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On the Correlations in Linearized Multivariate Stochastic Volatility Models

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Abstract

In the analysis of multivariate stochastic volatility models, many estimation procedures begin by transforming the data, taking the logarithm of the squared returns to obtain a linear state space model. A well-known series representation links the correlations between elements of the observation error in the actual and linearized forms of the model. This note derives a closed-form expression for the series and discusses its statistical implications. Additionally, it offers a new interpretation of the correlations in the linearized model.

Keywords: Multivariate volatility models, Quasi-maximum likelihood, Cross-correlation.

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1 Introduction

In the econometrics literature, considerable attention has been devoted to the study of time-varying volatility in financial assets. The insights from these analyses have had a substantial impact on practical applications. Notable examples include portfolio allocation (Aguilar and West 2000; Han 2006) and risk management (McNeil, Frey, & Embrechts, 2015), where the interplay between volatility components plays a crucial role.

A common approach to modeling these dynamics is through multivariate stochastic volatility (MSV) models; see the reviews by Asai, McAleer, and Yu (2006) and Chib, Omori, and Asai (2009). Let $y_t = (y_{1,t}, \ldots, y_{d,t})'$ denote the log returns of $d \in \mathbb{N}$ financial assets observed at time t, and let $x_t = (x_{1,t}, \ldots, x_{d,t})'$ represent their (unobserved) log variances. One of the most widely used MSV models, proposed by Harvey, Ruiz, and Shephard (1994), is given by

$$
\begin{aligned}
y_t &= \exp(0.5x_t) \odot \varepsilon_t^y, & \varepsilon_t^y &\sim \mathrm{N}(0, R), \\
x_{t+1} &= \mu + \Phi(x_t - \mu) + \varepsilon_t^x, & \varepsilon_t^x &\sim \mathrm{N}(0, \Sigma),
\end{aligned}
\tag{1}
$$

for $t = 1, \ldots, T$, where $T \in \mathbb{N}$, $\varepsilon_t^x, \varepsilon_t^y \in \mathbb{R}^d$, and the symbol $\odot$ denotes elementwise multiplication. The matrices Φ, Σ, and R are all $d \times d$, with R being a correlation matrix. The model is initialized as $x_1 \sim \mathrm{N}(\mu_1, \Sigma_1)$ for suitable choices of μ_1 and Σ_1.

The above specification is standard in multivariate volatility modeling and is commonly referred to as the "basic MSV model" (Asai et al., 2006; Chib et al., 2009). The model belongs to the class of nonlinear state space models (Durbin & Koopman, 2012), for which parameter and state estimation is typically challenging. A straightforward solution is to linearize the model by applying the transformation $\widetilde{y}_{i,t} = \log(y_{i,t}^2)$ for $i = 1, \ldots, d$, as proposed independently by Nelson (1988, Ch. 1) in the univariate case and by Harvey et al. (1994) in the multivariate setting. Using ι to denote a $d \times 1$ vector of ones, this yields

$$
\begin{aligned}
\widetilde{y}_t &= \omega\iota + x_t + \widetilde{\varepsilon}_t^y, & \widetilde{\varepsilon}_{i,t}^y &= \log|\varepsilon_{i,t}^y|^2 - \omega, \\
x_{t+1} &= \mu + \Phi(x_t - \mu) + \varepsilon_t^x, & \omega &= \mathbb{E}\log|\varepsilon_{i,t}^y|^2 \approx -1.27.
\end{aligned}
\tag{2}
$$

The linearization in (2) forms the basis of numerous estimation procedures for the basic MSV model and its extensions. It has been employed to estimate the parameters of (1) using the Kalman filter (Nelson 1988; Harvey et al. 1994; Ruiz 1994), the method of moments (Ahsan and Dufour 2021; Ahsan and Dufour 2024), and Markov chain Monte Carlo methods (De Jong and Shephard 1995; Kim, Shephard, and Chib 1998; Kastner and Frühwirth-Schnatter 2014; Phan, Wachter, Solomon, and Kahana 2019), among others. For extensions of the model, the transformation has been used to estimate specifications incorporating leverage (Harvey and Shephard 1996; Omori, Chib, Shephard, and Nakajima 2007; Catania 2022), long-memory (Harvey 1998; Breidt, Crato, and De Lima 1998), realized volatility (Asai, 2023), and factor volatility models (Han 2006; Kastner, Frühwirth-Schnatter, and Lopes 2017).

Given the widespread use of the linearizing transformation, it is important to obtain a thorough understanding of its properties and statistical implications. From a practical perspective, the main complicating factor of the linearized form in (2) is that the observation errors, $\widetilde{\varepsilon}_t^y$, are non-Gaussian. Specifically, they follow the log-chi-squared distribution

(Harvey et al., 1994), with mean zero and variance matrix

$$\mathbb{V}\mathrm{ar}\,[\widetilde{\varepsilon}_t^y] = \frac{\pi^2}{2}\widetilde{R},$$

where $\widetilde{R}$ represents the correlation matrix of $\widetilde{\varepsilon}_t^y$. The correlation matrices $\widetilde{R}$ and R, corresponding to the observation errors in the linearized and actual forms of the MSV model, respectively, are generally not equivalent. Harvey et al. (1994, Eq. 10) derived the following relationship between the (i,j)-th element of the matrix $\widetilde{R}$, with $i, j \in \{1, \dots, d\}$, and its corresponding element in R:

$$\left|\widetilde{R}_{i,j}\right| = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(n-1)!}{n(1/2)_n}R_{i,j}^{2n}, \tag{3}$$

where $(x)_n = x(x+1)\dots(x+n-1)$ is the rising factorial. The analysis of this relationship is the subject of this note.

