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Abstract

In this paper, we introduce a new class of cost allocation rules for rooted tree cost allocation problems that is based on the position value (Meessen 1988 and Borm et al. 1992). We further extend this by introducing a class of cost allocation rules that, besides these new position rules, also contains the fixed-fraction rules of Gudmundsson et al. (2024) and the permission values of Gilles et al. (1992), also known as upstream equal sharing rules in Ni and Wang (2007) and Dong et al. (2012). This last rule belongs to this class but is neither a position rule nor a fixed-fraction rule. We provide an axiomatic characterization of the new class of rules, and by additional axioms obtain axiomatizations of subclasses of rules. We argue that the axioms are specifically useful to motivate the application of these rules for smart contracts in, for example, blockchains.

Keywords: Cost Allocation, Sharing Sequentially Trigerred Losses, Position Fixed-Fraction rules, Differential Proportionality

JEL code: C71, D63, D85.

AMS subject classification: 91A12, 91A43.

1 Introduction

Blockchain technology is a decentralized and secure digital ledger system. It records transactions across a network of computers. Unlike traditional centralized systems, blockchain operates autonomously. This ensures tamper resistance, universal accessibility, and the verifiability of its entire data history. Transactions are stored in blocks that are chronologically linked, forming a continuous chain.

Smart contracts, built on blockchain technology, are self-executing agreements with terms directly written in code. These contracts have transformed various industries by automating transactions and enforcing terms without the need for intermediaries. In finance, for example, smart contracts have simplified international remittances. In supply chain management, they have improved traceability and accountability. Despite their benefits, the autonomous nature of smart contracts also presents challenges. These challenges arise particularly in cost allocation and damage compensation during disputes or unforeseen events.

Recent legal cases highlight the importance of effective dispute resolution frameworks. For instance, in *D'Aloia v Persons Unknown* (2024)[1], the High Court of England and Wales addressed issues of fraudulent misrepresentation in blockchain transactions. The case revealed the complexities of tracing digital assets and assigning liability in decentralized systems. Similarly, in the case of the Commodity Futures Trading Commission (CFTC) versus the decentralized autonomous organization (DAO) Ooki in 2023[2], the DAO was found liable for regulatory violations in futures trading. This raised questions about collective liability and governance in the DAO and decentralized finance sectors. These cases demonstrate the urgent need for robust frameworks to resolve disputes and allocate damages equitably in smart contract systems.

This paper introduces a class of general cost-sharing rules for smart contract systems and provides their axiomatic characterization. The characterization examines the compatibility of economic incentives for participants with fairness principles. It also evaluates the social desirability of smart contract systems by rigorously analyzing the normative principles underlying desirable cost-sharing rules in their design.

The general class of cost-sharing rules introduced in this paper is the class of *position fixed-fraction rules* that are governed by two parameters, both between zero and one, and offer a flexible and systematic solution for cost allocation in blockchain-based systems. The roles of the two parameters, α and β, are as follows:

[1]See D'Aloia v Persons Unknown & Others [2024] EWHC 2342 (Ch) (High Court of England and Wales, Chancery Division).

[2]See Commodity Futures Trading Commission v. Ooki DAO, No. 3:22-cv-05416-WHO (N.D. Cal. June 8, 2023).

- α (top-down dimension): This parameter controls the allocation of costs or responsibilities between top agents (e.g., initiators in blockchain-based smart contracts) and bottom agents (e.g., final stakeholders in smart contracts). By adjusting α, the relative responsibility between these groups can be balanced according to their roles.

- β (in-out dimension): This parameter adjusts the allocation between the two extreme agents and the intermediate agents, taking into account their relative importance and level of involvement within the system. The parameter β provides a mechanism to distribute costs or responsibilities among these agents while maintaining fairness across the network.

By combining these parameters, the position fixed-fraction rules enable a precise and adaptive cost-sharing mechanism suitable for various multi-agent scenarios, making it highly useful in smart contract applications. For example, in decentralized energy trading platforms within the electricity market, smart contracts can automatically distribute electricity costs among suppliers, grid system operators responsible for power management, and consumers. When applying the position fixed-fraction rules to such an energy trading platform, α serves as the parameter that determines cost allocation between suppliers and consumers, while β controls the cost allocation assigned to intermediate grid system operators. In blockchain-based supply chain platforms, smart contracts can similarly automate the distribution of logistics costs among producers, logistics providers, and retailers. When applying a position fixed-fraction rule to this supply chain platform, α determines the responsibility allocation between producers and retailers, while β adjusts the responsibilities caused by intermediate logistics providers.

How to appropriately describe cost-sharing rules for smart contract systems within program code is a highly practical and critical issue. The position fixed-fraction rules offer a theoretically straightforward implementation. A smart contract implementing a position fixed-fraction rule consists of the following three steps (naturally, if there is only one agent, that agent bears all the costs):

Step (i): Cost sharing between two agents: Initially, the cost-sharing rule for two agents (so there is no intermediary) is defined in the program code using the parameter α. (Notice that in this case there are no intermediaries, so the parameter β plays no role.) In an energy platform, for example, the top agent (Agent 1) corresponds to an electricity supplier and bears setup costs, while the bottom agent (Agent 2) represents a consumer. The electricity supply cost is allocated between Agents 1 and 2 using the parameter $\alpha \in [0, 1]$. The cost incurred by Agent 1 is the sum of the setup cost and the portion of the electricity supply cost weighted by α. Similarly, in a supply chain platform, the top agent (Agent 1) might represent an electricity producer and bears production costs, while the bottom agent (Agent 2) corresponds to a retailer. The logistics costs are allocated between Agents 1 and 2 using α. The cost incurred by Agent 1 is the sum of the production cost and the portion of the logistics cost weighted by α.

Step (ii): Cost sharing with an intermediary agent: Next, the program code defines the cost-sharing rule when a single intermediary agent exists between the top agent and the bottom agent. This scenario assumes that the intermediary's action is necessary for the bottom agent to consume services but that the intermediary itself cannot act without the top agent's activities on the smart contract. In an energy platform, this intermediary corresponds to a grid system operator, and in a supply chain platform, it corresponds to a logistics provider. To adjust the cost-sharing affected by the intermediary's presence, the parameter $\beta \in [0, 1]$ is introduced. Regardless of the value of β, the cost-sharing ratio between the top and bottom agents, as determined by α in Step (i), remains fixed, eliminating the need for renegotiation.

Step (iii): Extension to multiple intermediary agents: Finally, the rule is extended to accommodate cases with multiple intermediary agents, applying a principle of proportionality to update cost-sharing ratios. As the number of intermediary agents increases, the contributions of the top agent, intermediary agents, and the bottom agent change. In cases where one smart contract is implemented per block, this implies that adjustments to cost-sharing are needed as the linear blockchain grows longer. For the adjustments to cost-sharing, we introduce a proportionality principle, referred to as *differential proportionality*, ensuring that the proportional impact of adding a new blockchain node (and thereby increasing the chain length by one) is constant over the blockchain. This principle is incorporated into the smart contract code as a series of simultaneous equations. The cost-sharing ratios are automatically adjusted based on differential proportionality. This ensures that even as new agents join the blockchain system, there is no need for existing agents to renegotiate their cost-sharing arrangements.

The main result of this paper is that the class of position fixed-fraction rules can be fully characterized through differential proportionality and several fundamental properties introduced in Gudmundsson et al. (2024).[3] The two parameters α and β do not appear explicitly in the axioms, and can be seen as a "coordination mechanism" that resolves conflicts of interest within the blockchain system. The central axiom underlying these cost-sharing rules is differential proportionality, which reflects fairness and is straightforward to implement in smart contract systems.

[3] We remark that by sharing the cost only between the two extreme agents, in Gudmundsson et al. (2024) only α matters and β does not appear.

Two key points to emphasize about our axiomatization are the following. First, all position fixed-fraction rules satisfy *population monotonicity*, which is a key axiom in Gudmundsson et al. (2024) ensuring that participants are motivated to join the platform. Second, taking specific values for the a- and β-parameters give axiomatizations of specific classes of notable position fixed-fraction rules such as the fixed-fraction rules proposed by Gudmundsson et al. (2024) (for $\beta = 0$), the well-known *permission value* (van den Brink and Gilles 1996) (for $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$), in polluted river problems also known as the *upstream equal sharing rule* (Ni and Wang 2007), and a new class of rules, called the *position rules* (for $\alpha \in [0, 1]$) that is based on the position value introduced by Meessen (1988) and Borm et al. (1992) for communication graph games, and their generalization for directed networks in Gavilán et al. (2022).

The contribution of this study to economics and management science lies in providing a microeconomic foundation for designing cost-sharing rules that encourage participation and ensure fairness in blockchain-based platform businesses. In recent years, various blockchain platforms have introduced innovative business models utilizing smart contracts, such as *TradeLens* (operated by IBM and Maersk) and *LO3 Energy* (operated by the startup company Brooklyn Microgrid). However, many of these models have struggled to succeed. The primary reason for their failure is the lack of effective incentives for participation, which prevents platforms from scaling. As a result, these platforms fail to achieve economies of scale and cannot grow into market-attractive businesses. Additionally, smart contract systems often lack an equitable mechanism to balance participants' interests. This study identifies key properties that cost-sharing rules in blockchain-based platforms should possess.

Summarizing, we show that the position fixed-fraction rules, which integrate these properties, can be effectively implemented in smart contracts. As follows from the Steps (i)-(iii) above, the class of position fixed-fraction rules is parametrized by the two parameters $\alpha, \beta \in [0, 1]$. Therefore, in order to apply a specific position fixed-fraction rule, it is sufficient to choose to what extent responsibility is put (i) at the top or bottom in the sequence (the α-dimension), and (ii) at the (two) extremes or intermediaries in the sequence (the β-dimension). Although the second dimension makes the rules somewhat more difficult than the fixed-fraction rules of Gudmundsson et al. (2024) (who only consider the first dimension), we consider them still simple enough to be 'Measurable' and 'Realistic' in a smart contract and, moreover, give an extra dimension to the contract being 'Specific', 'Assignable' and 'Time-related'. In other words, we gain in having a more detailed smart contract at the cost of a little more complexity.

The structure of the paper is as follows. After an overview of related literature in Section 2, Section 3 contains preliminaries on line- and rooted-tree cost allocation problems and discusses some known cost allocation rules, specifically the fixed-fraction rules and the permission value/upstream equal sharing rule. In Section 4 we introduce the new axiom of differential proportionality and use it to axiomatically introduce the new class of position fixed-fraction rules. We show that, besides the fixed-fraction rules and the permission value/upstream equal sharing rule, this class contains an interesting subclass that is based on the position value. In Section 5, we consider variations of differential proportionality that characterize two special subclasses of position fixed-fraction rules. Both these classes do not contain all position rules or all fixed-fraction rules, but both classes do contain a position and a fixed-fraction rule. Moreover, the permission value/upstream equal sharing rule belongs to both classes. In Section 6 we provide game theoretic foundations of the position fixed-fraction rules. Section 7 extends our main results from line cost allocation problems to rooted tree cost allocation problems. Finally, Section 8 contains concluding remarks. All proofs can be found in the appendix.

2 Related literature

2.1 Smart contract literature

We briefly discuss how our paper relates to the recent game-theoretic literature on smart contracts and what contributions it makes to this body of research.

First, our paper generalizes the work of Gudmundsson et al. (2024) who proposed the class of fixed-fraction rules as rules for distributing recurring losses among agents transacting via smart contracts, and established its axiomatic foundation. We introduce a more general class of loss-sharing rules, called the position fixed-fraction rules, which includes the fixed-fraction rules as a special case, and we provide their axiomatic characterization. Since smart contracts are executed between nodes on a network, the position rules, which maintains a constant liability ratio between neighbouring nodes, can be considered a natural candidate for cost allocation. Analyzing a class of rules that includes both the fixed-fraction rules and the position rules reflects the flexibility of smart contract code and allows us to capture the principles of allocation that such programmability enables since it maintains a fixed liability ratio between a top node and its downstream nodes. By considering this family of rules, we can explicitly incorporate the flexibility that smart contract parameters, such as coefficients α and β, can be adjusted in response to the state of the blockchain environment.

Our paper also connects to the literature on clearing problems in decentralized financial networks based on blockchain technology. Csóka and Herings (2018) analyzed whether decentralized clearing is possible in financial networks composed of mutually indebted firms without a central authority, using only distributed and local information. They proposed a decentralized clearing process and demonstrated that it converges to a bankruptcy rule that assigns the smallest possible payment amount to creditors in the event of firm default. Since this decentralized process involves agents sequentially making payments based on local information, it can be implemented via blockchain technology. For instance, consider a linear financial network in which only adjacent firms have mutual obligations. In the decentralized clearing process, each firm observes the funds it receives from its upstream firm and makes payments to its downstream creditors accordingly. Thanks to blockchain technology, this sequential process can be written into smart contract code. In situations where the minimal payments to debtors coincide with the outcome of the position fixed-fraction rule, the parameters α and β can be interpreted within the framework of Csóka and Herings (2018).

Furthermore, our paper applies not only to cost-sharing problems via smart contracts but also to revenue-sharing scenarios. Hougaard et al. (2017) provided an axiomatic analysis of revenue allocation rules among agents organized in hierarchical ventures. They introduced a parametric family of rules, called the class of geometric rules according to which a fraction of the revenue earned by a lower-level agent is recursively transferred to upper-level agents at a fixed rate. These geometric rules can be implemented through smart contract code, but only allow for one parameter, limiting their adaptability to varying blockchain environments. Our paper, by contrast, highlights the potential of smart contracts to provide more flexible solutions for revenue sharing within hierarchical decision-making structures.

