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Reconciling marginalism with the core in two-sided markets with money

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Abstract

In two-sided markets with money, core stability and marginalism are often in conflict. We reconcile them with two main results.

First, we show that in the assignment game (Shapley & Shubik, 1972), the Banzhaf value is core stable if and only if the game is exact. This is surprising for two reasons: (i) the Banzhaf value is generally not efficient, and (ii) although exactness suffices for the Shapley value to be stable, it is not necessary. Consequently, stability of the Banzhaf value implies stability of the Shapley value, but not vice versa.

Second, we consider a family of intra-sector Shapley and Banzhaf values by applying each value separately to the game on the set of buyers assuming all sellers are available and to the game on the set of sellers assuming all buyers are available, and then taking any convex combination. We prove that all such convex combinations lie in the core if and only if the valuation matrix has a dominant diagonal. Under this condition, the equal-weight intra-sector Shapley and Banzhaf values coincide with the fair-division point. Together, these results deliver simple criteria under which marginalist solutions assign stable payoff vectors in the original market.

JEL classification: C71; C78; D47.

Keywords: Assignment problems; Stability; Marginalism; Banzhaf value; Shapley value; Fair-division point.

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1 Introduction

A *two-sided market with money* involves two finite sets of agents, where each of them is interested in making at most one trade with agents on the opposite side of the market. Each possible transaction in the market generates a potential monetary gain.[1] Examples of economic situations that can be modeled by such markets are interactions between buyers and sellers or firms and workers. Throughout this paper we will use the buyers-sellers example to introduce and illustrate the various notions and results. It is assumed that monetary transfers are possible, i.e. when a buyer and seller trade, the buyer can pay a price to the seller for obtaining the object that is originally owned by the seller. We assume that all agents have a quasi-linear utility function in the sense that a transaction between a buyer and a seller yields a utility gain for the seller that is equal to the price (monetary transfer) minus its reservation value and a utility gain for the buyer that is equal to its valuation for the object minus the price.

It is a well-known problem that (core) stability and marginalism are difficult to combine, not only in coalitional games in general, but also in two-sided markets with money. The *core* consists of the set of payoff vectors in a market that are *stable* in the sense that every coalition of buyers and sellers earns at least what they can earn by optimally trading among themselves. In this way, the core can be seen as the set of those payoff vectors where no coalition of players could, on their own, improve the payoff of all its members. This approach captures the coalitional competitive forces within the game, aligning with the descriptive side of cooperative game theory.[2] On the other hand, *marginalism* adopts a normative approach. It requires that each player should be paid according to its productivity in the game, that is, according to his or her marginal contributions to the different coalitions he or she may join. There are multiple solutions that reflect this principle, with the *Shapley value* (Shapley, 1953) being the most prominent.[3] The Shapley value is efficient, i.e., the sum of the individual payoffs adds up to the total worth generated by the grand coalition consisting of all market participants, but it is generally not core stable. This is also the case for two-sided markets with money.

To illustrate the tension between core stability and marginalism, consider a market with one seller who owns one item of an object, and two buyers (buyers 1 and 2) where the valuation of the buyer 1 for the object is 1 and the valuation[4] of buyer 2 is 2. Obviously, the optimal allocation is to assign the object to buyer 2 and generate a total trade surplus of 2 in the market. Since the seller and buyer 2 can also obtain this trade surplus on their own, in any core payoff vector buyer 1 will get zero. However, buyer 1 has a positive marginal contribution when it forms a coalition with the seller, and therefore every marginalist solution will assign a positive payoff to buyer 1. This also fits with the idea in Shapley & Shubik (1972) that buyer 1 has leverage in the game. Although it is not necessary to generate the surplus of 2, the presence of buyer 1 strengthens the bargaining position of the seller in its relation with buyer 2. This

[1]For ease of explanation, we sometimes use markets instead of two-sided markets with money.

[2]See Mas-Colell *et al.* (1995) for a wider description of the core.

[3]In fact, Young (1985) shows that the Shapley value is the only efficient, symmetric solution for cooperative TU-games that is determined by marginal contributions.

[4]Throughout the paper, we assume the sellers to have reservation value zero for the objects.

example also shows that, to make marginalism compatible with core stability, we either need to weaken at least one of these notions or define subclasses of two-sided markets where marginalism is compatible with core stability.

In the literature, support can be found for both principles. Therefore, the main objective of the underlying paper is to reconcile the core with marginalism. Specifically, we provide necessary and sufficient conditions for two types of marginalist solutions to be core stable for two-sided markets with money. Both types of solutions are based on the *assignment game* associated with each two-sided market with money as described by Shapley & Shubik (1972). In this game, the worth of any coalition that contains buyers as well as sellers equals the maximal trade surplus that can be obtained by allowing trade only between buyers and sellers within this coalition.[5] Hoffmann & Sudhölter (2007) prove that, if an assignment game is *exact*, that means that each coalition attains its worth at some core allocation, then the Shapley value payoff vector lies in the core. Solymosi & Raghavan (2001) show that exactness of the assignment game can be characterized directly in terms of the potential gains from pairwise trade in the assignment market, without using the coalitional function: an assignment game is exact if and only if the valuation matrix satisfies the *dominant diagonal* and *doubly dominant diagonal* properties. Whereas exactness, as shown by Hoffmann & Sudhölter (2007), is sufficient for the Shapley value to assign a payoff vector in the core, it is not necessary. In fact, to the best of our knowledge, there is no result in the literature, without recourse to replication or limiting arguments, showing a necessary and sufficient market condition for the Shapley value payoff vector to belong to the core of the assignment game.[6]

Building on those results, we explore the relationship between the core and marginalism by considering two types of solutions for two-sided markets. First, we consider the *Banzhaf value* (Banzhaf, 1965) of the assignment game being an alternative for the Shapley value that is also grounded on marginalism. Similar to the Shapley value, this solution allocates payoffs based on the marginal contributions of each player, but uses a different way of weighting marginal contributions to coalitions. Similar as the Shapley value, the Banzhaf value is generally not core stable. Even worse, the Banzhaf value is generally not efficient. This is also a main reason why, outside measuring political voting power, the Banzhaf value is rarely applied in economic allocation problems. Surprisingly, it turns out that exactness of the assignment game is necessary and sufficient for the Banzhaf value to be core stable. This result establishes a significant connection between marginalism and the core and, moreover, improves the result of Hoffmann & Sudhölter (2007) on the Shapley value. While exactness is a sufficient, but not necessary, condition for the Shapley value payoff vector to belong to the core, it is sufficient and necessary for the Banzhaf value payoff vector to lie in the core. This is very surprising because in the literature usually the Shapley value outperforms the Banzhaf value and therefore the Banzhaf value has often been overlooked in the economic literature in favor of the Shapley value.

[5] Coalitions that contain only buyers or only sellers cannot trade and thus their worth is zero.

[6] For the special case for markets with only two buyers and two sellers, there is a condition that is necessary and sufficient for the Shapley value payoff vector to be in the core.

The second type of solution that we consider are two classes of solutions that are based on the Shapley and Banzhaf values. In a two-sided market with money, the marginal contributions of the agents reflect the opposition of interest between both sides of the market. That is, the marginal contributions of an agent to different coalitions increase as new agents from the opposite sector enter the coalitions, and they decrease as new agents of its same sector enter the coalition (Shapley, 1962). In order to analyze this competition inside each sector, we consider two coalitional games that were already introduced by van den Brink & Pintér (2015). The first game, called the *sellers-come-first game*, is played only by the buyers assuming that all sellers are available. Similarly, the second game, called the *buyers-come-first game*, is played only by the sellers and it is assumed that all buyers are available in the market. Notice that the Shapley value of the sellers-come-first game rewards each buyer taking into account its leverage in the competition between buyers, and similarly does the Shapley value of the buyers-come-first game with the sellers.

Building on this, we introduce two families of single valued solutions: the *intra-sector Shapley values* and the *intra-sector Banzhaf values*. These solutions reward each buyer according to his/her contributions in the sellers-come-first game and each seller according to the buyers-come-first game. First, the *intra-sector Shapley value* assigns to every buyer (respectively seller) half of its Shapley value payoff in the buyers-come-first game (respectively sellers-come-first game). By efficiency of the Shapley value and the fact that all sellers together (respectively all buyers together) earn the total optimal trade gain in the full market, the intra-sector Shapley value is efficient. Doing the same for the Banzhaf value gives the intra-sector Banzhaf value which is not efficient for general assignment games. Similar to their general counterparts, the intra-sector Shapley and Banzhaf values do not coincide and are not, in general, core stable for the assignment game. Our second main result shows that just the dominant diagonal property (being a weaker condition than exactness of the assignment game which additionally requires the doubly dominant diagonal property) is necessary and sufficient for any convex combination of the two extreme buyers-come-first and sellers-come-first solutions to assign a core stable payoff vector. The intra-sector Shapley and Banzhaf value are the special cases where the average of the two extreme solutions is taken.

