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The Consistency Principle in the Reordering Problem

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The Consistency Principle in the Reordering Problem*

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Abstract

We investigate implications of the consistency principle for the reordering problem, also known as the queueing problem with an initial queue. The consistency principle specifies how an allocation rule should respond when an agent leaves the problem. We introduce four different consistency properties for the reordering problem and characterize three allocations rules, the pairwise equal-splitting rule (Curiel et al., 1989), the maximum price rule and the minimum price rule. *Balanced consistency* requires that for each pair of agents i and j, the impact on agent i's net utility when agent j leaves the initial queue and the agents behind her move forward by one position, should be equal to the impact on agent j's net utility when agent i leaves the initial queue and the agents behind her move forward by one position. *Balanced cost reduction* requires that if an agent leaves the initial queue and the agents behind her move forward by one position, then the total net utilities of the remaining agents should be reduced by the amount equal to the net utility of the departing agent. *Smallest-cost consistency* (respectively, *largest-cost consistency*) requires that if an agent with the smallest (respectively, largest) unit waiting cost leaves the initial queue and the agents behind her move forward by one position, then the net utilities of the remaining agents should not be affected. We show that either *balanced consistency* or *balanced cost reduction*, together with the three basic properties of *queue-efficiency*, *budget-balance* and *Pareto indifference*, characterizes the pairwise equal-splitting rule. On the other hand, together with the three basic properties, *smallest-cost consistency* characterizes the maximum price rule and *largest-cost consistency* the minimum price rule.

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1 Introduction

We are interested in the reordering problem, also known as the queueing problem with an initial queue, introduced in Curiel et al. (1989). Agents are lined up in an initial queue to have a service in a facility. Each agent needs the same amount of service time, which is normalized to one, but differs in the unit waiting cost. The facility can serve only one agent at a time. We are concerned with finding an order to serve agents and (monetary) transfers that are reasonable and motivate the agents to reorder themselves into an efficient queue which minimizes the aggregate waiting cost. Although the first-come-first-served rule is the most common, it is usually not efficient and the agents can be better off by switching positions with transfers. To motivate all agents to cooperate in reordering into an efficient queue, we design transfers among the agents such that no agent is worse off than in the initial queue. An allocation for a reordering problem consists of a reordered queue and a vector of transfers to the agents. A reordering rule (or a rule, for short) is a correspondence that associates with each reordering problem a set of feasible allocations. We assume that each agent has quasi-linear preferences so that each agent's net utility obtained by reordering the initial queue is defined as the sum of the transfer to her together with the difference in her total waiting cost between the initial queue and the reordered queue.

The reordering problem has been studied from various perspectives in recent literature. Curiel et al. (1989) and Hamers et al. (1996) adopt a normative viewpoint, examining the pairwise equal-splitting rule and the split core, respectively. The pairwise equal-splitting rule selects an efficient queue and allocates the cost-savings from swapping the positions of any two neighboring agents equally between them. The split core generalizes this rule by allowing all nonnegative allocations of the cost-savings obtained from such swaps. Gershkov and Schweinzer (2010) and Chun et al. (2017) analyze the reordering problem from an incentive viewpoint. Yang et al. (2019) take a strategic viewpoint to study the pairwise equal-splitting rule. Additionally, Tsay et al. (2025a, b) investigate the connected equal-splitting rule (Curiel et al., 1989),[1] which selects an efficient queue and allocates the cost-savings from swapping the positions of any two agents equally among themselves and all agents initially positioned between them, and the pairwise equal-splitting rule, respectively, from both normative and strategic viewpoints.

In this paper, we adopt a normative viewpoint to study the reordering problem, with a particular focus on the consistency principle, which specifies how an allocation rule

[1]This rule is called as the Shapley value in Curiel et al. (1989). We follow the terminology in Chun (2019).

should respond when an agent leaves the problem. An allocation rule is "consistent" if the recommendation it makes for each problem aligns with the recommendation it provides for each associated reduced problem, obtained by imagining some agents leaving with their assignments. This principle has been widely examined in various allocation problems.[2]

We formulate four different consistency properties for the reordering problem. *Balanced consistency* requires that for each pair of agents i and j, the impact on agent i's net utility when agent j leaves the initial queue and the agents behind her move forward by one position, should be equal to the impact on agent j's net utility when agent i leaves and the agents behind her move forward by one position. *Balanced cost reduction* requires that if an agent leaves the problem and the agents behind her move forward by one position, then the total net utilities of the remaining agents should be reduced by the amount equal to the net utility of the departing agent. *Smallest-cost consistency* (respectively, *largest-cost consistency*) requires that if an agent with the smallest (respectively, largest) unit waiting cost leaves the problem and the agents behind her move forward by one position, then the net utilities of the remaining agents should not be affected.

We present axiomatic characterizations of three allocations rules, the *pairwise equal-splitting rule,* the *maximum price rule* and the *minimum price rule.* In contrast with the pairwise equal-splitting rule, both the maximum and the minimum price rules select an efficient queue, and each rule allocates the entire cost-savings obtained by swapping the positions of any two agents to either one of the two agents. The maximum price rule chooses the largest amount of transfer between any two agents whose initial positions are swapped to form an efficient queue, ensuring that no one is worse off from the swapping. This amount is equal to the larger unit waiting costs of the two corresponding agents, implying that the entire cost-savings that can be obtained from these two agents swapping positions in case they are neighbours in the initial queue, are allocated to the agent with the smaller unit waiting cost among the two of them. On the other hand, the minimum price rule chooses the smallest amount of transfer between any two agents whose initial positions are swapped to form an efficient queue, ensuring that no one is worse off from the swapping. This amount is equal to the smaller unit waiting costs of the two corresponding agents, implying that the entire cost-savings that can be obtained from these two agents swapping positions in case they are neighbours in the initial queue, are allocated to the agent with the larger unit waiting cost among the two of them.

These three rules satisfy three basic properties widely used in the literature: *queue-efficiency*, *budget-balance* and *Pareto indifference. Queue-efficiency* requires that a rule should choose feasible allocations with an efficient queue. *Budget-balance* requires that a rule should choose feasible allocations whose sum of transfers is equal to zero. Finally, *Pareto indifference* requires that if a feasible allocation is chosen by a rule, then all other feasible allocations which give the same net utilities to all agents should be chosen by

[2]For comprehensive surveys on the consistency principle, see Thomson (2011, 2025).

the rule as well. We show that either *balanced consistency* or *balanced cost reduction*, together with the three basic properties of *queue-efficiency*, *budget-balance* and *Pareto indifference*, characterizes the pairwise equal-splitting rule. On the other hand, together with these three basic properties, *smallest-cost consistency* characterizes the maximum price rule and *largest-cost consistency* the minimum price rule.

In a related stream of literature, another type of problem, known as the queueing problem and studied in Maniquet (2003), Chun (2006), and Ju et al. (2014), considers a group of agents who must be served at a facility. These agents have different unit waiting costs, but there is no initial queue. Thus, each agent has the same "right" on being served first, second, etc. Again, the goal is to serve the agents in an efficient queue and a main question is how agents served earlier should compensate agents served later.[3] A rule for such problems assigns to each queueing problem (without an initial queue) an allocation consisting of a queue and transfers that are based only on the unit waiting costs since the agents have equal rights to any position in the queue. Similarly to the current paper, which focuses on the consistency principle in the reordering problem, Chun (2011) and van den Brink and Chun (2012) formulate different consistency properties and examine their implications in the queueing problem.

The paper is organized as follows. Section 2 introduces the reordering problem and the three basic properties of *queue-efficiency, budget-balance* and *Pareto indifference*. Section 3 defines the three rules and explains how the net utilities can be calculated for each rule. Section 4 introduces the properties of *balanced consistency* and *balanced cost reduction* and shows that either one of these two properties, together with the three basic properties, characterizes the pairwise equal-splitting rule. Section 5 introduces the properties of *smallest-* and *largest-cost consistency* and shows that together with the three basic properties, *smallest-cost consistency* characterizes the maximum price rule and *largest-cost consistency* the minimum price rule. Concluding remarks follow in Section 6.

2 Preliminaries

Let $\mathbb{N} \equiv \{1, 2, \ldots\}$ be an (infinite) universe of potential agents and $\mathcal{N}$ be the family of all non-empty finite subsets of $\mathbb{N}$ with a generic element N.[4] For each $N \in \mathcal{N}$, a *queue* σ is a permutation on N. Let $\Pi(N)$ be the set of all permutations of N. For each $N \in \mathcal{N}$, let $\theta = (\theta_i)_{i \in N} \in \mathbb{R}^N_{++}$ be the vector of unit waiting costs and $\sigma^0 = (\sigma^0_i)_{i \in N} \in \Pi(N)$ be the initial queue. Each agent $i \in N$ is characterized by two parameters: (θ_i, σ^0_i), where $\theta_i \in \mathbb{R}_{++}$ is her waiting cost per unit of time, or *unit waiting cost,* and $\sigma^0_i \in \{1, \ldots, |N|\}$ is her position in the initial queue. Each agent wants to receive a service at a service facility which can process only one agent at a time. Moreover, each agent needs the same amount of service time, which is normalized to one.

