012).

methods, namely the Local Responsibility Sharing (LRS) method and the Upstream Equal Sharing (UES) method, as the appropriate ways to implement the ATS and UTI doctrines, respectively. According to the LRS method, each agent is responsible for the cost of cleaning up the river segment within its own territory. On the other hand, the UES method is obtained by sharing the cost of cleaning up each segment equally among all upstream agents reflecting that, since pollution flows downstream along the river, these agents should also be held responsible for it. Dong et al. (2012) extend the two methods to rivers with multiple sources. Later on, van den Brink et al. (2018) model the problem as a game with a permission structure and derive new characterizations of the UES method as well as alternative methods to solve the problem. Alcalde-Unzu et al. (2015) incorporate pollution transfer rates to the model with a single spring and propose a different cost sharing rule using a quite different axiomatic approach, as it does not try to implement the existing international doctrines.

Our main contribution is the study of the possibilities for manipulation by collusion in this model. To the best of our knowledge, we are the first to consider this issue in the general polluted river problem; Yang et al. (2025) consider merge proofness in the context of their pollution claims problem, but only for rivers with a single source. Similarly to Yang et al. (2025), we take the river network as restricting the potential coalitions that may collude, namely we only allow certain connected coalitions in the directed graph to collude. In the polluted river problem, when such a coalition colludes, we collapse all its members into a single node, and add up all their cleaning costs into this node's outgoing edge. These changes may be motivated, for instance, by a desire to elude the coalition's responsibilities for pollution or a change in the jurisdictional level at which the cleaning costs are allotted (say, from the municipal to the provincial level). Either way, if such changes yield benefits for the colluding agents at the expense of other agents, they would introduce incentives for strategic behavior and lead to possibly unfair outcomes.[3] In particular, if we want to follow the UTI doctrine, agreements of this type should not harm any of the agents downstream of the colluding agents. We formalize this idea in a property we call collusion neutrality.

[3] Properties motivated by this practices have been used in many other related problems (Haller, 1994; Ju et al., 2007).

This property is satisfied by the naive LRS method, but not by the UES method. We introduce a new method that, similar to the UES method, shares the cost of cleaning up each segment among agents upstream and, similar to the LRS method, satisfies collusion neutrality. This method, which we refer to as the Upstream Recursive Equal Sharing (URES) method, challenges the idea of the UES method that cleaning costs should be shared equally among upstream agents. It is based on a recursive procedure that dilutes the responsibility of upstream agents: the further downstream the pollution, the less responsibility an agent must take for it.[4]

In addition to collusion neutrality, we impose three standard properties from the literature—efficiency, additivity, and the inessential agent property. Taken together, however, these four properties do not uniquely determine a cost sharing method, since they do not specify how the costs should be shared in two-agent river problems. To obtain a characterization of the new URES method, we therefore introduce a fifth property that applies only to clean sources, that is, nodes with no predecessors and zero pollution cost in the immediately downstream segment. The absence of pollution costs upstream may arise either because the agent located at the source generates no pollution or because all pollution has been transferred downstream. The URES method shares this cost equally between the source node and its successor. Under this assumption, both agents are assigned equal responsibility for downstream pollution, which can be interpreted as a weak implementation of UTI.

Alternatively, one could adopt the extreme assumption that the source node bears no responsibility at all, corresponding to an implementation of ATS. This approach is formalized by the no blind cost at sources property, which we use to obtain a parallel characterization of the LRS method. This result highlights a key distinction between the URES and LRS methods. While both satisfy the three standard axioms and collusion neutrality, they differ in how they allocate costs between a clean source and its immediate downstream neighbor. More generally, this distinction gives rise to a family of rules containing both the LRS and URES methods, parametrized by a nonnegative scalar that determines the ratio of payments

[4] This idea of decreasing responsibility the further away from the relevant segment is something that our approach has in common with the method introduced by Alcalde-Unzu et al. (2015), but the way responsibility is shared is different.

assigned to a clean source and its downstream neighbor. This parameter allows one to interpolate between UTI and ATS, and more broadly, to calibrate the strength with which UTI is interpreted.

As mentioned before, the LRS method is one extreme of this class, corresponding to a parameter value of zero, while the URES method occupies an intermediate position, with the parameter equal to one. We then consider the other extreme, obtained when the parameter tends to infinity, and show that this limit method allocates the entire cleaning costs to sources, which is formalized by the clean source full transfer property. Interestingly, the class of geometric rules proposed by Hougaard et al. (2017), which are developed for a different model,[5] have the same two extremes but are different otherwise. The last result of the paper characterizes the whole family of methods by means of the four properties used in the previous results, a symmetry property that only applies at sources, and the new clean source proportionality. The latter requires that clean sources share costs with their successors in the same ratio throughout the river network. It is worth stressing that, in this characterization, none of the axioms depends on the parameter used to generate the family.

The remainder of the paper is organized as follows. In Section 2 we formally introduce the model, the most common properties, and well-known cost sharing methods. Next, in Section 3 we introduce collusion neutrality and use it to characterize our new URES method. Section 4 presents a parallel characterization of the LRS method. Building upon these two characterizations, in Section 5 we derive and characterize a new class of collusion neutral cost sharing methods that includes the limit method mentioned above, for which we also provide a characterization. Finally, Section 6 concludes and points towards directions for future research. In the appendix we show logical independence of the axioms in each characterization.

2. Preliminaries

A loopless directed graph is a pair (N, D), where $N \subset \mathbb{N}$ is a finite set, whose elements are called *nodes*, and $D \subseteq \{(i, j) \in N \times N : i \neq j\}$ is a subset of ordered pairs of distinct nodes, where each element (i, j) is called a *directed edge*, which we will shortly refer to as

[5] Hougaard et al. (2017) study the allocation of revenue in a hierarchical venture.

an *edge*, from node i to node j. As long as there is no confusion about the set of nodes, we identify a graph (N, D) with its set of edges, D. For the remainder of this section, let (N, D) be a loopless directed graph. Given a subset of nodes $N' \subset N$, we denote by $D[N'] = D \cap (N' \times N')$ the set of edges with both endpoints in N'.

Given $i \in N$, we shall denote by $N_D^+(i) = \{j \in N : (i, j) \in D\}$ the set of nodes to which there is an edge from i. We refer to the nodes in $N_D^+(i)$ as the *successors* of i in D. A *bottom* node is one that has no successors; we denote the set of bottom nodes of D by $B(D) = \{i \in N : N_D^+(i) = \varnothing\}$. Similarly, we define by $N_D^-(i) = \{j \in N : (j, i) \in D\}$ the set of *predecessors* of i in D. We denote by $T(D) = \{i \in N : N_D^-(i) = \varnothing\}$ the set of *top* nodes of D, being the nodes which have no predecessors.

A *(directed) path* from $i \in N$ to $j \in N$ in D is a sequence of nodes $(i_0, \ldots, i_k)$, where $i_0 = i$, $i_k = j$, and $(i_l, i_{l+1}) \in D$ for every $l \in \{0, \ldots, k-1\}$. We denote by $N_D^{--}(i)$ the set of nodes from which there is a path ending at i, and call this the set of *superiors* of i in D.

We say that (N, D) is a *sink tree* if and only if there exists a node $b \in N$ such that (i) $B(D) = \{b\}$, (ii) $N_D^{--}(b) = N \setminus \{b\}$, and (iii) $\left|N_D^+(i)\right| = 1$ for all $i \in N \setminus \{b\}$. In words, a sink tree is a graph with a single bottom node to which there is a path from every other node, and all other nodes have exactly one successor, which we denote by i^+, i.e., $N_D^+(i) = \{i^+\}$. Then, for every $i, j \in N$, if there is a path from i to j, it is unique. Whenever such a path exists, we will denote its set of nodes by $[i, j]_D$.

A *polluted river problem* is a triple (N, D, c), where (N, D) is a sink tree and $c \in \mathbb{R}_+^N$ is an $|N|$-dimensional non-negative cost vector. In this setting, the sink tree represents a river network where water flows along the edges of D. Given $i \in N$, c_i represents the cost of cleaning the river segment between i and its sole successor.[6] In this context, we call the top nodes of D the *sources* of the river, and the sole bottom node its *mouth*. We denote by $\mathcal{R}$ the class of polluted river problems.

[6]In the case of the bottom node $b \in N$, which by construction does not have any successors, c_b represents the cost of cleaning the segment of the river from the last settlement it flows through and the body of water to which it leads (e.g. a sea or a lake). For convenience, we often speak about the cost of a node as the cost of cleaning the segment just below this node. We refer the reader to Ni and Wang (2007); Dong et al. (2012) for a more detailed description of the cost vector.

A *cost allocation* for $(N,\, D,\, c) \in \mathcal{R}$ is an $|N|$-dimensional non-negative vector $x \in \mathbb{R}^N_+$, where x_i represents the share in the total cost to be paid by node i. A *cost sharing method*, shortly referred to as a *method*, is a function ψ that maps every $(N,\, D,\, c) \in \mathcal{R}$ into a cost allocation for $(N,\, D,\, c)$.

Ni and Wang (2007) introduced the following two methods for single-source rivers. The *Local Responsibility Sharing* (LRS) method assigns to each node the cost of cleaning its own segment. The *Upstream Equal Sharing* (UES) method equally divides the cost of cleaning a segment among the nodes that are upstream from it. Dong et al. (2012) extended both methods to multiple source rivers.Formally, the LRS method assigns to every $(N, D, c) \in \mathcal{R}$, the cost allocation

$$\mathsf{LRS}_i\,(N,\, D,\, c) = c_i \qquad \text{for all } i \in N,$$

and the UES method assigns to every $(N, D, c) \in \mathcal{R}$, the cost allocation

$$\mathsf{UES}_i\,(N,\, D,\, c) = \sum_{j \in [i,\, b]_D} \frac{c_j}{\left|N_D^{--}(i) \cup i\right|} \qquad \text{for all } i \in N.$$[7]

We end this section by introducing some axioms a cost sharing method ψ may satisfy and characterizations for the LRS and UES methods from the literature.

Efficiency: For every $(N,\, D,\, c) \in \mathcal{R}$, $\sum_{i \in N} \psi_i\,(N,\, D,\, c) = \sum_{i \in N} c_i$.

Additivity: For every $(N,\, D,\, c)\,,\, (N,\, D,\, c') \in \mathcal{R}$, $\psi\,(N,\, D,\, c + c') = \psi\,(N,\, D,\, c) + \psi\,(N,\, D,\, c')$.

No blind cost: For every $(N,\, D,\, c) \in \mathcal{R}$ and $i \in N$, if $c_i = 0$, then $\psi_i\,(N,\, D,\, c) = 0$.

Inessential agent property: For every $(N,\, D,\, c) \in \mathcal{R}$ and $i \in N$ such that $c_j = 0$ for every $j \in [i,\, b]_D$, $\psi_i\,(N,\, D,\, c) = 0$.

Necessary agent property: For every $(N,\, D,\, c) \in \mathcal{R}$ and $i \in N$ with $c_j = 0$ for every $j \in N \setminus i$, $\psi_i\,(N,\, D,\, c) \geqslant \psi_j\,(N,\, D,\, c)$ for every $j \in N \setminus i$.

[7] We use the short-hand notation $S \cup i$ to denote $S \cup \{i\}$, where S is a generic set. Similarly, we will use $S \setminus i$ to denote $S \setminus \{i\}$.

Structural monotonicity: For every $(N, D, c) \in \mathcal{R}$ and $i, j \in N$, if $i \in N_D^-(j)$, then $\psi_i(N, D, c) \geqslant \psi_j(N, D, c)$.