We end this paper by comparing the solutions studied in this paper with the *fair-division point* introduced for two-sided markets with money by Thompson (1981), which is a core stable solution that also relies on marginalism. It is known that two extreme points of the core of an assignment game are the sellers optimal core allocation (being the core allocation that is best for all the sellers) and the buyers optimal core allocation (being the core allocation that is best for all the buyers). The fair-division-point of a two-sided market with money is the average of these two extreme core allocations and, by the core being a convex set, obviously belongs to the core. As shown by Demange (1982) and Leonard (1983), this reflects some kind of marginalism because the best core payoff of a market participant (buyer or seller) equals its marginal contribution to the market as a whole, and its worst core payoff is its marginal contribution to the coalition where the agent with whom it is matched has left the market. It turns out that the intra-sector Shapley and Banzhaf values lead to payoff vectors that coincide with the fair-division-point if the valuation matrix satisfies the dominant diagonal property. For the payoff vectors given by the 'classic' Banzhaf and Shapley values to coincide with the fair-division point, we need more. We introduce a notion of symmetry of a valuation matrix with as many buyers as sellers that states that the trade gain generated by matching a buyer with a seller is mirrored for similar transactions between corresponding agents on the other side of the market. We show that when the valuation matrix is symmetric and satisfies the dominant diagonal and doubly dominant diagonal properties, the Banzhaf value payoff vector and the Shapley value payoff vector do coincide with the fair-division point.

The paper is organized as follows. Section 2 contains preliminaries on two-sided markets with money, assignment games, marginalism and the dominant diagonal and doubly dominant diagonal properties. Section 3 provides necessary and sufficient conditions for the Banzhaf value to be core stable, while Section 4 answers the same question for the intra-sector Shapley and Banzhaf values. Section 5 provides conditions when these solutions coincide with the fair-division point. Finally, Section 6 contains concluding remarks.

Related literature

There is a large literature on the core and marginalism being two fundamental approaches in the analysis of economic and social problems from a coalitional perspective. For example, Mas-Colell *et al.* (1995) support the core by mentioning that "[t]he core tries to capture how the possible outcomes of a game may be shaped by coalitional competitive forces. It is the simplest solution concept in what could be called the descriptive side of cooperative game theory", while, for example, Hart (2012) mentions that marginality can be seen as "the value of a player is only a function of his marginal contributions".[7] However, as made clear in Young (1985), these two approaches are difficult, and in most cases impossible, to combine. Therefore, a natural question is under which conditions do marginalist solutions like the Shapley value or the Banzhaf value belong to the core? The literature on coalitional games has provided some sufficient conditions to show when the Shapley value payoff vector belongs to the core. One of these conditions is convexity, introduced by Shapley (1971). Another condition that is weaker than convexity is called average convexity (Inarra & Usategui, 1993). In a recent paper, Abe & Nakada (2023), study the relationship between the core and the Shapley value from a more geometric perspective.

In the context of two-sided markets, the non–emptiness of the core in assignment games was first proven by Shapley & Shubik (1972). Demange (1982) and Leonard (1983) study the two extreme points of the core in assignment games that identify the maximum core payoffs for buyers and sellers as their marginal contributions to the market. Thompson (1981) introduced the *fair-division point* for two-sided markets with money as the average of these two extreme core payoff vectors. By the core being a convex set, this fair-division point always belongs to the core. Núñez & Rafels (2002)

[7]For an economic approach to marginality in exchange and production economies, see for example, Ostroy (1984).

show that this fair-division point coincides with applying the τ-value of Tijs (1981) in the context of assignment games. In van den Brink *et al.* (2021) the buyers and sellers optimal core allocations are axiomatized using certain monotonicity axioms, while the fair-division point is axiomatized using a fairness axiom which requires equal impact on the payoffs of the market participants under certain modifications of the market.

The Shapley value payoff vector does not generally belong to the core of an assignment game. Hoffmann & Sudhölter (2007) show that exactness of the assignment game is a sufficient condition for the Shapley value payoff vector to belong to the core. Moreover, Solymosi & Raghavan (2001) shows that exactness of the game can be characterized by the dominant diagonal and doubly dominant diagonal properties of the valuation matrix. However, exactness is not necessary for the Shapley value to be in the core.

The Banzhaf value for coalitional games is based on the Banzhaf index for voting games (Banzhaf, 1965) which is generalized to coalitional TU-games by, e.g., Owen (1975) and Dubey & Shapley (1979). A main reason why the Banzhaf value is rarely applied in economics is that it is not efficient, i.e. it generally does not allocate the total surplus that can be obtained by all players cooperating together. As mentioned before, surprisingly it tuns out that exactness of the assignment game is necessary and sufficient for the Banzhaf value payoff vector to lie in the core.

In a recent paper, van den Brink & Pintér (2015) present the sellers-come-first and buyers-come-first games for two-sided markets with money. These concepts focus on marginalism within each sector, leading to a new solution concept, which we call intra-sector Shapley value. In this paper, we extrapolate this approach to the Banzhaf value to define the intra-sector Banzhaf value. We show that these intra-sector values can belong to the core under less restrictive conditions than their general counterparts.

In a more recent contribution, Luo *et al.* (2024) analyze the relationship between the Shapley value and competitive equilibria in large two-sided markets with money. They show that, as one takes infinitely many replicas of the underlying multiple-partners game (Sotomayor, 1992), the Shapley value payoff vector of the replicated market converges to a competitive-equilibrium payoff, and the same result holds for the Banzhaf value payoff vector. This "replica" approach has been studied before for similar indivisible-good markets, see e.g., Massó & Neme (2014).

Our paper and Luo *et al.* (2024) are complementary in the sense that, while the latter establishes (competitive-equilibrium) stability only in the limit of large replicas, we ask ourselves what market conditions guarantee that, in the original, unreplicated market, marginalism is compatible with (core) stability. By working directly on the given market (without replication), we identify verifiable conditions under which marginal-based solution concepts are stable.

Finally, for a recent survey of the Shapley value and Shapley-like solution concepts in the context of assignment games, we refer to Núñez & Rafels (2019).

2 Preliminaries

2.1 Two-sided markets with money

Consider a market situation between two disjoint finite sets of agents, e.g., buyers and sellers or workers and firms. In this paper, we simply refer to them as buyers and sellers. The set of buyers is denoted by B and the set of sellers by S, with $B \cap S = \emptyset$. In these markets, each agent in the market is allowed to make a transaction with one agent from the other side of the market. In other words, each seller owns one indivisible object for sale and each buyer wants to acquire at most one object. Each potential transaction between a buyer i and a seller j leads to a joint trade gain $a_{ij} \geq 0$ that could be divided between them.[8] Each agent has a quasi-linear utility function with respect to money.[9] We denote by $a = (a_{ij})_{(i,j)\in B\times S}$ the *valuation matrix* that captures all possible joint values resulting from all potential transactions. Thus, a *two-sided market with money* is defined by a triple (B, S, a) as described above.

Given a non-empty subset of buyers $B' \subseteq B$ and a non-empty subset of sellers $S' \subseteq S$, a *matching* μ between B' and S' is a subset of buyer-seller pairs $(i, j) \in B' \times S'$ such that each agent belongs to at most one pair. That is, a matching describes a set of partnerships between the two sectors. We write $\mathcal{M}(B', S')$ for the set of all matchings between B' and S'. Such a matching μ is *optimal* if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for any other $\mu' \in \mathcal{M}(B', S')$. We denote by $\mathcal{M}_a(B', S')$ the set of optimal matchings between B' and S'. We say that buyer $i \in B'$ and seller $j \in S'$ are *matched* by matching $\mu \in \mathcal{M}(B', S')$ on $B' \cup S'$ if $(i, j) \in \mu$. If there is no seller j, such that $(i, j) \in \mu$, then we say that buyer i is *unmatched*. Similarly, if there is no buyer i, such that $(i, j) \in \mu$, then we say that seller j is unmatched.

Remark 2.1. *Note that, since an optimal matching always exists, without loss of generality, the rows and columns of the valuation matrix can be reorganized so that an optimal matching for the whole market is now on the main diagonal. For notational purposes, given a buyer $i \in B$, we often use $i' \in S$ to denote this buyer's counterpart such that $a_{ii'}$ is located on the main diagonal of the valuation matrix.*

2.2 A cooperative game approach

A *cooperative transferable utility game*, shortly referred to as *cooperative game*, is a pair (N, v) where N is a finite set of *players*, and v is a real valued *characteristic function* which assigns to every *coalition* of players $S \subseteq N$, a real number $v(S)$ representing the welfare gain that the players in S can generate when cooperating.

For two-sided markets with money, Shapley & Shubik (1972) define the *assignment game* being the cooperative game where the set of players is $B \cup S$ and the worth of each

[8] We assume that the reservation value of each seller for its own object is zero.

[9] This means that the utility of a buyer i who trades with seller j equals its valuation a_{ij} minus the price it has to pay to agent j to get the object and equals zero if it does not trade, while the utility of a seller j who trades with buyer i equals the price buyer i pays to get the object and equals zero if it does not trade.

coalition is the maximum value that can be generated by matching buyers and sellers in these coalitions.