[3] For a comprehensive survey on the queueing problem, see Chun (2016).

[4] The cardinality of N is denoted by $|N|$.

For each $N \in \mathcal{N}$ and each $\sigma \in \Pi(N)$, let $P_i(\sigma) \equiv \{j \in N | \sigma_j < \sigma_i\}$ be the set of agents preceding agent i in σ, and $F_i(\sigma) \equiv \{j \in N | \sigma_j > \sigma_i\}$ the set of agents following her.

A *reordering problem,* or simply a *problem*, is a list (N, θ, σ^0), where $N \in \mathcal{N}$ is the set of agents, $\theta = (\theta_i)_{i \in N} \in \mathbb{R}^N_{++}$ is the vector of unit waiting costs and $\sigma^0 \in \Pi(N)$ is the initial queue. If there is no confusion, then we denote a reordering problem on N just as a pair (θ, σ^0) instead of (N, θ, σ^0). For each $N \in \mathcal{N}$, let $\mathcal{Q}^N$ be the class of all reordering problems for N.

For each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$, an *allocation* is a pair $(\sigma, t) \in \Pi(N) \times \mathbb{R}^N$, where $\sigma \in \Pi(N)$ is a reordered queue and $t \in \mathbb{R}^N$ a vector of (monetary) transfers. For each $i \in N$, σ_i denotes agent i's reordered position, and t_i the transfer to her. The pair (σ_i, t_i) is the *assignment* to agent i. It is most likely that the transfer is positive if an agent moves backward in the queue, and the transfer is negative if an agent moves forward in the queue. An allocation is *feasible* if the sum of transfers is non-positive, that is, for each $(\sigma, t) \in \Pi(N) \times \mathbb{R}^N$, $\sum_{i \in N} t_i \leq 0$. For each $(\theta, \sigma^0) \in \mathcal{Q}^N$, let $\mathcal{F}(\theta)$ be the set of all feasible allocations. Each agent is assumed to have quasi-linear preferences, so that agent i's utility under (σ, t) is:

$$u_i(\sigma, t; \theta) = -(\sigma_i - 1)\theta_i + t_i.$$

Taking account of the utility gain/loss that the agent i obtains compared to her position in the initial queue, her net utility is:

$$U_i(\sigma, t; \theta, \sigma^0) = u_i(\sigma, t; \theta) + (\sigma_i^0 - 1)\theta_i = (\sigma_i^0 - \sigma_i)\theta_i + t_i.$$

A queue σ is *queue-efficient* in (θ, σ^0) if it minimizes the sum of all agents' total waiting costs, that is, $\sigma \in \arg\min_{\sigma' \in \Pi(N)} \sum_{i \in N} \sigma'_i \theta_i$. As shown in Smith (1956), *queue-efficiency* can be achieved if the agents are served in a non-increasing order of their unit waiting costs. Clearly, an efficient queue does not depend on the initial queue. Moreover, it is essentially unique except for agents with the same unit waiting cost, who will be next to each other, but in any order. For each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$, let $\mathcal{E}(\theta)$ be the set of all efficient queues in (θ, σ^0).

A *reordering rule,* or simply a *rule,* is a correspondence ϕ that associates with each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$ a non-empty set $\phi(\theta, \sigma^0) \subseteq \mathcal{F}(\theta)$. For each $N \in \mathcal{N}$, each $N' \subset N$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, and each $i \in N$, let $\theta_{N'} \equiv (\theta_k)_{k \in N'}$ and $\theta_{-i} \equiv (\theta_k)_{k \in N \setminus \{i\}}$.

We now introduce three basic properties for rules: *queue-efficiency*, *budget-balance* and *Pareto indifference*. *Queue-efficiency* requires that a rule should choose feasible allocations with an efficient queue. *Budget-balance* requires that a rule should choose feasible allocations whose sum of transfers is equal to zero. Finally, *Pareto indifference* requires that if a feasible allocation is chosen by a rule, then all other feasible allocations which give the same net utilities to all agents should be chosen by the rule as well.

Queue-efficiency: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, and each $(\sigma, t) \in \phi(\theta, \sigma^0)$, $\sigma \in \mathcal{E}(\theta)$.

Budget-balance: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, and each $(\sigma, t) \in \phi(\theta, \sigma^0)$, $\sum_{i \in N} t_i = 0$.

Pareto indifference: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, and each $(\sigma', t') \in \mathcal{F}(\theta)$, if for each $i \in N$, $U_i(\sigma', t'; \theta, \sigma^0) = U_i(\sigma, t; \theta, \sigma^0)$, then $(\sigma', t') \in \phi(\theta, \sigma^0)$.

3 Three Rules

Following Curiel et al. (1989), we are interested in finding a fair way of serving agents by providing an incentive to switch positions of agents. Of course, it is natural to ask agents who move forward in the queue to compensate agents who move backward in the queue, which can be done by monetary transfers. We introduce three rules for the reordering problem: the pairwise equal-splitting rule, the maximum price rule, and the minimum price rule. Each rule chooses an efficient queue and determines different amounts of transfers to agents. They all satisfy the three basic properties.

We begin with the pairwise equal-splitting rule (Curiel et al. 1989; Tsay et al. 2025b) which selects an efficient queue and allocates the cost-savings obtained by swapping the positions of any two agents equally between them.

Pairwise equal-splitting rule, ϕ^P: For each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$,

$$\phi^P(\theta, \sigma^0) = \left\{ (\sigma^P, t^P) \in \mathcal{F}(\theta) \;\middle|\; \begin{array}{c} \sigma^P \in \mathcal{E}(\theta), \text{ and for each } i \in N, \\ t_i^P = \sum_{j \in F_i(\sigma^0) \cap P_i(\sigma^P)} \frac{\theta_i + \theta_j}{2} - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^P)} \frac{\theta_i + \theta_j}{2} \end{array} \right\}.$$

It is easy to check that each agent ends up with the same utility under any allocation chosen by the pairwise equal-splitting rule. For each $x \in \mathbb{R}$, let $(x)^+ \equiv \max(x, 0)$. Then, for each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$ and each $(\sigma^P, t^P) \in \phi^P(\theta, \sigma^0)$, the utility of agent i is given by:

$$\begin{aligned} u_i^P(\theta, \sigma^0) &\equiv -(\sigma_i^P - 1)\theta_i + t_i^P \\ &= -(\sigma_i^P - 1)\theta_i + \frac{1}{2}\Big[\sum_{j \in F_i(\sigma^0) \cap P_i(\sigma^P)} (\theta_i + \theta_j) - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^P)} (\theta_i + \theta_j) \Big] \end{aligned}$$

and her net utility is:

$$
\begin{aligned}
& U_i^P(\theta,\sigma^0) \\
\equiv\ & \left(\sigma_i^0-\sigma_i^P\right)\theta_i + t_i^P \\
=\ & \left(\sigma_i^0-\sigma_i^P\right)\theta_i + \frac{1}{2}\Big[\sum_{j\in F_i(\sigma^0)\cap P_i(\sigma^P)}(\theta_i+\theta_j) - \sum_{j\in P_i(\sigma^0)\cap F_i(\sigma^P)}(\theta_i+\theta_j)\Big] \\
=\ & \left(\sigma_i^0-\sigma_i^P\right)\theta_i + \frac{1}{2}\Big[\sum_{j\in F_i(\sigma^0)}(\theta_i+\theta_j) - \sum_{j\in F_i(\sigma^0)\cap F_i(\sigma^P)}(\theta_i+\theta_j) \\
& \qquad - \sum_{j\in F_i(\sigma^P)}(\theta_i+\theta_j) + \sum_{j\in F_i(\sigma^0)\cap F_i(\sigma^P)}(\theta_i+\theta_j)\Big] \qquad (1) \\
=\ & \left(\sigma_i^0-\sigma_i^P\right)\theta_i + \frac{1}{2}\Big[(|N|-\sigma_i^0)\theta_i + \sum_{j\in F_i(\sigma^0)}\theta_j - (|N|-\sigma_i^P)\theta_i - \sum_{j\in F_i(\sigma^P)}\theta_j\Big] \\
=\ & \left(\sigma_i^0-\sigma_i^P\right)\frac{\theta_i}{2} - \sum_{j\in F_i(\sigma^P)}\frac{\theta_j}{2} + \sum_{j\in F_i(\sigma^0)}\frac{\theta_j}{2} \\
=\ & \sum_{j\in P_i(\sigma^0)}\frac{\theta_i}{2} - \sum_{j\in P_i(\sigma^P)}\frac{\theta_i}{2} - \sum_{j\in F_i(\sigma^P)}\frac{\theta_j}{2} + \sum_{j\in F_i(\sigma^0)}\frac{\theta_j}{2} \\
=\ & \sum_{j\in P_i(\sigma^0)\setminus P_i(\sigma^P)}\frac{\theta_i}{2} + \sum_{j\in P_i(\sigma^0)\cap P_i(\sigma^P)}\frac{\theta_i}{2} - \sum_{j\in P_i(\sigma^P)\setminus P_i(\sigma^0)}\frac{\theta_i}{2} - \sum_{j\in P_i(\sigma^P)\cap P_i(\sigma^0)}\frac{\theta_i}{2} \\
& - \sum_{j\in F_i(\sigma^P)\setminus F_i(\sigma^0)}\frac{\theta_j}{2} - \sum_{j\in F_i(\sigma^P)\cap F_i(\sigma^0)}\frac{\theta_j}{2} + \sum_{j\in F_i(\sigma^0)\setminus F_i(\sigma^P)}\frac{\theta_j}{2} + \sum_{j\in F_i(\sigma^0)\cap F_i(\sigma^P)}\frac{\theta_j}{2} \\
=\ & \sum_{j\in P_i(\sigma^0)\setminus P_i(\sigma^P)}\frac{\theta_i}{2} - \sum_{j\in P_i(\sigma^P)\setminus P_i(\sigma^0)}\frac{\theta_i}{2} - \sum_{j\in F_i(\sigma^P)\setminus F_i(\sigma^0)}\frac{\theta_j}{2} + \sum_{j\in F_i(\sigma^0)\setminus F_i(\sigma^P)}\frac{\theta_j}{2} \\
=\ & \sum_{j\in P_i(\sigma^0)\setminus P_i(\sigma^P)}\frac{\theta_i-\theta_j}{2} + \sum_{j\in F_i(\sigma^0)\setminus F_i(\sigma^P)}\frac{\theta_j-\theta_i}{2} \qquad (2) \\
=\ & \sum_{j\in P_i(\sigma^0)}\frac{(\theta_i-\theta_j)^+}{2} + \sum_{j\in F_i(\sigma^0)}\frac{(\theta_j-\theta_i)^+}{2}, \qquad (3)
\end{aligned}
$$

where Eq. (1) follows from

$F_i(\sigma^0)\cap P_i(\sigma^P) = F_i(\sigma^0)\setminus(F_i(\sigma^0)\cap F_i(\sigma^P))$ and $P_i(\sigma^0)\cap F_i(\sigma^P) = F_i(\sigma^P)\setminus(F_i(\sigma^0)\cap F_i(\sigma^P))$,

Eq. (2) from

$$F_i(\sigma^P)\setminus F_i(\sigma^0) = P_i(\sigma^0)\setminus P_i(\sigma^P) \text{ and } P_i(\sigma^P)\setminus P_i(\sigma^0) = F_i(\sigma^0)\setminus F_i(\sigma^P),$$

and Eq. (3) from

$$[j\in P_i(\sigma^0)\cap P_i(\sigma^P) \Rightarrow \theta_i-\theta_j\le 0] \text{ and } [j\in F_i(\sigma^0)\cap F_i(\sigma^P)\Rightarrow \theta_j-\theta_i\le 0].$$

Instead of the 'equal split' idea behind the pairwise equal-splitting rule, the maximum and the minimum price rules select an efficient queue, but each rule allocates the entire cost-savings obtained by swapping the positions of any two agents to either one of the two agents. The maximum price rule chooses the largest amount of transfer between any two agents whose initial positions are swapped to form an efficient queue, ensuring that no one is worse off from the swapping. This amount is equal to the larger unit waiting cost of the two corresponding agents, implying that the entire cost-savings that can be obtained from these two agents switching positions in case they are neighbours in the initial queue, are allocated to the agent with the smaller unit waiting cost among the two of them.

Maximum price rule, ϕ^M: For each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$,

$$\phi^M(\theta,\sigma^0) = \left\{ (\sigma^M, t^M) \in \mathcal{F}(\theta) \;\middle|\; \begin{array}{c} \sigma^M \in \mathcal{E}(\theta), \text{ and for each } i \in N, \\ t_i^M = \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^M)} \theta_i \end{array} \right\}.$$

Note that each agent ends up with the same utility under any allocation chosen by the maximum price rule. Thus, for each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$ and each $(\sigma^M, t^M) \in \phi^M(\theta, \sigma^0)$, the utility of agent i is given by:

$$\begin{aligned} u_i^M(\theta,\sigma^0) &\equiv -(\sigma_i^M - 1)\theta_i + t_i^M \\ &= -(\sigma_i^M - 1)\theta_i + \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^M)} \theta_i, \end{aligned}$$

and her net utility is:

$$
\begin{aligned}
& U_i^M(\theta, \sigma^0) \\
\equiv\ & \left(\sigma_i^0 - \sigma_i^M\right)\theta_i + t_i^M \\
=\ & \sigma_i^0\theta_i - \sigma_i^M\theta_i + \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^M)} \theta_i \\
=\ & \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j + \sum_{j \in P_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^M)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^M)} \theta_i \\
=\ & \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in F_i(\sigma^0)} \theta_i + \sum_{j \in F_i(\sigma^M)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^M)} \theta_i \qquad (4) \\
=\ & \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in F_i(\sigma^0)} \theta_i + \sum_{j \in F_i(\sigma^M) \setminus P_i(\sigma^0)} \theta_i \\
=\ & \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in F_i(\sigma^0)} \theta_i + \sum_{j \in F_i(\sigma^M) \cap F_i(\sigma^0)} \theta_i \qquad (5) \\
=\ & \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in F_i(\sigma^0) \setminus F_i(\sigma^M)} \theta_i \\
=\ & \sum_{j \in P_i(\sigma^M) \cap F_i(\sigma^0)} \theta_j - \sum_{j \in F_i(\sigma^0) \cap P_i(\sigma^M)} \theta_i \qquad (6) \\
=\ & \sum_{j \in F_i(\sigma^0) \cap P_i(\sigma^M)} (\theta_j - \theta_i) \\
=\ & \sum_{j \in F_i(\sigma^0)} (\theta_j - \theta_i)^+, \qquad (7)
\end{aligned}
$$

where Eq. (4) follows from

$$\sum_{j \in P_i(\sigma)} \theta_i = (|N| - 1)\theta_i - \sum_{j \in F_i(\sigma)} \theta_i \text{ for each } \sigma \in \Pi(N),$$

Eq. (5) from $F_i(\sigma^M) \setminus P_i(\sigma^0) = F_i(\sigma^M) \cap F_i(\sigma^0)$, Eq. (6) from $F_i(\sigma^0) \setminus F_i(\sigma^M) = F_i(\sigma^0) \cap P_i(\sigma^M)$, and Eq. (7) from $j \in F_i(\sigma^0) \setminus P_i(\sigma^M) \Rightarrow \theta_j - \theta_i \leq 0$.

On the other hand, the minimum price rule chooses the smallest amount of transfer between any two agents whose initial positions are swapped to form an efficient queue, ensuring that no one is worse off from the swapping. This amount is equal to the smaller unit waiting cost of the two corresponding agents, implying that the entire cost-savings that can be obtained from these two agents switching positions in case they are neighbours in the initial queue, are allocated to the agent with the larger unit waiting cost among the two of them.

Minimum price rule, ϕ^m: For each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$,

$$\phi^m(\theta,\sigma^0) = \left\{ (\sigma^m, t^m) \in \mathcal{F}(\theta) \;\middle|\; \begin{array}{c} \sigma^m \in \mathcal{E}(\theta), \text{ and for each } i \in N, \\ t_i^m = \sum_{j \in P_i(\sigma^m) \cap F_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \end{array} \right\}.$$

As in the other two rules, each agent ends up with the same utility under any allocation chosen by the minimum price rule. Thus, for each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$ and each $(\sigma^m, t^m) \in \phi^m(\theta, \sigma^0)$, the utility of agent i is given by:

$$\begin{aligned} u_i^m(\theta,\sigma^0) &\equiv -(\sigma_i^m - 1)\theta_i + t_i^m \\ &= -(\sigma_i^m - 1)\theta_i + \sum_{j \in P_i(\sigma^m) \cap F_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j, \end{aligned}$$

and her net utility is:

$$\begin{aligned} U_i^m(\theta,\sigma^0) &\equiv \left(\sigma_i^0 - \sigma_i^m\right)\theta_i + t_i^m \\ &= \sigma_i^0\theta_i - \sigma_i^m\theta_i + \sum_{j \in P_i(\sigma^m) \cap F_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \\ &= \sum_{j \in P_i(\sigma^m) \cap F_i(\sigma^0)} \theta_i + \sum_{j \in P_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^m)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \\ &= -\sum_{j \in P_i(\sigma^m) \setminus F_i(\sigma^0)} \theta_i + \sum_{j \in P_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \\ &= -\sum_{j \in P_i(\sigma^m) \cap P_i(\sigma^0)} \theta_i + \sum_{j \in P_i(\sigma^0)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \qquad (8) \\ &= \sum_{j \in P_i(\sigma^0) \setminus P_i(\sigma^m)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \\ &= \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_i - \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} \theta_j \qquad (9) \\ &= \sum_{j \in P_i(\sigma^0) \cap F_i(\sigma^m)} (\theta_i - \theta_j) \\ &= \sum_{j \in P_i(\sigma^0)} (\theta_i - \theta_j)^+. \qquad (10) \end{aligned}$$

where Eqs. (8) and (9) follow from $P_i(\sigma^m) \setminus F_i(\sigma^0) = P_i(\sigma^m) \cap P_i(\sigma^0)$ and $P_i(\sigma^0) \setminus P_i(\sigma^m) = P_i(\sigma^0) \cap F_i(\sigma^m)$, respectively, and Eq. (10) from

$$j \in P_i(\sigma^0) \setminus F_i(\sigma^m) \Rightarrow \theta_i - \theta_j \leq 0.$$