Theorem 2.1 (Dong et al., 2012)**.** The LRS method is the only cost sharing method that satisfies efficiency, additivity and no blind cost.

Theorem 2.2 (van den Brink et al., 2018)**.** The UES method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, the necessary agent property and structural monotonicity.

3. The Upstream Recursive Equal Sharing method

In this section we introduce a new cost sharing method that, similar to the UES method, allocates the cost of a river segment among the agents upstream, but does not share the costs evenly. Instead, it reflects that responsibility decreases as the distance to the relevant segment increases. Our method has this feature in common with the method introduced by Alcalde-Unzu et al. (2015), although they consider a slightly different model. We discuss these connections later on.

In the URES method, which we introduce below, the share of pollution costs downstream of an agent depends on the structure of the river in the sense that (i) the closer an agent is to a downstream segment, the larger its share of that segment's pollution cost, and (ii) in rivers with multiple springs, the costs are divided so that, at any confluence, each tributary jointly pays the same share of the cleaning costs for the segments downstream of it. We formally define our new method as follows.

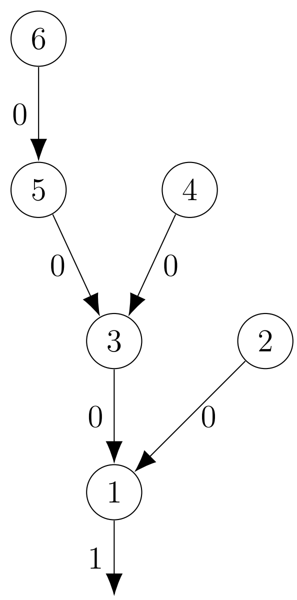
Definition 3.1. The *Upstream Recursive Equal Sharing* (URES) method assigns to every $(N, D, c) \in \mathcal{R}$, the cost allocation given by[8]

$$\mathsf{URES}_i(N, D, c) = \sum_{j \in [i,b]_D} \frac{c_j}{\prod\limits_{k \in [i,j]_D} \left(1 + \left|N_D^-(k)\right|\right)} \qquad \text{for all } i \in N. \tag{1}$$

[8]Notice that $1 + \left|N_D^-(k)\right| = \left|N_D^-(k) \cup k\right|$ $\forall k \in N$. Although the right-hand side expression would bring the notation closer to that of the UES method, we favor the form in (1), since it will make connections of the URES method with the class of solutions to be defined in Section 5 more clear.

The following example illustrates the URES method and compares the resulting allocation with that of the LRS and UES methods.

Example 3.2. Consider a river where there is only one non-zero cleaning cost, equal to one and located at the mouth of the river, node 1 (see Figure 1). The river network itself has two tributaries that flow into its mouth. Notice that the URES method allocates the same total costs to each tributary and the mouth itself. Indeed, the mouth is held responsible for one third of the cost, while another third is collectively allocated to nodes 3 through 6, which are the nodes upstream of the "left" tributary, and the final third is assigned to node 2, the only node in the course of the "right" tributary.



i	c_i	LRS_i	UES_i	URES_i
1	1	1	$\frac{1}{6}$	$\frac{1}{3}$
2	0	0	$\frac{1}{6}$	$\frac{1}{3}$
3	0	0	$\frac{1}{6}$	$\frac{1}{9}$
4	0	0	$\frac{1}{6}$	$\frac{1}{9}$
5	0	0	$\frac{1}{6}$	$\frac{1}{18}$
6	0	0	$\frac{1}{6}$	$\frac{1}{18}$

Figure 1: An application of the LRS, UES and URES methods to a polluted river problem.

This pattern repeats itself across the river network: node 4 pays $\frac{1}{9}$, as do the nodes 5 and 6 together who form the "left" tributary flowing into node 3, and node 4, which on its own forms the "right" tributary of node 3. $\square$

In general, for every $i \in N$ and predecessor $j \in N_D^-(i)$, the URES method assigns to i the same share in the cleaning costs of its downstream segments as to j and its superiors, i.e. $N_D^{--}(j) \cup j$. In particular, if $\left|N_D^-(i)\right| > 1$, then each tributary that joins the river at i pays the same share of the downstream costs. Moreover, this holds for any polluted river problem, where the allocation of the total cost is done by adding the allocations for each separate segment's cost, not only those where there is a unique non-zero cleaning cost. These observations suggest an algorithm to compute the URES method, which we will also use in the proofs of our axiomatizations below.

Proposition 3.3. Let Ψ be the function that assigns to every polluted river problem $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

Step 1: Set $N^1 = N$, $D^1 = D$, $c^1 = c$, and $\Psi_b(N, D, c) = \frac{c_b^1}{1+\left|N_D^-(b)\right|}$, where $b \in N$ is the mouth of the river.

Step $m \geqslant 2$: Set $N^m = N^{m-1} \setminus B(D^{m-1})$ and $D^m = D^{m-1}[N^m]$. If $N^m = \varnothing$, stop. Otherwise, define $c^m \in \mathbb{R}^{N^m}$ by[9]

$$c_i^m = \begin{cases} c_i^{m-1} + \frac{c_{i^+}^{m-1}}{1+\left|N_D^-(i^+)\right|} & \text{if } i \in B(D^m) \\ c_i^{m-1} & \text{otherwise.} \end{cases}$$

Set $\Psi_i(N, D, c) = \frac{c_i^m}{1+\left|N_D^-(i)\right|}$ for every $i \in B(D^m)$. Go to step $m+1$.

The resulting cost sharing method is the URES method, that is, $\Psi = \mathsf{URES}$.

Proof. Let $(N, D, c) \in \mathcal{R}$. First of all, note that the algorithm is guaranteed to stop in a finite number of steps, since N is finite and, at each step $m \geqslant 1$, if $N^m \neq \varnothing$, then $|N^{m+1}| < |N^m|$. Hence, Ψ is well-defined.

It follows from the algorithm and the definition of the URES method that for the mouth $b \in B(D)$, $\Psi_b(N, D, c) = \frac{c_b^1}{1+\left|N_D^-(b)\right|} = \frac{c_b}{1+\left|N_D^-(b)\right|} = \mathsf{URES}_b(N, D, c)$. Notice that $|[b, b]_D| = |\{b\}| = 1$, and $|[i, b]_D| > 1$ for all $i \in N \setminus b$. Inductively, suppose that $\Psi_i(N, D, c) = \mathsf{URES}_i(N, D, c)$ for all $i \in N$ with $|[i, b]_D| \leqslant r$ for some $r \geqslant 1$, and suppose there is some $j \in N$ such that $|[j, b]_D| = r + 1$. We will show that $\Psi_j(N, D, c) = \mathsf{URES}_j(N, D, c)$.

Observe that, for every $m \geqslant 1$, $|[i, b]_D| = m$ if and only if $i \in B(D^m)$, and the algorithm

[9] Note that for $m > 1$, $B(D^m)$ may contain more than one element.

sets $\Psi_i(N, D, c) = \frac{c_i^m}{1+\left|N_D^-(i)\right|}$ at step m for such i. In particular, since $\left|[j, b]_D\right| = r+1$,

$$\begin{aligned}\Psi_j(N, D, c) &= \frac{c_j^{r+1}}{1+\left|N_D^-(j)\right|} \\ &= \frac{1}{1+\left|N_D^-(j)\right|}\left(c_j^r + \frac{c_{j^+}^r}{1+\left|N_D^-(j^+)\right|}\right) \\ &= \frac{1}{1+\left|N_D^-(j)\right|}\left(c_j + \Psi_{j^+}(N, D, c)\right),\end{aligned}$$

where the second equality follows from the definition of c^{r+1} in the algorithm, and the final equality uses that, by construction, $c_k^r = c_k$ for every $k \in N$ such that $\left|[k, b]_D\right| > r$, hence $c_j^r = c_j$, and $j^+ \in B(D^r)$. By the induction hypothesis, $\Psi_{j^+}(N, D, c) = \mathsf{URES}_{j^+}(N, D, c)$, since $\left|[j^+, b]_D\right| = \left|[j, b]_D\right| - 1 = r$. Then,

$$\begin{aligned}\Psi_j(N, D, c) &= \frac{c_j}{1+\left|N_D^-(j)\right|} + \frac{\mathsf{URES}_{j^+}(N, D, c)}{1+\left|N_D^-(j)\right|} \\ &= \frac{c_j}{1+\left|N_D^-(j)\right|} + \frac{1}{1+\left|N_D^-(j)\right|}\sum_{k\in[j^+, b]_D}\frac{c_k}{\prod\limits_{l\in[j^+, k]_D}\left(1+\left|N_D^-(l)\right|\right)} \\ &= \frac{c_j}{1+\left|N_D^-(j)\right|} + \sum_{k\in[j^+, b]_D}\frac{c_k}{\prod\limits_{l\in[j, k]_D}\left(1+\left|N_D^-(l)\right|\right)} \\ &= \sum_{k\in[j, b]_D}\frac{c_k}{\prod\limits_{l\in[j, k]_D}\left(1+\left|N_D^-(l)\right|\right)} = \mathsf{URES}_j(N, D, c),\end{aligned}$$

where the second and final equalities follow from the definition of the URES method. $\square$

As we have seen, the URES method distributes the cleaning cost of each segment equally among the node immediately upstream of the segment and each of the tributaries (if any) that join the river at that point. This principle is then applied recursively to share this joint cost among the agents located upstream. For this reason, we call it the Upstream Recursive Equal Sharing method.

Towards a characterization of the URES method, we introduce two axioms a cost sharing method ψ might satisfy. The first new axiom is the main motivation for our paper, while the second one will be relaxed in later sections. Before introducing the first one, we will need some additional notation.

Given $(N, D, c) \in \mathcal{R}$ and $i \in N$, suppose i and its superiors decide to collude and act as a single entity. This induces a new polluted river problem $(N_D^{i*}, D^{i*}, c^{i*}) \in \mathcal{R}$, where

$N_D^{i*} = N \setminus N_D^{--}(i)$, $D^{i*} = D[N_D^{i*}]$, and the cleaning cost vector c^{i*} is such that the cleaning cost of each segment in D^{i*} remains the same, except for the one that originates at i, which now also takes into account the cleaning costs of all its upstream segments; that is, for $j \in N_D^{i*}$,

$$c_j^{i*} = \begin{cases} \sum\limits_{k \in N_D^{--}(i) \cup i} c_k & \text{if } j = i \\ c_j & \text{otherwise.} \end{cases}$$

Collusion neutrality requires that, if a node reaches an agreement to represent the nodes upstream of it in the polluted river problem, then this agreement does not change the total cost allocated to these nodes.

Collusion neutrality: For every $(N, D, c) \in \mathcal{R}$ and $i \in N$, $\psi_i\left(N_D^{i*}, D^{i*}, c^{i*}\right) = \sum\limits_{j \in N_D^{--}(i) \cup i} \psi_j\left(N, D, c\right)$.

As we discussed in the introduction, this axiom relates to the UTI doctrine for settling international disputes. According to this doctrine, a country may not alter the natural conditions on its own territory to the disadvantage of a downstream country. Since we only consider that all agents upstream a given point can collude, we argue that these kind of manipulations should not affect the payoffs to the rest of agents. Note that collusion neutrality demands that the colluding coalition cannot unilaterally override the given allocation to their benefit. If efficiency is also satisfied, the new property implies that this kind of strategic behavior cannot affect the *collective* cost allocated to the agents not participating in the agreement. However, by itself, the property does not preclude the alteration of the allocation to any of these agents individually. We elaborate on this later on in Remark 3.5.