We conclude this part by arguing how our paper contributes to addressing the potential limitations of smart contracts in relation to previous research. Although smart contracts are automated execution codes, full automation of contractual obligations is difficult unless there is a way to generate and verify legal evidence of contract fulfillment (Gans 2019). However, smart contracts excel at automating payments and recording transaction data. As Gans (2019) emphasizes, incentive design that mitigates these limitations is essential, and smart contracts with automatic escrow functions triggering payment upon confirmation of fulfillment are indispensable. Our paper offers a normative analysis of the cost-sharing problem in such escrow-based smart contracts, addressing fairness among contract participants. In this sense, our work complements the growing research on incentive design in smart contracts (e.g., Gans 2019, Brzustowski et al. 2023).

2.2 Cost-sharing and game theoretic literature on networks

Our proposed position fixed-fraction rules have applications to a wide range of cost-sharing problems beyond smart contracts. For instance, Ni and Wang (2007) and Dong et al.

(2012) introduced the upstream equal sharing rule for the polluted river model where the river structure is represented by a line, respectively rooted tree directed graph, with costs assigned to each agent to clean the river at its territory. According to the upstream equal sharing rule the cost of each segment is equally attributed to herself and all upstream agents. It turns out that this rule belongs to the family of position fixed-fraction rules, although it is neither a fixed-fraction rule nor a position rule. In the context of tort liability, when the causal relationships among tortfeasors are represented by a directed graph on a line or a tree structure, both the Shapley value and the Nucleolus have been characterized from a legal perspective (Dehez and Ferey 2013, Ferey and Dehez 2016, Oishi et al. 2023). Since their model is nearly identical to that of Gudmundsson et al. (2024), our characterization of the position fixed-fraction rule can naturally be situated within this line of research, thereby establishing connections to these legal applications.

Our model can be related to various solution concepts of cooperative games defined on networks represented by graphs. Myerson (1977) introduced communication graph games where the players in a cooperative game belong to a communication structure that is represented by an undirected graph. He also introduced the Myerson value which applies the Shapley value (Shapley 1953) to a restricted game where only connected coalitions can generate worth. Since the introduction of the Myerson value, many solution concepts have been introduced for cooperative games in which the coalitional worth is defined on the underlying network. One such solution is the position value (Meessen 1988, Borm et al. 1992), which is obtained by first computing the Shapley value of the associated link game and then splitting each link's Shapley value equally between the two agents on the link. Originally introduced and analyzed by Borm et al. (1992) for networks represented by undirected, cycle-free graphs, the position value has been characterized for more general networks that allow cycles by Slikker (2005), and has been extended to directed graphs by Gavilán et al. (2022) where a directed link's Shapley value need not be shared equally between the two agents on the link. The position value can also be regarded as an index for evaluating the importance of nodes and edges in a network (Algaba and Saavedra-Nieves 2024). This line of research on position values can be related not only to the axiomatization of the position fixed-fraction rule, but also suggests the possibility of interpreting the position fixed-fraction rules as indices for assessing the influence of agents in social networks. For example, in Gómez et al. (2003) "centrality" in a network is defined as the difference between a node's Shapley value before and after communication restrictions are imposed. If, however, the Shapley value is replaced with the value assigned by a position fixed-fraction rule, it would be possible to introduce a new notion of centrality for nodes that is expressed in terms of multiple parameters.

Our model is based on additive (or inessential) games and 2-games—i.e., games in which the stand-alone worth is zero and the worth of any other coalition equals the sum of the worths of all two-player subcoalitions (van den Nouweland et al. 1996). Inessential games with a hierarchy have been analyzed as peer group games in Brânzei et al. (2002) and extended to games on rooted-tree permission structures in van den Brink et al. (2017) being a special case of games with an arbitrary permission structure of van den Brink and Gilles (1996). In particular, by studying additive games and 2-games, we can apply our framework to similar revenue-sharing problems, such as the distribution of broadcasting revenues in soccer leagues (Bergantinos and Moreno-Ternero 2020)[4]. When distributing organizational revenues via smart contracts, if the revenue structure can be represented as a permission game constructed from a k-game, then the position fixed-fraction rule would be applicable.

3 Preliminaries

3.1 The model

An *n-line cost allocation problem*, shortly refered to as an *n-problem*, is a pair (N, c) where $N = \{1, \dots n\} \subset \mathbb{N}$ is a set of agents that is linearly ordered, and $c \in \mathbb{R}^n_+$ is a nonnegative vector of costs where c_i is the cost associated to agent $i \in N$. It is assumed without loss of generality that the agents are linearly ordered according to the labels $1, 2, \dots n$ with 1 being the most *upstream agent* and n being the most *downstream agent*.[5] We denote the set of all n-line cost allocation problems by $\mathcal{L}^n$, and by $\mathcal{L} = \bigcup_{n \in \mathbb{N}} \mathcal{L}^n$ we denote the set of all problems with any set of agents N.

A *cost allocation* for n-problem $(N, c) \in \mathcal{L}^n$ is a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Here, x_i is the contribution or cost share of agent i in the total cost $\sum_{i=1}^n c_i$. A *cost allocation rule*, or *rule* for short, is a mapping $\varphi\colon \mathcal{L} \to \bigcup_{n \in \mathbb{N}} \mathbb{R}^n_+$ satisfying $\varphi(N, c) \in \mathbb{R}^n_+$ if $(N, c) \in \mathcal{L}^n$, and assigns a cost allocation to every n-problem $(N, c) \in \mathcal{L}$. We will often write c and $\varphi(c)$ instead of (N, c) or $\varphi(N, c)$, respectively, when the set of agents is clear from the context.

Examples of this model from the literature are (i) the sequentially triggered loss model in Gudmundsson et al. (2024) where c_i is the loss caused by agent i which is triggered by the agents $j < i$ before her, (ii) the polluted river problem of Ni and Wang (2007) where the linear order represents the flow of an international river and c_i is the cost of cleaning the polluted river at country i where the pollution might be created by all agents $j \leq i$ together, and (iii) the joint liability problem of Dehez and Ferey (2013) and Dehez and

[4]This 2-game setup can be generalized to k-games ($k \geq 2$).

[5]Formally, this means that the linear order $\succ$ on N is defined by $i \succ j$ if and only if $i < j$.

Ferey (2013) where the order is a sequence of agents who, by doing wrongful acts, cause a damage where $j < i$ means that agent j's wrongful act is necessary for agent i to be in the position to do its wrongful and c_i is the direct damage caused by the wrongful act of agent i. A main question in all these examples is how to allocate the total sum of costs over the individual agents. One straightforward possibility is that each agent i just pays its 'own' cost c_i. However, in all these applications, it makes sense that also agents upstream of agent i share in the cost c_i of agent i: in the sequentially triggered loss and joint liability problem this makes sense since the loss or damage created by agent i could not be done without the actions of agents $j < i$, and in the polluted river problem the pollution at agent i might be the result of agents $j < i$ upstream of i polluting the river. Since the cost allocation rules that we consider in this paper could be applied to each of these applications, we write our model and results in neutral terms as a line-cost sharing problem.

3.2 Known cost allocation rules

We recall some known cost allocation rules for n-line cost allocation problems.[6]

1. The *upstream equal sharing rule* (Ni and Wang 2007) or *permission value* (van den Brink and Gilles 1996) allocates each cost c_i equally among agent i and all its upstream agents: it is the rule UES given by

$$UES_i(N, c) = \sum_{j=i}^{n} \frac{c_j}{j} \text{ for all } i \in N.$$

2. The *local responsibility rule* (Ni and Wang 2007) or *direct liability rule* (Gudmundsson et al. 2024) assigns each cost c_i fully to agent i: it is the rule LR given by

$$LR_i(N, c) = c_i \text{ for all } i \in N.$$

3. The *top rule* (van den Brink et al. 2017) also known as *full-transfer* rule (Hougaard et al. 2017) or *indirect liability rule* (Gudmundsson et al. 2024) assigns all the cost to the top agent 1: it is the rule TOP given by

$$TOP_i(N, c) = \begin{cases} \sum_{j \in N} c_j & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

[6] We recall that this model and some of the rules are known in the literature under different names.

4. For every $\lambda \in [0,1]$, the corresponding λ fixed-fraction rule (Gudmundsson et al. 2024) assigns a fraction λ of the cost c_i to agent i and assigns the remaining fraction to the top agent: it is the rule F^λ given by

$$F_i^\lambda(N,c) = \begin{cases} c_1 + (1-\lambda)\displaystyle\sum_{j=i+1}^{n} c_j & \text{if } i = 1 \\ \lambda c_i & \text{if } i \neq 1 \end{cases}$$

Notice that (i) according to the fixed-fraction rules, the intermediary agents $i \geq 2$ do not contribute to the cost c_j, $j > i$, (ii) the local responsibility and top rules are special fixed-fraction rules for $\lambda = 1$, respectively $\lambda = 0$, and (iii) the upstream equal sharing rule is not a fixed-fraction rule.

3.3 Axioms

We recall axioms for a rule φ defined on $\mathcal{L}$ from Gudmundsson et al. (2024).

Axiom 3.1 (Additivity) *For each (N,c) and $(N,c') \in \mathcal{L}$, $\varphi(c) + \varphi(c') = \varphi(c + c')$.*

Axiom 3.2 (Zero truncation) *For each $(N,c) \in \mathcal{L}$ and $i \in \{1,\dots,n\}$, $\varphi_i(c) = \varphi_i((c,0))$, where $(c,0) \in \mathbb{R}^{n+1}$ is given by $(c_1,...,c_n,0)$.*

Implicitly, they assume efficiency and nonnegativity.

Axiom 3.3 (Efficiency) *For each $(N,c) \in \mathcal{L}$, $\sum_{i\in N} \varphi_i(c) = \sum_{i\in N} c_i$.*

Axiom 3.4 (Nonnegativity) *For each $(N,c) \in \mathcal{L}$ and $i \in \{1,\dots,n\}$, $\varphi_i(c) \geq 0$.*

By additivity, it is sufficient to know the allocations that the rule assigns to problems where only one cost is positive. Additionally requiring zero-truncation, it turns out that it is sufficient to know the allocations that the rule assigns to elementary cost vectors $e_n \in \mathcal{L}^n$, given by $(e_n)_n = 1$ and $(e_n)_i = 0$ for all $n \in \mathbb{N}$ and $i \in \{1,\dots n-1\}$.

Proposition 3.5 (Gudmundsson et al. 2024) *An allocation rule φ defined on $\mathcal{L}$ satisfies efficiency, nonnegativity, additivity and zero truncation if and only if, for each $(N,c) \in \mathcal{L}^n$ and $i \in N$, $\varphi_i(c) = \sum_{j=i}^{n} c_j \cdot \varphi_i(e_j)$.*[7]

[7]The reader will observe that $\varphi_i(e_j)$, for $j = i,...,n$, are the allocations for agent i in cost allocation problems with different number of agents. Specifically, agents $\{1,...,j\}$ participate in the e_j problem.

The next axiom is population monotonicity and reflects that sharing the same elementary unit cost with more agents should not increase the contributions of each of the original agents. Notice that extending the problem with one more agent, each existing agent can be compared with the agent at the same position from the top (i.e. comparing the share of agent i in e_n with that of agents i in e_{n+1}), but can also be compared with the agent at the same position from the bottom (i.e. comparing the share of agent i in e_n with that of agents $i+1$ in e_{n+1}).

Axiom 3.6 (Population monotonicity) *For each* (N, e_n), $(N \cup \{n+1\}, e_{n+1}) \in \mathcal{L}$ *with* $n \in N = \{1, \dots, n\}$*:*

(i) $\varphi_1(e_n) \geq \varphi_1(e_{n+1})$

(ii) $\varphi_i(e_n) \geq \varphi_i(e_{n+1})$ *and* $\varphi_i(e_n) \geq \varphi_{i+1}(e_{n+1})$ *for all* $1 < i < n$

(iii) $\varphi_n(e_n) \geq \varphi_{n+1}(e_{n+1})$.

Finally, they introduce merging proofness that reflects that going from a two-agent problem to a three-agent problem, the new non-bottom agents (agents 1 and 2) do not contribute more than the original non-bottom agent (agent 1), and also the new non-top agents (agents 2 and 3) do not contribute more than the original non-top agent (agent 2).

Axiom 3.7 (Merging proofness) *For* $(\{1,2\}, e_2)$, $(\{1,2,3\}, e_3)$ *and* $i \in \{1,2\}$*, we have* $\varphi_i(e_3) + \varphi_{i+1}(e_3) \leq \varphi_i(e_2)$.

The above axioms characterize the class of fixed-fraction rules.

Theorem 3.8 (Gudmundsson et al. 2024) *An allocation rule* φ *defined on* $\mathcal{L}$ *satisfies efficiency, nonnegativity, additivity, zero truncation, population monotonicity and merging proofness if and only if there exists a* $\lambda \in [0,1]$ *such that* $\varphi = F^{\lambda}$.

4 Differential proportionality and the class of position fixed-fraction rules

While a main advantage of the fixed-fraction rules is their simplicity, which makes them suitable to implement in smart contracts, a disadvantage is that they do not allow to assign any cost share to intermediary agents who clearly do play a role in, for example (i) sequentially triggered losses in blockchains, (ii) transfering polution in a river, or (iii) a sequence of wrongful acts leading to a damage in a joint liability problem. Therefore, our goal in the underlying paper is to extend the class of fixed-fraction rules in such a way that it allows positive contributions of intermediaries but still be simple enough to be implementable in a smart contract.