4 Balanced Consistency and Balanced Cost Reduction

Myerson (1980) introduces a notion of fairness in cooperation, the *balanced contributions principle*, which requires that for any two agents i and j, the departure of agent i has the same effect on the payoff of agent j as the departure of agent j has on the payoff of agent i. By applying this principle to the queueing problem (without an initial queue), van den Brink and Chun (2012) propose a property of allocation rules in the queueing problem to consider the impact on the utilities of other agents due to the departure of one agent. This property requires that for any two agents, the impact on the utility due to the departure of one agent i on the utility of another agent $j \neq i$, should be equal to the impact of the departure of agent j on the utility of agent i. Similarly, in the reordering problem, the balanced contribution principle implies that for any two agents, the net utility gain of one agent from the departure of the other should be equal to that of the latter agent from the departure of the former. *Balanced consistency* embodies this principle for reordering rules and requires that for each reordering problem and each pair of agents i and j, the impact on agent i's net utility when agent j leaves the initial queue and the agents behind her move forward by one position, should be equal to the impact on agent j's net utility when agent i leaves the initial queue and the agents behind her move forward by one position. For each $\sigma \in \Pi(N)$ and each $i \in N$, let $\sigma^{0|-i} \in \Pi(N \setminus \{i\})$ be the queue obtained after the departure of agent i defined by $\sigma^{0|-i}_k = \sigma^0_k$ if $k \in P_i(\sigma^0)$ and $\sigma^{0|-i}_k = \sigma^0_k - 1$ if $k \in F_i(\sigma^0)$.

Balanced consistency: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each pair $\{i, j\} \subseteq N$ with $i \neq j$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, each $(\sigma', t') \in \phi(\theta_{-j}, \sigma^{0|-j})$, and each $(\sigma'', t'') \in \phi(\theta_{-i}, \sigma^{0|-i})$,

$$U_i(\sigma, t; \theta, \sigma^0) - U_i(\sigma', t'; \theta_{-j}, \sigma^{0|-j}) = U_j(\sigma, t; \theta, \sigma^0) - U_j(\sigma'', t''; \theta_{-i}, \sigma^{0|-i}). \qquad (11)$$

On the other hand, for the queueing problem (without an initial queue), van den Brink and Chun (2012) take another viewpoint by considering the impact on the total utility of all other agents when one agent leaves a problem. In the queueing problem, the total waiting cost of all other agents decreases due to the departure of an agent. This implies that the presence of an agent imposes a negative externality on other agents. They formulate *balanced cost reduction* in the queueing problem that requires the utility of each agent to be equal to the total externality she imposes on the other agents with her presence. By applying this idea to reordering problems, *balanced cost reduction* requires that if an agent leaves the initial queue and the agents behind her move forward by one position, then the total net utility of the remaining agents should be reduced by an amount equal to the net utility of the departing agent assigned to her before her departure.

Balanced cost reduction: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each $i \in N$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, and each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$,

$$\sum_{j \neq i} \Big[U_j(\sigma, t; \theta, \sigma^0) - U_j(\sigma', t'; \theta_{-i}, \sigma^{0|-i}) \Big] = U_i(\sigma, t; \theta, \sigma^0). \tag{12}$$

We now investigate the implications of *balanced consistency* and *balanced cost reduction* in the reordering problem. In particular, we show that either *balanced consistency* or *balanced cost reduction*, together with the three basic properties of *queue-efficiency*, *budget-balance* and *Pareto indifference*, characterizes the pairwise equal-splitting rule.

4.1 Balanced consistency and the pairwise equal-splitting rule

Our first result characterizes the pairwise equal-splitting rule by the properties of *queue-efficiency*, *budget-balance*, *Pareto indifference*, and *balanced consistency*.[5]

Theorem 1 The pairwise equal-splitting rule is the only rule satisfying *queue-efficiency*, *budget-balance*, *Pareto indifference*, and *balanced consistency*.

Proof. Since it is obvious that the pairwise equal-splitting rule satisfies *queue-efficiency*, *budget balance* and *Pareto indifference*, we only show that it satisfies *balanced consistency*. Let $N \in \mathcal{N}$, $\big(\theta, \sigma^0\big) \in \mathcal{Q}^N$, and $i, j \in N$ be such that $i \in P_j(\sigma^0)$. By Eq. (3), for each $(\sigma^*, t^*) \in \phi^P(\theta, \sigma^0)$ and each $(\sigma', t') \in \phi^P(\theta_{-j}, \sigma^{0|-j})$,

$$\begin{aligned} & U_i^P(\theta, \sigma^0) - U_i^P(\theta_{-j}, \sigma^{0|-j}) \\ = \ & \Big[\sum_{k \in P_i(\sigma^0)} \frac{(\theta_i - \theta_k)^+}{2} + \sum_{k \in F_i(\sigma^0)} \frac{(\theta_k - \theta_i)^+}{2} \Big] \\ & - \Big[\sum_{k \in P_i(\sigma^{0|-j})} \frac{(\theta_i - \theta_k)^+}{2} + \sum_{k \in F_i(\sigma^{0|-j})} \frac{(\theta_k - \theta_i)^+}{2} \Big] \\ = \ & \frac{(\theta_j - \theta_i)^+}{2} \end{aligned} \tag{13}$$

where Eq. (13) follows from $i \in P_j(\sigma^0) \Rightarrow P_i(\sigma^0) = P_i(\sigma^{0|-j})$ and $F_i(\sigma^0) = F_i(\sigma^{0|-j}) \cup \{j\}$.

[5] In the appendix, we provide examples to demonstrate that all properties listed in each of the following theorems are independent.

Similarly, for each $(\sigma'', t'') \in \phi^P(\theta_{-i}, \sigma^{0|-i})$,

$$\begin{aligned}
& U_j^P(\theta, \sigma^0) - U_j^P(\theta_{-i}, \sigma^{0|-i}) \\
= & \Big[\sum_{k \in P_j(\sigma^0)} \frac{(\theta_j - \theta_k)^+}{2} + \sum_{k \in F_j(\sigma^0)} \frac{(\theta_k - \theta_j)^+}{2} \Big] \\
& - \Big[\sum_{k \in P_j(\sigma^{0|-i})} \frac{(\theta_j - \theta_k)^+}{2} + \sum_{k \in F_j(\sigma^{0|-i})} \frac{(\theta_k - \theta_j)^+}{2} \Big] \\
= & \frac{(\theta_j - \theta_i)^+}{2}, \qquad (14)
\end{aligned}$$

where Eq. (14) follows from $i \in P_j(\sigma^0) \Rightarrow P_j(\sigma^0) = P_j(\sigma^{0|-i}) \cup \{i\}$ and $F_j(\sigma^0) = F_j(\sigma^{0|-i})$. Therefore, by Eqs. (13) and (14), the pairwise equal-splitting rule satisfies *balanced consistency*.

Conversely, let ϕ be a rule satisfying the four properties. We show that $\phi = \phi^P$ by induction on $|N|$.

Case 1: $|N| = 2$. Let $N = \{i, j\}$ with $i \neq j$. Then, for each $(\theta, \sigma^0) \in \mathcal{Q}^N$, we may assume without loss of generality that $i \in P_j(\sigma^0)$. Note that *budget-balance* guarantees that $\phi(\theta_{-i}, \sigma^{0|-i}) = (\sigma^{0|-i}, 0)$ and $\phi(\theta_{-j}, \sigma^{0|-j}) = (\sigma^{0|-j}, 0)$, and thus $U_i\left(\sigma^{0|-j}, 0; \theta_{-j}, \sigma^{0|-j}\right) = U_j\left(\sigma^{0|-i}, 0; \theta_{-i}, \sigma^{0|-i}\right) = 0$. Then, for each $(\sigma, t) \in \phi(\theta, \sigma^0)$,

$$\begin{aligned}
U_i\left(\sigma, t; \theta, \sigma^0\right) &= U_i\left(\sigma, t; \theta, \sigma^0\right) - U_i\left(\sigma^{0|-j}, 0; \theta_{-j}, \sigma^{0|-j}\right) \\
&= U_j\left(\sigma, t; \theta, \sigma^0\right) - U_j\left(\sigma^{0|-i}, 0; \theta_{-i}, \sigma^{0|-i}\right) \qquad (15) \\
&= U_j\left(\sigma, t; \theta, \sigma^0\right), \qquad (16)
\end{aligned}$$

where Eq. (15) follows from *balanced consistency*.