Definition 2.2. *Given a two-sided market with money* (B, S, a)*, the corresponding* ***assignment game*** *is denoted by* $(B \cup S, w_a)$ *where the set of players is* $B \cup S$ *and the worth of each coalition* $T \subseteq B \cup S$ *is* $w_a(T) = 0$ *if* $T \cap B = \emptyset$ *or* $T \cap S = \emptyset$ *and, otherwise,*

$$w_a(T) = \max_{\mu \in \mathcal{M}(T \cap B, T \cap S)} \left\{ \sum_{\{i,j\} \in \mu} a_{ij} \right\}.$$

We refer to the worth $w_a(T)$ as the optimal trade gain in coalition $T \subseteq B \cup S$. Specifically, $w_a(B \cup S)$ is the optimal trade gain in the full market. Note that for any coalition $T \subseteq B \cup S$ with $i \in T \cap B$ and $j \in T \cap S$, such that $(i, j) \in \mu$ where $\mu \in \mathcal{M}_a(T \cap B, T \cap S)$, the following trivially holds:

$$w_a(T) = w_a(T \setminus \{i, j\}) + a_{ij} = w_a(T \setminus \{i, j\}) + w_a(\{i, j\}). \tag{1}$$

Definition 2.3. *Given a two-sided market with money* (B, S, a) *and its corresponding assignment game,* $(B \cup S, w_a)$*, the set of* imputations*,* $I(w_a)$*, consists of those payoff vectors* $(u, v) \in \mathbb{R}^B \times \mathbb{R}^S$ *that are*

(i) efficient*:* $\sum_{i \in B} u_i + \sum_{j \in S} v_j = w_a(B \cup S)$*, and*

(ii) individually rational*:* $u_i \geq w_a(\{i\}) = 0$ *for all* $i \in B$ *and* $v_j \geq w_a(\{j\}) = 0$ *for all* $j \in S$*.*

An imputation splits the worth that is generated in the whole market among all agents in such a way that each agent is guaranteed to receive what he/she can achieve on his/her own. In the context of two-sided markets with money, an imputation (u, v) belongs to the core of the assignment game $(B \cup S, w_a)$ if it is stable in the sense that every coalition of buyers and sellers earns at least what they can earn by optimally trading among themselves. In other words, the imputation cannot be blocked by any coalition of market participants.

Definition 2.4. *Given a two-sided market with money,* (B, S, a)*, and its corresponding assignment game,* $(B \cup S, w_a)$*, the set of* core *allocations,* $C(w_a)$*, consists of those imputations* $(u, v) \in I(w_a)$ *that are coalitionally stable:*

$$\text{for all } T \subseteq B \cup S, \sum_{i \in B \cap T} u_i + \sum_{j \in S \cap T} v_j \geq w_a(T).$$

Shapley & Shubik (1972) shows that the core of the assignment game is always non-empty and it is enough to require pairwise stability:

$$C(w_a) = \{(u, v) \in I(w_a) \mid u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in B \times S\}. \tag{2}$$

Notice that (2) implies that if μ is an optimal matching for (B, S, a) and $(u, v) \in C(w_a)$, then

$$u_i + v_j = a_{ij} \text{ for all } (i, j) \in \mu, \tag{3}$$

and any core payoff to unmatched agents is zero. This means that core allocations exclude third-party payment; all the transfers are made inside the buyer-seller pairs that trade an object. The classical notion of the core can then be simplified to pairwise stability. This refinement reduces the number of inequalities that must be considered when determining if a payoff vector lies in the core.

Regarding the mathematical properties of the core of an assignment game, it is known that it has a complete lattice structure with two outstanding elements, the *buyers-optimal* core allocation $(\overline{u}, \underline{v})$, where each buyer attains his/her maximum core payoff; and the *sellers-optimal* core allocation $(\underline{u}, \overline{v})$, where each sellers attains his/her maximum core payoff. Demange (1982) and Leonard (1983) show that the maximum core payoff of an agent in an assignment game is its marginal contribution to the whole market:

$$\begin{aligned} \overline{u}_i &= w_a(B \cup S) - w_a((B \setminus \{i\}) \cup S) \text{ for all } i \in B, \\ \overline{v}_j &= w_a(B \cup S) - w_a((B \cup (S \setminus \{j\})) \text{ for all } j \in S. \end{aligned}$$

2.3 Marginalism

In the context of coalitional games, the marginal contribution of a player to a coalition is defined as the additional value the player brings to the coalition upon joining, measured as the difference in the worth of a coalition before and after the player becomes part of it. In general, the idea of marginalism is to allocate a payoff to individual players in a way that reflects their marginal contributions. In the following paragraphs, we introduce three solutions that rely on marginalism. The first two are solutions that can generally be applied to coalitional games, while the last one is more tailored to the case of two-sided market situations.

A *payoff vector* of a game on player set $N(= B \cup S)$ is a $|N|$-dimensional real vector which components are the individual (utility) payoffs of the players in the game. A *solution* for games (respectively markets) is a function that assigns a payoff vector to every game (respectively market). The Shapley value (Shapley, 1953) is the solution that assigns to every coalitional game (N, v) the payoff vector that allocates to each player a weighted average of its marginal contributions over all possible orders of formation of the grand coalition. More precisely, the Shapley value assigns to every game (N, v) the payoff vector defined by

$$Sh_i(v) = \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N| - |T| - 1)!}{|N|!} \left(v(T \cup \{i\}) - v(T)\right) \text{ for all } i \in N.$$

While, in general, the Shapley value payoff vector $Sh(v) = (Sh_i(v))_{i \in N}$ does not belong to the core of the coalitional game (N, v), it is a standard-bearer of the idea of marginalism in coalitional games.

Another solution that clearly reflects the idea of marginalism is the Banzhaf value (Banzhaf, 1965). This solution is similar to the Shapley value; however, it differs in the weighting scheme, assigning to every coalitional game the payoff vector that allocates to each player the average of its marginal contributions across all possible coalitions they might join. More specific, the Banzhaf value assigns to every game (N, v) the payoff vector defined by

$$\beta_i(v) = \frac{1}{2^{|N|-1}} \sum_{T \subseteq N \setminus \{i\}} (v(T \cup \{i\}) - v(T)) \text{ for all } i \in N.$$

The Banzhaf value reflects the player's ability to change the worth of a coalition, emphasizing their potential impact on coalitions rather than on the order in which they are formed. Just like the Shapley value, the Banzhaf value payoff vector $\beta(v) = (\beta_i(v))_{i \in N}$ does not belong to the core of the game (N, v). Even worse, in general it is not even efficient.

In the context of two-sided markets with money, a key question arises when comparing the Shapley value and the Banzhaf value: under which scenarios is the latter more adequate than the former (or vice versa)? Our first theorem will offer an answer to that question, but before stating our result, we recall the fair-division point.

For the case of two-sided markets with money, the fair-division point was introduced by Thompson (1981) as the average of the buyers-optimal and the sellers-optimal payoff vectors. More precisely, the fair-division point for market (B, S, a) is the payoff vector that assigns to every buyer $i \in B$, the payoff

$$\tau_i(a) = \frac{1}{2}(\overline{u}_i(a) + \underline{u}_i(a)),$$

and to every seller $j \in S$, the payoff

$$\tau_j(a) = \frac{1}{2}(\underline{v}_j(a) + \overline{v}_j(a)).$$

Notice that the fair-division point is a payoff vector that is directly defined on the valuation matrix a, and therefore is denoted by $\tau(a)$. Unlike the payoff vectors resulting from the Shapley and Banzhaf values, the fair-division point, by definition, belongs to the core of the corresponding game since it is the average of two core payoff vectors and the core of a game is a convex set. The fair-division point represents a compromise between the interests of buyers and sellers, ensuring an equitable division of the total value generated by the market, see van den Brink *et al.* (2021).

2.4 Exact assignment games: the dominant and doubly dominant diagonal properties

Hoffmann & Sudhölter (2007) show that *exactness* of an assignment game is a sufficient condition for the Shapley value payoff vector to belong to the core.[10] Solymosi &

[10] A coalitional game (N, v) is exact if for each coalition $S \subseteq N$ there exists a core element x such that $\sum_{k \in S} x_k = v(S)$.

Raghavan (2001) show that for a two-sided market with money with as many buyers as sellers,[11] the associated assignment game is exact if and only if the valuation matrix satisfies the dominant diagonal and doubly dominant diagonal properties.

Definition 2.5. *Consider a two-sided market with money (B, S, a). The valuation matrix a satisfies the*

1. *Dominant diagonal property: if for any optimal matching $\mu \in \mathcal{M}(B, S)$ and for each $(i, j) \in \mu$, it holds that $a_{ij} \geq a_{ik}$ and $a_{ij} \geq a_{tj}$ for all $k \in S$ and all $t \in B$.*
2. *Doubly dominant diagonal property: if for any optimal matching $\mu \in \mathcal{M}(B, S)$ and for each $(i, j) \in \mu$, it holds that $a_{ij} + a_{tk} \geq a_{ik} + a_{tj}$ for all $k \in S$ and all $t \in B$.*

We can interpret these conditions as follows. First, a valuation matrix satisfies the dominant diagonal property if in any optimal matching, it is feasible for each agent to make a transaction with one of her/his most preferred agents of the opposite sector. Second, the doubly dominant diagonal property requires that an optimally matched pair together with another buyer-seller pair cannot improve their total value by interchanging partners.