To introduce our second new axiom, consider a segment with zero cleaning cost at a source of the river. As mentioned in the introduction, this does not necessarily mean that the agent located in that source has not polluted, as it is possible that this pollution has been transferred downstream. Without further information on the origin of the pollution, it seems reasonable to assume that the agent at the source should pay the same as its successor.

Clean source property: For every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_t\left(N, D, c\right) = \psi_{t^+}\left(N, D, c\right)$, where t^+ is the sole successor of t in D.

We argue that this clean source property also follows the UTI doctrine, which implies that agents should bear some responsibility for cleaning the downstream segments. Specifically, it holds the source equally responsible for the pollution at its downstream neighbor. In Section 5 we consider alternative interpretations of UTI, where responsibility is not distributed evenly between these two agents. We also point out that, under additivity, the clean source property is weaker than upstream symmetry, which is used by Dong et al. (2012) to axiomatize the UES method and requires that, if there is only one agent with non-zero cost, then this agent and all upstream agents pay the same.[10]

Together with the known axioms of efficiency, additivity and the inessential agent property, the two new axioms characterize the URES method.

Theorem 3.4. The URES method is the unique cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, and the clean source property.

Proof. It is straightforward from Definition 3.1 that the URES method satisfies additivity. The inessential agent property is also trivially satisfied, as agents only pay shares of their own cost and costs downstream. To check that the URES method satisfies the clean source property, consider $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$ such that $c_t = 0$. By Definition 3.1 and the null cost at node t, it follows that $\frac{\mathsf{URES}_t(N,D,c)}{\mathsf{URES}_{t^+}(N,D,c)} = \frac{1}{1+\left|N_D^-(t)\right|}$, which equals one because, by definition, $t \in T(D)$ implies $N_D^-(t) = \varnothing$.

To show efficiency, we proceed by induction on the *depth* of the network. Given a sink tree (N, D), we define its depth as $\mathsf{depth}\,(N, D) = \max_{i \in N} |[i, b]_D|$. If $(N, D, c) \in \mathcal{R}$ is such that $\mathsf{depth}\,(N, D) = 1$, then $|N| = 1$, and efficiency is trivially satisfied. Inductively, suppose the URES method is efficient on problems $\left(\overline{N}, \overline{D}, \overline{c}\right) \in \mathcal{R}$ where $\mathsf{depth}\left(\overline{N}, \overline{D}\right) \leqslant d$ for some $d \geqslant 1$, and let $(N, D, c) \in \mathcal{R}$ be such that $\mathsf{depth}\,(N, D) = d + 1$. We will show that $\sum_{i \in N} \mathsf{URES}_i\,(N, D, c) = \sum_{i \in N} c_i$.

For every $i \in N$, define $N_D^{i-} = N_D^{--}(i) \cup i$ and $D^{i-} = D\left[N_D^{i-}\right]$. Consider $(N^2, D^2, c^2) \in \mathcal{R}$

[10]Formally, upstream symmetry requires that, for every $(N, D, c) \in \mathcal{R}$ and $i \in N$ with $c_j = 0$ for every $j \in N \setminus i$, $\psi_i\,(N, D, c) = \psi_j\,(N, D, c)$ for every $j \in N_D^{--}(i)$. We remark that for our axiomatization it is sufficient to require the clean source property only for cost vectors with one positive cost.

generated by the algorithm in Proposition 3.3.[11] We claim that, for every $i \in N^2 = N \setminus b$, $\mathsf{URES}_i\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right) = \mathsf{URES}_i\,(N,\, D,\, c)$, where j is the only predecessor of b in $[i,\, b]_D$.

First of all, note that $N_D^{j-} \subseteq N^2$, and $\left(N_D^{j-},\, D^{j-}\right)$ is a sink tree with sink j. Hence, $\mathsf{URES}_i\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right)$ is well-defined. Also notice that for every $l \in N_D^{j-}$ we have $N^-_{D^{j-}}(l) = N^-_D(l)$, and for every $k \in N^{--}_{D^{j-}}(l)$, $[k,\, l]_{D^{j-}} = [k,\, l]_D$. Thus,

$$\mathsf{URES}_i\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right) = \sum_{k \in [i,\, j]_{D^{j-}}} \frac{c^2_k}{\prod\limits_{l \in [i,\, k]_{D^{j-}}} \left(1 + \left|N^-_{D^{j-}}(l)\right|\right)} = \sum_{k \in [i,\, j]_D} \frac{c^2_k}{\prod\limits_{l \in [i,\, k]_D} \left(1 + \left|N^-_D(l)\right|\right)},$$

and, by the definition of the cost vector c^2,

$$\begin{aligned}
\mathsf{URES}_i\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right) &= \sum_{\substack{k \in [i,\, j]_D \\ k \neq j}} \frac{c^2_k}{\prod\limits_{l \in [i,\, k]_D} \left(1 + \left|N^-_D(l)\right|\right)} + \frac{c^2_j}{\prod\limits_{l \in [i,\, j]_D} \left(1 + \left|N^-_D(l)\right|\right)} \\
&= \sum_{\substack{k \in [i,\, j]_D \\ k \neq j}} \frac{c_k}{\prod\limits_{l \in [i,\, k]_D} \left(1 + \left|N^-_D(l)\right|\right)} + \frac{c_j + \frac{c_b}{1 + \left|N^-_D(b)\right|}}{\prod\limits_{l \in [i,\, j]_D} \left(1 + \left|N^-_D(l)\right|\right)} \\
&= \sum_{k \in [i,\, j]_D} \frac{c_k}{\prod\limits_{l \in [i,\, k]_D} \left(1 + \left|N^-_D(l)\right|\right)} + \frac{c_b}{\prod\limits_{l \in [i,\, b]_D} \left(1 + \left|N^-_D(l)\right|\right)} \\
&= \mathsf{URES}_i\,(N,\, D,\, c)\,,
\end{aligned}$$

where in the penultimate equality we use that $j \in N^-_D(b)$, and the final equality follows from the definition of the URES method.

Since $\mathsf{depth}\left(N_D^{j-},\, D^{j-}\right) < d + 1$, the induction hypothesis applies to yield

$$\sum_{i \in N_D^{j-}} \mathsf{URES}_i\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right) = \sum_{i \in N_D^{j-}} c^2_i.$$

Then,

$$\begin{aligned}
\sum_{i \in N} \mathsf{URES}_i\,(N,\, D,\, c) &= \mathsf{URES}_b\,(N,\, D,\, c) + \sum_{i \in N \setminus b} \mathsf{URES}_i\,(N,\, D,\, c) \\
&= \frac{c_b}{1 + \left|N^-_D(b)\right|} + \sum_{j \in N^-_D(b)} \sum_{i \in N_D^{j-}} \mathsf{URES}_i\,(N,\, D,\, c)
\end{aligned}$$

[11] Note that since $d \geqslant 1$, by hypothesis $\mathsf{depth}\,(N,\, D) = d + 1 > 1$, so $N^2 = N \setminus b$ is non-empty.

$$= \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{j\in N_D^-(b)} \sum_{i\in N_D^{j-}} \mathsf{URES}_i \left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right)$$

$$= \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{j\in N_D^-(b)} \sum_{i\in N_D^{j-}} c_i^2,$$

and, by definition of c^2,

$$\sum_{i\in N} \mathsf{URES}_i\,(N,\, D,\, c) = \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{j\in N_D^-(b)} \left(c_j + \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{i\in N_D^{--}(j)} c_i \right)$$

$$= \frac{c_b}{1+\left|N_D^-(b)\right|} + \left|N_D^-(b)\right| \frac{c_b}{1+\left|N_D^-(b)\right|} + \sum_{i\in N\setminus b} c_i$$

$$= \sum_{i\in N} c_i.$$

To show collusion neutrality, let $(N,\, D,\, c) \in \mathcal{R}$ and $i \in N$. Note that if $i \in B(D)$, that is, i is the mouth of the river, collusion neutrality follows directly from efficiency. If $i \in T(D)$, collusion neutrality is trivial.

Suppose then $i \in N \setminus (T(D) \cup B(D))$ is neither a source nor the mouth of the river, and consider the problem $(N_D^{i*},\, D^{i*},\, c^{i*})$, where i and its superiors collude. The key observation is that, for every $j \in N^{i*}$, the path from j to the mouth of the river contains the same nodes in network D^{i*} as it does in D, that is, $[j,\, b]_{D^{i*}} = [j,\, b]_D$. Furthermore, if $j \in N_D^{i*} \setminus i$, for each $k \in [j,\, b]_D$, $N_D^-(k) = N_{D^{i*}}^-(k)$. Finally, since for every $j \in N_D^{i*} \setminus i$, $c_j^{i*} = c_j$, the definition of the URES method in (1) implies that

$$\mathsf{URES}_j\,(N,\, D,\, c) = \mathsf{URES}_j \left(N_D^{i*},\, D^{i*},\, c^{i*}\right) \qquad \text{for every } j \in N_D^{i*} \setminus i. \tag{2}$$

Collusion neutrality follows from this observation and the fact that the URES method is efficient, as previously shown.

To show uniqueness, let ψ be a cost sharing method satisfying the properties in the statement, and let $(N,\, D,\, c) \in \mathcal{R}$. If $|N| = 1$, efficiency implies that ψ is uniquely defined. Let $n \geqslant 1$ be such that ψ is uniquely determined on problems with up to n nodes, and suppose $|N| = n + 1$.

Due to additivity, it suffices to show uniqueness for problems where only one segment has non-zero cleaning cost. Suppose $i \in N$ is such that $c_i > 0$ and for every $j \in N \setminus i$, $c_j = 0$.

The inessential agent property implies that

$$\psi_j(N,\, D,\, c) = 0 \qquad \text{for every } j \in N \setminus (N_D^{--}(i) \cup i). \tag{3}$$

Then, if $N_D^{--}(i) = \varnothing$, that is, $i \in T(D)$, $\psi(N,\, D,\, c)$ is uniquely determined by efficiency and (3).

Therefore, suppose $i \in N \setminus T(D)$. The clean source property implies that

$$\psi_j(N,\, D,\, c) = \psi_{j^+}(N,\, D,\, c) \qquad \text{for every } j \in N_D^{--}(i) \cap T(D). \tag{4}$$

By collusion neutrality, we obtain

$$\sum_{k \in N_D^{--}(j) \cup j} \psi_k(N,\, D,\, c) = \psi_j\left(N_D^{j*},\, D^{j*},\, c^{j*}\right) \qquad \text{for every } j \in (N_D^{--}(i) \cup i) \setminus T(D). \tag{5}$$

By construction, $\left|N_D^{j*}\right| \leqslant n$ for every $j \in (N_D^{--}(i) \cup i) \setminus T(D)$ and thus the induction hypothesis implies that, for each such j, $\psi\left(N_D^{j*},\, D^{j*},\, c^{j*}\right)$ is uniquely determined.[12] Hence, the right-hand side of each equation (5) is a constant.

In summary, (4) and (5) together provide $\left|N_D^{--}(i) \cap T(D)\right| + \left|(N_D^{--}(i) \cup i) \setminus T(D)\right| = \left|N_D^{--}(i) \cup i\right|$ equations in the same number of unknowns $x_j = \psi_j(N,\, D,\, c)\,,\ j \in N_D^{--}(i) \cup i$. Therefore, together with (3), to end the proof it is necessary and sufficient to show that the equations are linearly independent.

First, observe that the set of equations of the form (4) is linearly independent, since each of them contains a variable associated to a top node, which does not appear in any other equation of this form.

Now, let $d(k) = \max\limits_{t \in T(D) \cap N_D^{--}(k)} \left|[t,\, k]_D\right| - 1$ be the maximum length of a path from a top node of D to $k \in N$. We will show that for every $k \in N_D^{--}(i) \cup i$, the system of equations (4) and (5) with $j \in N_D^{--}(k) \cup k$ are linearly independent. If $d(k) = 0$, then $k \in T(D)$, and the result is trivial, since there is only one equation.