Moreover, by *queue-efficiency* and *budget-balance*,

$$U_i\left(\sigma, t; \theta, \sigma^0\right) + U_j\left(\sigma, t; \theta, \sigma^0\right) = (\theta_j - \theta_i)^+. \qquad (17)$$

Therefore, Eqs. (3), (16) and (17) together imply that

$$U_i\left(\sigma, t; \theta, \sigma^0\right) = U_j\left(\sigma, t; \theta, \sigma^0\right) = \frac{(\theta_j - \theta_i)^+}{2} = U_j^P\left(\theta, \sigma^0\right) = U_i^P\left(\theta, \sigma^0\right).$$

By *Pareto indifference*, we conclude that $\phi(\theta, \sigma^0) = \phi^P(\theta, \sigma^0)$.

Case 2: $|N| > 2$. Proceeding by induction, suppose that the net utilities of agents are uniquely determined for $|N| = 2, \ldots, n-1$. We show that they are also uniquely determined for $|N| = n$.

Given $(\theta,\sigma^0)\in\mathcal{Q}^N$, let $i,j\in N$ be such that $j\neq i$. By *balanced consistency*, for each $(\sigma^*,t^*)\in\phi(\theta,\sigma^0)$,

$$U_i\left(\sigma^*,t^*;\theta,\sigma^0\right)-U_i\left(\sigma',t';\theta_{-j},\sigma^{0|-j}\right)=U_j\left(\sigma^*,t^*;\theta,\sigma^0\right)-U_j\left(\sigma'',t'';\theta_{-i},\sigma^{0|-i}\right),\tag{18}$$

where $(\sigma',t')\in\phi(\theta_{-j},\sigma^{0|-j})$ and $(\sigma'',t'')\in\phi(\theta_{-i},\sigma^{0|-i})$. By the induction hypothesis and Eq. (3), for each $j\neq i$,

$$U_i\left(\sigma',t';\theta_{-j},\sigma^{0|-j}\right)=\begin{cases}U_i^P(\theta,\sigma^0) & \text{if } j\in P_i(\sigma^0)\cap P_i(\sigma^*),\\ U_i^P(\theta,\sigma^0)+\frac{\theta_j-\theta_i}{2} & \text{if } j\in P_i(\sigma^0)\cap F_i(\sigma^*),\\ U_i^P(\theta,\sigma^0)+\frac{\theta_i-\theta_j}{2} & \text{if } j\in F_i(\sigma^0)\cap P_i(\sigma^*),\\ U_i^P(\theta,\sigma^0) & \text{if } j\in F_i(\sigma^0)\cap F_i(\sigma^*).\end{cases}\tag{19}$$

By summing up Eq. (18) over all $j\neq i$ and adding $U_i(\sigma^*,t^*;\theta,\sigma^0)$ to both sides, we obtain that

$$\begin{aligned}
& nU_i\left(\sigma^*,t^*;\theta,\sigma^0\right)\\
= & \sum_{j\in N}U_j\left(\sigma^*,t^*;\theta,\sigma^0\right)-\sum_{j\neq i}U_j\left(\sigma'',t'';\theta_{-i},\sigma^{0|-i}\right)+\sum_{j\neq i}U_i\left(\sigma',t';\theta_{-j},\sigma^{0|-j}\right)\\
= & \sum_{k\in N}\sum_{j\in P_k(\sigma^0)}(\theta_k-\theta_j)^+-\sum_{k\in N\setminus\{i\}}\sum_{j\in P_k(\sigma^0)\cap(N\setminus\{i\})}(\theta_k-\theta_j)^+\\
& +(n-1)U_i^P(\theta,\sigma^0)+\sum_{j\in P_i(\sigma^0)\cap F_i(\sigma^*)}\frac{\theta_j-\theta_i}{2}+\sum_{j\in F_i(\sigma^0)\cap P_i(\sigma^*)}\frac{\theta_i-\theta_j}{2} \qquad (20)\\
= & \sum_{k\in N}\Big[\sum_{j\in P_k(\sigma^0)}(\theta_k-\theta_j)^+-\sum_{j\in P_k(\sigma^0)\cap(N\setminus\{i\})}(\theta_k-\theta_j)^+\Big]+\sum_{j\in P_i(\sigma^0)}(\theta_i-\theta_j)^+\\
& +(n-1)U_i^P(\theta,\sigma^0)-\sum_{j\in P_i(\sigma^0)}\frac{(\theta_i-\theta_j)^+}{2}-\sum_{j\in F_i(\sigma^0)}\frac{(\theta_j-\theta_i)^+}{2}\\
= & \sum_{k\in F_i(\sigma^0)}(\theta_k-\theta_i)^++\sum_{j\in P_i(\sigma^0)}(\theta_i-\theta_j)^+\\
& +(n-1)U_i^P(\theta,\sigma^0)-\sum_{j\in P_i(\sigma^0)}\frac{(\theta_i-\theta_j)^+}{2}-\sum_{j\in F_i(\sigma^0)}\frac{(\theta_j-\theta_i)^+}{2}\\
= & (n-1)U_i^P(\theta,\sigma^0)+\sum_{j\in P_i(\sigma^0)}\frac{(\theta_i-\theta_j)^+}{2}+\sum_{j\in F_i(\sigma^0)}\frac{(\theta_j-\theta_i)^+}{2}\\
= & nU_i^P(\theta,\sigma^0), \qquad (21)
\end{aligned}$$

where Eq. (20) follows since (i) *queue-efficiency* and *budget-balance* of ϕ imply that

$$\sum_{j\in N}U_j\left(\sigma^*,t^*;\theta,\sigma^0\right)=\sum_{k\in N}\sum_{j\in P_k(\sigma^0)}(\theta_k-\theta_j)^+$$

and

$$\sum_{j\neq i} U_j\left(\sigma'', t''; \theta_{-i}, \sigma^{0|-i}\right) = \sum_{k\in N\setminus\{i\}} \sum_{j\in P_k(\sigma^0)\cap(N\setminus\{i\})} (\theta_k - \theta_j)^+,$$

(ii) Eq. (19) implies that

$$\sum_{j\neq i} U_i\left(\sigma', t'; \theta_{-j}, \sigma^{0|-j}\right) = (n-1)U_i^P(\theta,\sigma^0) + \sum_{j\in P_i(\sigma^0)\cap F_i(\sigma^*)} \frac{\theta_j-\theta_i}{2} + \sum_{j\in F_i(\sigma^0)\cap P_i(\sigma^*)} \frac{\theta_i-\theta_j}{2},$$

and Eq. (21) follows from Eq. (3). Finally, Eq. (21) implies that

$$U_i\left(\sigma^*, t^*; \theta, \sigma^0\right) = U_i^P(\theta,\sigma^0),$$

and by *Pareto indifference*, we conclude that $\phi(\theta,\sigma^0) = \phi^P(\theta,\sigma^0)$. ■

4.2 Balanced cost reduction and the pairwise equal-splitting rule

Next, we show that *balanced cost reduction,* together with the three basic properties of *queue-efficiency*, *budget-balance* and *Pareto indifference*, characterizes the pairwise equal-splitting rule.

Theorem 2 The pairwise equal-splitting rule is the only rule satisfying *queue-efficiency*, *budget-balance*, *Pareto indifference*, and *balanced cost reduction.*

Proof. Since it is obvious that the pairwise equal-splitting rule satisfies *queue-efficiency*, *budget-balance* and *Pareto indifference*, it suffices to show that it satisfies *balanced cost reduction.* Let $N \in \mathcal{N}$, $\left(\theta,\sigma^0\right) \in \mathcal{Q}^N$, and $i \in N$. Then, for each $(\sigma^*, t^*) \in \phi^P(\theta,\sigma^0)$ and each $(\sigma', t') \in \phi^P(\theta_{-j}, \sigma^{0|-j})$,

$$\begin{aligned}
&\sum_{j\neq i}\left[U_j^P\left(\theta,\sigma^0\right) - U_j^P\left(\theta_{-i},\sigma^{0|-i}\right)\right] \\
=\ & \sum_{j\neq i}\Big[\sum_{k\in P_j(\sigma^0)} \frac{(\theta_j-\theta_k)^+}{2} + \sum_{k\in F_j(\sigma^0)} \frac{(\theta_k-\theta_j)^+}{2}\Big] \\
&\qquad -\sum_{j\neq i}\Big[\sum_{k\in P_j(\sigma^{0|-i})} \frac{(\theta_j-\theta_k)^+}{2} + \sum_{k\in F_j(\sigma^{0|-i})} \frac{(\theta_k-\theta_j)^+}{2}\Big] \\
=\ & \sum_{j\in P_i(\sigma^0)} \frac{(\theta_i-\theta_j)^+}{2} + \sum_{j\in F_i(\sigma^0)} \frac{(\theta_j-\theta_i)^+}{2} \\
=\ & U_i^P\left(\theta,\sigma^0\right). \qquad (22)
\end{aligned}$$

where Eq. (22) follows from Eq. (3).