Hoffmann & Sudhölter (2007) show that an assignment game being exact, i.e. the valuation matrix satisfying both the dominant diagonal and doubly dominant diagonal properties, is a sufficient condition for the Shapley value payoff vector to belong to the core. By means of a counterexample they also show that exactness of the game is not necessary for the Shapley value payoff vector to belong to the core.

In the next section we will see that, surprisingly, exactness is necessary and sufficient for the Banzhaf value to be core stable in assignment games.

3 The Banzhaf value and the core

Our first objective is to contribute to the understanding of how the Banzhaf value compares to the Shapley value, especially given that the former has often been overlooked in favor of the latter. A main reason why the Banzhaf value is almost not applied, except as a measure of political voting power, is that it is not efficient, i.e. in general it does not exactly allocate the total worth that can be obtained by all players cooperating together.[12] First, we show that also for assignment markets the Banzhaf value is not efficient, as illustrated by the next example.

Example 3.1. *Consider a two-sided market with money (B, S, a) with two buyers, $B = \{1, 2\}$, and two sellers, $S = \{1', 2'\}$ and the valuation matrix given by*

$$a = \begin{pmatrix} 7 & 2 \\ 8 & 4 \end{pmatrix}.$$

[11] We will assume the two-sided market with money has as many buyers as sellers, but this can be done with no loss of generality just adding dummy agents, that produce zero valuations, in the short side of the market.

[12] Since measuring political voting power is mainly about how the power of different agents relate to each other, efficiency is not an issue in political voting power measurement.

In this matrix, row i gives the valuations of buyer i for the objects of the two sellers, and column j gives the valuations of the two buyers for the object owned by seller j. Since the unique optimal matching matches buyer 1 to seller $1'$ and buyer 2 to seller $2'$, the total surplus that can be generated in the market, and thus in the game $(B \cup S, w_a)$, is $w_a(B \cup S) = 7 + 4 = 11$. The Banzhaf value allocates the following payoff vector $\beta(w_a) = (\frac{19}{8}, \frac{27}{8}, \frac{39}{8}, \frac{13}{8}) \in \mathbb{R}^{|B|} \times \mathbb{R}^{|S|}$ and thus

$$\sum_{i \in B} \beta_i(w_a) + \sum_{j \in S} \beta_j(w_a) = \frac{98}{8} = 12.25 > 11 = w_a(B \cup S).$$

This shows that the Banzhaf value of an assignment game may not be efficient, and in order to be applied, a subsidy from an outsider is needed.

Before stating our first theorem, we show that if a buyer and seller are optimally matched in the overall market, they will remain optimally matched even when we consider submarkets on smaller coalitions, as long as the conditions of dominant and doubly dominant diagonals are satisfied.

Proposition 3.2. *Let (B, S, a) be a two-sided market with money and $\mu \in \mathcal{M}_a(B, S)$. Assume that the valuation matrix a satisfies the dominant diagonal and doubly dominant diagonal properties. Take any pair of agents $i \in B$ and $i' \in S$ such that $(i, i') \in \mu$, and any coalition $T \subseteq (B \cup S) \setminus \{i, i'\}$. Then, there is at least one matching $\mu' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$ where $(i, i') \in \mu'$.*

Proof. Take any $\mu'' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$. Since there always exists an optimal matching, it is sufficient to prove that, if there is an optimal matching μ'' with $(i, i') \notin \mu''$, then there is an optimal matching μ' with $(i, i') \in \mu'$. Therefore, suppose that there is an optimal matching μ'' with $(i, i') \notin \mu''$. We consider the following four cases.

<u>Case a:</u> (i and i' are matched) Let $(i, k) \in \mu''$ and $(l, i') \in \mu''$ with $k \neq i'$ and $l \neq i$. Define $\mu' = (\mu'' \setminus \{(i, k), (l, i')\}) \cup \{(i, i'), (l, k)\}$. To show that μ' is also an optimal matching, note that

$$\sum_{(b,s) \in \mu''} a_{bs} = \sum_{(b,s) \in \mu'' \setminus \{(i,k),(l,i')\}} a_{bs} + a_{ik} + a_{li'} \leq a_{ii'} + a_{lk} + \sum_{(b,s) \in \mu'' \setminus \{(i,k),(l,i')\}} a_{bs} = \sum_{(b,s) \in \mu'} a_{bs},$$

where the inequality holds because of the doubly dominant diagonal property (implying that $a_{ik} + a_{li'} \leq a_{ii'} + a_{lk}$). Since $\mu'' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$, this inequality implies that $\mu' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$ as desired.

<u>Case b:</u> (i is matched, i' is not matched) Let $(i, k) \in \mu''$ with $k \neq i'$, and i' is not matched. Define $\mu' = (\mu'' \setminus \{(i, k)\}) \cup \{(i, i')\}$. To show that μ' is also an optimal matching, note that

$$\sum_{(b,s) \in \mu''} a_{bs} = \sum_{(b,s) \in \mu'' \setminus \{(i,k)\}} a_{bs} + a_{ik} \leq a_{ii'} + \sum_{(b,s) \in \mu'' \setminus \{(i,k)\}} a_{bs} = \sum_{(b,s) \in \mu'} a_{bs},$$

where the inequality holds because of the dominant diagonal property (implying that $a_{ik} \leq a_{ii'}$). Since $\mu'' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$, this inequality implies that $\mu' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$ as desired.

Case c: (i is not matched, i' is matched) Taking $(l, i') \in \mu''$ with $l \neq i$ and i being not matched, this case proceeds analogously as the previous case b.

Case d: (i and i' are both not matched) Define $\mu' = \mu'' \cup \{(i, i')\}$, and note that

$$\sum_{(b,s)\in\mu''} a_{bs} \leq a_{ii'} + \sum_{(b,s)\in\mu''} a_{bs} = \sum_{(b,s)\in\mu'} a_{bs}.$$

Since $\mu'' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$, this inequality implies that $\mu' \in \mathcal{M}_a((T \cup \{i\}) \cap B, (T \cup \{i'\}) \cap S)$ as desired.

This completes the proof. □

The next result shows that the dominant diagonal and doubly dominant diagonal properties together form a necessary and sufficient condition for the Banzhaf value payoff vector to be in the core of the assignment game. This is very surprising since (i) as shown before the Banzhaf value is generally not efficient also not for two-sided assignment games, (ii) this characterizes a class of markets where the Banzhaf value is not only efficient but even stable, and (iii) this improves the result of Hoffmann & Sudhölter (2007) that these are sufficient but not necessary conditions for the Shapley value to be in the core.

Theorem 3.3. *Let (B, S, a) be a two-sided market with money and $(B \cup S, w_a)$ the corresponding assignment game. The Banzhaf value belongs to the core, $\beta(w_a) \in C(w_a)$, if and only if the valuation matrix a satisfies the dominant diagonal and the doubly dominant diagonal properties.*

Proof. Let $N = B \cup S$ and $n = |B \cup S|$. Before starting the proofs of the "if" and "only if" parts, recall that without loss of generality, we may assume that there is an optimal matching for the whole market located on the main diagonal of the valuation matrix, see Remark 2.1. Further, for any matched buyer-seller pair (i, i') on the main diagonal, we have

$$\begin{aligned}
\beta_i(w_a) + \beta_{i'}(w_a) &= \frac{1}{2^{n-1}} \sum_{T\subseteq N\setminus\{i\}} (w_a(T \cup \{i\}) - w_a(T)) + \frac{1}{2^{n-1}} \sum_{T\subseteq N\setminus\{i'\}} (w_a(T \cup \{i'\}) - w_a(T)) \\
&= \frac{1}{2^{n-1}} \left(\sum_{T\subseteq N\setminus\{i,i'\}} (w_a(T \cup \{i\}) - w_a(T) + w_a(T \cup \{i,i'\}) - w_a(T \cup \{i'\})) \right) \\
&+ \frac{1}{2^{n-1}} \left(\sum_{T\subseteq N\setminus\{i,i'\}} (w_a(T \cup \{i'\}) - w_a(T) + w_a(T \cup \{i,i'\}) - w_a(T \cup \{i\})) \right) \\
&= \frac{1}{2^{n-2}} \sum_{T\subseteq N\setminus\{i,i'\}} (w_a(T \cup \{i,i'\}) - w_a(T)) . \qquad (4)
\end{aligned}$$

The "if" part. Suppose that (B, S, a) is a two-sided market with money such that the valuation matrix a satisfies the dominant diagonal and the doubly dominant diagonal properties, then we show that $\beta(w_a) \in C(w_a)$. Let $\beta_t(w_a)$ denote the Banzhaf value for any agent $t \in B \cup S$. From the description of the core (2) and equation (3), it is sufficient to show that for all $i \in B$ and all $j \in S \setminus \{i'\}$ with $a_{ii'}$ belonging to the main diagonal:

i) $\beta_i(w_a) + \beta_{i'}(w_a) = a_{ii'}$, and

ii) $\beta_i(w_a) + \beta_j(w_a) \geq a_{ij}$.