Proceeding by induction, suppose that, for some $d \geqslant 0$, for every $k \in N_D^{--}(i) \cup i$ with $d(k) \leqslant d$, all equations (4) and (5) with $j \in N_D^{--}(k) \cup k$ are linearly independent. Suppose there is some $l \in N_D^{--}(i) \cup i$ such that $d(l) = d + 1$. By the induction hypothesis, for every

[12] Note that if $j = i$, equation (5) is also implied by efficiency.

$r \in N_D^-(l)$, the equations (4) and (5) with $j \in N_D^{--}(r) \cup r$ are linearly independent, since for every $r \in N_D^-(l)$, $d(r) \leqslant d$. Furthermore, for any pair $r, s \in N_D^-(l) \setminus T(D)$, $r \neq s$, the sets of equations (4) and (5) associated with the nodes in $N_D^{--}(r) \cup r$ and $N_D^{--}(s) \cup s$ are linearly independent from each other, since they involve disjoint sets of variables. Hence, the equations for $j \in N_D^{--}(l) \setminus \left(N_D^-(l) \cap T(D)\right)$ are linearly independent.

Note that the variables x_j with $j \in \left(N_D^-(l) \cap T(D)\right) \cup l$ only appear in the equations associated with these nodes. Thus, it only remains to be shown that the resulting set of equations is linearly independent, since it is independent from the rest of equations at issue. The equations associated with $j \in N_D^-(l) \cap T(D)$ are of the form (4), which we have shown to be independent. The last equation of the form (5), associated with l, also involves variables x_r with $r \in N_D^{--}(l) \setminus T(D)$, which do not appear in the previous equations, hence the result follows.

Summarizing, it is shown then that, for every $k \in N_D^{--}(i) \cup i$, the set of equations (4) and (5) with $j \in N_D^{--}(k) \cup k$ is linearly independent. In particular, this is true for $j = i$, which is what we needed to prove. □

Remark 3.5. Earlier, we remarked that collusion neutrality reflects the UTI doctrine since, under efficiency, it implies that collusion of an agent with all its superiors does not change the total contribution of all other agents together. A stronger interpretation of UTI is to require that such a collusion does not have an impact on the contribution of any of the other agents individually. On its turn, under efficiency, this collusion invariance[13] axiom implies collusion neutrality. Since the URES method also satisfies collusion invariance, as seen in the derivation of (2) in the proof of Theorem 3.4, we can replace collusion neutrality by collusion invariance in the characterization of the URES method.

In contrast with the UES methods, the newly defined URES method dilutes the responsibility of a node to clean up a downstream segment as its distance from the segment increases. In a related model, instead of the river network, Alcalde-Unzu et al. (2015) use the cost vector

[13]Formally, a cost sharing method ψ satisfies collusion invariance if, for every $(N, D, c) \in \mathcal{R}$ and $i \in N$, $\psi_j\left(N_D^{i*}, D^{i*}, c^{i*}\right) = \psi_j(N, D, c)$ for every $j \in N \setminus \left(N_D^{--}(i) \cup i\right)$.

to determine the responsibility of each node. In their model, the *transfer rate* of pollution, that is, the share of pollution transferred from one segment of the river to the next, plays an important role. In fact, the specification of a problem in their model requires an interval that reflects a social planner's initial knowledge of the transfer rate. The solution proposed by Alcalde-Unzu et al. (2015), named the *Upstream Responsibility* method, estimates the transfer rate in a given polluted river network by means of the cost vector[14] and computes the responsibility of each node using this information.

On the other hand, in a slightly more general model than polluted river problems, Hougaard et al. (2017) introduce the class of *geometric* cost sharing methods. When applied to polluted river problems, the elements in this class also dilute the responsibility of a node for downstream costs as the distance increases, similarly to the URES method. The difference lies in how rivers with multiple springs are treated, hence, in how downstream costs are shared among tributaries that join the river at the confluence. We defer this discussion to Section 6, where we compare the geometric methods to a newly defined class of methods.

Whereas the UES method does not satisfy collusion neutrality, the LRS method does. However, the clean source property is satisfied by the UES method, but not by the LRS method. As we see in the next section, the LRS method satisfies an alternative clean source property which can be used to axiomatize this method.

4. A new axiomatization of the LRS method

From the properties used to characterize the URES method in Theorem 3.4, the LRS method satisfies all except the clean source property. However, this property is just a specific way to interpret the UTI doctrine. It implies that, if a segment at the source of the river is clean, then the source node is to be held equally responsible for the pollution downstream as its immediate successor. The UES method extends this interpretation to the whole network so that the responsibility for cleaning each segment is evenly distributed among the agents upstream of it.

[14] The transfer rate is estimated as the average between the lower bound of the given interval and the minimum ratio of the cleaning costs of a node and one of its predecessors. If this minimum is greater than the upper bound of the interval, the midpoint of the initial interval is taken instead.

On the other hand, Ni and Wang (2007) argue that the LRS method is an implementation of the Absolute Territorial Sovereignty (ATS) principle. Indeed, Ni and Wang (2007) regard ATS as "a statement that people living in a segment of the river have an absolute sovereignty to ask any polluter located within said segment to pay the costs of cleaning pollutants [in the segment]". Hence, irrespective of the network, the responsibility for the costs of cleaning pollutants in a segment should be assigned to the node where this segment originates, which Ni and Wang (2007) formalized into the no blind cost axiom, as recalled in the preliminaries of this paper, and which requires that agents with zero cost pay zero.

We propose a significant weakening of the no blind cost property by applying it at the sources of the river only.

No blind cost at sources: For every $(N,\, D,\, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_t\,(N,\, D,\, c) = 0$.

As such, no blind cost at sources serves as a variant of the clean source property that relaxes the no blind cost property and the ATS doctrine in the same way that the clean source property relaxes some of the principles of the UES method and the UTI doctrine, and applies them only at the top of the river network. The following result shows that, by replacing the clean source property by no blind cost at sources in Theorem 3.4, we obtain a new characterization for the LRS method.

Theorem 4.1. The LRS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, and no blind cost at sources.

Proof. It is straightforward from the definitions that the LRS method satisfies the five properties. To show uniqueness, it suffices to follow the same process described in the proof of Theorem 3.4, just changing the part that uses the clean source property. To be specific, at the induction step of the proof, consider a polluted river problem $(N,\, D,\, c) \in \mathcal{R}$ where $c_j = 0\ \forall j \in N \setminus i$ and $N_D^{--}(i) \neq \varnothing$. Now, instead of (4), which was implied by the clean source property, in this proof no blind cost at sources implies that

$$\psi_j\,(N,\, D,\, c) = 0 \qquad \text{for every } j \in N_D^{--}(i) \cap T(D). \tag{6}$$

Replacing (4) by (6), together with (5), which follows from collusion neutrality, similarly to the proof of Theorem 3.4, we obtain a set of $\left|N_D^{--}(i) \cup i\right|$ equations with as many variables. The same arguments as in the proof of Theorem 3.4 can be used to show that the equations are linearly independent, which ends the proof. $\square$

5. A class of collusion neutral cost sharing methods

As mentioned earlier, the new characterization of the LRS method we present in the previous section provides some insight on how to derive a class of cost sharing methods that satisfy collusion neutrality. The key lies in the differences between the URES and LRS methods.

5.1. The α-URS methods

In summary, from a normative point of view, we have seen that the two methods only differ in how they allocate costs at the top of the river network: the URES method satisfies the clean source property reflecting that any source is as responsible as its successor for downstream pollution, while the LRS method satisfies no blind cost at sources which reflects that sources bear no responsibility for downstream pollution.[15] This characterizes the different doctrines where the LRS method follows the ATS doctrine by assigning a zero payment to a clean source, while the URES method follows the UTI doctrine by applying an equal sharing principle at the top of the river network with a clean source.

To make a trade-off between ATS and UTI, we introduce a parameter that requires that, in case of zero cleaning costs at a source, this source pays a fraction $\alpha \geqslant 0$ of the payments of its unique successor.

α-clean source property: Given $\alpha \geqslant 0$, for every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_t(N, D, c) = \alpha\psi_{t^+}(N, D, c)$.

[15] Collusion neutrality extends these properties to tributaries: given a tributary that flows into the river at node i, the URES method assigns to the tributary and i the same responsibility for downstream costs, while the LRS method contends that the tributary bears no responsibility for downstream pollution.

It is clear that for $\alpha = 0$ the axiom becomes the no blind cost at sources property, and for $\alpha = 1$ it becomes the clean source property.

We will see that, for each $\alpha \geqslant 0$, replacing the clean source property or the no blind cost at sources property in Theorem 3.4 or Theorem 4.1, respectively, by the α-clean source property, we characterize the α-URS method given by the following definition.

Definition 5.1. Let $\alpha \geqslant 0$. The *α-Upstream Recursive Sharing (α-URS)* method assigns to each $(N, D, c) \in \mathcal{R}$ the cost allocation given by

$$\mathsf{URS}_i^{\alpha}(N, D, c) = \sum_{j \in [i, b]_D} \alpha^{\left|[i, j]_D\right| - 1} \frac{c_j}{\prod\limits_{k \in [i, j]_D} \left(1 + \alpha \left|N_D^-(k)\right|\right)} \qquad \text{for all } i \in N. \tag{7}$$

It is straightforward to see that the α-URS methods for $\alpha = 0$ and $\alpha = 1$ are the LRS method and the URES method, respectively. Before giving the axiomatization of the α-Upstream Recursive Sharing methods, we illustrate several of these methods.

Example 5.2. Figure 2 depicts the allocations prescribed by some members of the class for the polluted river problem given in Example 3.2.

i	$\mathsf{URS}_i^0 = \mathsf{LRS}_i$	$\mathsf{URS}_i^{\frac{1}{2}}$	$\mathsf{URS}_i^1 = \mathsf{URES}_i$	URS_i^2	URS_i^{∞}
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	0
2	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$
3	0	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{2}{25}$	0
4	0	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{4}{25}$	$\frac{1}{4}$
5	0	$\frac{1}{24}$	$\frac{1}{18}$	$\frac{4}{75}$	0
6	0	$\frac{1}{48}$	$\frac{1}{18}$	$\frac{8}{75}$	$\frac{1}{4}$

Figure 2: The allocations given by some α-URS methods applied to Example 3.2.

Indeed, we see that the $\frac{1}{2}$-URS method assigns to node 2 half the payment as to node 1; the tributary formed by nodes 3 through 6 also collectively pays half as much as node 1. Similarly, under the 2-URS method, the payment of node 2 and the sum of payments of nodes 3 through

6 are double the payment of node 1. Furthermore, $\mathsf{URS}_5^{\frac{1}{2}} + \mathsf{URS}_6^{\frac{1}{2}} = \mathsf{URS}_4^{\frac{1}{2}} = \frac{1}{16} = \frac{1}{2} \cdot \mathsf{URS}_3^{\frac{1}{2}}$ and $\mathsf{URS}_6^2 + \mathsf{URS}_5^2 = \mathsf{URS}_4^2 = \frac{4}{25} = 2 \cdot \mathsf{URS}_3^2$. This is in line with the idea that, for every node i, the α-URS method assigns α times as much responsibility for downstream pollution to each tributary leading to i than to i itself. □

We can generalize Proposition 3.3 to compute any α-URS method. As in the case of the URES method, the algorithm will be useful to show some of the properties satisfied by the α-URS methods.