4.1 Differential proportionality: an axiomatization

First, we introduce a new axiom which applies a proportionality principle with respect to changes in the cost shares of agents when the problem is extended with one more agent, and we compare the same cost shares as done in population monotonicity. Specifically, we require that, if $n \geq 2$, extending the elementary cost problem from e_n to e_{n+1}, the change in shares compared to the original shares is the same for every agent $i < n$, i.e. there exists a $K \in \mathbb{R}$ such that

$$\frac{\varphi_i(e_n) - \varphi_i(e_{n+1})}{\varphi_i(e_n)} = K \text{ for all } 1 \leq i < n \tag{4.1}$$

This compares the cost shares of every agent with the agent in the same position from the top. We can make a similar comparison looking bottom-up and compare the shares of agent $i > 1$ in e_n with that of agent $i+1$ in e_{n+1} (both being at the $(n-i+1)$-th position from the bottom) by requiring that the K in (4.1) also satisfies that:

$$\frac{\varphi_i(e_n) - \varphi_{i+1}(e_{n+1})}{\varphi_i(e_n)} = K \text{ for all } 1 < i \leq n \tag{4.2}$$

Notice that these fractions are only well-defined if the denominator is positive. To avoid zero in the denominator, we take cross-products and formalize this proportionality principle in the following axiom where, similar as in the definition of population monotonicity (Axiom 3.6), we state these comparisons distinguishing three type of agents: the top (agent 1), the intermediate agents ($1 < i < n$) and the bottom ($i = n$), .

Axiom 4.1 (Differential proportionality) *Let $n \geq 2$. There exists $K \in \mathbb{R}$ such that*

(i) $\varphi_1(e_n) - \varphi_1(e_{n+1}) = K\varphi_1(e_n)$

(ii) *(a)* $\varphi_i(e_n) - \varphi_i(e_{n+1}) = K\varphi_i(e_n)$ *and*

(b) $\varphi_i(e_n) - \varphi_{i+1}(e_{n+1}) = K\varphi_i(e_n)$ *for all* $1 < i < n$

(iii) $\varphi_n(e_n) - \varphi_{n+1}(e_{n+1}) = K\varphi_n(e_n)$

In this axiom, we compare the proportional change in the cost shares of agents in the old situation e_n with that of the corresponding agent in the new situation e_{n+1}, where an agent is added at the bottom of the line. What is the 'corresponding' agent depends on whether one looks top-down or bottom-up. Therefore, in (i) we compare the share of the top, being agent 1 in both situations, with each other. Since for agent n, looking from the bottom, the new bottom agent is $n+1$, in (iii) we compare the share of agent n in the original situation with that of agent $n+1$ in the new situation. Regarding every intermediate agent $1 < i < n$, we compare the share of i with (a) that of i being the agent in this position looking top-down, and (b) that of $i+1$ being the agent in this position looking bottom-up.

Differential proportionality can be interpreted as an entry robustness condition: when the platform grows by one additional node, the proportional adjustment in existing participants shares is uniform, which makes the contract predictable and eliminates the need for repeated renegotiation as the system scales.

Notice that population monotonicity and differential proportionality compare the shares of the same agent pairs with each other, but make different requirements. It turns out that replacing population monotonicity and merging proofness in Theorem 3.8 by differential proportionality property, characterizes a new class of rules that contains all fixed-fraction rules.

Theorem 4.2 *An allocation rule φ on $\mathcal{L}$ satisfies efficiency, nonnegativity, additivity, zero truncation and differential proportionality if and only if there exist $\alpha, \beta \in [0,1]$ such that $\varphi = \varphi^{\alpha,\beta}$ with $\varphi^{\alpha,\beta}$ given by*

(i): $\varphi_1^{\alpha,\beta}(\{1\}, c_1) = c_1$,

(ii): $\varphi_i^{\alpha,\beta}(\{1,2\}, c) = \begin{cases} c_1 + (1-\alpha)c_2 & \text{if } i = 1 \\ \alpha c_2 & \text{if } i = 2, \end{cases}$

(iii): *For (N, c) with $n \geq 3$,*[8]

$$\varphi_i^{\alpha,\beta}(N,c) = \begin{cases} c_1 + (1-\alpha)c_2 + \sum_{j=3}^{n} \left(\frac{(1-\alpha)(1-\beta)}{1+(j-3)\beta} \right) c_j & \text{if } i = 1 \\ \alpha c_2 + \sum_{j=3}^{n} \left(\frac{\beta}{1+(j-3)\beta} \right) c_j & \text{if } i = 2 \\ \frac{\alpha(1-\beta)}{1+(j-3)\beta} c_i + \sum_{j=i+1}^{n} \left(\frac{\beta}{1+(j-3)\beta} \right) c_j & \text{if } 3 \leq i \leq n-1 \\ \frac{\alpha(1-\beta)}{1+(j-3)\beta} c_i & \text{if } i = n. \end{cases}$$

As mentioned at the end of the introduction, all proofs can be found in the Appendix. In the appendix it is also shown that the axioms in Theorem 4.2 are logically independent.

For $\alpha, \beta \in [0,1]$ we refer to the rule $\varphi^{\alpha,\beta}$ as the (α,β)-position fixed-fraction rule, and to the class of rules $\Phi = \{\varphi^{\alpha,\beta} \mid \alpha, \beta \in [0,1]\}$ as the class of *position fixed-fraction rules*.

Specific parameter settings give specific rules that we mentioned in Section 3.2, see Table 1. Specifically, fixed-fraction rules are obtained if $\beta = 0$. Among the fixed-fraction

[8]Notice that the case $3 \leq i \leq n-1$ only occurs if $n \geq 4$.

$\varphi^{\alpha,0},\ \alpha \in [0,1]$	fixed-fraction rule F^{α}
$\varphi^{1,0}$	local responsibility rule LR
$\varphi^{0,0}$	top rule TOP
$\varphi^{\alpha,\frac{1}{2}},\ \alpha \in [0,1]$	position rule π^{α}
$\varphi^{\frac{1}{2},\frac{1}{3}}$	permission rule/upstream equal sharing rule PV/UES

Table 1: Special cases of position position fixed-fraction rules.

rules, we obtain the local responsibility rule F^1 for $\beta = 0,\ \alpha = 1$, and the top rule F^0 for $\beta = \alpha = 0$. Although it is not a fixed-fraction rule, the permission value/upstream equal sharing rule belongs to this new class and is obtained for $\alpha = \frac{1}{2},\ \beta = \frac{1}{3}$.

This theorem also makes clear how by setting the parameters α and β, the cost allocation is determined. For one-agent cost allocation problems, the rule $\varphi^{\alpha,\beta}$ assigns the full cost to the only agent on the problem. The first parameter α enters in two-agent problems, where any nonnegative allocation between the two agents is possible; the shares of agents 1 and 2 being $1-\alpha$ and α, respectively. The second parameter β enters in three-agent problems, where it is the share of the only intermediate agent, while the shares of the top and bottom agents are modified to $(1-\alpha)(1-\beta)$ and $\alpha(1-\beta)$, respectively. Continuing with more than three agents, the shares are fixed by applying the differential proportionality principle. So, fixing α (for two agent problems) and β (for three agents problems), differential proportionality fixes the shares for all problems.

Extreme cases in the top-bottom dimension are $\alpha = 0$ implying that the share of any agent j in c_j is zero, and $\alpha = 1$ implying that the share of the top agent in any cost, except its own cost c_1, is zero. Extreme cases in the in-out dimension are $\beta = 0$ implying that the intermediate agents get zero, and $\beta = 1$ implying that the top agent 1 and 'bottom' agent j are assigned a zero share in the cost c_j (if $n \geq 3$).

Notice that, as a corollary of Theorem 4.2, Proposition 3.5 is applicable to all position fixed-fraction rules.

Corollary 4.3 *Let $\alpha, \beta \in [0,1]$. For each $(N,c) \in \mathcal{L}^N$, and $i \in \{1,\ldots,n\}$,*

$$\varphi_i^{\alpha,\beta}(c) = \sum_{j=i}^{n} c_j \cdot \varphi_i^{\alpha,\beta}(e_j).$$

As mentioned before, our new axiom of differential proportionality pairwise compares the payoffs of the same agents as population monotonicity.[9] Although there is no logical

[9] Instead of the inequalities in population monotonicity, the equalities in differential proportionality compare the shares of agents in a different way.

relation between these two axioms, see our discussion in the concluding remarks Section 8, we want to remark that all position fixed-fraction rules satisfy population monotonicity.

Proposition 4.4 *For all $\alpha, \beta \in [0,1]$, $\varphi^{\alpha,\beta} \in \Phi$ satisfies population monotonicity.*

4.2 The position rules

An interesting observation is that taking $\alpha \in [0,1]$ and $\beta = \frac{1}{2}$, we obtain a class of rules $\pi^\alpha = \varphi^{\alpha,\frac{1}{2}}$ that can be written as

$$\pi_i^\alpha(N,c) = \begin{cases} c_1 + \sum_{j=2}^{n} \frac{(1-\alpha)}{j-1} c_j & \text{if } i = 1 \\ \frac{\alpha}{i-1} c_i + \sum_{j=i+1}^{n} \frac{c_j}{j-1} & \text{if } i \neq 1 \end{cases}$$

For $\alpha \in [0,1]$ we refer to the rule π^α as the α-position rule, and to the class of rules $\Pi = \{\pi^\alpha \mid \alpha \in [0,1]\}$ as the class of *position rules*. As we will see in Section 6, these rules are related to the position value for communication graph games (see Meessen 1988, see Borm et al. 1992). Besides assigning the cost c_1 to the top agent 1, for a given $\alpha \in [0,1]$ these position rules assign an equal share $\frac{c_n}{n-1}$ to all intermediary agents, and also assigns the same share to the top 1 and bottom agent n together, with the top agent getting a share $(1-\alpha)$ from this (so getting $\frac{(1-\alpha)c_n}{n-1}$) and agent n a share α (so getting $\frac{\alpha c_n}{n-1}$).

The position rules can be seen as some kind of counterpart for the fixed-fraction rules. Compared to the fixed-fraction rules, the position rules assign higher shares to the intermediate agents $1 < i < j$ in the cost c_j for every agent j. According to every position fixed-fraction rule, all intermediary agents $1 < i < j$ get the same share in cost c_j, which is also equal to the total share assigned to agents 1 and j together. The extreme cases assign zero to the top agent and equal shares to the other agents (for $\alpha = 1$), or zero to agent j and equal shares to the other agents (for $\alpha = 0$). Taking $\alpha = \frac{1}{2}$, yields equal shares for the top agent 1 and agent j, which thus equals half the share of each intermediate agents $1 < i < j$ in the cost c_j. So, whereas the fixed-fraction rules share the cost c_j fully between the top agent 1 and the bottom agent j, the position rules assign to the intermediate agents a share at least as high as the top and bottom agents.

We illustrate specific position fixed-fraction rules in Table 2, where we compute the shares according to several allocation rules in this class for the four-agent problem (N,c) with $N = \{1,2,3,4\}$ and $c = e_4$.

π^0	π^1	$\pi^{1/2}$	π^α	F^λ	$\varphi^{\alpha,\beta}$
$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{1-\alpha}{3}$	$1-\lambda$	$\frac{(1-\alpha)(1-\beta)}{1+\beta}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{\beta}{1+\beta}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{\beta}{1+\beta}$
0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{\alpha}{3}$	λ	$\frac{\alpha(1-\beta)}{1+\beta}$

Table 2: Example: e_4

4.3 Extended differential proportionality for $n \geq 1$ and the midpoint position fixed-fraction rules

Similar as differential proportionality, Gudmundsson et al. (2024) apply population monotonicity for problems with $n \geq 2$. It is clear from Theorems 3.8 and 4.2 that the position fixed-fraction rules do not satisfy merging proofness. Gudmundsson et al. (2024) using merging proofness to determine the shares for the transition from $n = 1$ to $n = 2$.[10] We can deal with this by going one step further and apply differential proportionality also to the transition from e_1 to e_2. It turns out that, if we require extended differential proportionality[11] for $n \geq 1$, we characterize the class of position fixed-fraction rules with $\alpha = \frac{1}{2}$, where agents 1 and j get the same share in c_j. This class obviously contains the *midpoint* fixed-fraction and position rules $F^{1/2}$ and $\pi^{1/2}$. Gudmundsson et al. (2024) motivate why the midpoint fixed-fraction rule is a focal point within the class of fixed-fraction rules. Theorem 4.5 below gives an extra motivation for the *midpoint position fixed-fraction rules*. (Besides this, we give attention to the midpoint rules in the game theoretic analysis in Section 6.)

Without giving an explicit definition, extended differential monotonicity for $n \geq 1$ is defined the same as differential proportionality in Axiom 4.1, but requiring this also for $n = 1$ (the transition from e_1 to e_2). So, besides the equations in Axiom 4.1, K also must satisfy:

$$\varphi_1(e_1) - \varphi_1(e_2) = K\varphi_1(e_1), \tag{4.3}$$

and

$$\varphi_1(e_1) - \varphi_2(e_2) = K\varphi_1(e_1). \tag{4.4}$$

Extending differential proportionality with the case $n = 1$, the next theorem characterizes the midpoint position fixed-fraction rules. In this case we even do not need nonnegativity.

[10] We make some remarks on merging proofness in the concluding remarks (Section 8.2) and Appendix C.

[11] Notice that population monotonicity is also satisfied for $n \geq 1$.