To prove i), notice that, since the valuation matrix satisfies the dominant diagonal and the doubly dominant diagonal properties, by Proposition 3.2 and equation (1), we have for any coalition $T \subseteq N \setminus \{i, i'\}$, that

$$w_a(T \cup \{i, i'\}) = w_a(T) + a_{ii'}.$$

Then, with (4) we have that

$$\beta_i(w_a) + \beta_{i'}(w_a) = \frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i, i'\}} a_{ii'} = a_{ii'},$$

where the last equality holds since $|\{T \mid T \subseteq N \setminus \{i, i'\}\}| = 2^{n-2}$. This shows that condition i) is satisfied.

To prove point ii), take any pair $(i, j) \in B \times S$ such that $j \neq i'$. Then, with a similar reasoning as in equation (4), we have

$$\beta_i(w_a) + \beta_j(w_a) = \frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i, j\}} \left(w_a(T \cup \{i, j\}) - w_a(T) \right). \tag{5}$$

By superadditivity of the assignment game, for any coalition $T \subseteq N \setminus \{i, j\}$ it holds that $w_a(T \cup \{i, j\}) \geq w_a(T) + a_{ij}$. As a direct consequence,

$$\frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i, j\}} \left(w_a(T \cup \{i, j\}) - w_a(T) \right) \geq \frac{1}{2^{n-2}} \cdot 2^{n-2} a_{ij} = a_{ij}, \tag{6}$$

showing that condition ii) is satisfied.

This completes the proof of the "if" part.

The "only if" part. Consider an assignment game $(B \cup S, w_a)$, such that its Banzhaf value payoff vector $\beta(w_a) = (\beta_t(w_a))_{t \in B \cup S}$ belongs to the core of $(B \cup S, w_a)$. We prove that the valuation matrix a satisfies the dominant diagonal and the doubly dominant diagonal properties. Without loss of generality, assume that an optimal matching for the whole market is on the main diagonal of the valuation matrix, see Remark 2.1.

First, we will show that the dominant diagonal property must be satisfied. Note that, as $\beta(w_a)$ belongs to the core, for any buyer-seller pair (i, i') where $a_{ii'}$ belongs to the main diagonal, using (4) and (3) it must hold that:

$$\beta_i(w_a) + \beta_{i'}(w_a) = \frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i, i'\}} \left(w_a(T \cup \{i, i'\}) - w_a(T) \right) = a_{ii'}. \tag{7}$$

Suppose, by way of contradiction that the dominant diagonal property is not satisfied, i.e., there is a pair $(i,j) \in B \times S$ with $i' \neq j$ such that

$$a_{ii'} < a_{ij}. \tag{8}$$

Because of the previous inequality (8), we have

$$w_a(\{i,i',j\}) - w_a(\{j\}) = a_{ij} - 0 = a_{ij}, \tag{9}$$

which means with (4) that

$$\beta_i(w_a) + \beta_{i'}(w_a) = \frac{1}{2^{n-2}} \sum_{T\subseteq N\setminus\{i,i'\},\ T\neq\{j\}} (w_a(T\cup\{i,i'\}) - w_a(T)) + \frac{1}{2^{n-2}} a_{ij}. \tag{10}$$

On the one hand, expressions (7), (8) and (10) imply that

$$\frac{1}{2^{n-2}} \sum_{T\subseteq N\setminus\{i,i'\},\ T\neq\{j\}} (w_a(T\cup\{i,i'\}) - w_a(T))$$

$$= \frac{1}{2^{n-2}} \sum_{T\subseteq N\setminus\{i,i'\}} (w_a(T\cup\{i,i'\}) - w_a(T)) - \frac{1}{2^{n-2}} w_a(\{i,i',j\}) + \frac{1}{2^{n-2}} w_a(\{j\})$$

$$= a_{ii'} - \frac{1}{2^{n-2}} a_{ij} < \left(1 - \frac{1}{2^{n-2}}\right) a_{ii'} = \left(\frac{2^{n-2}-1}{2^{n-2}}\right) a_{ii'}, \tag{11}$$

where the first equality follows by rearranging terms, the second equality follows from (7) and (9) and the inequality follows from (8). On the other hand, for every coalition $T \subseteq N \setminus \{i,i'\}$ such that $T \neq \{j\}$, by superadditivity we have that $w_a(T\cup\{i,i'\}) \geq w_a(T) + a_{ii'}$. This contradicts inequality (11) and thus also contradicts our initial assumption that the Banzhaf value payoff vector belongs to the core. Thus, the dominant diagonal property must be satisfied.

Second, we prove that if the Banzhaf value payoff vector belongs to the core of $(B\cup S, w_a)$, then the valuation matrix a satisfies the doubly dominant diagonal property. Note that, as $\beta(w_a)$ belongs to the core, expression (7) must be satisfied. Suppose, by way of contradiction that the doubly dominant diagonal condition is not satisfied, i.e., there are agents $i,l \in B$ with $i \neq l$ and $i', j \in S$ with $i' \neq j$ satisfying

$$a_{ii'} + a_{lk} < a_{ik} + a_{li'}, \tag{12}$$

where $a_{ii'}$ belongs to the main diagonal of the valuation matrix and none of the other entries does. Because of the previous inequality (12), we have that

$$w_a(\{i,i',l,k\}) - w_a(\{l,k\}) = a_{ik} + a_{li'} - a_{lk} > a_{ii'}, \tag{13}$$

which means with (4) that

$$\beta_i(w_a) + \beta_{i'}(w_a) = \frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i,i'\},\ T \neq \{l,k\}} (w_a(T \cup \{i,i'\}) - w_a(T)) + \frac{1}{2^{n-2}}(a_{ik} + a_{li'} - a_{lk}). \tag{14}$$

Similar as above for the dominant diagonal property (see (11)), expressions (7), (13) and (14) imply that

$$\frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i,i'\},\ T \neq \{l.k\}} (w_a(T \cup \{i,i'\}) - w_a(T))$$

$$= \frac{1}{2^{n-2}} \sum_{T \subseteq N \setminus \{i,i'\}} (w_a(T \cup \{i,i'\}) - w_a(T)) - \frac{1}{2^{n-2}} w_a(\{i,i',l,k\}) + \frac{1}{2^{n-2}} w_a(\{l,k\})$$

$$= a_{ii'} - \frac{1}{2^{n-2}}(a_{ik} + a_{li'}) + \frac{1}{2^{n-2}} a_{lk} < \left(1 - \frac{1}{2^{n-2}}\right) a_{ii'} = \left(\frac{2^{n-2}-1}{2^{n-2}}\right) a_{ii'}, \tag{15}$$

which follows similar as (11) where the first equality follows by rearranging terms, but the second equality now follows from (14) and (9) and the inequality follows from (12). On the other hand, again similar as the proof of the dominant diaginal property above, for every coalition $T \subseteq N \setminus \{i,i'\}$ such that $T \neq \{l,k\}$, by superadditivity we have that $w_a(T \cup \{i,i'\}) \geq w_a(T) + a_{ii'}$. This contradicts inequality (15) and thus also contradicts our initial assumption that the Banzhaf value payoff vector belongs to the core. Thus, the doubly dominant diagonal property must be also satisfied.

This completes the proof of the "only if" part. □

Our previous result establishes a significant link between marginalism and the core and highlights a key difference between the Banzhaf value and the Shapley value. While exactness is a sufficient condition for the Shapley value to be core stable, it is a sufficient but also necessary condition for the Banzhaf value payoff vector to be an element of the core.

In light of our previous result, a natural question is: if the payoff vectors associated with the Banzhaf and Shapley values belong to the core, do they coincide with the fair-division point? The following example answers this question in the negative.

Example 3.4. *Consider a two-sided market with money (B, S, a) with three buyers, $B = \{1, 2, 3\}$ and three sellers, $S = \{1', 2', 3'\}$ and the following valuation matrix*

$$a = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 1 & 4 & 7 \end{pmatrix}.$$

Note that it satisfies the dominant diagonal property and the doubly dominant diagonal property, and as a consequence, the payoff vectors associated with the Banzhaf and Shapley values belong to the core of the game. In fact, buyer 1, represented by the first *row in the matrix, receives $\beta_1(w_a) = 2.41$ when the Banzhaf solution concept is applied, but when the Shapley value is considered, we obtain $Sh_1(w_a) = 2.43$. However, the fair-division point allocates $\tau_1(a) = 2.5$ to buyer 1. So, in this example, the Banzhaf value allocates a worse payoff to buyer 1 than the fair-division point.*

In this example, the fair-division point equally splits the trade gain $a_{11'}$ between buyer 1 and seller $1'$ each of them getting 2.5. Note that buyer 1 cannot generate any trade gain when matched to either seller 2' or seller 3', meaning that his marginal contribution to sellers 2' and 3' is zero. On the other hand, while seller 1' cannot generate any trade gain when matched with buyer 2, this seller could make a positive trade gain with buyer 3. Given this, from a fair perspective grounded on marginalism, seller 1' should receive a higher payoff than buyer 1. In this example, this is consistent with the allocation derived from the Banzhaf and Shapley values, but not with the fair-division point.

We come back to the comparison between the Shapley value, Banzhaf value and fair-division point in Section 5, where we provide sufficient conditions for them to coincide. But first we introduce another class of solutions for assignment markets that may combine marginalism with core stability.