Proposition 5.3. Given $\alpha \geqslant 0$, let Ψ^α be the function that assigns to every polluted river problem $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

Step 1: Set $N^1 = N$, $D^1 = D$, $c^1(\alpha) = c$, and $\Psi_b^\alpha(N, D, c) = \frac{c_b^1(\alpha)}{1+\alpha\left|N_D^-(b)\right|}$, where $b \in N$ is the mouth of the river.

Step $m \geqslant 2$: Set $N^m = N^{m-1} \setminus B\left(D^{m-1}\right)$ and $D^m = D^{m-1}\left[N^m\right]$. If $N^m = \varnothing$, stop. Otherwise, define $c^m(\alpha) \in \mathbb{R}^{N^m}$ by

$$c_i^m(\alpha) = \begin{cases} c_i^{m-1}(\alpha) + \frac{1}{\left|N_D^-(i^+)\right|}\left(c_{i^+}^{m-1}(\alpha) - \frac{c_{i^+}^{m-1}(\alpha)}{1+\alpha\left|N_D^-(i^+)\right|}\right) & \text{if } i \in B\left(D^m\right) \\ c_i^{m-1}(\alpha) & \text{otherwise.} \end{cases}$$

Set $\Psi_i^\alpha(N, D, c) = \frac{c_i^m(\alpha)}{1+\alpha\left|N_D^-(i)\right|}$ for every $i \in B\left(D^m\right)$. Go to step $m+1$.

The resulting cost sharing method is the α-URS method, that is, $\Psi^\alpha = \mathsf{URS}^\alpha$.

The proof of Proposition 5.3 is relegated to the appendix as it follows the same arguments as that of Proposition 3.3. Similarly, the proof for Theorem 5.4 below uses the same arguments as Theorem 3.4 and can also be found in the appendix.

Theorem 5.4. Let $\alpha \geqslant 0$. The α-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, and the α-clean source property.

This result characterizes each α-URS method where the difference between these methods is restricted to the α-clean source property, i.e. how to share the cost of a successor of a clean source between them. A next step would be to characterize the class of all α-URS methods without any axiom that depends on α. But before doing that we note that, whereas the LRS method is one extreme method satisfying the most ATS type of clean source property, the URS method is not an extreme but an intermediate method obtained for $\alpha = 1$.

5.2. The Limit Recursive Sharing method

To further exploit the UTI doctrine, we can take values of α greater than one. In fact, the most UTI type of method is obtained by letting α go to infinity. As we will see, this yields the following method.

Definition 5.5. The *Limit Recursive Sharing (∞-URS)* method assigns to each $(N, D, c) \in \mathcal{R}$ the cost allocation given by

$$\mathsf{URS}_i^{\infty}(N, D, c) = \begin{cases} \sum\limits_{j \in [i, b]_D} \frac{c_j}{\prod\limits_{k \in [i^+, j]_D} \left| N_D^-(k) \right|} & \text{if } i \in T(D) \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

This method fits naturally within the class of α-URS methods, as it is the limit of these methods as α grows to infinity. We prove this formally in the appendix.

Proposition 5.6. The ∞-URS method satisfies $\mathsf{URS}^{\infty} = \lim\limits_{\alpha \to \infty} \mathsf{URS}^{\alpha}$.

It is easy to see that among the axioms used in the characterization for the α-URS methods in Theorem 5.4, the ∞-URS method satisfies all except the α-clean source property for any $\alpha \geqslant 0$. Instead, it satisfies the following axiom that requires that the successor of a clean source pays zero.

Clean source full-transfer: For every $(N, D, c) \in \mathcal{R}$ and $t \in T(D)$, if $c_t = 0$, then $\psi_{t^+}(N, D, c) = 0$.

This property states that all the responsibility lies in the sources of the river. It follows an extreme interpretation of the UTI doctrine and makes sense in situations where pollution travels far with high probability. Even if there is uncertainty about the origin of the pollution, all the cost is *transferred* to the sources of the river.

Thus, clean source full-transfer takes the opposite view from no blind cost at sources regarding how costs are divided at the top of the river network. Namely, no blind cost at sources implies that clean sources need not pay any cleaning costs, while clean source full-transfer states that their successors pay nothing.

In the characterization of the ∞-URS method, we will also use the following axiom, which is a weak symmetry requirement. In words, it imposes that any two sources with zero cleaning cost pay the same.

Clean source symmetry: For every $(N,\, D,\, c) \in \mathcal{R}$, if $s,\, t \in T(D)$ have the same successor, that is, $s^+ = t^+$, and $c_s = c_t = 0$, then $\psi_t\,(N,\, D,\, c) = \psi_s\,(N,\, D,\, c)$.

As it is the case with several axioms we have introduced, clean source symmetry only applies to the top of the river network. It is worth noting that all of the cost sharing methods discussed throughout the paper satisfy clean source symmetry. The ∞-URS method is characterized by replacing the α-clean source property in Theorem 5.4 by the clean source full-transfer property and adding clean source symmetry.

Theorem 5.7. The ∞-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source full-transfer, and clean source symmetry.

Proof. It is immediate from Definition 5.5 that the ∞-URS method satisfies additivity, the inessential agent property, clean source symmetry and the clean source full-transfer property.

Furthermore, since every α-URS method is efficient, by the sum law of limits, for every $(N,\, D,\, c) \in \mathcal{R}$,

$$\begin{aligned}\sum_{i \in N} \mathsf{URS}_i^{\infty}\,(N,\, D,\, c) &= \sum_{i \in N} \lim_{\alpha \to \infty} \mathsf{URS}_i^{\alpha}\,(N,\, D,\, c) \\ &= \lim_{\alpha \to \infty} \sum_{i \in N} \mathsf{URS}_i^{\alpha}\,(N,\, D,\, c) = \lim_{\alpha \to \infty} \sum_{i \in N} c_i = \sum_{i \in N} c_i,\end{aligned}$$

showing that the ∞-URS method satisfies efficiency.

Similarly, since the α-URS methods are collusion neutral, for any $(N,\, D,\, c) \in \mathcal{R}$ and

$i \in N$, it follows that

$$\sum_{j \in N_D^{--}(i) \cup i} \mathsf{URS}_j^{\infty}(N, D, c) = \sum_{j \in N_D^{--}(i) \cup i} \lim_{\alpha \to \infty} \mathsf{URS}_j^{\alpha}(N, D, c) = \lim_{\alpha \to \infty} \sum_{j \in N_D^{--}(i) \cup i} \mathsf{URS}_j^{\alpha}(N, D, c)$$
$$= \lim_{\alpha \to \infty} \mathsf{URS}_i^{\alpha}\left(N_D^{i*}, D^{i*}, c^{i*}\right) = \mathsf{URS}_i^{\infty}\left(N_D^{i*}, D^{i*}, c^{i*}\right).$$

showing that the ∞-URS method satisfies collusion neutrality.

As in Theorems 4.1 and 5.4, to show uniqueness we proceed as in Theorem 3.4 changing (4), which follows from the clean source property. In this case, for every $j \in N_D^{--}(i) \cap T(D)$, clean source full-transfer and clean source symmetry imply

$$\psi_{j^+}(N, D, c) = 0, \tag{9}$$

and

$$\psi_j(N, D, c) = \psi_k(N, D, c) \text{ for every } k \in T_j(D), \tag{10}$$

respectively, where $T_j(D) = \{t \in T(D) : \ t^+ = j^+\}$ is the set of top nodes of D with the same successor as j. We use (9) and (10) as the equations associated with node j. Note that these provide $1 + (|T_j(D)| - 1) = |T_j(D)|$ linearly independent equations. Notice that for any two agents k, j with $k \in T_j(D)$, the equations associated to j and k coincide. With this change, we can apply the inductive argument as in Theorem 3.4, using (9) and (10) instead of (4), and the result follows. □

5.3. Axiomatization of the class of α-URS-methods for $\alpha \geqslant 0$ and $\alpha = \infty$

As a limiting case of the α-URS methods, the ∞-URS method shares some properties with those methods. In particular, although it does not satisfy the α-clean source property for any $\alpha \geqslant 0$, it still allocates downstream costs between a source and its successor in the same ratio across all networks. This is a key feature we observe in the class of all α-URS methods. We formalize this in the following property.

Clean source proportionality: For every $(N, D, c), (N', D', c') \in \mathcal{R}$ and $t \in T(D)$, $s \in T(D')$ such that $c_t = c'_s = 0$, $\psi_t(N, D, c) \cdot \psi_{s^+}(N', D', c') = \psi_{t^+}(N, D, c) \cdot \psi_s(N', D', c')$.

This property settles a particular way to allocate downstream costs at the top of each river network; collusion neutrality extends it so that it applies at any point of a river network, in particular when distributing responsibilities between a tributary and the confluence. As for the responsibilities of different tributaries, clean source symmetry guarantees that they are the same for tributaries of length one that join the river at the same point. As before, collusion neutrality extends this symmetry so that any two tributaries that join the river at the same point are responsible for the same share of the downstream cleaning costs.

These observations lead to the following characterization of the class of α-URS methods including the ∞-URS method.

Theorem 5.8. A cost sharing method satisfies efficiency, additivity, the inessential agent property, collusion neutrality, clean source symmetry, and clean source proportionality if and only if it is an α-URS method with $\alpha \geqslant 0$ or $\alpha = \infty$.

Proof. By Theorems 5.4 and 5.7 we already know that URS^{∞} and every URS^{α} with $\alpha \geqslant 0$ satisfy efficiency, additivity, the inessential agent property and collusion neutrality; we also know that the ∞-URS method satisfies clean source symmetry. Now, let $\alpha \geqslant 0$. Since the α-clean source property implies clean source symmetry for every $\alpha \geqslant 0$, URS^{α} satisfies clean source symmetry. To show clean source proportionality, notice that the α-clean source property also implies that for every $(N, D, c), (N', D', c') \in \mathcal{R}$ and $t \in T(D)$, $s \in T(D')$ with $c_t = c'_s = 0$,

$$\begin{aligned} \mathsf{URS}^{\alpha}_t(N, D, c) \cdot \mathsf{URS}^{\alpha}_{s^+}(N', D', c') &= \alpha \mathsf{URS}^{\alpha}_{t^+}(N, D, c) \cdot \mathsf{URS}^{\alpha}_{s^+}(N', D', c') \\ &= \mathsf{URS}^{\alpha}_s(N', D', c') \cdot \mathsf{URS}^{\alpha}_{t^+}(N, D, c). \end{aligned}$$

Similarly, for the ∞-URS method, due to the clean source full-transfer property,

$$\mathsf{URS}^{\infty}_t(N, D, c) \cdot \mathsf{URS}^{\infty}_{s^+}(N', D', c') = 0 = \mathsf{URS}^{\infty}_s(N', D', c') \cdot \mathsf{URS}^{\infty}_{t^+}(N, D, c).$$

Thus, we conclude that the requirement of clean source proportionality is satisfied in all cases.

To show uniqueness, let ψ be a cost sharing method that satisfies the six properties and consider a polluted river problem with just two nodes and a clean source, i.e. $(N, D, c) \in \mathcal{R}$, with $N = \{t, b\}$, $D = \{(t, b)\}$, $c_t = 0$, and $c_b > 0$. If $\psi_b(N, D, c) > 0$, then there is some $\alpha \geqslant 0$ such that $\psi_t(N, D, c) = \alpha \psi_b(N, D, c)$. By clean source proportionality, this implies that ψ satisfies the α-clean source property; in turn, by Theorem 5.4, $\psi = \mathsf{URS}^{\alpha}$.