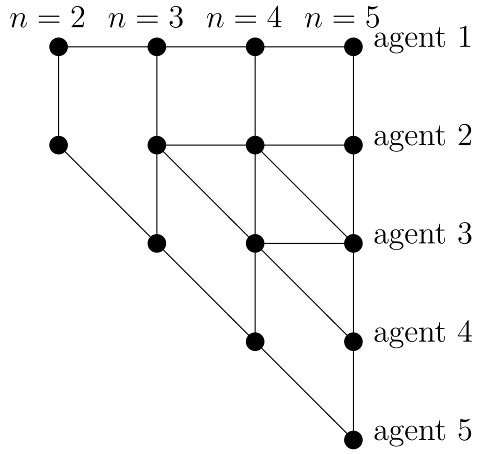

Figure 1: Comparisons differential proportionality: position fixed-fraction rules

Theorem 4.5 *An allocation rule φ on $\mathcal{L}$ satisfies efficiency, additivity, zero truncation and extended differential proportionality for $n \geq 1$ if and only if there exists $\beta \in [0,1]$ such that $\varphi = \varphi^{\frac{1}{2},\beta}$.*

5 Special classes of position fixed-fraction rules: Top-down and Bottom-up differential proportionality

The characterization in Theorem 4.2 'mirrors' the one for the fixed-fraction rules in Gudmundsson et al. (2024) (see Theorem 3.8), in the sense that differential proportionality compares the same pairs of agent cost shares as population monotonicity, but with a different comparison. This is vizualized in Figure 1 (where the vertical lines are the line graphs for $n = 2, 3, 4$ and 5, and the horizontal and diagonal lines reflect the comparisons that we make).

Although the proof of Theorem 4.2 is in the appendix, here we sketch how uniqueness of the cost shares is determined by induction since this also illustrates how these rules can be implemented in smart contracts. For the case $n = 2$, any sharing of the cost between the two agents is allowed and just expressed as $1-\alpha$ and α for agents 1 and 2, respectively. Assuming that the cost shares are determined for problems with n agents, notice that for $n \geq 3$, going from e_n to e_{n+1}, the horizontal and diagonal lines in Figure 1 represent more equations (specifically $2(n-1)$ equations) than unkowns (specifically $n+1$ unkowns). Although these equations are not independent, in the proof in the appendix we show that this system contains $n+1$ independent equations, and thus the $n+1$ unkowns are uniquely determined. However, for $n = 2$ we see there are only two equations, and thus the $n+1 = 3$

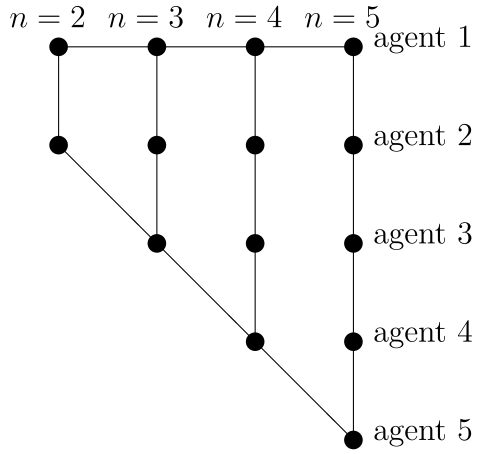

Figure 2: Comparisons between top and bottom agents

unkowns are not unique. Gudmundsson et al. (2024) use merging proofness to deal with the transition from e_2 to e_3. In our case, we keep this open to the extent that differential proportionality keeps the ratios between the shares of agents 1 and 3 fixed, but allows any nonnegative share for agent 2 as reflected by the parameter β entering the rule for $n = 3$. From there onwards, i.e. for $n \geq 3$, differential proportionality (together with the other axioms) does the work. So, under differential proportionality, we only need to set the parameters for $n = 2$ (α) and $n = 3$ (β) to determine the cost shares for all problems. This simple structure makes the position fixed-fraction rules very useful to be implemented in smart contracts.

Next, we explain the comparisons and how they come from the top-down and bottom-up approach. First, observe that for any $n \geq 2$, differential proportionality compares the cost shares of the top agent 1 in e_n and e_{n+1}, and the shares of the bottom agent n in e_n with that of the bottom agent $n + 1$ in e_{n+1}, see Figure 2.

Besides these comparisons, for $n \geq 3$, for every transition going from e_n to e_{n+1}, the cost share for every agent $1 < i < n$ is compared with the cost shares of agents e_i (in the same position top-down, see Figure 3) as well as e_{i+1} (in the same position bottom-up, see Figure 4).

Splitting the top-down from the bottom-up comparisons, but keeping the comparisons of the two tops (agent 1 in both cases) and bottoms (agents n respectively $n + 1$), we consider the following two variations of differential proportionality where, for every transition going from e_n to e_{n+1}, we can compare the proportional changes in shares of

(i) agent i with i (top-down approach, see Figure 3), or

(ii) agent i with $i + 1$ (bottom-up approach, see Figure 4).

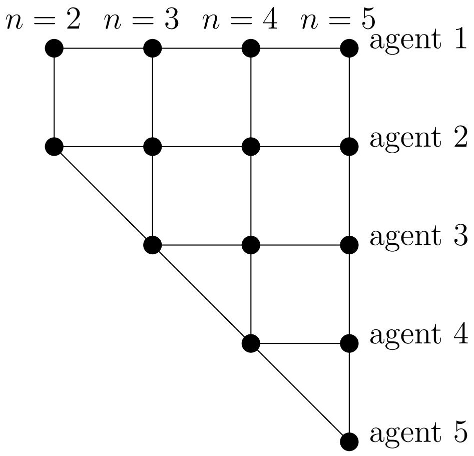

Figure 3: Comparisons top-down

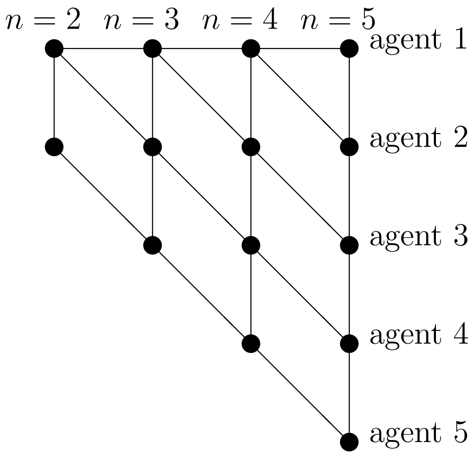

Figure 4: Comparisons bottom-up

$\varphi^{0,0}$	F^0
$\varphi^{1,\frac{1}{2}}$	π^1
$\varphi^{\frac{1}{2},\frac{1}{3}}$	UES

Table 3: Special cases satisfying top-down differential proportionality.

5.1 Top-down differential proportionality

First, considering the top-down approach, we define the alternative differential proportionality axiom where, for every transition, we compare the shares of agents on the same level from the top (and as always the share of agent n in e_n with that of agent $n+1$ in e_{n+1}), see Figure 3.

Axiom 5.1 (Top-down differential proportionality) *An allocation rule φ on $\mathcal{L}$ satisfies top-down differential proportionality if for $n \geq 2$ there exists $K \in \mathbb{R}$ such that*

(i) $\varphi_i(e_n) - \varphi_i(e_{n+1}) = K\varphi_i(e_n)$ *for all* $1 \leq i \leq n$*;*

(ii) $\varphi_n(e_n) - \varphi_{n+1}(e_{n+1}) = K\varphi_n(e_n)$.

Compared with the original differential proportionality (Axiom 4.1), in top differential proportionality, besides the comparison of the top with the top ($\varphi_1(e_n)$ with $\varphi_1(e_{n+1})$) and the comparison of the bottom with the bottom ($\varphi_n(e_n)$ with $\varphi_{n+1}(e_{n+1})$), the share of every agent $i \in \{1, \dots, n\}$ in e_n is compared with that of agent i in e_{n+1}. It turns out that replacing differential proportionality in Theorem 4.2 by top-down differential proportionality characterizes a subclass of position fixed-fraction rules containing π^0 and F^1.

Theorem 5.2 *An allocation rule φ defined on $\mathcal{L}$ satisfies efficiency, nonnegativity, additivity, zero truncation and top-down differential proportionality if and only if $\varphi = \varphi^{\alpha,\beta} \in \Phi$ with $\alpha, \beta \in [0, 1]$ such that $\beta = \frac{\alpha}{1+\alpha}$.*

The class of rules characterized in Theorem 5.2 contains the extreme position rule π^1 (for $\alpha = 1$, and thus $\beta = \frac{1}{2}$), the extreme FF rule F^0 (for $\alpha = 0$, and thus $\beta = 0$) and the upstream equal sharing rule (for $\alpha = \frac{1}{2}$, and thus $\beta = \frac{1}{3}$), see Table 3.

Notice that this class of position fixed-fraction rules satisfying top-down differential proportionality is ambiguous with respect to the top-down dimension since it contains the extreme top fixed-fraction rule F^0 as well as the extreme bottom position rule π^1.

$\varphi^{1,0}$	F^1
$\varphi^{0,\frac{1}{2}}$	π^0
$\varphi^{\frac{1}{2},\frac{1}{3}}$	UES

Table 4: Special cases satisfying bottom-up differential proportionality.

5.2 Bottom-up differential proportionality

Similar, we define bottom-up differential proportionality comparing for every transition from e_n to e_{n+1} the proportional changes in shares of agent i with $i+1$ (and as always the share of agent 1 in e_n with that of agent 1 in e_{n+1}), see Figure 4.

Axiom 5.3 (Bottom-up differential proportionality) *An allocation rule φ on $\mathcal{L}$ satisfies bottom-up differential proportionality if for $n \geq 2$ there exists $K \in \mathbb{R}$ such that*

(i) $\varphi_1(e_n) - \varphi_1(e_{n+1}) = K\varphi_1(e_n)$

(ii) $\varphi_i(e_n) - \varphi_{i+1}(e_{n+1}) = K\varphi_i(e_n)$ *for* $1 \leq i \leq n$.

Compared with the original differential proportionality (Axiom 4.1) in Section 4, in bottom-up differential proportionality, besides the comparison of the share of top with that of the top ($\varphi_1(e_n)$ with $\varphi_1(e_{n+1})$) and the comparison of the bottom with the bottom (($\varphi_n(e_n)$ with $\varphi_{n+1}(e_{n+1})$), every agent $i \in \{1, \ldots, n\}$ in e_n is compared with agent $i+1$ who is the same distance from the bottom in e_{n+1}. It turns out that replacing differential proportionality in Theorem 4.2 by bottom-up differential proportionality characterizes a subclass of position fixed-fraction rules containing π^0 and F^1.

Theorem 5.4 *An allocation rule φ on $\mathcal{L}$ satisfies efficiency, nonnegativity, additivity, zero truncation and bottom-up differential proportionality if and only if $\varphi = \varphi^{\alpha,\beta} \in \Phi$ with $\alpha, \beta \in [0,1]$ such that $\beta = \frac{1-\alpha}{2-\alpha}$.*

The class of rules characterized in Theorem 5.4 contains the other extreme position rule π^0 (for $\alpha = 0$, and thus $\beta = \frac{1}{2}$), the extreme fixed-fraction rule F^1 (for $\alpha = 1$, and thus $\beta = 0$) and again the upstream equal sharing rule (for $\alpha = \frac{1}{2}$, and thus $\beta = \frac{1}{3}$), see Table 4.

In this section, we characterized two subclasses of position fixed-fraction rules with variations of differential proportionality.[12] In the next section, we complement this with a game theoretic foundation.

[12]We remark that further subclasses can be characterized by applying differential proportionality from $n \geq 1$, but we will not go into these details here.

6 Game theoretic foundation

Cooperative games are often used in cost and profit allocation problems. Gudmundsson et al. (2024) give a cooperative game foundation for the midpoint fixed-fraction rule $F^{1/2}$. In this section, we give a unified cooperative game foundation for all position fixed-fraction rules. We first give some preliminaries on cooperative games.

6.1 Cooperative TU-games and graph games

A cooperative game with transferable utility (or a TU-game for short) is a pair (N, v) in which $N = \{1, \ldots, n\}$ is the set of players and $v \colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, is the characteristic function. In this model, $v(S) \in \mathbb{R}$ is the cost (in cost games) or profit (in profit games), that the $s = |S|$ members of the *coalition* $S \in 2^N = \{S \mid S \subseteq N\}$ jointly have to pay, respectively can obtain by cooperating. If there is no confusion about the player set, we simple write a game (N, v) by its characteristic function v and refer to it as game v.

For each $S \subseteq N \neq \emptyset$, the unanimity game (N, u_S) is given by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Every game v can be written as a linear combination of unanimity games in a unique way.[13] That is, for every game (N, v) there exist coordinates $\{\Delta_v(S)\}_{\emptyset \neq S \subseteq N}$ (called the *Harsanyi dividends* (Harsanyi 1959)) such that

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S.$$

An *allocation rule* for TU-games is a mapping that assigns to every game (N, v) an n-dimensional vector which components are the individual cost shares or payoffs assigned to the players. Taking a permutation π of the set of players[14], the corresponding *marginal vector* is given by

$$m_i^\pi(v) = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) \text{ for all } i \in N,$$

where $P_i(\pi) = \{j \in N \mid \pi(j) < \pi(i)\}$ is the set of players that enter before i according to permutation π.

[13] In other words, the family of unanimity games $\{(N, u_S)\}_{\emptyset \neq S \subseteq N}$ forms a basis for the vector space of TU-games.

[14] A permutation $\pi \colon N \to N$ is a bijection that assigns to every player i an entrance number $\pi(i)$. The idea here is that, assuming that the players enter one by one, of $\pi(i) = k$ then player i is the k-th player to enter.

The *Shapley value* (Shapley 1953) is the allocation rule Sh that assigns to every game the average of all marginal vectors over all $n!$ permutations, and thus assigns to every player $i \in N$ in any game (N, v), the payoff

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)),$$

where $\Pi(N)$ denotes the collection of all permutations of the player set N.