4 Intra-sector marginalism and the core

In van den Brink & Pintér (2015), the authors introduce another solution concept for assignment games that is based on marginalism. This solution focuses on two coalitional games, played by the agents of one sector, under the assumption that all the agents of the opposite sector are available. The *buyers-come-first* game associated with a market (B, S, a) is the game (S, w_a^B) with the set of sellers S as the player set and the worth of coalitions of sellers defined by

$$w_a^B(S') = w_a(B \cup S') \text{ for all } S' \subseteq S.$$

Thus, in this game the worth of every coalition of sellers equals the total trade gain that is attainable by this set of sellers assuming that all buyers are present and available to trade with.

Similarly, the *sellers-come-first* game associated with a market (B, S, a) is the game (B, w_a^S) with the set of buyers B as the player set and the worth of coalitions of buyers defined by

$$w_a^S(B') = w_a(S \cup B') \text{ for all } B' \subseteq B.$$

The solution proposed in van den Brink & Pintér (2015), which we refer to as the intra-sector Shapley value, allocates to each player in an assignment game $(B \cup S, w_a)$ a weighted average of its marginal contributions to all coalitions of agents in its own sector, assuming that all agents of the other sector are already present. More precisely, in any assignment game $(B \cup S, w_a)$, the intra-sector Shapley value Sh^I assigns to any buyer half of its Shapley value payoff in the sellers-come-first game, and assigns to every seller half of its Shapley value payoff in the buyers-come-first game. Thus, it is defined by

$$Sh_i^I(w_a) = \frac{1}{2} \sum_{T \subseteq B \setminus \{i\}} \frac{|T|!(|B|-|T|-1)!}{|B|!} \left(w_a^S(T \cup \{i\}) - w_a^S(T)\right) \text{ for every } i \in B,$$

and

$$Sh_j^I(w_a) = \frac{1}{2} \sum_{T \subseteq S \setminus \{j\}} \frac{|T|!(|S|-|T|-1)!}{|S|!} \left(w_a^B(T \cup \{j\}) - w_a^B(T)\right) \text{ for every } j \in S.$$

Notice that, since $w_a^S(B) = w_a^B(S) = w_a(B \cup S)$, by taking half of the Shapley values in the buyers-come-first, respectively sellers-come-first, game the intra-sector Shapley value Sh^I is efficient. Denoting the payoff vector resulting from the intra-sector Shapley value by $Sh^I(w_a)$, this can also be expressed as $Sh^I(w_a) = (\frac{1}{2}(Sh(w_a^S)), \frac{1}{2}(Sh(w_a^B)))$, and thus $Sh_i^I(w_a) = \frac{1}{2}Sh_i(w_a^S)$ for all $i \in B$ and $Sh_j^I(w_a) = \frac{1}{2}Sh_j(w_a^B)$ for all $j \in S$.[13]

Applying this same approach using the Banzhaf value, we also define the *intra-sector Banzhaf value* as the solution β^I that in every assignment game $(B \cup S, w_a)$ assigns to any buyer half of its Banzhaf value payoff in the sellers-come-first game, and assigns to every seller half of its Banzhaf value payoff in the buyers-come-first game, and thus is defined by

$$\beta_i^I(w_a) = \frac{1}{2^{|B|}} \sum_{T \subseteq B \setminus \{i\}} \left(w_a^S(T \cup \{i\}) - w_a^S(T)\right) \text{ for every } i \in B,$$

and

$$\beta_j^I(w_a) = \frac{1}{2^{|S|}} \sum_{T \subseteq S \setminus \{j\}} \left(w_a^B(T \cup \{j\}) - w_a^B(T)\right) \text{ for every } j \in S.$$

We denote by $\beta^I(w_a)$ the payoff vector resulting from the intra-sector Banzhaf value, which can also be expressed as $\beta^I(w_a) = (\frac{1}{2}(\beta(w_a^S)), \frac{1}{2}(\beta(w_a^B)))$. Although the intra-sector Banzhaf value need not be efficient, to stay parallel with the definition of the intra-sector Shapley value, also here we assign to every buyer, respectively seller, half of its Banzhaf value in the sellers-come first, respectively buyers-come-first, games, which explains the terms $\frac{1}{2} \cdot \frac{1}{2^{|B|-1}} = \frac{1}{2^{|B|}}$, respectively $\frac{1}{2} \cdot \frac{1}{2^{|S|-1}} = \frac{1}{2^{|S|}}$.

These two solution concepts reflect the idea of marginalism, but applying them to the two sectors of the market separately, assuming that all agents of the other sector are already present in the market. One might say that these games model the competition in one sector of the market. Just like the Shapley value does not coincide with the Banzhaf value for general assignment games, the same is true for their intra-sector versions as shown in the following example.

[13] For the ease of exposition, in some cases we say intra-sector Shapley value to refer to the payoff vector as just defined. The same applies for the intra-sector Banzhaf value.

Example 4.1. *Consider a two-sided market with money* (B, S, a) *with three buyers,* $B = \{1, 2, 3\}$ *and three sellers,* $S = \{1', 2', 3'\}$ *and valuation matrix*

$$a = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 2 & 5 \end{pmatrix}.$$

The intra-sector Banzhaf value allocates $\frac{5}{8}$ *to buyer 1, whereas the intra-sector Shapley value allocates* $\frac{7}{12}$ *to the same buyer, showing that these two solutions, in general, do not coincide with each other. In addition, it is easily seen that, in any core allocation, buyer 1 receives a payoff of zero, as his marginal contribution to the whole market is precisely zero.*

This example also shows that neither the intra-sector Shapley value nor the intra-sector Banzhaf value belong to the core of the corresponding game. In fact, whenever a buyer i is optimally matched to a seller j and their joint value is $a_{ij} = 0$, they will receive a payoff of zero in every core payoff vector, as mentioned in the previous example. However, if this buyer makes a strictly positive contribution in a submarket of its sector game, then the application of the intra-sector Shapley value or the intra-sector Banzhaf value would result in a strictly positive payoff for the buyer in question, which thus deviates from any core allocation.

Although the intra-sector Shapley value generally fails to lie in the core, the dominant diagonal property turns out to be sufficient to obtain core stability. In fact, we can go further and introduce a family of intra-sector Shapley values by taking any convex combination of the buyer-come-first and seller-come-first Shapley values

Definition 4.2. *Let* $\alpha \in [0, 1]$. *The* α*-intra-sector Shapley value is defined as* $Sh^{\alpha}(w_a) = (\alpha(Sh(w_a^S), (1-\alpha)Sh(w_a^B))$ *for any given assignment game* $(B \cup S, w_a)$.

It turns out that the dominant diagonal property is necessary and sufficient for any α-intra-sector Shapley value to belong to the core. Since this is a weaker condition than exactness of the game (which also requires the doubly dominant diagonal property) this makes the α-intra-sector Shapley values more attractive than the Shapley value for two-sided assignment games when one is concerned about core stability.

Theorem 4.3. *Let* (B, S, a) *be a two-sided market with money and* $(B \cup S, w_a)$ *be the corresponding assignment game. For any* $\alpha \in [0, 1]$, $Sh^{\alpha}(w_a) \in C(w_a)$ *if and only if the valuation matrix* a *satisfies the dominant diagonal property.*

Proof. The "only if" part. Suppose that for any $\alpha \in [0, 1]$, $Sh^{\alpha}(w_a) = (\alpha(Sh(w_a^S), (1-\alpha)Sh(w_a^B))$ is in the core of (B, S, a). We prove that a satisfies the dominant diagonal property. Without loss of generality, assume that an optimal matching is located over the main diagonal of the valuation matrix, see Remark 2.1.

Because the core is a convex set, it is sufficient to consider the two extreme cases: $\alpha = 1$ and $\alpha = 0$. Take any $\mu \in \mathcal{M}_a(B, S)$. Consider first the case $\alpha = 1$. Then, by assumption $Sh^{\alpha}(w_a) = Sh^1(w_a) = (Sh(w_a^S), 0)$ belongs to the core. Notice that $Sh^1_{j'}(w_a) = 0$ for all $j' \in S$. Since, $(Sh(w_a^S), 0)$ is in the core and $Sh^1_{i'}(w_a) = 0$, we have from (3) that for each buyer $i \in B$, $Sh^1_i(w_a) = Sh_i(w^S_a) = a_{ii'}$. Take now any $k \in S$. Since $(Sh(w^S_a), 0)$ is in the core, we have that $Sh^1_i(w_a) + Sh^1_k(w_a) \geq a_{ik}$ for all $i \in B$. Moreover, since $Sh^1_i(w_a) = a_{ii'}$ and $Sh^1_k(w_a) = 0$ for all $k \in S$, we conclude that $a_{ii'} \geq a_{ik}$ for all $k \in S$, and thus the dominant diagonal property is satisfied in this case. Considering the case $\alpha = 0$, by similar arguments as the case $\alpha = 1$ and using the fact that the payoff vector $Sh^\alpha(w_a) = Sh^0(w_a) = (0, Sh(w^B_a))$ belongs to the core, it can be shown that for any seller $j' \in S$, $Sh^0_{j'}(w_a) = a_{jj'} \geq a_{tj'}$ for all $t \in B$. This shows that also in this case, the valuation matrix a satisfies the dominant diagonal property.