Therefore, the pairwise equal-splitting rule satisfies *balanced cost reduction.*

Conversely, let ϕ be a rule which satisfies the four properties. We show that $\phi = \phi^P$ by induction on $|N|$.

Case 1: $|N| = 2$. Let $N = \{i, j\}$ with $i \neq j$. Note that, similarly to the proof of Theorem 1, *budget-balance* guarantees that $\phi(\theta_{-j}, \sigma^{0|-j}) = (\sigma^{0|-j}, 0)$. Then, for each $(\theta, \sigma^0) \in \mathcal{Q}^N$ and each $(\sigma, t) \in \phi(\theta, \sigma^0)$,

$$\begin{aligned} U_i\left(\sigma, t; \theta, \sigma^0\right) &= U_i\left(\sigma, t; \theta, \sigma^0\right) - U_i\left(\sigma^{0|-j}, 0; \theta_{-j}, \sigma^{0|-j}\right) \\ &= U_j\left(\sigma, t; \theta, \sigma^0\right), \end{aligned} \tag{23}$$

where Eq. (23) follows from *balanced cost reduction*. Moreover, by *queue-efficiency* and *budget-balance*,

$$U_i\left(\sigma, t; \theta, \sigma^0\right) + U_j\left(\sigma, t; \theta, \sigma^0\right) = \begin{cases} (\theta_i - \theta_j)^+ & \text{if } j \in P_i(\sigma^0), \\ (\theta_j - \theta_i)^+ & \text{if } j \in F_i(\sigma^0). \end{cases} \tag{24}$$

Since Eqs. (3), (23) and (24) together imply that

$$U_i\left(\sigma, t; \theta, \sigma^0\right) = U_j\left(\sigma, t; \theta, \sigma^0\right) = U_j^P\left(\theta, \sigma^0\right) = U_i^P\left(\theta, \sigma^0\right),$$

by *Pareto indifference*, we conclude that $\phi(\theta, \sigma^0) = \phi^P(\theta, \sigma^0)$.

Case 2: $|N| > 2$. Proceeding by induction, suppose that the net utilities of agents are uniquely determined for $|N| = 2, \ldots, n-1$. We show that they are also uniquely determined for $|N| = n$.

For each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each $i \in N$, each $(\sigma^*, t^*) \in \phi(\theta, \sigma^0)$, and each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$, by *balanced cost reduction*, the induction hypothesis and Eq. (3), we have

$$\begin{aligned} & U_i\left(\sigma^*, t^*; \theta, \sigma^0\right) \\ =\ & \sum_{j \neq i} \left[U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) - U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right) \right] \\ =\ & \sum_{j \neq i} \left[U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) - U_j^P\left(\theta_{-i}, \sigma^{0|-i}\right) \right] \\ =\ & \sum_{j \neq i} U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) - \sum_{j \neq i} U_j^P\left(\theta, \sigma^0\right) + \sum_{j \in P_i(\sigma^0)} \frac{(\theta_i - \theta_j)^+}{2} + \sum_{j \in F_i(\sigma^0)} \frac{(\theta_j - \theta_i)^+}{2} \\ =\ & \sum_{j \neq i} U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) - \sum_{j \neq i} U_j^P\left(\theta, \sigma^0\right) + U_i^P\left(\theta, \sigma^0\right) \end{aligned} \tag{25}$$

By *queue-efficiency* and *budget balance*, Eq. (25) implies that

$$\begin{aligned} 2U_i\left(\sigma^*, t^*; \theta, \sigma^0\right) &= \sum_{j \in N} U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) - \sum_{j \in N} U_j^P\left(\theta, \sigma^0\right) + 2U_i^P\left(\theta, \sigma^0\right) \\ &= 2U_i^P(\theta, \sigma^0), \end{aligned}$$

and therefore,

$$U_i\left(\sigma^*, t^*; \theta, \sigma^0\right) = U_i^P(\theta, \sigma^0).$$

Finally, by *Pareto indifference*, we conclude that $\phi(\theta, \sigma^0) = \phi^P(\theta, \sigma^0)$. ■

Remark 1 By exploring the implications of *balanced consistency* and *balanced cost reduction* in the reordering problem, we characterize the pairwise equal-splitting rule which shares the cost-savings obtained from switching the positions of any two neighboring agents *equally* between them. Although each agent has an "unequal" initial position in the queue, these two properties characterize the pairwise equal-splitting rule which seems to be fair for any two neighboring agents switching their positions. On the other hand, in the queueing problem, these axioms characterize the minimal transfer rule (Maniquet, 2003) which benefits an agent with a larger unit waiting cost. Although the queueing problem assumes that all agents have "equal" rights on each position in the queue, these two properties characterize the minimal transfer rule which is more beneficial to an agent with a larger unit waiting cost.

Remark 2 Tsay et al. (2025a) formulate another property, *balanced reduction of agents,* by applying Myersons (1980) balanced contributions principle to the reordering problem in a different way. Suppose that for each reordering problem and each pair of agents i and j such that agent i is positioned behind agent j in the initial queue, if agent j leaves from the initial queue, then agent i cannot switch positions with any other agents initially positioned after agent j and if agent i leaves from the initial queue, then agent j cannot switch positions with any other agents initially positioned before agent i. Then, this property requires that the impact on any one agent's net utility, resulting from another agent's departure from the initial queue, should be equal for any two agents. Moreover, they show that together with *queue-efficiency, budget-balance* and *Pareto indifference, balanced reduction of agents* characterizes the connected equal-splitting rule (Curiel et al. 1989).

5 Smallest- and Largest-Cost Consistency

Now, we introduce two alternative consistency properties and investigate their implications in the reordering problem. Each of these two properties requires the invariance of the net utilities of the remaining agents when a certain agent leaves the initial queue. Specifically, the first property, *smallest-cost consistency,* requires that if an agent with the smallest unit waiting cost leaves the initial queue and the agents behind her move forward by one position, then the net utilities of all remaining agents should not be affected.

Smallest-cost consistency: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each agent $i \in N$ with $\theta_i = \min_{k \in N} \theta_k$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$, and each

$j \in N \setminus \{i\}$,

$$U_j(\sigma, t; \theta, \sigma^0) = U_j(\sigma', t'; \theta_{-i}, \sigma^{0|-i}). \tag{26}$$

The second property, *largest-cost consistency,* requires that if an agent with the largest unit waiting cost leaves the initial queue and the agents behind her move forward by one position, then the net utilities of all remaining agents should not be affected.

Largest-cost consistency: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each $i \in N$ with $\theta_i = \max_{k \in N} \theta_k$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$, and each $j \in N \setminus \{i\}$,

$$U_j(\sigma, t; \theta, \sigma^0) = U_j(\sigma', t'; \theta_{-i}, \sigma^{0|-i}). \tag{27}$$

The next two results show that the maximum price rule and the minimum price rule can be characterized by *smallest-cost consistency* and *largest-cost consistency*, respectively, together with the three basic properties of *queue-efficiency*, *budget-balance* and *Pareto indifference.*

5.1 Smallest-cost consistency and the maximum price rule

We first characterize the maximum price rule by the properties of *queue-efficiency*, *budget-balance*, *Pareto indifference,* and *smallest-cost consistency.*

Theorem 3 The maximum price rule is the only rule satisfying *queue-efficiency*, *budget-balance*, *Pareto indifference* and *smallest-cost consistency.*

Proof. Since it is obvious that the maximum price rule satisfies *queue-efficiency*, *budget-balance* and *Pareto indifference,* we only show that it satisfies *smallest-cost consistency.* For each $N \in \mathcal{N}$, each $\left(\theta, \sigma^0\right) \in \mathcal{Q}^N$, each $i \in N$ such that $\theta_i = \min_{k \in N} \theta_k$, and each $j \in N \setminus \{i\}$, by Eq. (7), we have

$$\begin{aligned}
U_j^M\left(\theta, \sigma^0\right) &= \sum_{k \in F_j(\sigma^0)} (\theta_k - \theta_j)^+ \\
&= \begin{cases} \sum_{k \in F_j(\sigma^{0|-i})} (\theta_k - \theta_j)^+ & \text{if } \sigma_j^0 > \sigma_i^0 \\ (\theta_i - \theta_j)^+ + \sum_{k \in F_j(\sigma^{0|-i})} (\theta_k - \theta_j)^+ & \text{if } \sigma_j^0 < \sigma_i^0 \end{cases} \\
&= \sum_{k \in F_j(\sigma^{0|-i})} (\theta_k - \theta_j)^+ \qquad (28) \\
&= U_j^M\left(\theta_{-i}, \sigma^{0|-i}\right),
\end{aligned}$$

where Eq. (28) follows from $\theta_i = \min_{k \in N} \theta_k$. Therefore, the maximum price rule satisfies *smallest-cost consistency.*

Conversely, let ϕ be a rule which satisfies the four properties. We show that $\phi = \phi^M$ by induction on $|N|$.