Equivalently, this can be written as

$$Sh_i(N, v) = \sum_{i \in S \subseteq N} \frac{\Delta_v(S)}{s}, \quad i \in N$$

showing that the Shapley value equally allocates the Harsanyi dividend of any coalition S equally over the players in S.

Games can be restricted by (undirected or directed) graphs. An *undirected graph* is a pair (N, γ) with $N = \{1, 2, \ldots, n\}$ being a set of *nodes* and $\gamma \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$, being a set of *edges* or *links*. A *graph game* is a triple (N, v, γ), where (N, v) is a TU-game and (N, γ) is a graph which set of nodes corresponds to the set of players in the game.

Specifically, we speak about a line-graph, respectively a line-graph game on $N = \{1, \ldots, n\}$ if $\gamma = \gamma_L = \{\{i, i+1\} \mid i \in \{1, \ldots n-1\}\}$. The *equal gain splitting rule* (introduced by Curiel et al. 1993, 1994 for one-machine sequencing problems and generalized to line-graph games in van den Brink et al. 2007) is the rule EGS that assigns to every line-graph game the average of the two marginal vectors where the players enter in the order from 1 to n, and the reverse order from n to 1, i.e.

$$EGS(N, v, \gamma_L) = \frac{1}{2}(m^{\pi^u}(v) + m^{\pi^\ell}(v)),$$

where $\pi^u(i) = i$ and $\pi^\ell(i) = n - i + 1$ for all $i \in \{1, \ldots n\}$.

A *directed graph* or a *digraph* is a pair (N, D) where $N = \{1, 2, \ldots, n\}$ is a (finite) set of nodes and $D \subseteq N \times N$ is a binary relation on N. Each *directed edge* or *arc* $(i, j) \in D$ is an ordered pair of nodes where i is the *tail* and j is the *head*. The *outdegree* of a node is the number of edges where it is the tail, while the *indegree* of a node is the number of edges where it is head. A *directed graph game* is a triple (N, v, D) in which (N, v) is a TU-game and (N, D) is a directed graph, the nodes in the digraph being the players in the game.

Specifically, we speak about a directed line-graph, respectively a directed line-graph game on $N = \{1, \ldots, n\}$ if $D = D_L = \{(i, i+1) \mid i \in \{1, \ldots n-1\}\}$. The *permission value* *PV* (Gilles et al. 1992)[15] assigns to every directed-line graph game the Shapley value of

[15] We remark that the permission value is defined for arbitrary digraphs.

the game where every player needs its upstream agents to be in a coalition, and is given by $PV(N, v, D_L) = Sh(v^P)$ with $v^P(S) = v(\{i \in S \mid [j < i \Rightarrow j \in S]\}$ for all $S \subseteq N$.

Gavilán et al. (2022) generalize the position value that is introduced by Meessen (1988) and Borm et al. (1992) for communication graph games, to the setting of directed communication situations. For the more specific directed line graph games, these boil down to the following. The *out-oriented* (respectively *in-oriented*) position value π^0 (respectively π^1) allocates for every directed line-graph game (N, v, D_L), the Harsanyi dividends of any coalition proportional to the relative out-degree (respectively in-degree) of all the agents in the directed connected hull of that coalition in D_L[16]. For $\alpha \in [0, 1]$, the corresponding α-position value is obtained as the convex combination $\pi^\alpha(N, v, D_L) = (1-\alpha)\pi^0(N, v, D_L) + \alpha\pi^1(N, v, D_L)$.

6.2 Game theoretic cost allocation rules

Gudmundsson et al. (2024) gave a game theoretic foundation of the fixed-fraction rule $F^{1/2}$ by applying the Shapley value to the veto-game v_c given by

$$v_c(S) = \begin{cases} \sum_{j \in S} c_j & \text{if } 1 \in S \\ 0 & \text{otherwise.} \end{cases}$$

In this game it is necessary and sufficient for the top agent 1 to cooperate with agent j to generate the cost c_j. Using the Harsanyi dividends (see Expression (6.1)), this game can be written as

$$v_c = c_1 u_{\{1\}} + \sum_{j \in N \setminus \{1\}} c_j u_{\{1,j\}}, \tag{6.6}$$

with $u_{\{1,j\}}$ being the unanimity game (see (6.5)) of coalition $\{1, j\}$. Since v_c is a so-called convex big boss game (see Muto et al. 1988, Theorem 4.5), the Shapley value coincides with other solutions such as the nucleolus (Schmeidler 1969) or τ-value (Tijs 1981).

It can easily be verified that we also obtain this fixed-fraction rule by applying the equal gain splitting rule to this game restricted on the corresponding line-graph γ_L.

It turns out that we can also obtain the other rules using this game, but applying different solutions for (undirected or directed) graph games. Specifically, (i) we obtain the midpoint fixed-fraction rule $F^{1/2}$ by applying the equal gains splitting rule to game v_c restricted on the corresponding line-graph γ_L, (ii) we obtain any of the fixed-fraction rules by taking the corresponding convex combination of the two 'extreme' marginal vectors in v_c, (iii) we obtain the permission value/upstream equal sharing rule by applying the

[16]Formally, for directed line-graphs D_L the directed connected hull of a coalition $T = \{i_1, i_2, ..., i_t\}$, where $i_1 < i_2 < ... < i_t$, is $[T] = \{j_1, j_2, ..., j_{k(t)}\}$ with $j_1 = i_1, j_{k(t)} = i_t$ and $j_{l+1} = j_l + 1$ for $l = 1, ..., k(t) - 1$.

permission value to game v_c restricted to the digraph D_L, and (iv) we obtain the position rules $\pi^\alpha,\ \alpha \in [0,1]$, by applying the α-position value to game v_c restricted to digraph D_L.

Theorem 6.1 *For every $(N,c) \in \mathcal{L}$,*

(i) $F^{1/2}(N,c) = EGS(N, v_c, \gamma_L)$.

(ii) *for all* $\lambda \in [0,1]$, $F^\lambda(N,c) = \lambda m^{\pi^u}(v_c) + (1-\lambda) m^{\pi^\ell}(v_c)$.

(iii) $UES(N,c) = PV(N, v_c, D_L)$.

(iv) *for all* $\alpha \in [0,1]$[17] ,

$$\pi_i^\alpha(N,c) = \begin{cases} c_1 + \pi_1^\alpha(N, \hat{v}_c, D_L) & \text{if } i = 1 \\ \pi_i^\alpha(N, \hat{v}_c, D_L) & \text{if } i > 1. \end{cases}$$

where $\hat{v}_c$ is the zero-normalization[18] *of v_c.*

Remark 6.2 The proofs of parts (iii) and (iv) of this theorem use the fact that the restriction of game v_c on directed line-graph D_L is the game $(v_c)^P$ given by

$$(v_c)^P = \sum_{j \in N} c_j u_{\{1,2,\ldots,j\}}. \tag{6.7}$$

It can be verified that in all four statements of Theorem 6.1, the game v_c can be replaced by the restricted game $(v_c)^P$. Moreover, the upstream equal sharing method can be obtained directly by applying the Shapley value to this game: $UES(N,c) = Sh(N,(v_c)^P)$. Therefore, this can be seen as the counterpart of the midpoint fixed-fraction rule $F^{1/2}$ which, as shown by Gudmundsson et al. (2024) is obtained by applying the Shapley value to the game (N, v_c): $F^{1/2}(N,c) = Sh(N, v_c)$.

7 Rooted tree model

The results in the previous sections can easily be extended to rooted tree cost allocation problems. A rooted tree on $N = \{1, \ldots, n\}$ with root 1 is a digraph $D \subseteq N \times N$ such that (i) $P_D(1) = \emptyset$, and (ii) for every $i \in N \setminus \{1\}$ there is exactly one directed path[19] from 1 to i. Given a digraph $D \subseteq N \times N$, let $S_D(i) = \{j \in N \mid (i,j) \in D\}$ be the set of *successors* of $i \in N$, and $P_D(i) = \{j \in N \mid (j,i) \in D\}$ be the set of *predecessors* of $i \in N$. Further,

[17] Taking account of the fact that the position values are defined only for zero-normalized games, we assign c_1 fully to the top player 1.

[18] The *zero-normalization* $\widehat{v}$ of game v is given by $\widehat{v}(S) = v(S) - \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$.

[19] A *directed path* in D is a sequence of nodes $(i_1, \ldots, i_p)$ such that $(i_k, i_{k+1}) \in D$ for all $k \in \{1, \ldots p-1\}$.

for every $i \in N$, let $\widehat{S}_D(i) = \{j \in N \mid \text{there is a directed path from } i \text{ to } j\}$ be the set of *subordinates* of $i \in D$, and $\widehat{P}_D(i) = \{j \in N \mid \text{there is a directed path from } j \text{ to } i\}$ be the set of *superiors* of $i \in D$. Finally, we introduce the notation $\widehat{S}^0_D(i) = \widehat{S}_D(i) \cup \{i\}$, and $\widehat{P}^0_D(i) = \widehat{P}_D(i) \cup \{i\}$.

7.1 Model and rules

A *rooted tree* cost allocation problem is a triple (N, c, D) with

- $N = \{1, 2, ...n\}$ being a set of agents,
- $c \in \mathbb{R}^n_+$ a nonnegative cost vector, and
- $D \subset N \times N$ a rooted tree with root $1 \in N$, and $(i, i) \notin D$ for all $i \in N$.

We denote the class of all rooted tree cost allocation problems on N by $\mathcal{C}^N$, and the class of all rooted tree cost allocation problems by $\mathcal{C}$. A cost allocation rule (or rule for short) for rooted tree cost allocation problems is a mapping φ that assigns to every rooted tree cost allocation problem $(N, c, D) \in \mathcal{C}$ an allocation $\varphi(N, c, D) \in \mathbb{R}^n$ of the total cost $\sum_{j \in N} c_j$.

The class of position fixed-fraction rules can be extended for rooted tree cost allocation problems as follows: For $\alpha, \beta \in [0, 1]$, the (α, β)-position fixed-fraction rule on $\mathcal{C}$ is the rule $\varphi^{\alpha,\beta}$ given by

- $\varphi^{\alpha,\beta}_1(\{1\}, c_1) = c_1$;
- If $n > 1$:

$$\varphi^{\alpha,\beta}_i(N, c, D) = \begin{cases} c_1 + \sum\limits_{j \in S_D(1)} (1-\alpha)c_j + \sum\limits_{j \in \widehat{S}_D(1) \setminus S_D(1)} \dfrac{(1-\alpha)(1-\beta)}{1 + (|\widehat{P}_D(j)| - 2)\beta} c_j & \text{if } i = 1 \\[2ex] \alpha c_i + \sum\limits_{j \in \widehat{S}_D(i)} \dfrac{\beta}{1 + (|\widehat{P}_D(j)| - 2)\beta} c_j & \text{if } i \in S_D(1) \\[2ex] \frac{\alpha(1-\beta)}{1+(|\widehat{P}_D(j)|-2)\beta} c_i + \sum\limits_{j \in \widehat{S}_D(i)} \dfrac{\beta}{1 + (|\widehat{P}_D(j)| - 2)\beta} c_j & \text{if } i \in \widehat{S}_D(1) \setminus S_D(1) \end{cases}$$

Special cases (corresponding to the special cases considered for line-cost allocation problems before with the same values for the parameters α and β, see Table 1) are:

1. Taking $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$ gives the upstream equal sharing rule (Dong et al. 2012) or permission value van den Brink and Gilles 1996) being the rule UES given by

$$UES_i(N, c, D) = \sum_{j \in \widehat{S}_D^0(i)} \frac{c_j}{|\widehat{P}_D^0(j)|} \text{ for all } i \in N.$$

2. Taking $\alpha = 1$ and $\beta = 0$ gives the local responsibility rule (Dong et al. 2012) being the rule LR given by

$$LR_i(N, c, D) = c_i \text{ for all } i \in N.$$

3. Taking $\alpha = \beta = 0$ gives the top rule (van den Brink et al. 2017 also known as full-transfer rule in Hougaard et al. 2017) or indirect liability rule (Gudmundsson et al. 2024) being the rule TOP given by

$$TOP_i(N, c, D) = \begin{cases} \sum_{j \in N} c_j & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

4. Taking $\alpha = \lambda \in [0, 1]$ and $\beta = 0$ gives the corresponding λ-fixed-fraction rule Gudmundsson et al. 2024) being the rule F^λ given by

$$F_i^\lambda(N, c, D) = \begin{cases} c_1 + (1 - \lambda) \sum_{j \in N \setminus \{1\}} c_j & \text{if } i = 1 \\ \lambda c_i & \text{if } i \neq 1 \end{cases}$$

5. Taking $\alpha \in [0, 1]$ and $\beta = \frac{1}{2}$ gives the corresponding α-position rule being the rule π^α given by

$$\pi_i^\alpha(N, c, D) = \begin{cases} c_1 + \sum_{j \in N \setminus \{1\}} \frac{(1-\alpha)}{|\widehat{P}_D(j)|} c_j \text{ if } i = 1 \\ \\ \frac{\alpha}{|\widehat{P}_D(i)|} c_i + \sum_{j \in \widehat{S}_D(i)} \frac{c_j}{|\widehat{P}_D(j)|} \text{ if } i \neq 1 \end{cases}$$

Similar as for line-cost allocation problems, notice that the local responsibility and top rules are fixed-fraction rules for $\alpha = \lambda = 1$, respectively $\alpha = \lambda = 0$. The upstream equal sharing rule is neither a fixed-fraction rule nor a position rule, but it but is the position fixed-fraction rule for $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$.

7.2 Axioms

The axiomatizations in Sections 3 and 4 can straightforward be extended to rooted tree problems by considering a rooted tree as a union of lines and realizing that according to any position fixed-fraction rule, the cost c_j in a rooted tree is allocated over the agents on the path from root 1 to agent j.