The "if" part. Let a satisfy the dominant diagonal property. We prove that for any $\alpha \in [0,1]$, $Sh^\alpha(w_a) = (\alpha(Sh(w^S_a), (1-\alpha)Sh(w^B_a))$ is in the core of (B, S, a). Similar as in the "only if" part, because the core is a convex set, it is sufficient to consider the two extreme cases: $\alpha = 1$ and $\alpha = 0$. Take any $\mu \in \mathcal{M}_a(B,S)$. First consider the case that $\alpha = 1$. For any $i \in B$, let us denote by $i' \in S$ the seller such that $a_{ii'}$ is in the main diagonal. By assumption of the dominant diagonal property, $a_{ii'} \geq a_{ij}$ for all $j \in S$. Again because of the dominant diagonal property, for every $B' \subseteq B$, it holds that

$$w^S_a(B') = w_a(B' \cup S) = \sum_{i \in B'} a_{ii'}. \tag{16}$$

As an immediate consequence of (16), for any $t \in B$ and any $B' \subseteq B \setminus \{t\}$, we have that

$$w^S_a(B' \cup \{t\}) = w_a((B' \cup \{t\}) \cup S) = \sum_{i \in B' \cup \{t\}} a_{ii'} = \sum_{i \in B'} a_{ii'} + a_{tt'}.$$

Hence,

$$w^S_a(B' \cup \{t\}) - w^S_a(B') = a_{tt'}.$$

Then, the marginal contribution of any buyer i to any coalition $B' \subseteq B \setminus \{i\}$ in the sellers-come-first situation is equal to $a_{ii'}$. We deduce then that $Sh_i(w^S_a) = a_{ii'}$. Therefore, $Sh^\alpha(w_a) = (Sh(w^S_a), 0)$ is the buyers-optimal core allocation $(\overline{u}(a), \underline{v}(a))$ for (B, S, a). In a similar way, we can show that $Sh^\alpha(w_a) = (0, Sh(w^B_a))$ is the sellers-optimal core allocation when $\alpha = 0$.

By the convexity of the core, any convex combination of $(Sh(w^S_a), 0)$ and $(0, Sh(w^B_a))$ is also a core element, which completes the proof. □

In a similar way, we can define the class of all convex combinations of the buyer-come-first and seller-come-first Banzhaf values.

Definition 4.4. *Let $\alpha \in [0,1]$. The α-intra-sector Banzhaf value is defined as $\beta^\alpha(w_a) = (\alpha(\beta(w^S_a), (1-\alpha)\beta(w^B_a))$ for any given assignment game $(B \cup S, w_a)$.*

A similar result as Theorem 4.3 holds for all α-intra-sector Banzhaf values. We state the following result without proof because it follows analogous arguments as in the proof of our previous result regarding the intra-sector Shapley value.

Theorem 4.5. *Let (B, S, a) be a two-sided market with money and $(B \cup S, w_a)$ be the corresponding assignment game. For any $\alpha \in [0,1]$, $\beta^\alpha(w_a) \in C(w_a)$ if and only if the valuation matrix a satisfies the dominant diagonal property.*

5 Coincidence with the fair-division point

As part of the proofs of Theorems 4.3 and 4.5, we showed that whenever a valuation matrix a satisfies the dominant diagonal property, then both the intra-sector Shapley and Banzhaf values split the joint surplus $a_{ii'}$ equally among buyer i and seller i' if (i, i') is a matched buyer-seller pair in an optimal matching. Since the fair-division point splits the trade surplusses in the same way, as a corollary we have that the dominant diagonal property also guarantees that both the intra-sector Shapley and Banzhaf values coincide with the fair-division point of the assignment game.

Corollary 5.1. *Let (B, S, a) be a two-sided market with money and $(B \cup S, w_a)$ be the corresponding assignment game. If the valuation matrix a satisfies the dominant diagonal property, then $(\frac{1}{2}(Sh(w_a^S), \frac{1}{2}Sh(w_a^B)) = ((\frac{1}{2}(\beta(w_a^S), \frac{1}{2}\beta(w_a^B))) = \tau(a)$.*

Theorem 3.3 and the main result of Hoffmann & Sudhölter (2007), Theorem 3.1, imply that when the valuation matrix satisfies the dominant diagonal and doubly dominant diagonal properties, then the payoff vectors associated with the Shapley and Banzhaf values are both in the core of the assignment game. However, as shown by Example 3.4, they do not necessarily coincide when they are in the core, and even more, they do not coincide with the fair division point. A natural question is then, under which conditions the payoff vectors associated with the Shapley and Banzhaf values coincide with the fair division point. Before providing an answer to such a question, we need to introduce a notion of symmetry for a two-sided market with money.

A valuation matrix is symmetric when there are as many buyers as sellers and the value generated by matching a buyer with a seller is mirrored for similar transactions between corresponding agents on the other side of the market. In other words, the valuation matrix is symmetric when, for any two buyers i and k, the value of matching buyer i with seller k' is equal to the value of matching buyer k with seller i'.

Definition 5.2. *Let (B, S, a) be a two-sided market with money, we say that the valuation matrix a is symmetric if $|B| = |S|$ and for every $i, k \in B$ with $i \neq k$, $a_{ik'} = a_{ki'}$.*

This notion of symmetry reflects some sort of evenness in the market. It can be interpreted as a market in which the relationship between buyers and sellers is reciprocal in terms of the value they create and the gains from trade are not dependent on the specific roles of buyers or sellers, but rather on their positions on their side of the market. This notion of a symmetric matrix does not imply that a buyer and a seller are symmetric as players in a coalitional game. That is, while the values in the matrix reflect symmetry, this does not suggest that the roles or contributions of the buyers and sellers in the market are identical or interchangeable.

Proposition 5.3. *Let (B, S, a) be a two-sided market with money and $(B \cup S, w_a)$ be the corresponding assignment game. If the valuation matrix a is symmetric and satisfies the dominant diagonal and the doubly dominant diagonal properties, then $\beta(w_a) = Sh(w_a) = \tau(a)$.*

Proof. First, assume that a valuation matrix a is symmetric and satisfies the dominant diagonal and doubly dominant diagonal properties. Second, assume that an optimal matching is located on the main diagonal, see Remark 2.1. Third, it can be deduced from expression (1) in Solymosi & Raghavan (2001), that under the dominant diagonal property, the fair-division point allocates to each buyer $i \in B$, $\tau_i(a) = \frac{1}{2}a_{ii'}$ and the other half $\frac{1}{2}a_{ii'}$ goes to the corresponding seller i', $\tau_{i'}(a) = \tau_i(a) = \frac{1}{2}a_{ii'}$. Fourth, we will prove $\beta(w_a) = \tau(a)$, and in the fifth step, we will finish the proof by showing that $\beta(w_a) = \tau(a) = Sh(w_a)$.

To show the fourth step, as the valuation matrix satisfies the dominant diagonal and doubly dominant diagonal properties, $\beta(w_a) \in C(w_a)$. As described by expression (2), we know that $\beta_i(w_a) + \beta_{i'}(w_a) = a_{ii'}$ for each pair $(i, i') \in B \times S$, where $a_{ii'}$ belongs to the main diagonal. What remains to show is $\beta_i(w_a) = \beta_{i'}(w_a)$. This can be proven using "symmetric" coalitions. Take any buyer i, any coalition $T \subseteq (B \setminus \{i\}) \cup S$ and the seller i' so that $a_{ii'}$ is on the main diagonal. Note that we can define $T' \subseteq B \cup (S \setminus \{i'\})$ by

$$T' = \{j \in B \mid j' \in T\} \cup \{k' \in S \mid k \in T\}.$$

Clearly, we have $|T| = |T'|$. Since the matrix a is symmetric, the marginal contribution of i to any coalition T equals that of i' to T'. Hence, the average contributions of buyer i and seller i' are the same, and thus we have that:

$$\beta_i(w_a) = \beta_{i'}(w_a),$$

which implies that the Banzhaf value payoff vector coincides with the fair-division point.

Finally, as fifth step we prove that for a symmetric matrix that satisfies the dominant diagonal and doubly dominant diagonal properties, the Shapley value payoff vector coincides with the fair-division. As the Shapley value payoff vector belongs to the core when the dominant and doubly dominant diagonal properties hold, similarly to the Banzhaf value, it suffices to show that

$$Sh_i(w_a) = Sh_{i'}(w_a),$$

for buyer i and seller i', where $a_{ii'}$ belongs to the main diagonal. Recall that the Shapley value considers all permutations of agents and computes each agent's expected marginal contribution when joining a coalition. Take any buyer i and any permutation of agents π, assume that buyer i is in position l. Now, take seller i' and define the permutation of agents π' where, for all $k \in \{1, 2, \ldots, n\}$,

$$\pi'(k) = \begin{cases} j \in B & \text{if} \quad \pi(k) = j' \in S, \\ t' \in S & \text{if} \quad \pi(k) = t \in B. \end{cases}$$

Since the matrix a is symmetric, the marginal contribution of buyer i to his predecessors in π is equal to that of i' to π'. As all permutations are equally likely, we have that

$$Sh_i(w_a) = Sh_{i'}(w_a).$$

Thus the payoff vectors associated with the Shapley and Banzhaf values coincide with the fair-division point. □

Symmetry is a natural and easily verifiable condition under which the fair-division point of an assignment game coincides with both payoff vectors associated with the

Shapley and Banzhaf values, respectively. Our next result complements a related result from Hoffmann & Sudhölter (2007) (their Proposition 3.5), where the authors characterize the class of two-buyer, two-seller markets for which the Shapley value payoff vector lies in the core. Specifically, the following result characterizes the exact conditions under which the fair-division point coincides with both payoff vectors associated with the Shapley and Banzhaf values in assignment games with two buyers and two sellers.