If $\psi_b\,(N,\,D,\,c) = 0$, then by efficiency $\psi_t\,(N,\,D,\,c) = c_b > 0$. By clean source proportionality, for every $(N,\,D,\,c) \in \mathcal{R}$ and $t \in T(D)$, $\psi_{t^+}\,(N,\,D,\,c) = 0$. That is, ψ satisfies the clean source full-transfer property and, by Theorem 5.7, $\psi = \mathsf{URS}^{\infty}$. $\square$

6. Concluding remarks

In this paper, we study the effects of collusion in the distribution of costs to clean a polluted river network. Our main contribution is the introduction of collusion neutrality, a new property that eliminates incentives for coalitions of agents to manipulate the cost allocation by acting together. This property implements the UTI doctrine in the following sense: under efficiency, it prevents upstream agents to harm those downstream via collusion. We define a new method that satisfies collusion neutrality, the URES method, and characterize it using this property. There are many more collusion neutral cost sharing methods; we identify a family that includes the URES and LRS methods, the α-URS methods. Each of these methods can be computed with a recursive procedure that distributes the cleaning cost of each segment of the river between the node immediately upstream of it and the rest of its upstream agents; the exact distribution depends on the value of the parameter α.

Normatively, the differences between the methods are reflected by the α-clean source property, which requires that each clean source pays a fraction $\alpha \geqslant 0$ of the cost paid by its successor. For different values of $\alpha \geqslant 0$, the α-clean source property reflects how strictly UTI is implemented: the higher the value of α, the higher the payments assigned to upstream agents. Furthermore, the different elements of the class can be used to deal with different types of pollutants. If the pollutant at issue is likely to travel far from its origin (for instance, microplastics, oil and other chemicals), then a method that attributes less responsibility to agents closer to the origin of pollution is more desirable. This corresponds to α-URS methods with a high value of α, or, in the most extreme case, the ∞-URS. Conversely, a method with a low α is preferable when attributing responsibilities for pollutants that accumulate near their origin (rubble, heavy metals, macroplastics, etc.).

Our class of cost sharing methods bears resemblance to the so-called geometric rules introduced by Hougaard et al. (2017). These rules are presented as revenue allocations in hierarchical ventures, but can also be applied to the polluted river problem.[16] In this context, both the geometric rules and the α-URS methods assign a fraction of the cleaning cost of a segment to the node where it originates, and share the rest among the agents upstream of it. The two families differ in their treatment of tributaries: at any confluence i, the share of the downstream costs allocated to i by a given geometric rule is always the same; an α-URS method, in contrast, divides the costs so that the ratio between the payments of the tributaries and the payments of i is constant. Arguably, this implies that the α-URS methods are more sensitive to the structure of the network, as the local payment depends on the number of predecessors of a node. It would be interesting to further study the application of geometric rules to the polluted river problem, and compare them axiomatically to the α-URS methods.

Finally, we remark that the URES method can be obtained as the Shapley levels value (Winter, 1989) of a cooperative game. This insight contributes to the game theoretical foundations of the various methods. Dong et al. (2012) showed that the LRS method is the Shapley value of the game that assigns to each coalition the sum of their cleaning costs. Later, van den Brink et al. (2018) showed that the UES method can be obtained by applying the Shapley value to the permission restricted game of the aforementioned game with the river network as the permission structure.[17] It can be shown that the URES method coincides with the Shapley levels value of the permission restricted game with a levels structure derived from the river network using a mapping introduced by Álvarez-Mozos et al. (2017). Thus, the URES method refines the UES method, in that it uses the river network to (i) define a restricted game, and (ii) construct a levels structure, which are then used to compute the cost allocation. For future work, we aim to further study the relations between the class of α-URS methods and solutions of cooperative games.

References

Alcalde-Unzu, J., Gómez-Rúa, M., Molis, E., 2015. Sharing the costs of cleaning a river: the upstream responsibility rule. Games and Economic Behavior 90, 134–150.

[16]Conversely, the α-URS methods can be extended to the more general setting studied by Hougaard et al. (2017).

[17]In order to use the river network as a permission structure, the direction of the edges has to be reversed.

Álvarez-Mozos, M., van den Brink, R., van der Laan, G., Tejada, O., 2017. From hierarchies to levels: new solutions for games with hierarchical structure. International Journal of Game Theory 46, 1089–1113.

Ambec, S., Ehlers, L., 2008. Sharing a river among satiable agents. Games and Economic Behavior 64, 35–50.

Ambec, S., Sprumont, Y., 2002. Sharing a river. Journal of Economic Theory 107, 453–462. doi:10.1006/jeth.2001.2949.

Ansink, E., Weikard, H.P., 2012. Sequential sharing rules for river sharing problems. Social Choice and Welfare 38, 187–210.

Barrett, S., 1994. Conflict and Cooperation in Managing International Water Resources. Policy Research Working Paper Series 1303. The World Bank, Washington.

van den Brink, R., He, S., Huang, J., 2018. Polluted river problems and games with a permission structure. Games and Economic Behavior 108, 182–205.

van den Brink, R., van der Laan, G., Moes, N., 2012. Fair agreements for sharing international rivers with multiple springs and externalities. Journal of Environmental Economics and Management 63, 388–403.

Dong, B., Ni, D., Wang, Y., 2012. Sharing a polluted river network. Environmental Resource Economics 53, 367–387.

Gudmundsson, J., Hougaard, J.L., Ko, C.Y., 2019. Decentralized mechanisms for river sharing. Journal of Environmental Economics and Management 94, 67–81.

Haller, H., 1994. Collusion properties of values. International Journal of Game Theory 23, 261–281.

Hougaard, J.L., Moreno-Ternero, J.D., Tvede, M., Østerdal, L.P., 2017. Sharing the proceeds from a hierarchical venture. Games and Economic Behavior 102, 98–110.

Hung, M.F., Shaw, D., 2005. A trading-ratio system for trading water pollution discharge permits. Journal of Environmental Economics and Management 49, 83–102.

Ju, B.G., Miyagawa, E., Sakai, T., 2007. Non-manipulable division rules in claim problems and generalizations. Journal of Economic Theory 132, 1–26.

Kilgour, D.M., Dinar, A., 1995. Are stable agreements for sharing international river waters now possible? Policy Research Working Paper Series 1474. The World Bank, Washington.

van der Laan, G., Moes, N., 2016. Collective decision making in an international river pollution model. Natural Resource Modeling 29, 374–399.

Martínez, R., Moreno-Tenero, J., 2025. The allocation of riparian water rights. Environmental and Resource Economics 88, 3841–3872.

Ni, D., Wang, Y., 2007. Sharing a polluted river. Games and Economic Behavior 60, 176–186.

Öztürk, Z.E., 2020. Fair social orderings for the sharing of international rivers: A leximin based approach. Journal of Environmental Economics and Management 101, 102302.

Rieu-Clarke, A., Moynihan, R., Magsig, B.O., 2012. UN Watercourses Convention: user's guide. IHP-HELP Centre for Water Law, Policy and Science.

Winter, E., 1989. A value for cooperative games with levels structure of cooperation. International Journal of Game Theory 18, 227–240.

Yang, Y., Ansink, E., Gudmindsson, J., 2025. How to pollute a river if you must. Journal of Environmental Economics and Management 130, 103105.

Appendix A. Missing proofs for Section 5

Proposition 5.3. Given $\alpha \geqslant 0$, let Ψ^α be the function that assigns to every polluted river problem $(N, D, c) \in \mathcal{R}$ the output of the following algorithm:

Step 1: Set $N^1 = N$, $D^1 = D$, $c^1(\alpha) = c$, and $\Psi^\alpha_b(N, D, c) = \frac{c^1_b(\alpha)}{1+\alpha\left|N^-_D(b)\right|}$, where $b \in N$ is the mouth of the river.

Step $m \geqslant 2$: Set $N^m = N^{m-1} \setminus B(D^{m-1})$ and $D^m = D^{m-1}[N^m]$. If $N^m = \varnothing$, stop. Otherwise, define $c^m(\alpha) \in \mathbb{R}^{N^m}$ by

$$c_i^m(\alpha)=\begin{cases} c_i^{m-1}(\alpha)+\frac{1}{\left|N_D^-(i^+)\right|}\left(c_{i^+}^{m-1}(\alpha)-\frac{c_{i^+}^{m-1}(\alpha)}{1+\alpha\left|N_D^-(i^+)\right|}\right) & \text{if } i\in B(D^m) \\ c_i^{m-1}(\alpha) & \text{otherwise.}\end{cases}$$

Set $\Psi_i^\alpha(N,\,D,\,c)=\frac{c_i^m(\alpha)}{1+\alpha\left|N_D^-(i)\right|}$ for every $i\in B(D^m)$. Go to step $m+1$.

The resulting cost sharing method is the α-URS method, that is, $\Psi^\alpha=\mathsf{URS}^\alpha$.

Proof. Let $\alpha\geqslant 0$. By the same reasoning used in Proposition 3.3, the algorithm is guaranteed to stop in a finite number of steps, and so Ψ^α is well-defined. It also follows from the algorithm and the definition of the α-URS method that $\Psi_b^\alpha(N,\,D,\,c)=\frac{c_b^1(\alpha)}{1+\alpha\left|N_D^-(b)\right|}=\frac{c_b}{1+\alpha\left|N_D^-(b)\right|}=URS_b^\alpha(N,\,D,\,c)$.

Suppose $\Psi_i^\alpha(N,\,D,\,c)=\mathsf{URS}_i^\alpha(N,\,D,\,c)$ for every $i\in N$ with $\left|[i,\,b]_D\right|\leqslant r$ for some $r\geqslant 1$. The base case $r=1$ holds since $\left|[b,\,b]_D\right|=|\{b\}|=1$ and no node $i\in N\setminus b$ satisfies $\left|[i,\,b]_D\right|\leqslant 1$. Now, suppose there is some $j\in N$ be such that $\left|[j,\,b]_D\right|=r+1$. We will show that $\Psi_j^\alpha(N,\,D,\,c)=\mathsf{URS}_j^\alpha(N,\,D,\,c)$. As in Proposition 3.3, for every $m\geqslant 1$, if $\left|[i,\,b]_D\right|=m$, then $i\in B(D^m)$, and the algorithm sets $\Psi_i^\alpha(N,\,D,\,c)=\frac{c_i^m(\alpha)}{1+\alpha\left|N_D^-(i)\right|}$ at step m. In particular, since $|[j,b]_D|=r+1$,

$$\begin{aligned}\Psi_j^\alpha(N,\,D,\,c)&=\frac{c_j^{r+1}(\alpha)}{1+\alpha\left|N_D^-(j)\right|}\\&=\frac{1}{1+\alpha\left|N_D^-(j)\right|}\left(c_j^r(\alpha)+\frac{1}{\left|N_D^-(j^+)\right|}\left(c_{j^+}^r(\alpha)-\frac{c_{j^+}^r(\alpha)}{1+\alpha\left|N_D^-(j^+)\right|}\right)\right)\\&=\frac{1}{1+\alpha\left|N_D^-(j)\right|}\left(c_j^r(\alpha)+\frac{\alpha c_{j^+}^r(\alpha)}{1+\alpha\left|N_D^-(j^+)\right|}\right)\\&=\frac{1}{1+\alpha\left|N_D^-(j)\right|}\left(c_j+\alpha\Psi_{j^+}^\alpha(N,\,D,\,c)\right),\end{aligned}$$

where the second equality is by definition of $c^{r+1}(\alpha)$ and for the final equality we use that $c_k^r(\alpha)=c_k$ for every $k\in N$ with $\left|[k,\,b]_D\right|>r$. In particular, $c_j^r(\alpha)=c_j$. By the induction hypothesis, $\Psi_{j^+}^\alpha(N,\,D,\,c)=\mathsf{URS}_{j^+}^\alpha(N,\,D,\,c)$, since $\left|[j^+,\,b]_D\right|=\left|[j,\,b]_D\right|-1=r$. Then, using the definition of the α-URS method,