Corollary 7.1 *For every rooted tree problem (N, c, D) and $\alpha, \beta \in [0, 1]$, we have $\varphi_i^{\alpha,\beta}(N, c, D) = \sum_{j \in \{i\} \cup \widehat{S}_D(i)} c_j \varphi_i^{\alpha,\beta}(\{j\} \cup \widehat{S}_D^{-1}(j), e_j)$.*

The two additional axioms used by Gudmundsson et al. (2024) for rooted tree problems are the following. Tree additivity is a straightforward extension of additivity as is also used in, e.g. Dong et al. (2012) and, for general games, in van den Brink and Gilles (1996).

Axiom 7.2 (Tree additivity) *For every rooted tree (N, D) and cost vectors $c, c' \in \mathbb{R}^n$, it holds that $\varphi(N, c, D) + \varphi(N, c', D) = \varphi(N, c + c', D)$.*

For rooted tree (N, D), $j \in N = \{1, \ldots, n\}$ such that $S_D(j) = \emptyset$, let the rooted tree D^{j+} be given by $D^{j+} = D \cup \{(j, n + 1)\}$, and cost vector c^+ be given by $c^+_{n+1} = 0$ and $c^+_i = c_i$ for all $i \in N$. Tree zero truncation requires that adding an agent with zero cost at the bottom of the tree does not change the shares of the existing agents.

Axiom 7.3 (Tree zero truncation) *For every rooted tree (N, D), $j \in N = \{1, \ldots, n\}$ such that $S_D(j) = \emptyset$, it holds that $\varphi_i(N, c, D) = \varphi_i(N \cup \{n + 1\}, c^+, D^{j+})$ for all $i \in N$.*

To extend differential proportionality, for rooted tree (N, D), $n \geq 2$, and $j \in N = \{1, \ldots, n\}$ such that $S_D(j) = \emptyset$, let $\widehat{c}^+ \in \mathbb{R}^{n+1}$ be given by $\widehat{c}^+_{n+1} = c_j$, $c^+_j = 0$, and $c^+_i = c_i$ for all $i \in N \setminus \{j\}$.

Axiom 7.4 (Differential proportionality) *Let $n \geq 2$. For every rooted tree (N, D) and $j \in N = \{1, \ldots, n\}$ such that $S_D(j) = \emptyset$, suppose without loss of generality that $\widehat{P}_D(j) = \{j_1, j_2, \ldots, j_p\}$ with $j_1 = 1$ and $j_k \in S_D(j_{k-1})$ for all $k \in \{2, \ldots p\}$. There exists $K \in \mathbb{R}$ such that*

(i) $\varphi_1(N, e_j, D) - \varphi_1(N \cup \{n + 1\}, e_{n+1}, D^{j^+}) = K\varphi_1(N, e_j, D)$

(ii) *(a)* $\varphi_{j_k}(N, e_j, D) - \varphi_{j_k}(N \cup \{n + 1\}, e_{n+1}, D^{j^+}) = K\varphi_{j_k}(N, e_j, D)$ *and*

(b) $\varphi_{j_k}(N, e_j, D) - \varphi_{j_{k+1}}(N \cup \{n+1\}, e_{n+1}, D^{j^+}) = K\varphi_{j_k}(N, e_j, D)$ *for all* $j_k \in \widehat{P}_D(j) \setminus \{1\}$

(iii) $\varphi_j(N, e_j, D) - \varphi_{n+1}(N \cup \{n + 1\}, e_{n+1}, D^{j^+}) = K\varphi_j(N, e_j, D)$

Theorem 4.2 can be extended to rooted tree problems by extending zero truncation and additivity by the above two axioms. The obvious proof is omitted.

Theorem 7.5 *Allocation rule φ on rooted tree problems satisfies efficiency, nonnegativity, rooted tree additivity, tree zero truncation and differential proportionality if and only if there exist $\alpha, \beta \in [0,1]$ such that $\varphi = \varphi^{\alpha,\beta} \in \Phi$.*

Notice that this theorem extends Theorem 4.2 from line- to rooted tree cost allocation problems, but also extends the axiomatization of the fixed-fraction rules for rooted tree problems in Gudmundsson et al. (2024) to the more larger class of position fixed-fraction rules.

Remark 7.6 *We remark that Gudmundsson et al. (2024) use a modified problem on rooted trees, where instead of costs associated with the agents, the costs are associated to the links, i.e. they consider a $n \times n$ cost matrix $\widehat{C}$ where the cost $\widehat{c}_{ij} = 0$ if $(i,j) \notin D$. Replacing this cost matrix by the cost vector*

$$c_j = \begin{cases} c_1 + \sum_{k \in S_D(1)} \widehat{c}_{1,k} & \text{if } j = 1 \\ \sum_{k \in S_D(j)} \widehat{c}_{j,k} & \text{if } j \neq 1 \end{cases}$$

our position fixed-fraction rules for $\beta = 0$ give their fixed-fraction rules for rooted tree problems.

8 Concluding remarks

8.1 Smart contracts

Gudmundsson et al. (2024) motivate why their fixed-fraction rules are suitable to implement in smart contracts for blockchains.[20] We argue that the position fixed-fraction rules, although more elaborate, are very well suitable to be used in a smart contract. The great benefit of position fixed-fraction rules is that, besides the top-down dimension, one can also take the in-out dimension into account.

As mentioned in the introduction, the smart contract is determined by specifying these three elements: (i) sharing between two agents (by α), (ii) determining the intermediary share in a three agent problem with one intermediary (by β), and (iii) determining which are the relevant pairs for comparison when extending the problem with one agent and applying the differential proportionality principle.

[20] For a description of smart contracts in blockchains and why fixed-fraction rules are suitable to allocate sequentially triggered losses, we refer to their paper.

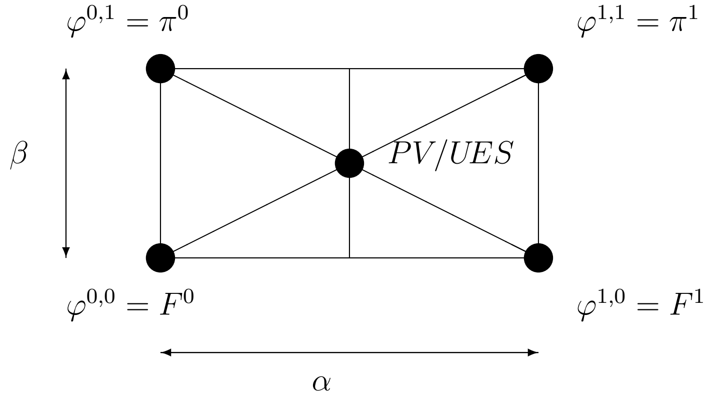

Figure 5: Classes of rules

One can consider the two parameters α and β that fix this type of contract as two 'control buttons' that both can be turned from 0 to 1, see Figure 5. The α-button sets the up-down dimension with, for example π^0 and F^0 on one extreme and π^1 and F^1 on the other extreme. The β-button sets the in-out dimension with, for example the position rules on one extreme and the fixed-fraction rules on the other extreme. As also illustrated in Figure 5, the permission/upstream equal sharing rule can be seen as a compromise along several dimensions.

8.2 Merging proofness

As mentioned before, since all position fixed-fraction rules satisfy all the axioms in Theorem 3.8 except merging proofness, the fixed-fraction rules are the only position fixed-fraction rules that satisfy merging proofness. Notice that merging proofness consists of two conditions, one for $i = 2$ and one for $i = 1$. In Appendix C we show that other position fixed-fraction rules still satisfy one of the two conditions in merging proofness. Specifically, a position fixed-fraction rule satisfies the first part of merging proofness (for $i = 1$) if and only if it is a fixed-fraction rule or a rule that assigns a zero share to every agent j in its own cost c_j (such as the position rule π^0). On the other hand, a position fixed-fraction rule satisfies the second part of merging proofness (for $i = 2$) if and only if it is a fixed-fraction rule or a rule that assigns zero share to the top agent 1 in any c_j, $j > 1$ (such as the position rule π^1). Other position rules do not satisfy any of the two merging proofness inequalities. One idea for future research is to combine merging proofness with other variations of differential proportionality (or the top-down or bottom-up versions of Section 5). Another idea for future research is to focus on the position rules as an interesting class of rules in itself. For example, looking for axiomatic characterizations by finding an alternative for merging proofness.

Gudmundsson et al. (2024) mention that using their fixed-fraction rules ensures some stability with respect to 'sybil attacks' (see e.g. Douceur 2002) in the sense that it is not beneficial to manipulate the blockchain by creating artificial losses, i.e. adding an artificial zero loss agent somewhere in the blockchain[21]. Specifically, according the fixed-fraction rules such a manipulation has no effect for the existing agents. Notice that merging proofness itself allows a manipulating agent to pay lower cost shares, when extending from a 2- to a 3-loss problem, and thus is in contradiction with stability against sybil attacks. Only in the special case where the weak inequality is an equality this does not occur, which is the case for the fixed-fraction rules. As mentioned, since our position fixed-fraction rules satisfy all the axioms of Theorem 4.2 except merging proofness, they cannot satisfy this axiom (besides the fixed-fraction rules). More specific, they satisfy the 'opposite' splitting proofness where the inequality goes in the other direction, and which is consistent with stability against sybil attacks. In fact, this *sybil attack proofness* holds for any agent in any n-loss problem. It is related to a weak version of *reallocation proofness* in claims problems which, in the context of n-loss problems, means that adding an artificial agent between agents i and $i+1$ such that (i) the total cost of agent i and this new agent is equal to the original cost of agent i, and (ii) all other agents keep the same cost, does not benefit agent i.

A plan for future research is to consider sybil attack proofness as an axiom to characterize rules for n-loss problems. Another future research question is to replace differential proportionality in Theorem 4.2 by population monotonicity and characterize the class of rules that satisfy these axioms. Note that this is equivalent to removing merging proofness from the axiomatization in Theorem 3.8. This results in a class of rules between the fixed-fraction and position fixed-fraction rules, see Figure 6 which pictures logical relations under the assumptions that the rule satisfies efficiency, nonnegativity, additivity and zero truncation. From this figure we see clearly that within this class of rules, differential proportionality (giving the position fixed-fraction rules) is weaker than population monotonicity plus merging proofness (giving the fixed-fraction rules), but stronger than only population monotonicity.[22] As mentioned in the paragraph before Proposition 4.4,

[21]For an n-loss problem $(l_1, \dots, l_i, l_{i+1}, \dots, l_n)$, such a manipulation by agent i would occur if by agent i artificially adding a zero loss agent immediately behind it (so creating the $(n+1)$-loss problem $(l_1, \dots, l_i, 0, l_{i+1}, \dots, l_n)$) this is not beneficial for agent i in the sense that the sum of the cost shares of agent i and the newly created zero-loss agent is not lower than the original cost share of agent i.

[22]This can be seen from the alternative rule in which agent 1 pay its own cost, and every other agent pays half of its own cost and the other half is equally shared among its upstream agents. This rule, which is an

without any further assumption on a rule, there is no logical relation between differential proportionality and population monotonicity.[23]

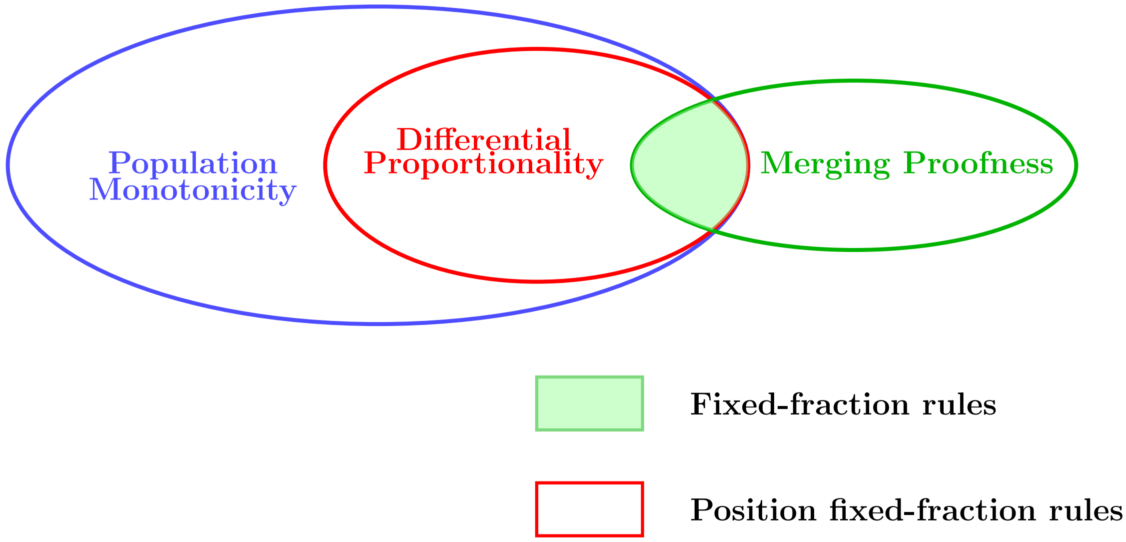

Figure 6: Logical relations under efficiency, nonnegativity, additivity and zero truncation

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equal upstream responsibility method as introduced in Li et al. (2023), satisfies efficiency, nonnegativity, additivity, zero truncation and population monotonicity but does not satisfy differential proportionality.