Case 1: $|N| = 2$. Let $N = \{i, j\}$ with $i \neq j$. For each $(\theta, \sigma^0) \in \mathcal{Q}^N$, we may assume without loss of generality that $\theta_i \leq \theta_j$. By *smallest-cost consistency,* for each $(\sigma, t) \in \phi(\theta, \sigma^0)$ and each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$,

$$U_j\left(\sigma, t; \theta, \sigma^0\right) \;=\; U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right). \tag{29}$$

Next, by *budget-balance*, $t' = 0$ and then $U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right) = 0$. Therefore, Eq. (29) implies that $U_j\left(\sigma, t; \theta, \sigma^0\right) = 0 = U_j^M\left(\theta, \sigma^0\right)$. Moreover, by *queue-efficiency* and *budget-balance*, we have

$$\begin{aligned} U_i\left(\sigma, t; \theta, \sigma^0\right) &= U_i\left(\sigma, t; \theta, \sigma^0\right) + U_j\left(\sigma, t; \theta, \sigma^0\right) \\ &= \begin{cases} \theta_j - \theta_i & \text{if } \sigma_i^0 < \sigma_j^0 \\ 0 & \text{if } \sigma_i^0 > \sigma_j^0 \end{cases} \\ &= U_i^M\left(\theta, \sigma^0\right). \end{aligned}$$

Therefore, by *Pareto indifference,* we conclude that $\phi(\theta, \sigma^0) = \phi^M(\theta, \sigma^0)$.

Case 2: $|N| > 2$. Proceeding by induction, suppose that the net utilities of agents are uniquely determined for $|N| = 2, \ldots, n-1$. We show that they are also uniquely determined for $|N| = n$.

Given $(\theta, \sigma^0) \in \mathcal{Q}^N$, let $i \in N$ be such that $\theta_i = \min_{k \in N} \theta_k$. By *smallest-cost consistency*, for each $(\sigma^*, t^*) \in \phi(\theta, \sigma^0)$, each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$, and each $j \in N \backslash \{i\}$,

$$U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) = U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right). \tag{30}$$

By the induction hypothesis, Eq. (30) implies that for each $j \in N \setminus \{i\}$,

$$\begin{aligned} U_j\left(\sigma^*, t^*; \theta, \sigma^0\right) &= U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right) \\ &= U_j^M(\theta_{-i}, \sigma^{0|-i}) \\ &= U_j^M(\theta, \sigma^0), \end{aligned} \tag{31}$$

where Eq. (31) follows from *smallest-cost consistency* of the maximum price rule. Moreover, by *queue-efficiency* and *budget-balance* of ϕ^M and ϕ, and Eq. (31), we have

$$\begin{aligned} \sum_{k \in N} U_k^M\left(\theta, \sigma^0\right) &= \min_{\sigma \in \Pi(N)} \sum_{k \in N} \left(\sigma_k^0 - \sigma_k\right)\theta_k \\ &= \sum_{k \in N} U_k\left(\sigma^*, t^*; \theta, \sigma^0\right) \\ &= \sum_{k \in N \backslash \{i\}} U_k^M(\theta, \sigma^0) + U_i\left(\sigma^*, t^*; \theta, \sigma^0\right), \end{aligned} \tag{32}$$

which implies that

$$U_i\left(\sigma^*, t^*; \theta, \sigma^0\right) = U_i^M(\theta, \sigma^0). \tag{33}$$

Therefore, by *Pareto indifference*, we conclude that $\phi(\theta, \sigma^0) = \phi^M(\theta, \sigma^0)$. ■

5.2 Largest-cost consistency and the minimum price rule

Next, we characterize the minimum price rule by the properties of *queue-efficiency*, *budget-balance*, *Pareto indifference*, and *largest-cost consistency*.

Theorem 4 The minimum price rule is the only rule satisfying *queue-efficiency*, *budget-balance*, *Pareto indifference,* and *largest-cost consistency*.

Proof. Since it is obvious that the minimum price rule satisfies *queue-efficiency*, *budget-balance* and *Pareto indifference,* we only show that it satisfies *largest-cost consistency*. For each $N \in \mathcal{N}$, each $\left(\theta, \sigma^0\right) \in \mathcal{Q}^N$, each $i \in N$ such that $\theta_i = \max_{k \in N} \theta_k$, and each $j \in N \setminus \{i\}$, by Eq. (10), we have

$$\begin{aligned}
U_j^m\left(\theta, \sigma^0\right) &= \sum_{k \in P_j(\sigma^0)} (\theta_j - \theta_k)^+ \\
&= \begin{cases} \sum_{k \in P_j(\sigma^{0|-i})} (\theta_j - \theta_k)^+ & \text{if } \sigma_j^0 < \sigma_i^0 \\ (\theta_j - \theta_i)^+ + \sum_{k \in P_j(\sigma^{0|-i})} (\theta_j - \theta_k)^+ & \text{if } \sigma_j^0 > \sigma_i^0 \end{cases} \\
&= \sum_{k \in P_j(\sigma^{0|-i})} (\theta_j - \theta_k)^+ \qquad (34) \\
&= U_j^m\left(\theta_{-i}, \sigma^{0|-i}\right)
\end{aligned}$$

where Eq. (34) follows from $\theta_i = \max_{k \in N} \theta_k$. Therefore, the minimum price rule satisfies *largest-cost consistency*.

Conversely, let ϕ be a rule satisfying the four properties. We show that $\phi = \phi^m$ by induction on $|N|$.

Case 1: $|N| = 2$. Let $N = \{i, j\}$ with $i \neq j$. For each $(\theta, \sigma^0) \in \mathcal{Q}^N$, we may assume without loss of generality that $\theta_i \geq \theta_j$. By *largest-cost consistency*, for each $(\sigma, t) \in \phi(\theta, \sigma^0)$ and each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$,

$$U_j\left(\sigma, t; \theta, \sigma^0\right) \;=\; U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right). \tag{35}$$

By *budget-balance*, $t' = 0$ and thus, $U_j\left(\sigma', t'; \theta_{-i}, \sigma^{0|-i}\right) = 0$. Therefore, Eq. (35) implies that $U_j\left(\sigma, t; \theta, \sigma^0\right) = 0 = U_j^m\left(\theta, \sigma^0\right)$. On the other hand, by *queue-efficiency* and *budget-balance*, we have

$$\begin{aligned} U_i\left(\sigma,t;\theta,\sigma^0\right) &= U_i\left(\sigma,t;\theta,\sigma^0\right)+U_j\left(\sigma,t;\theta,\sigma^0\right) \\ &= \begin{cases} \theta_i-\theta_j & \text{if } \sigma_j^0<\sigma_i^0 \\ 0 & \text{if } \sigma_j^0>\sigma_i^0 \end{cases} \\ &= U_i^m\left(\theta,\sigma^0\right). \end{aligned}$$

Therefore, by *Pareto indifference*, we conclude that $\phi(\theta,\sigma^0)=\phi^m(\theta,\sigma^0)$.

Case 2: $|N|>2$. Proceeding by induction, suppose that the net utilities of agents are uniquely determined for $|N|=2,\dots,n-1$. We show that they are also uniquely determined for $|N|=n$.