Proposition 5.4. *Let (B, S, a) be a two-sided market with money and $(B \cup S, w_a)$ be the corresponding assignment game with $|B| = |S| = 2$. Then, $\beta(w_a) = Sh(w_a) = \tau(a)$ if and only if the valuation matrix a satisfies the dominant diagonal property and $a_{12'} = a_{21'}$.*

Proof. The "if part" is a direct consequence of Proposition 5.3 and the fact that the doubly dominant diagonal property has no meaning if $|B| = |S| = 2$.

Now, we show the "only if part". First, assume that the Banzhaf value payoff vector coincides with the fair-division point. Because the fair-division point is always in the core, the Banzhaf value payoff vector must also lie in the core. Then, by Theorem 3.3, the valuation matrix a must satisfy the dominant diagonal property.

Under the dominant diagonal property and in a market with two buyers and two sellers, as mentioned in Proposition 5.3, the fair-division point allocates to each buyer $i \in B$ a payoff of $\tau_i(a) = \frac{1}{2}a_{ii'}$ and the other half $\frac{1}{2}a_{ii'}$ goes to the corresponding seller i'. Hence, $\beta_i(w_a) = \beta_{i'}(w_a)$.

We now show that $a_{12'} = a_{21'}$ must hold. Suppose, by way of contradiction, that $a_{12'} \neq a_{21'}$. Without loss of generality, consider the market (B, S, a) with two buyers, $B = \{1, 2\}$ and two sellers, $S = \{1', 2'\}$ and valuation matrix:

$$a = \begin{pmatrix} b & c \\ d & e \end{pmatrix},$$

with $b \geq c$, $b \geq d$, $e \geq c$, $e \geq d$ and $c > d$. The matrix satisfies the dominant diagonal property, but $a_{12'} \neq a_{21'}$. The Banzhaf value payoff vector for buyer 1 and seller 1 can be expressed as[14]

$$\beta_1(w_a) = \frac{4b - d + c}{8}, \qquad \beta_{1'}(w_a) = \frac{4b - c + d}{8}.$$

Since $c > d$, we get $\beta_1(w_a) > \beta_{1'}(w_a)$, contradicting our assumption that the Banzhaf value equals the fair-division point. Hence, $a_{12'} = a_{21'}$ must hold.

Now, we show that under the same assumptions, the Shapley value payoff vector also coincides with the fair-division point and the Banzhaf value payoff vector under the same conditions.

Since we are assuming that the Banzhaf value payoff vector equals the fair-division point, the dominant diagonal property holds by Theorem 3.3. As before, under this property, the fair-division point assigns $\tau_i(a) = \frac{1}{2}a_{ii'}$ to any buyer $i \in B$ and $\tau_{i'}(a) = \frac{1}{2}a_{ii'}$

[14]This can be seen by averaging the marginal contributions $w_a(1) - w_a(\emptyset) = 0$, $w_a(1,2) - w_a(2) = 0$, $w_a(1,1') - w_a(1') = b$, $w_a(1,2') - w_a(2') = c$, $w_a(1,1',2) - w_a(1',2) = b - d$, $w_a(1,1',2') - w_a(1',2') = 0$, $w_a(1,2,2') - w_a(2,2') = b$ and $wa_(1,1',2,2') - w_a(1',2,2') = b$.

to the corresponding seller. Thus it must be that $Sh_i(w_a) = Sh_{i'}(w_a)$ to any buyer $i \in B$ and the corresponding seller.

We now show that $a_{12'} = a_{21'}$ must also hold. Suppose, by contradiction that $a_{12'} \neq a_{21'}$. Without loss of generality, consider the previous market (B, S, a) with two buyers, $B = \{1, 2\}$ and two sellers, $S = \{1', 2'\}$ and valuation matrix:

$$a = \begin{pmatrix} b & c \\ d & e \end{pmatrix},$$

with $b \geq c$, $b \geq d$, $e \geq c$, $e \geq d$ and $c > d$. If we calculate the Shapley value for buyer 1 and seller 1, we have

$$\beta_1(w_a) = \frac{1}{2}b + \frac{1}{12}(c - d), \qquad \beta_{1'}(w_a) = \frac{1}{2}b + \frac{1}{12}(d - c).$$

Since $c > d$, we get $Sh_1(w_a) > Sh_{1'}(w_a)$, contradicting our assumption that the Shapley value payoff vector equals the fair-division point and the Banzhaf value payoff vector. Hence, $a_{12'} = a_{21'}$ must hold. Therefore, both the dominant diagonal property and $a_{12'} = a_{21'}$ are necessary for the coincidence of the fair-division point with both payoff vectors associated with the Shapley and Banzhaf values. This completes the proof. □

6 Concluding remarks

This paper reconciles marginalism with core stability in the context of two-sided markets with money. Our first main result shows that the Banzhaf value payoff vector is guaranteed to be within the core if and only if the valuation matrix satisfies both the dominant diagonal and doubly dominant diagonal properties. It highlights a distinct difference between the Banzhaf and Shapley values: while those conditions are sufficient, but not necessary for the Shapley value to lie in the core, they are both necessary and sufficient for the Banzhaf value to be core stable.

Additionally, we introduced the concept of intra-sector solutions, specifically the intra-sectors Shapley value and the intra-sectors Banzhaf value, which account for the competition within each side of the market. We showed that all these solutions assign a payoff vector in the core if and only if the valuation matrix satisfies the dominant diagonal property.

Two main general conclusions that can be drawn from these results are that there are applications, such as two-sided markets with money, where (i) the stability of the Banzhaf value implies the stability of the Shapley value, but not the other way around, and (ii) modifications of the Shapley or Banzhaf value using the characteristics of the application, such as the division of market participants in buyers and sellers, are stable in a wider set of markets than the 'standard' solutions are.

As a future project, we propose to provide axiomatic characterizations of the Banzhaf value and the intra-sector solutions for classes of two-sided markets with money or assignment games. The Banzhaf value is often axiomatized using axioms that relate to collusion or amalgamation, see e.g Haller (1994) and van den Brink (2012). However, a difficulty with such axioms for assignment games is that a collusion or amalgamation of two players in an assignment game may result in a game that is not an assignment game. Regarding the intra-sector Shapley value, this solution has many axioms in common with the Shapley value. For example, it satisfies Pareto optimality (also known as efficiency), the null player property, the equal treatment property and additivity, axioms that characterize the Shapley value on the domain of all coalitional games with transferable utility. That is, on the domain of assignment games, the aforementioned axioms characterize neither the Shapley value nor its intra-sector version.

Also related to this problem, Núñez & Rafels (2009) show that to any assignment game we can associate a unique exact assignment game on the same player set which core is a translation of the core of the initial game. Hence, the translation of the Shapley and Banzhaf values of this exact representative would lead to a core selection of the assignment game, which could also be compared axiomatically with the Shapley and Banzhaf values of the assignment game.

The underlying paper considers possibilities to combine marginalism with core stability. Another economic principle is egalitarianism as reflected by the equal division solution that allocates the worth of the grand coalition equally over all players, and is axiomatized in van den Brink (2007). There are various classes of solutions that combine the Shapley value with the equal division solution such as the class of egalitarian Shapley values introduced in Joosten (1996) and axiomatized using a weak version of monotonicity in van den Brink & Funaki (2013) and Casajus & Huettner (2014). This raises several related research questions for assignment games. Since we showed that the Banzhaf value is efficient for exact assignment games, we might consider to study combinations of the Shapley value as well as the Banzhaf value with the equal division solution for assignment games. An intermediate form of equal division is pairwise equal division where the valuation of a buyer for the object of a seller is equally split between this seller and buyer if they are matched in an (assume unique) optimal matching. From the proof of Proposition 5.3, we know that if the valuation matrix is symmetric and satisfies the dominant and doubly dominant diagonal properties, this allocation is the fair division point and equals the payoff vector assigned by the Shapley and Banzhaf values.

As shown by Csóka *et al.* (2009), another example of exact games are the risk allocation games with no aggregate uncertainty. Contrary to the exact games studied in the underlying paper, for these games the Shapley value does not assign a core stable payoff vector. As noted by Csóka & Pintér (2016), it is even impossible to have a core stable and marginalist (in the sense of satisfying Young (1985)'s strong monotonicity) allocation rule for this class of exact games.

Finally, besides axiomatization, another question for future research is to find a strategic implementation for the solutions considered in this paper, specifically for assignment games. A first possibility is to see if the strategic implementation of the Shapley value in Pérez-Castrillo & Wettstein (2001) can be modified for this context.

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