[23]This can be seen by taking any position fixed-fraction rule which assigns a positive cost share to the top agent 1, for example $\varphi^{\frac{1}{2},\frac{1}{2}}$ and, (i) consider the alternative rule that assigns to every agent i the negative $-\varphi_i^{\frac{1}{2},\frac{1}{2}}(N,c)$ which satisfies differential proportionality but does not satisfy population monotonicity, or (ii) consider the rule where agent 1 gets n times its position fixed-fraction share, $n \cdot \varphi_1^{\frac{1}{2},\frac{1}{2}}(N,c)$ and all other agents $i \neq 1$ get their position fixed-fraction cost share $\varphi_i^{\frac{1}{2},\frac{1}{2}}(N,c)$, which satisfies population monotonicity but does not satisfy differential proportionality.

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Appendix A: Proofs

This appendix contains all proofs of the paper.

Proof of Theorem 4.2

Before giving the proof of Theorem 4.2, we want to make the following remark.

Remark 8.1 *By rearranging terms in the definition of differential proportionality (Axiom 4.1), applying the distributive property to $\varphi_1(e_n)$ in (i), to $\varphi_i(e_n)$ in (ii) and to $\varphi_n(e_n)$ in (iii) and setting $K^* = 1 - K$, the expressions in (i), (ii) and (iii) can be simplified as follows*

(i') $\varphi_1(e_{n+1}) = K^*\varphi_1(e_n)$.

(ii') *(a)* $\varphi_i(e_{n+1}) = K^*\varphi_i(e_n)$ *and (b)* $\varphi_{i+1}(e_{n+1}) = K^*\varphi_i(e_n)$ *for all* $1 < i < n$.

(iii') $\varphi_{n+1}(e_{n+1}) = K^*\varphi_n(e_n)$.

Proof of Theorem 4.2
(If) Let $\alpha, \beta \in [0, 1]$. It is obvious that $\varphi = \varphi^{\alpha,\beta}$ satisfies efficiency, nonnegativity, additivity and zero truncation. Next, we show that it satisfies differential proportionality.
If $n = 2$

$$\varphi_i(e_2) = \begin{cases} 1-\alpha & \text{if } i = 1 \\ \alpha & \text{if } i = 2 \end{cases} \quad \text{and } \varphi_i(e_3) = \begin{cases} (1-\alpha)(1-\beta) & \text{if } i = 1 \\ \beta & \text{if } i = 2 \\ \alpha(1-\beta) & \text{if } i = 3, \end{cases} \tag{8.8}$$

and thus (i'), (ii') and (iii') in Remark 8.1 are satisfied for $K^* = 1-\beta$ (or $K = 1-K^* = \beta$ in Axiom 4.1).

If $n \geq 3$,

$$\varphi_i(e_n) = \begin{cases} \frac{(1-\alpha)(1-\beta)}{1+(n-3)\beta} & i = 1 \\ \frac{\beta}{1+(n-3)\beta} & 2 \leq i \leq n-1 \\ \frac{\alpha(1-\beta)}{1+(n-3)\beta} & i = n \end{cases} \quad \text{and } \varphi_i(e_{n+1}) = \begin{cases} \frac{(1-\alpha)(1-\beta)}{1+(n-2)\beta} & i = 1 \\ \frac{\beta}{1+(n-2)\beta} & 2 \leq i \leq n \\ \frac{\alpha(1-\beta)}{1+(n-2)\beta} & i = n+1. \end{cases} \tag{8.9}$$

In this case (i'), (ii') and (iii') in Remark 8.1 hold for $K^* = \frac{1+(n-3)\beta}{1+(n-2)\beta}$.

(Only if) Now, suppose that allocation rule φ satisfies efficiency, nonnegativity, additivity, zero truncation and differential proportionality. We show that there exists an $\alpha, \beta \in [0, 1]$ such that $\varphi = \varphi^{\alpha,\beta}$.

If $n = 1$ then efficiency implies that $\varphi_1(e_1) = 1$.
If $n = 2$ then efficiency implies that there exists an $\alpha \in \mathbb{R}$ such that $\varphi_1(e_2) = 1 - \alpha$ and $\varphi_2(e_2) = \alpha$. Nonnegativity implies that $\alpha \in [0, 1]$.

Proceeding by induction, suppose that $\varphi(e_{n'}) = \varphi^{\alpha,\beta}(e_{n'})$ for $n' \leq n$.
By differential proportionality and using Remark 8.1, there exits $K^* \in \mathbb{R}$ such that
(i') for $i = 1$,

$$\varphi_1(e_{n+1}) = K^*\varphi_1(e_n) = K^*\frac{(1-\alpha)(1-\beta)}{1+(n-3)\beta}. \tag{8.10}$$

(ii') for $1 < i < n$,

$$\varphi_i(e_{n+1}) = \varphi_{i+1}(e_{n+1}) = K^*\varphi_i(e_n) = K^*\frac{\beta}{1+(n-3)\beta}. \tag{8.11}$$

(iii') for $i = n + 1$,

$$\varphi_{n+1}(e_{n+1}) = K^*\varphi_n(e_n) = K^*\frac{\alpha(1-\beta)}{1+(n-3)\beta}. \tag{8.12}$$

With efficiency, after a simple algebraic manipulation, with (8.10)-(8.12), we have

$$1 = \sum_{i=1}^{n+1}\varphi_i(e_{n+1}) = \varphi_1(e_{n+1}) + \sum_{i=2}^{n}\varphi_i(e_{n+1}) + \varphi_{n+1}(e_{n+1}) = K^*\left(\frac{1+(n-2)\beta}{1+(n-3)\beta}\right),$$

and thus $K^* = \frac{1+(n-3)\beta}{1+(n-2)\beta}$.

Substituting K^* in (8.10)-(8.12) gives $\varphi_i(e_{n+1}) = \begin{cases} \frac{(1-\alpha)(1-\beta)}{1+(n-2)\beta} & \text{if } i = 1 \\ \frac{\beta}{1+(n-2)\beta} & \text{if } 2 \leq i \leq n \\ \frac{\alpha(1-\beta)}{1+(n-2)\beta} & \text{if } i = n+1. \end{cases} = \varphi_i^{\alpha,\beta}(e_{n+1})$.
By Proposition 3.5, the shares $\varphi(c)$ are uniquely determined for every c. □

Proof Proposition 4.4

Let $\alpha, \beta \in [0, 1]$.

Case $n = 1$:
(i) $\varphi_1^{\alpha,\beta}(e_1) = 1 \geq 1 - \alpha = \varphi_1^{\alpha,\beta}(e_2)$
(iii) $\varphi_1^{\alpha,\beta}(e_1) = 1 \geq \alpha = \varphi_2^{\alpha,\beta}(e_2)$.
Notice that Condition (ii) of population monotonicity need not be verified if $n = 1$.

Case $n = 2$:

(i) $\varphi_1^{\alpha,\beta}(e_2) = 1-\alpha \geq (1-\alpha)(1-\beta) = \varphi_1^{\alpha,\beta}(e_3)$, and
(iii) $\varphi_2^{\alpha,\beta}(e_2) = \alpha = \varphi_3^{\alpha,\beta}(e_3)$.
Notice that also in case $n = 2$, Condition (ii) of population monotonicity need not be verified.

Case $n \geq 3$: We have

$$\varphi_i^{\alpha,\beta}(e_n) = \begin{cases} \frac{(1-\alpha)(1-\beta)}{1+(n-3)\beta} & i = 1 \\ \frac{\beta}{1+(n-3)\beta} & 1 < i < n \\ \frac{\alpha(1-\beta)}{1+(n-3)\beta} & i = n. \end{cases}$$

which clearly decreases for a fixed i with n. Then

$$\varphi_1^{\alpha,\beta}(e_n) \geq \varphi_1^{\alpha,\beta}(e_{n+1}) \text{ and } \varphi_i^{\alpha,\beta}(e_n) \geq \varphi_i^{\alpha,\beta}(e_{n+1}) \text{ for } 1 < i < n.$$

Moreover,

$$\varphi_i^{\alpha,\beta}(e_n) = \frac{\beta}{1+(n-3)\beta} \geq \frac{\beta}{1+(n-2)\beta} = \varphi_{i+1}^{\alpha,\beta}(e_{n+1}) \text{ for } 1 < i < n$$

and

$$\varphi_n^{\alpha,\beta}(e_n) = \frac{\alpha(1-\beta)}{1+(n-3)\beta} \geq \frac{\alpha(1-\beta)}{1+(n-2)\beta} = \varphi_{n+1}^{\alpha,\beta}(e_{n+1}).$$

which completes the proof. □

Proof of Theorem 4.5

Requiring differential proportionality also for $n = 1$, similar as in Remark 8.1, we should additionally require

$$\varphi_1(e_2) = K^*\varphi_1(e_1) \text{ and } \varphi_2(e_2) = K^*\varphi_1(e_1), \tag{8.13}$$

being the replacements of Equations (4.3) and (4.4).

(If) It follows from Theorem 4.2 that the rules $\varphi^{\frac{1}{2},\beta}$, $\beta \in [0,1]$ satisfy efficiency, nonnegativity, additivity, zero truncation and differential proportionality for $n \geq 2$.
differential proportionality for $n = 1$ follows since $\varphi_1^{\frac{1}{2},\beta}(e_1) = 1$ and $\varphi_1^{\frac{1}{2},\beta}(e_2) = \frac{1}{2} = \varphi_2^{\frac{1}{2},\beta}(e_2)$ for all $\beta \in [0,1]$, and thus $K^* = \frac{1}{2}$ in (8.13).

(Only if) Now, suppose that allocation rule φ satisfies efficiency, nonnegativity, additivity, zero truncation, and extended differential proportionality for $n \geq 1$.

If $n = 1$ then, similar as in the uniqueness proof of Theorem 4.2, efficiency implies that $\varphi_1(e_1) = 1$.

If $n = 2$ then differential proportionality for $n = 1$ implies that $\varphi_1(e_2) = \varphi_2(e_2) = K^*\varphi_1(e_1)$. Using efficiency we have $\varphi_1(e_2) = \varphi_2(e_2) = \frac{1}{2}$.

From here, for $n \geq 3$ we can continue the proof by induction similar as in the proof of Theorem 4.2 and get uniqueness. □

Proof of Theorem 5.2

Before giving the proof of Theorem 5.2, we want to make the following remark.

Remark 8.2 *Notice that, similar as with differential proportionality and Remark 8.1, conditions (i) and (ii) in Axiom 5.1 can be written in term of $K^* = 1 - K$ in the following way:*

(i") $\varphi_i(e_{n+1}) = K^*\varphi_i(e_n)$ *for all* $1 \leq i \leq n$*;*

(ii") $\varphi_{n+1}(e_{n+1}) = K^*\varphi_n(e_n)$.

Proof of Theorem 5.2

(If) Let $\alpha, \beta \in [0, 1]$. By Theorem 4.2, any $\varphi = \varphi^{\alpha,\beta}$ satisfies efficiency, nonnegativity, additivity and zero truncation. Next we show that for $\beta = \frac{\alpha}{1+\alpha}$, it satisfies top differential proportionality.

From (8.8) for $\varphi = \varphi^{\alpha,\frac{\alpha}{1+\alpha}}$ we have:

if $n = 2$, $\varphi_i(e_2) = \begin{cases} 1-\alpha & \text{if } i = 1 \\ \alpha & \text{if } i = 2 \end{cases}$ and $\varphi_i(e_3) = \begin{cases} \frac{(1-\alpha)}{(1+\alpha)} & \text{if } i = 1 \\ \frac{\alpha}{(1+\alpha)} & \text{if } i = 2, 3 \end{cases}$,

and thus (i") and (ii") in Remark 8.2 are satisfied for $K^* = \frac{1}{1+\alpha}$ (or $K = 1 - K^* = \frac{\alpha}{1+\alpha}$ in Axiom 5.1).

If $n \geq 3$, from (8.9) we have

$$\varphi_i(e_n) = \begin{cases} \frac{(1-\alpha)}{1+(n-2)\alpha} & \text{if } i = 1 \\ \frac{\alpha}{1+(n-2)\alpha} & \text{if } 2 \leq i \leq n \end{cases} \quad \text{and } \varphi_i(e_{n+1}) = \begin{cases} \frac{(1-\alpha)}{1+(n-1)\alpha} & \text{if } i = 1 \\ \frac{\alpha}{1+(n-1)\alpha} & \text{if } 2 \leq i \leq n+1. \end{cases} \tag{8.14}$$

In this case (i") and (ii") in Remark 8.2 hold for $K^* = \frac{1+(n-2)\alpha}{1+(n-1)\alpha}$.

(Only if) Now, suppose that allocation rule φ satisfies efficiency, nonnegativity, additivity, zero truncation and top-down differential proportionality. We show that there exist $\alpha \in [0, 1]$ such that $\varphi = \varphi^{\alpha,\frac{\alpha}{1+\alpha}}$.

If $n = 1$ then efficiency implies that $\varphi_1(e_1) = 1 = \varphi_1^{\alpha,\frac{\alpha}{1+\alpha}}(e_1)$.

If $n = 2$ then efficiency implies that there exists an $\alpha \in \mathbb{R}$ such that $\varphi_1(e_2) = 1 - \alpha$ and $\varphi_2(e_2) = \alpha$. Nonnegativity implies that $\alpha \in [0, 1]$, and thus, in this case also $\varphi(e_2) = \varphi^{\alpha,\frac{\alpha}{1+\alpha}}(e_2)$.

If $n \geq 3$, proceeding by induction, suppose that $\varphi(e_{n'}) = \varphi^{\alpha,\frac{\alpha}{1+\alpha}}(e_{n'})$ for $n' \leq n$.