$$\Psi_j^\alpha(N,\,D,\,c)=\frac{c_j}{1+\alpha\left|N_D^-(j)\right|}+\frac{\alpha\mathsf{URS}_{j^+}^\alpha(N,\,D,\,c)}{1+\alpha\left|N_D^-(j)\right|}$$

$$= \frac{c_j}{1+\alpha\left|N_D^-(j)\right|} + \frac{\alpha}{1+\alpha\left|N_D^-(j)\right|} \sum_{k \in [j^+,\, b]_D} \alpha^{\left|[j^+,\, k]_D\right|-1} \frac{c_k}{\prod\limits_{l \in [j^+,\, k]_D} \left(1+\alpha\left|N_D^-(l)\right|\right)}$$

$$= \frac{c_j}{1+\alpha\left|N_D^-(j)\right|} + \sum_{k \in [j^+,\, b]_D} \alpha^{\left|[j^+,\, k]_D\right|} \frac{c_k}{\prod\limits_{l \in [j,\, k]_D} \left(1+\alpha\left|N_D^-(l)\right|\right)}$$

$$= \frac{c_j}{1+\alpha\left|N_D^-(j)\right|} + \sum_{k \in [j^+,\, b]_D} \alpha^{\left|[j,\, k]_D\right|-1} \frac{c_k}{\prod\limits_{l \in [j,\, k]_D} \left(1+\alpha\left|N_D^-(l)\right|\right)}$$

$$= \sum_{k \in [j,\, b]_D} \alpha^{\left|[j,\, k]_D\right|-1} \frac{c_k}{\prod\limits_{l \in [j,\, k]_D} \left(1+\alpha\left|N_D^-(l)\right|\right)} = \mathsf{URS}_j^\alpha\left(N,\, D,\, c\right),$$

where the second and last equalities follow from the definition of the α-URS method. □

Theorem 5.4. Let $\alpha \geqslant 0$. The α-URS method is the only cost sharing method that satisfies efficiency, additivity, the inessential agent property, collusion neutrality, and the α-clean source property.

Proof. Let $\alpha \geqslant 0$. The proof follows closely that of Theorem 3.4. It is clear from Definition 5.1 that the α-URS method is additive and satisfies the inessential agent property. For the α-clean source property, let $(N,\, D,\, c) \in \mathcal{R}$ and $s \in T(D)$ with $c_s = 0$. By Definition 5.1 and the fact that $N_D^-(s) = \varnothing$,

$$\mathsf{URS}_s^\alpha\left(N,\, D,\, c\right) = \sum_{j \in [s,\, b]_D} \alpha^{\left|[s,\, j]_D\right|-1} \frac{c_j}{\prod\limits_{k \in [s,\, j]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)}$$

$$= \sum_{j \in [s^+,\, b]_D} \alpha^{\left|[s,\, j]_D\right|-1} \frac{c_j}{\prod\limits_{k \in [s^+,\, j]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)}$$

$$= \alpha \sum_{j \in [s^+,\, b]_D} \alpha^{\left|[s^+,\, j]_D\right|-1} \frac{c_j}{\prod\limits_{k \in [s^+,\, j]_D} \left(1+\alpha\left|N_D^-(k)\right|\right)}$$

$$= \alpha \mathsf{URS}_{s^+}^\alpha\left(N,\, D,\, c\right).$$

As in Theorem 3.4, we show efficiency by induction over the depth of the network; it is trivial that the α-URS method is efficient on problems where the river network has depth 1. Suppose that the α-URS method is efficient on problems $\left(\overline{N},\, \overline{D},\, \overline{c}\right) \in \mathcal{R}$ where $\mathsf{depth}\left(\overline{N},\, \overline{D}\right) \leqslant d$ for some $d \geqslant 1$, and let $(N,\, D,\, c) \in \mathcal{R}$ be such that $\mathrm{depth}\,(N,\, D) = d+1$.

By using analogous arguments to those used in Theorem 3.4, we will see that

$$\mathsf{URS}_i^\alpha\left(N,\, D,\, c\right) = \mathsf{URS}_i^\alpha\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}(\alpha)\right) \text{ for every } i \in N \setminus b,$$

where j is the only predecessor of b in the path $[i,\, b]_D$, $N_D^{j-} = N_D^{--}(j) \cup j$, $D^{j-} = D\left[N_D^{j-}\right]$, and $c^2(\alpha)$ is the cost vector generated at the second step of the algorithm in Proposition 5.3. Indeed, since $N_{D^{j-}}^{-}(l) = N_D^{-}(l)\ \forall l \in N_D^{j-}$, and $[k,\, l]_{D^{j-}} = [k,\, l]_D\ \ \forall k \in N_{D^{j-}}^{--}(l)$,

$$\mathsf{URS}_i^\alpha\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}(\alpha)\right) = \sum_{k\in[i,\,j]_{D^{j-}}} \alpha^{\left|[i,\,k]_{D^{j-}}\right|-1} \frac{c_k^2(\alpha)}{\prod\limits_{l\in[i,\,k]_{D^{j-}}} \left(1+\alpha\left|N_{D^{j-}}^{-}(l)\right|\right)}$$

$$= \sum_{k\in[i,\,j]_D} \alpha^{\left|[i,\,k]_D\right|-1} \frac{c_k^2(\alpha)}{\prod\limits_{l\in[i,\,k]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)},$$

and, by the definition of $c^2(\alpha)$,

$$\begin{aligned}
\mathsf{URS}_i^\alpha\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right) &= \sum_{\substack{k\in[i,\,j]_D\\ k\neq j}} \alpha^{\left|[i,\,k]_D\right|-1} \frac{c_k^2(\alpha)}{\prod\limits_{l\in[i,\,k]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)} \\
&\quad + \alpha^{\left|[i,\,j]_D\right|-1} \frac{c_j^2(\alpha)}{\prod\limits_{l\in[i,\,j]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)} \\
&= \sum_{\substack{k\in[i,\,j]_D\\ k\neq j}} \alpha^{\left|[i,\,k]_D\right|-1} \frac{c_k}{\prod\limits_{l\in[i,\,k]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)} \\
&\quad + \alpha^{\left|[i,\,j]_D\right|-1} \frac{c_j + \frac{\alpha c_b}{1+\alpha\left|N_D^{-}(b)\right|}}{\prod\limits_{l\in[i,\,j]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)} \\
&= \sum_{k\in[i,\,j]_D} \alpha^{\left|[i,\,k]_D\right|-1} \frac{c_k}{\prod\limits_{l\in[i,\,k]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)} \\
&\quad + \alpha^{\left|[i,\,b]_D\right|-1} \frac{c_b}{\prod\limits_{l\in[i,\,b]_D} \left(1+\alpha\left|N_D^{-}(l)\right|\right)} \\
&= \mathsf{URS}_i^\alpha(N,\, D,\, c),
\end{aligned}$$

where in the penultimate equality we use that $j \in N_D^{-}(b)$ and the final equality follows from the definition of the α-URS method.

Since $\mathsf{depth}\left(N_D^{j-},\, D^{j-}\right) < d+1$, by the induction hypothesis URS^α is efficient on $\left(N_D^{j-},\, D^{j-},\, c^2_{|N_D^{j-}}\right)$. Then, again by the same reasoning as in Theorem 3.4,

$$\sum_{i\in N} \mathsf{URS}_i^\alpha(N,\, D,\, c) = \frac{c_b}{1+\alpha\left|N_D^{-}(b)\right|} + \sum_{j\in N_D^{-}(b)} \sum_{i\in N_D^{j-}} c_i^2(\alpha).$$

Finally, by the definition of $c^2(\alpha)$,

$$\begin{aligned}\sum_{i\in N}\mathsf{URS}_i^{\alpha}(N,\,D,\,c) &= \frac{c_b}{1+\alpha\left|N_D^-(b)\right|}+\sum_{j\in N_D^-(b)}\left(c_j+\frac{\alpha c_b}{1+\alpha\left|N_D^-(b)\right|}+\sum_{i\in N_D^{--}(j)}c_i\right)\\ &= \frac{c_b}{1+\alpha\left|N_D^-(b)\right|}+\alpha\left|N_D^-(b)\right|\frac{c_b}{1+\alpha\left|N_D^-(b)\right|}+\sum_{i\in N\setminus b}c_i\\ &= \sum_{i\in N}c_i.\end{aligned}$$

Again, the same logic we applied to the URES method in Theorem 3.4 shows that for every $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$ we have $\mathsf{URS}_j^{\alpha}(N,\,D,\,c)=\mathsf{URS}_j^{\alpha}(N_D^{i*},\,D^{i*},\,c^{i*})$ for every $j\in N_D^{i*}\backslash i$. Since α-URS method is also efficient, it follows that it satisfies collusion neutrality.

For uniqueness, it suffices to follow the process we described in the proof of Theorem 3.4, replacing (4) by

$$\psi_j(N,\,D,\,c)=\alpha\psi_{j^+}(N,\,D,\,c)\qquad\text{for every } j\in N_D^{--}(i)\cap T(D),$$

which follows from the α-clean source property. □

Proposition 5.6. The ∞-URS method satisfies $\mathsf{URS}^{\infty}=\lim_{\alpha\to\infty}\mathsf{URS}^{\alpha}$.

Proof. First of all we show that for every $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$, the limit $\lim_{\alpha\to\infty}\mathsf{URS}_i^{\alpha}(N,\,D,\,c)$ exists. Let $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$. If $i\in T(D)$, that is, $N_D^-(i)=\varnothing$, then $\lim_{\alpha\to\infty}\frac{1}{1+\alpha\left|N_D^-(i)\right|}=1$. Otherwise, if $i\in N\setminus T(D)$, we have $\lim_{\alpha\to\infty}\frac{1}{1+\alpha\left|N_D^-(i)\right|}=0$, and $\lim_{\alpha\to\infty}\frac{\alpha}{1+\alpha\left|N_D^-(i)\right|}=\frac{1}{\left|N_D^-(i)\right|}$. By the product law of limits, for every $j\in[i,\,b]_D$,

$$\begin{aligned}\lim_{\alpha\to\infty}\alpha^{\left|[i,\,j]_D\right|-1}\prod_{k\in[i,\,j]_D}\frac{1}{1+\alpha\left|N_D^-(k)\right|} &= \left(\lim_{\alpha\to\infty}\frac{1}{1+\alpha\left|N_D^-(i)\right|}\right)\cdot\prod_{k\in[i^+,\,j]_D}\lim_{\alpha\to\infty}\frac{\alpha}{1+\alpha\left|N_D^-(k)\right|}\\ &= \begin{cases}\prod_{k\in[i^+,\,j]_D}\frac{1}{\left|N_D^-(k)\right|} & \text{if } i\in T(D)\\ 0 & \text{otherwise.}\end{cases}\end{aligned}\tag{A.1}$$

By the sum law of limits, it follows that $\lim_{\alpha\to\infty}\mathsf{URS}_i^{\alpha}(N,\,D,\,c)$ exists for every $(N,\,D,\,c)\in\mathcal{R}$ and $i\in N$ and it satisfies

$$\begin{aligned}\lim_{\alpha\to\infty}\mathsf{URS}_i^{\alpha}(N,\,D,\,c) &= \lim_{\alpha\to\infty}\sum_{j\in[i,\,b]_D}c_j\cdot\alpha^{\left|[i,\,j]_D\right|-1}\prod_{k\in[i,\,j]_D}\frac{1}{1+\alpha\left|N_D^-(k)\right|}\\ &= \sum_{j\in[i,\,b]_D}c_j\cdot\lim_{\alpha\to\infty}\left(\alpha^{\left|[i,\,j]_D\right|-1}\prod_{k\in[i,\,j]_D}\frac{1}{1+\alpha\left|N_D^-(k)\right|}\right).\end{aligned}$$

With (A.1) and noting that $k \in [i^+, b]_D$ implies that $k \in N \setminus T(D)$, if $i \in T(D)$, then $\lim_{\alpha\to\infty} \mathsf{URS}_i^\alpha(N, D, c) = \sum_{j\in[i,b]_D} \frac{c_j}{\prod_{k\in[i^+,j]_D} |N_D^-(k)|} = \mathsf{URS}_i^\infty(N, D, c)$. Also, if $i \in N \setminus T(D)$, then $\lim_{\alpha\to\infty} \mathsf{URS}_i^\alpha(N, D, c) = \sum_{i\in N} c_j \cdot 0 = 0 = \mathsf{URS}_i^\infty(N, D, c)$. □

Appendix B. Logical independence of axioms

The following cost sharing methods show that the properties in Theorem 3.4 are logically independent.