Given $(\theta,\sigma^0)\in\mathcal{Q}^N$, let $i\in N$ be such that $\theta_i=\max_{k\in N}\theta_k$. By *largest-cost consistency*, for each $(\sigma^*,t^*)\in\phi(\theta,\sigma^0)$, each $(\sigma',t')\in\phi(\theta_{-i},\sigma^{0|-i})$ and each $j\in N\setminus\{i\}$,

$$U_j\left(\sigma^*,t^*;\theta,\sigma^0\right)=U_j\left(\sigma',t';\theta_{-i},\sigma^{0|-i}\right). \tag{36}$$

By the induction hypothesis, Eq. (36) implies that for each $j\in N\setminus\{i\}$,

$$\begin{aligned} U_j\left(\sigma^*,t^*;\theta,\sigma^0\right) &= U_j\left(\sigma',t';\theta_{-i},\sigma^{0|-i}\right) \\ &= U_j^m(\theta_{-i},\sigma^{0|-i}) \\ &= U_j^m(\theta,\sigma^0), \end{aligned} \tag{37}$$

where Eq. (37) follows from *largest-cost consistency* of the minimum price rule. Moreover, by *queue-efficiency* and *budget-balance* of ϕ^m and ϕ, and Eq. (37), we have

$$\begin{aligned} \sum_{k\in N}U_k^m\left(\theta,\sigma^0\right) &= \min_{\sigma\in\Pi(N)}\sum_{k\in N}\left(\sigma_k^0-\sigma_k\right)\theta_k \\ &= \sum_{k\in N}U_k\left(\sigma^*,t^*;\theta,\sigma^0\right) \\ &= \sum_{k\in N\setminus\{i\}}U_k^m(\theta,\sigma^0)+U_i\left(\sigma^*,t^*;\theta,\sigma^0\right), \end{aligned}$$

which implies that

$$U_i\left(\sigma^*,t^*;\theta,\sigma^0\right)=U_i^m(\theta,\sigma^0). \tag{38}$$

Therefore, by *Pareto indifference*, we conclude that $\phi(\theta,\sigma^0)=\phi^m(\theta,\sigma^0)$. ■

Remark 3 We can strengthen the property of *smallest-cost consistency* to *smaller-cost consistency,* which compares the unit waiting costs of each pair of agents by requiring that for each problem and each pair of agents i and j such that $\theta_i \leq \theta_j$, agent j's net utility is unaffected if agent i leaves the initial queue and the agents behind her move forward by one position.

Smaller-cost consistency: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each pair i, $j \in N$ with $\theta_i \leq \theta_j$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, and each $(\sigma', t') \in \phi(\theta_{-i}, \sigma^{0|-i})$,

$$U_j(\sigma, t; \theta, \sigma^0) = U_j(\sigma', t'; \theta_{-i}, \sigma^{0|-i}).$$

Since the maximum price rule satisfies *smaller-cost consistency,* Theorem 3 still holds when *smallest-cost consistency* is replaced by *smaller-cost consistency.*

Similarly, *largest-cost consistency* can be strengthened to *larger-cost consistency,* which compares the unit waiting costs of each pair of agents by requiring that for each problem and each pair of agents i and j such that $\theta_i \leq \theta_j$, agent i's net utility is unaffected if agent j leaves the initial queue and the agents behind her move forward by one position.

Larger-cost consistency: For each $N \in \mathcal{N}$, each $(\theta, \sigma^0) \in \mathcal{Q}^N$, each pair $i, j \in N$ with $\theta_i \geq \theta_j$, each $(\sigma, t) \in \phi(\theta, \sigma^0)$, and each $(\sigma', t') \in \phi(\theta_{N\setminus\{j\}}, \sigma^{0|N\setminus\{j\}})$,

$$U_j(\sigma, t; \theta, \sigma^0) = U_j(\sigma', t'; \theta_{-i}, \sigma^{0|-i}).$$

Since the minimum price rules satisfies *larger-cost consistency,* Theorem 4 still holds when *largest-cost consistency* is replaced by *larger-cost consistency.*

6 Concluding Remarks

First, we discuss whether our results can be generalized to sequencing problems with an initial queue in which each agent is characterized by three parameters: the amount of service time, the unit waiting cost, and the position in the initial queue. As shown in Smith (1956), an efficient queue can be obtained by ordering agents in a non-decreasing order of their urgency indices defined as the ratio of her unit waiting cost to her service time. As it turns out, our two results for the pairwise equal-splitting rule continue to hold for sequencing problems with an initial order. Furthermore, our results for the maximum and the minimum price rules remain valid if the properties of *smallest-cost* and *largest-cost consistency* are modified to *smallest-index* and *largest-index consistency.* These properties require that when an agent with the smallest or largest urgency index leaves the initial queue and the agents behind her move forward by one position, the net utilities of all remaining agents should remain unaffected.

We conclude this paper by mentioning an alternative approach to the reordering problem. Curiel et al. (1989) and van den Brink et al. (2007) try to solve the reordering problem by applying a cooperative game theoretic approach. In this approach, each agent is assigned with the net utility in the reordered queue, instead of a pair consisting of a position in the reordered queue and the amount of transfer as in our reordering problem. To do this, reordering problems should be mapped into *reordering games* (or (cost-saving) sequencing games in Curiel et al. (1989)) by appropriately defining a worth of each coalition. Curiel et al. (1989) and van den Brink et al. (2007) define the worth of a coalition consisting of consecutive agents in the queue as the cost-savings that these agents can obtain by reordering themselves into an efficient queue, while the worth of any nonconsecutive coalition as the sum of its maximally consecutive components. Moreover, van den Brink et al. (2007) represent an initial queue as a line-graph and formulate various properties assuming a deletion of some link: (i) *upper equivalence,* which requires that if a link between any two agents is deleted, then the net utilities of all agents in front of the deleted link should not be affected, (ii) *lower equivalence,* which requires that if a link between any two agents is deleted, then the net utilities of all agents at the back of the deleted link should not be affected, and (iii) *equal-loss property,* which requires that the impact of the deletion of some link on the total net utilities of the agents in front of the deleted link should be equal to the impact of the deletion of this link on the total net utilities of the agents at the back of the deleted link. Note that if a link is deleted, then agents cannot reorder their positions over the deleted link. They show that together with *component efficiency,*[6] the minimum price rule is the only rule satisfying *upper equivalence,* the maximum price rule the only rule satisfying *lower equivalence,* and the pairwise equal-splitting rule the only rule satisfying *equal-loss property.*[7] Whereas all these axioms are formulated by varying the set of links with the fixed set of agents, our consistency axioms are concerned with how the rule should respond to the variations in the set of agents.

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[6]This is a game theoretic axiom, that imposes a requirement similar to the combination of *queue-efficiency* and *budget balance* in our context.

[7]Since all our rules are essentially single-valued in utility terms, *Pareto indifference* is no longer required in the reordering game.

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Appendix. Independence of the Properties in Theorems 1-4

In this appendix, we demonstrate that the four properties listed in each theorem are independent. For each property, we provide an example of a rule that satisfies the other three properties but fails to satisfy the specified property.

Example 1. (Dropping *queue-efficiency.*) Let ϕ^{sq} be the status-quo rule, which chooses all allocations such that each agent receives her utility at the initial position. This implies that each agent has zero net utility. Formally, for each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$,

$$\phi^{sq}(\theta, \sigma^0) = \left\{ (\sigma, t) \;\middle|\; \begin{array}{c} \sum_{i \in N} (\sigma_i - 1)\theta_i = \sum_{i \in N} (\sigma_i^0 - 1)\theta_i, \text{ and} \\ \text{for each } i \in N, \, t_i = (\sigma_i - \sigma_i^0)\theta_i \end{array} \right\}.$$

This status-quo rule satisfies all properties listed in Theorems 1 to 4 except *queue-efficiency.*

Example 2. (Dropping *budget-balance.*) Let ϕ^{nr} be the no-rent rule, which chooses all allocations with efficient queues, but each agent receives zero net utility. Formally, for each $N \in \mathcal{N}$ and each $(\theta, \sigma^0) \in \mathcal{Q}^N$,

$$\phi^{nr}(\theta, \sigma^0) = \left\{ (\sigma, t) \;\middle|\; \begin{array}{c} \sigma \in \mathcal{E}(\theta), \text{ and for each } i \in N, \\ t_i = (\sigma_i - \sigma_i^0)\theta_i \end{array} \right\}.$$

This no-rent rule satisfies all properties listed in Theorems 1 to 4 except *budget-balance.*

Example 3. (Dropping *Pareto indifference.*) For each $N \in \mathcal{N}$, let $\succ^N$ be a priority order on N and $\succ \equiv (\succ^N)_{N \in \mathcal{N}}$ be the family of priority orders on $\mathcal{N}$. Let $\phi^{P,\succ}$ be a single-valued selection of the pairwise equal-splitting rule complying with $\succ$. Then, this rule satisfies all four properties listed in Theorems 1 and 2 except *Pareto indifference.*

Next, let $\phi^{M,\succ}$ be a single-valued selection of the maximum price rule complying with $\succ$. Then, this rule satisfies all four properties listed in Theorem 3 except *Pareto indifference.* Finally, in the same manner, a single-valued selection of the maximum price rule satisfies all four properties listed in Theorem 4 except *Pareto indifference.*

Example 4. (Dropping *balanced consistency.*) The maximum or the minimum price rule satisfies all four properties listed in Theorem 1 except *balanced consistency.*

Example 5. (Dropping *balanced cost reduction.*) The maximum or the minimum price rule satisfies all four properties listed in Theorem 2 except *balanced cost reduction.*

Example 6. (Dropping *smallest-cost consistency.*) The minimum price or the pairwise equal-splitting rule satisfies all four properties listed in Theorem 3 except *smallest-cost consistency.*

Example 7. (Dropping *largest-cost consistency.*) The maximum price or the pairwise equal-splitting rule satisfies all four properties listed in Theorem 4 except *largest-cost consistency.*