Using Remark 8.2, top-down differential proportionality implies that there exists a $K^* \in \mathbb{R}$ such that $\varphi_i(e_{n+1}) = \begin{cases} K^*\varphi_i(e_n) & \text{if } 1 \leq i \leq n \\ K^*\varphi_n(e_n) & \text{if } i = n+1 \end{cases}$
which with the induction hypothesis and the expression of $\varphi^{\alpha,\frac{\alpha}{1+\alpha}}(e_n)$ in (8.14) gives

$$\varphi_i(e_{n+1}) = \begin{cases} K^*\frac{1-\alpha}{1+(n-2)\alpha} & \text{if } i = 1 \\ K^*\frac{\alpha}{1+(n-2)\alpha} & \text{if } 2 \leq i \leq n+1. \end{cases} \tag{8.15}$$

With efficiency and after some simple algebraic manipulations we have

$$1 = \sum_{i=1}^{n+1} \varphi_i(e_{n+1}) = \varphi_1(e_{n+1}) + \sum_{i=2}^{n+1} \varphi_i(e_{n+1}) = K^* \left(\frac{1+(n-1)\alpha}{1+(n-2)\alpha} \right)$$

and thus $K^* = \frac{1+(n-2)\alpha}{1+(n-1)\alpha}$.

Substituting in (8.15) gives $\varphi_i(e_{n+1}) = \begin{cases} \frac{1-\alpha}{1+(n-1)\alpha} & \text{if } i = 1 \\ \frac{\alpha}{1+(n-1)\alpha} & \text{if } 2 \leq i \leq n+1 \end{cases}$ which equals $\varphi_i^{\alpha,\frac{\alpha}{1+\alpha}}(e_{n+1})$.

By Proposition 3.5, the shares $\varphi(c)$ are uniquely determined for every c. □

Proof of Theorem 5.4

Before giving the proof of Theorem 5.4, we want to make the following remark.

Remark 8.3 *Notice that, similar as with differential proportionality in Remark 8.1 (and with top-differential proportionality in Remark 8.2, conditions (i) and (ii) in Axiom 5.3 can be written, using $K^* = 1 - K$ in the following way:*

(i$''''$) $\varphi_1(e_{n+1}) = K^*\varphi_1(e_n)$

(ii$''''$) $\varphi_{i+1}(e_{n+1}) = K^*\varphi_i(e_{n+1})$ *for* $1 \leq i \leq n$.

Proof of Theorem 5.4

(If) Let $\alpha, \beta \in [0,1]$. By Theorem 4.2, any $\varphi = \varphi^{\alpha,\beta}$ satisfies efficiency, nonnegativity, additivity and zero truncation. Next we show that for $\beta = \frac{1-\alpha}{2-\alpha}$, it satisfies bottom-up differential proportionality.
From (8.8) for $\varphi = \varphi^{\alpha,\frac{1-\alpha}{2-\alpha}}$ we have:
if $n = 2$, $\varphi_i(e_2) = \begin{cases} 1-\alpha & \text{if } i = 1 \\ \alpha & \text{if } i = 2 \end{cases}$ and $\varphi_i(e_3) = \begin{cases} \frac{1-\alpha}{2-\alpha} & \text{if } i = 1,2 \\ \frac{\alpha}{2-\alpha} & \text{if } i = 3 \end{cases}$
and thus (i‴) and (ii‴) in Remark 8.3 are satisfied for $K^* = \frac{1}{2-\alpha}$ (or $K = 1 - K^* = \frac{1-\alpha}{2-\alpha}$ in Axiom 5.3).

If $n \geq 3$ from (8.9),

$$\varphi_i(e_n) = \begin{cases} \frac{(1-\alpha)}{n-1-(n-2)\alpha} & \text{if } 1 \leq i \leq n-1 \\ \frac{\alpha}{n-1-(n-2)\alpha} & \text{if } i = n \end{cases} \tag{8.16}$$

and $\varphi_i(e_{n+1}) = \begin{cases} \frac{(1-\alpha)}{n-(n-1)\alpha} & \text{if } 1 \leq i \leq n \\ \frac{\alpha}{n-(n-1)\alpha} & \text{if } i = n+1 \end{cases}$
In this case (i‴) and (ii‴) hold for $K^* = \frac{n-1-(n-2)\alpha}{n-(n-1)\alpha}$.

(Only if) Now, suppose that allocation rule φ satisfies efficiency, nonnegativity, additivity, zero truncation and bottom-up differential proportionality. We show that there exists $\alpha \in [0,1]$ such that $\varphi = \varphi^{\alpha,\frac{1-\alpha}{2-\alpha}}$.
If $n = 1$ then efficiency implies that $\varphi_1(e_1) = 1 = \varphi_1^{\alpha,\frac{1-\alpha}{2-\alpha}}(e_1)$.

If $n = 2$ then efficiency implies that there exists an $\alpha \in \mathbb{R}$ such that $\varphi_1(e_2) = 1 - \alpha$ and $\varphi_2(e_2) = \alpha$. Nonnegativity implies that $\alpha \in [0,1]$, and thus, in this case also $\varphi(e_2) = \varphi^{\alpha,\frac{1-\alpha}{2-\alpha}}(e_2)$.

If $n \geq 3$, proceeding by induction, suppose that $\varphi(e_{n'}) = \varphi^{\alpha,\frac{1-\alpha}{2-\alpha}}(e_{n'})$ for $n' \leq n$.

Using Remark 8.3, bottom-up differential proportionality implies that there exists a $K^* \in \mathbb{R}$ such that $\varphi_i(e_{n+1}) = \begin{cases} K^*\varphi_i(e_n) & \text{if } i = 1 \\ K^*\varphi_{i-1}(e_{n+1}) & \text{if } 2 \leq i \leq n+1 \end{cases}$
which from the expression of $\varphi^{\alpha,\frac{1-\alpha}{2-\alpha}}(e_n)$ in (8.16) gives

$$\varphi_i(e_{n+1}) = \begin{cases} K^* \frac{1-\alpha}{n-1-(n-2)\alpha} & \text{if } 1 \leq i \leq n \\ K^* \frac{\alpha}{n-1-(n-2)\alpha} & \text{if } i = n+1. \end{cases} \tag{8.17}$$

With efficiency and after some simple algebraic manipulations we have

$$1 = \sum_{i=1}^{n+1} \varphi_i(e_{n+1}) = \sum_{i=1}^{n} \varphi_i(e_{n+1}) + \varphi_{n+1}(e_{n+1}) = K^* \left(\frac{n-(n-1)\alpha}{n-1-(n-2)\alpha} \right)$$

and thus $K^* = \frac{n-1-(n-2)\alpha}{n-(n-1)\alpha}$.

Substituting in (8.17) gives $\varphi_i(e_{n+1}) = \begin{cases} \frac{1-\alpha}{n-(n-1)\alpha} & \text{if } 1 \le i \le n \\ \frac{\alpha}{n-(n-1)\alpha} & \text{if } i = n+1 \end{cases}$ which is equal to $\varphi_i^{\alpha, \frac{1-\alpha}{2-\alpha}}(e_{n+1})$.

By Proposition 3.5, the shares $\varphi(c)$ are uniquely determined for every c. □

Proof of Theorem 6.1

(i) and (ii): This follows straightforward from additivity and the fact that for game v_{e_n}, the marginal vector corresponding to the order $\sigma(i) = i$ for all $i \in N$ assigns the full cost of $(e_n)_n = 1$ to agent n (and zero to the other agents), while the marginal vector corresponding to the order $\sigma(i) = n - i + 1$ for all $i \in N$, assigns the full cost of $(e_n)_n = 1$ to agent 1 (and zero to the other agents).

(iii) This follows directly from Remark 6.2 and noting that the Harsanyi dividends of game $(v_c)^P$ are given by

$$\Delta_{(v_c)^P}(T) = \begin{cases} c_i & \text{if } T = \{1, \dots, i\} \\ 0 & \text{otherwise,} \end{cases}$$

and the Shapley value allocating every Harsanyi dividend equally over the players in the corresponding unanimity coalition.

(iv) Since the out-degrees of the nodes in the directed line-graph $D_L = \{(i, i+1) \mid i \in \{1, \dots n-1\}\}$ are given by $out_n(D_l) = 0$ and $out_i(D_L) = 1$ for all $i \in \{1, \dots, n-1\}$, the out-oriented position value is given by $\pi^0(N, v_c, D_L) = (\frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}, 0)$.
Similar, since the in-degrees of the nodes in the directed line-graph D_L are given by $in_1(D_l) = 0$ and $in_i(D_L) = 1$ for all $i \in \{2, \dots, n\}$, the in-oriented position value is given by $\pi^1(N, v_c, D_L) = (0, \frac{1}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1})$.
Statement(iv) then follows straightforward from the definition of the position value π^α, $\alpha \in [0, 1]$. □

Appendix B: Logical independence

Logical independence of the axioms in Theorem 4.2 can be shown by the following alternative allocation rules.

- The zero rule given by $\varphi_i^{zero}(N,c) = 0$ for all $i \in N$ satisfies additivity, nonnegativity, zero truncation and differential proportionality. It does not satisfy efficiency.

- For $\alpha, \beta \in [0,1]$, the rule given by $\varphi(N,c) = \varphi^{\alpha,\beta}(N,c)$ if $c_j = 0$ for all $j > 1$, and otherwise (i.e. if $c_j > 0$ for some $i > 1$) be given by

$$\varphi_i(N,c) = \begin{cases} \varphi_i^{\alpha,\beta}(N,c) - \sum_{j \in N} c_j & \text{if } i = 1 \\ \varphi_i^{\alpha,\beta}(N,c) + \sum_{j \in N} c_j & \text{if } i = 2 \\ \varphi_i^{\alpha,\beta}(N,c) & \text{if } i \geq 3 \end{cases}$$

  satisfies efficiency, additivity, zero truncation, and differential proportionality. It does not satisfy nonnegativity.

- The rule given by

$$\varphi(N,c) = \begin{cases} \pi^0(N,c) & \text{if } \sum_{i \in N} c_i \leq 10 \\ \pi^1(N,c) & \text{if } \sum_{i \in N} c_i > 10 \end{cases}$$

  satisfies efficiency, nonnegativity, zero truncation, and differential proportionality. It does not satisfy additivity.

- The equal division rule given by $\varphi_i^{equal}(N,c) = \frac{\sum_{j \in N} c_j}{n}$ for all $i \in N$ satisfies efficiency, additivity, and differential proportionality. It does not satisfy zero truncation.

- The rule given by

$$\varphi(N,c) = \sum_{j \in N} \varphi(N, c_j e_j)$$

  with

$$\varphi(N, c_j e_j) = \begin{cases} \pi^0(N, c_j e_j) & \text{if } j \leq 10 \\ \pi^1(N, c_j e_j) & \text{if } j > 10 \end{cases}$$

  satisfies efficiency, nonnegativity, additivity, and zero truncation. It does not satisfy differential proportionality.

Appendix C: Merging proofness

Since all position fixed-fraction rules satisfy all the axioms in Theorem 3.8 except merging proofness, the fixed-fraction rules are the only position fixed-fraction rules that satisfy merging proofness.

Corollary 8.4 *$\varphi^{\alpha,\beta} \in \Phi$ satisfies merging proofness if and only if $\beta = 0$.*

It turns out that an allocation rule in Φ satisfies the first part of merging proofness (for $i = 1$, referred to as merging proofness 1) if and only if it is a fixed-fraction rule or a rule that assigns a zero share to every agent j in its own cost c_j. On the other hand, an allocation rule in Φ satisfies the second part of merging proofness (for $i = 2$, referred to as merging proofness 2) if and only if it is a fixed-fraction rule or a rule that assigns zero share to the top agent 1 in any c_j, $j > 1$.

Proposition 8.5 *Let $\alpha, \beta \in [0, 1]$.*
Allocation rule $\varphi^{\alpha,\beta} \in \Phi$ satisfies merging proofness 1 if and only if $\alpha = 0$ or $\beta = 0$.
Allocation rule $\varphi^{\alpha,\beta} \in \Phi$ satisfies merging proofness 2 if and only if $\alpha = 1$ or $\beta = 0$.

Proof
Merging proofness 1 ($i = 1$) means that

$$\varphi_1(e_3) + \varphi_2(e_3) \leq \varphi_1(e_2) \tag{8.18}$$

Since $\varphi_1^{\alpha,\beta}(e_3) = (1-\alpha)(1-\beta) = 1 - \alpha - \beta + \alpha\beta$, $\varphi_2^{\alpha,\beta}(e_3) = \beta$ and $\varphi_1^{\alpha,\beta}(e_2) = 1 - \alpha$, merging proofness 1 is equivalent to

$$1 - \alpha - \beta + \alpha\beta + \beta \leq 1 - \alpha \Leftrightarrow \alpha\beta \leq 0 \Leftrightarrow \alpha = 0 \text{ or } \beta = 0.$$

Merging proofness 2 ($i = 2$) means that

$$\varphi_2(e_3) + \varphi_3(e_3) \leq \varphi_2(e_2) \tag{8.19}$$

Since $\varphi_3^{\alpha,\beta}(e_3) = \alpha(1-\beta) = \alpha - \alpha\beta$ and $\varphi_2^{\alpha,\beta}(e_2) = \alpha$, merging proofness 2 is equivalent to

$$\beta + \alpha - \alpha\beta \leq \alpha \Leftrightarrow (1-\alpha)\beta \leq 0 \Leftrightarrow \alpha = 1 \text{ or } \beta = 0.$$

□

Notice that Proposition 8.5 implies that the position rule π^0 satisfies merging proofness 1, while the position rule π^1 satisfies merging proofness 2. Other position rules do not satisfy any of the two merging proofness inequalities.

Corollary 8.6 *(i) π^0 satisfies merging proofness 1*
(ii) π^1 satisfies merging proofness 2.
(iii) π^α, $\alpha \in (0, 1)$, does not satisfy merging proofness 1 nor merging proofness 2.