(1) The cost sharing method that assigns zero to every node, for every polluted river problem, satisfies all properties in Theorem 3.4 but efficiency.

(2) For every $(N, D, c) \in \mathcal{R}$, define $B_D^c = \{i \in N : \; c_i > 0 \text{ and } c_j = 0 \text{ for all } j \in [i, b]_D\}$ as the set of agents with positive cost such that all its downstream agents have zero cost. Further, define $\bar{c}^D \in \mathbb{R}_+^N$ by

$$\bar{c}_i^D = \begin{cases} \sum_{j\in N_D^{--}(i)\cup i} c_j & \text{if } i \in B_D^c \\ 0 & \text{otherwise.} \end{cases} \tag{B.1}$$

as the cost vector where all agents in B_D^c have as cost the sum of the costs of itself plus all its upstream agents, while all other agents have cost zero.

Define the cost sharing method $\overline{\mathsf{URES}}$ by $\overline{\mathsf{URES}}(N, D, c) = \mathsf{URES}(N, D, \bar{c}^D)$ for every $(N, D, c) \in \mathcal{R}$. This method satisfies all properties in Theorem 3.4 but additivity.

(3) Consider the following modification of the URES method. For every $(N, D, c) \in \mathcal{R}$, define $\tilde{c}^D \in \mathbb{R}_+^N$ by

$$\tilde{c}_i = \begin{cases} \sum_{j\in N} c_j & \text{if } i = b \\ 0 & \text{otherwise.} \end{cases} \tag{B.2}$$

Let $\widetilde{\mathsf{URES}}(N, D, c) = \mathsf{URES}(N, D, c')$ for every $(N, D, c) \in \mathcal{R}$. The $\widetilde{\mathsf{URES}}$ method satisfies all properties in Theorem 3.4 but the inessential agent property.

(4) The UES method satisfies all properties in Theorem 3.4 but collusion neutrality.

(5) The LRS method satisfies all properties in Theorem 3.4 but the clean source property.

The following cost sharing methods show that the properties in Theorem 4.1 are logically independent.

(1) The cost sharing method that assigns zero to every node, for every polluted river problem, satisfies all properties in Theorem 4.1 but efficiency.

(2) For every $(N, D, c) \in \mathcal{R}$ consider the cost vector $\overline{c}^D$ as defined by (B.1).
Define the cost sharing method $\overline{\mathsf{LRS}}$ by $\overline{\mathsf{LRS}}(N, D, c) = \mathsf{LRS}(N, D, \overline{c}^D)$ for every $(N, D, c) \in \mathcal{R}$. This method satisfies all properties in Theorem 4.1 but additivity.

(3) The method that for every polluted river problem assigns all cleaning costs to the bottom node satisfies all properties in Theorem 4.1 but the inessential agent property.

(4) Consider the following modification of the UES method, call it $\widetilde{\mathsf{UES}}$:

$$\widetilde{\mathsf{UES}}_i\,(N,\,D,\,c) = \begin{cases} c_i & \text{if } i \in T(D) \\ \sum\limits_{j \in [i,b]_D} \frac{c_j}{\left|\left(N_D^{--}(j) \cup j\right) \setminus T(D)\right|} & \text{otherwise.} \end{cases}$$

The $\widetilde{\mathsf{UES}}$ method satisfies all properties in Theorem 4.1 but collusion neutrality.

(5) The URES method satisfies all properties in Theorem 4.1 but no blind cost at sources.

For a given $\alpha \geqslant 0$, the following cost sharing methods show that the properties in Theorem 5.4 are logically independent.

(1) The cost sharing method that for every $(N,\,D,\,c) \in \mathcal{R}$ assigns $\frac{1}{2}\mathsf{URS}_i^{\alpha}\,(N,\,D,\,c)$ to each $i \in N$ satisfies all properties in Theorem 5.4 but efficiency.

(2) For every $(N, D, c) \in \mathcal{R}$ consider the cost vector $\overline{c}^D$ as defined by (B.1).
Define the cost sharing method $\overline{\mathsf{URS}}^{\alpha}$ by $\overline{\mathsf{URS}}^{\alpha}(N, D, c) = \mathsf{URS}^{\alpha}(N, D, \overline{c}^D)$ for every $(N, D, c) \in \mathcal{R}$. This method satisfies all properties in Theorem 5.4 but additivity.

(3) For every $(N,\,D,\,c) \in \mathcal{R}$, consider the cost vector $\widetilde{c}^D$ as defined by (B.2).
Define the cost sharing method $\widetilde{\mathsf{URS}}^{\alpha}$ by $\widetilde{\mathsf{URS}}^{\alpha}\,(N,\,D,\,c) = \mathsf{URS}^{\alpha}\left(N,\,D,\,\widetilde{c}^D\right)$ for every $(N,\,D,\,c) \in \mathcal{R}$. The $\widetilde{\mathsf{URS}}^{\alpha}$ method satisfies all the properties in Theorem 5.4 except the inessential agent property.

(4) Consider the following modification of the UES method, call it $\widetilde{\mathsf{UES}}^{\alpha}$:

$$\widetilde{\mathsf{UES}}_i^{\alpha}(N,\,D,\,c)=\begin{cases} c_i+\frac{\alpha}{1+\alpha\left|N_D^-(i^+)\cap T(D)\right|}\sum\limits_{j\in[i^+,\,b]_D}\frac{c_j}{\left|\left(N_D^{-\,-}(j)\cup j\right)\setminus T(D)\right|} & \text{if } i\in T(D)\\ \frac{1}{1+\alpha\left|N_D^-(i)\cap T(D)\right|}\sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\left|\left(N_D^{-\,-}(j)\cup j\right)\setminus T(D)\right|} & \text{if } N_D^-(i)\cap T(D)\neq\varnothing\\ \sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\left|\left(N_D^{-\,-}(j)\cup j\right)\setminus T(D)\right|} & \text{otherwise.}\end{cases}$$

The $\widetilde{\mathsf{UES}}^{\alpha}$ method satisfies all properties in Theorem 5.4 but collusion neutrality.

(5) For any $\beta\geqslant 0$, $\beta\neq\alpha$, the β-URS method satisfies all properties in Theorem 5.4 but the α-clean source property.

The following cost sharing methods show that the properties in Theorem 5.7 are logically independent.

(1) The cost sharing method that assigns zero to every node, for every polluted river problem, satisfies all properties in Theorem 5.7 but efficiency.

(2) For every $(N,D,c)\in\mathcal{R}$ consider the cost vector $\overline{c}^D$ as defined by (B.1).
Define the cost sharing method $\overline{\mathsf{URS}}^{\infty}$ by $\overline{\mathsf{URS}}^{\infty}(N,D,c)=URS^{\infty}(N,D,\overline{c}^D)$ for every $(N,D,c)\in\mathcal{R}$. This method satisfies all properties in Theorem 5.7 but additivity.

(3) For every $(N,\,D,\,c)\in\mathcal{R}$, consider the cost vector $\widetilde{c}^D$ as defined by (B.2).
Define the cost sharing method $\widetilde{\mathsf{URS}}^{\infty}$ by $\widetilde{\mathsf{URS}}^{\infty}(N,\,D,\,c)=\mathsf{URS}^{\infty}\left(N,\,D,\,\widetilde{c}^D\right)$ for every $(N,\,D,\,c)\in\mathcal{R}$. The $\widetilde{\mathsf{URS}}^{\infty}$ method satisfies all the properties in Theorem 5.4 except the inessential agent property.

(4) The method $\mathsf{URS}^{\infty,eq}$ defined by

$$\mathsf{URS}_i^{\infty,eq}(N,\,D,\,c)=\begin{cases}\sum\limits_{j\in[i,\,b]_D}\frac{c_j}{\left|\left(N_D^{-\,-}(j)\cup j\right)\cap T(D)\right|} & \text{if } i\in T(D)\\ 0 & \text{otherwise,}\end{cases}$$

satisfies all properties in Theorem 5.7 but collusion neutrality.

(5) For any $\alpha\geqslant 0$, the α-URS method satisfies all properties in Theorem 5.7 but clean source full-transfer.

(6) Let $s_D : N \to N$ be a (selector) function satisfying (i) $s_D(i) = i$ if $i \in T(D)$, (ii) $s_D(i) \in N_D^-(i)$ if $i \in N \setminus T(D)$, and (iii) for every $j \in N$ and $i \in N \setminus N_D^{--}(j)$, if $s_D(i) \in N \setminus N_D^{--}(j)$, then $s_{D^{j*}}(i) = s_D(i)$. For every $(N, D, c) \in \mathcal{R}$, consider the cost vector $\widehat{c}^D \in \mathbb{R}_+^N$ defined by $\widehat{c}_i^D = \sum_{\substack{j \in N \\ s_D(j)=i}} c_j$.

The method $\widehat{\mathsf{URS}}^\infty$ given by $\widehat{\mathsf{URS}}^\infty (N, D, c) = \mathsf{URS}^\infty \left(N, D, \widehat{c}^D\right)$ for every $(N, D, c) \in \mathcal{R}$ satisfies all properties in Theorem 5.7 but clean source symmetry.

Regarding Theorem 5.8, for efficiency, additivity, the inessential agent property, collusion neutrality and clean source symmetry, the counterexamples for Theorem 5.7 above show each of these axioms is logically independent from the rest. Moreover, all these methods satisfy clean source proportionality. Finally, the cost sharing method γ that for every $(N, D, c) \in \mathcal{R}$ assigns to each $i \in N$

$$\gamma_i (N, D, c) = \begin{cases} c_i + \sum\limits_{\substack{j \in [i, b]_D \\ j \neq i}} \left(\frac{1}{2}\right)^{\left|[i,j]_D\right|-1} \dfrac{c_j}{\prod\limits_{\substack{k \in [i,j]_D \\ k \neq i}} \left|N_D^-(k)\right|} & \text{if } i \in T(D) \\ \frac{1}{2} \left(c_i + \sum\limits_{\substack{j \in [i, b]_D \\ j \neq i}} \left(\frac{1}{2}\right)^{\left|[i,j]_D\right|-1} \dfrac{c_j}{\prod\limits_{\substack{k \in [i,j]_D \\ k \neq i}} \left|N_D^-(k)\right|} \right) & \text{otherwise,} \end{cases}$$

satisfies all properties in Theorem 5.8 except for clean source proportionality.[18]

[18] The cost sharing method γ defined above is one of the geometric rules introduced by Hougaard et al. (2017). Any other geometric rule would also provide the desired counterexample, except for the LRS method and the limit method, which are also in our class.