2 Closed-form expression

The following result provides a closed-form expression for the series representation above.

Theorem 1. *Consider the correlations $R_{i,j}$ and $\widetilde{R}_{i,j}$ for $i, j \in \{1, \dots, d\}$ as in (3). Then,*

$$\left|\widetilde{R}_{i,j}\right| = \frac{4}{\pi^2}\arcsin^2(R_{i,j}), \tag{4}$$

where $\arcsin^2(x)$ *denotes the square of the inverse sine function.*

Proof. From (3) it is seen that (4) is equivalent to

$$2\arcsin^2(x) = \sum_{n=1}^{\infty}\frac{(n-1)!}{n(1/2)_n}x^{2n} \tag{5}$$

for $|x| \le 1$, since $R_{i,j} \in [-1, 1]$. We start by noting the similarity between the right-hand side in (5) and the final expression in

$$\begin{aligned}
\arcsin^2(x) &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{\binom{2n}{n}n^2} && (6)\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}}{\frac{2n!}{(n!)^2}n^2}x^{2n}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}\left[(n-1)!\right]^2}{2n!}x^{2n}\\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n-1}\left[(n-1)!\right]^2}{n\cdot(2n-1)!}x^{2n}, && (7)
\end{aligned}$$

where the power series representation in (6) for $|x| \leq 1$ is well known (e.g., Chen, 2006, Eq. 1). Substituting (7) in (5) and comparing the terms, we note that the desired result follows if it can be shown that

$$\frac{2^{2n-1}(n-1)!}{(2n-1)!} = \frac{1}{(1/2)_n} \tag{8}$$

holds for all $n \in \mathbb{N}$. To this end, first write

$$(1/2)_n = \prod_{m=0}^{n-1}(m+1/2) = \prod_{m=0}^{n-1}\left(\frac{2m+1}{2}\right) = 2^{-n}\prod_{m=0}^{n-1}(2m+1),$$

so that (8) is equivalent to the equality in

$$H_\ell(n) \equiv \frac{2^{n-1}(n-1)!}{(2n-1)!} = \left(\prod_{m=0}^{n-1} 2m+1\right)^{-1} \equiv H_r(n).$$

We will demonstrate that the above holds for all $n \in \mathbb{N}$ through induction. For $n = 1$, we have that $H_\ell(1) = 1 = H_r(1)$. Suppose that $H_\ell(n) = H_r(n)$ holds for some $n \in \mathbb{N}$. Then,

$$H_\ell(n+1) = \frac{2^n n!}{(2n+1)!} = \frac{2n}{(2n+1)2n}H_\ell(n) = \frac{1}{2n+1}H_r(n) = H_r(n+1),$$

which shows that $H_\ell(n) = H_r(n)$ for all $n \in \mathbb{N}$, completing the proof.

■

The power series in (6) is commonly presented in the literature with either a radius of convergence $|x| \leq 1$ (Chen, 2006, p. 364) or $|x| < 1$ (e.g., Lehmer 1985 pp. 452-453, Eq. 13; Chen 2006, Eq. 5), depending on the proof. Notably, the result in (4) for the special cases $R_{i,j} = \pm 1$ can also be established by demonstrating that $|R_{i,j}| = 1 \implies |\widetilde{R}_{i,j}| = 1$ through probabilistic reasoning. For completeness, the details of this argument are provided in Appendix A.

Figure 1 shows the absolute correlation $|\widetilde{R}_{i,j}|$ as a function of the corresponding correlation $R_{i,j}$ based on (4). For comparison, the figure also presents the partial sum approximation based on (3) with 10^5 terms. The large number of terms was needed to obtain visual convergence for values of $|R_{i,j}|$ near one, which illustrates some of the numerical benefits of the closed-form expression.

Theorem 1 also enables the derivation of the following result, which offers a new interpretation of the absolute correlation value $|\widetilde{R}_{i,j}|$, linking it directly to the corresponding observation errors $\varepsilon^y_{i,t}$ and $\varepsilon^y_{j,t}$ in the actual MSV model.

Proposition 1.

$$|\widetilde{R}_{i,j}| = \left\{\mathbb{E}\left[\operatorname{sgn}(\varepsilon^y_{i,t}\varepsilon^y_{j,t})\right]\right\}^2,$$

where $\operatorname{sgn}(x) = x/|x|$ *if* $x \neq 0$, *and* $\operatorname{sgn}(x) = 0$ *otherwise.*

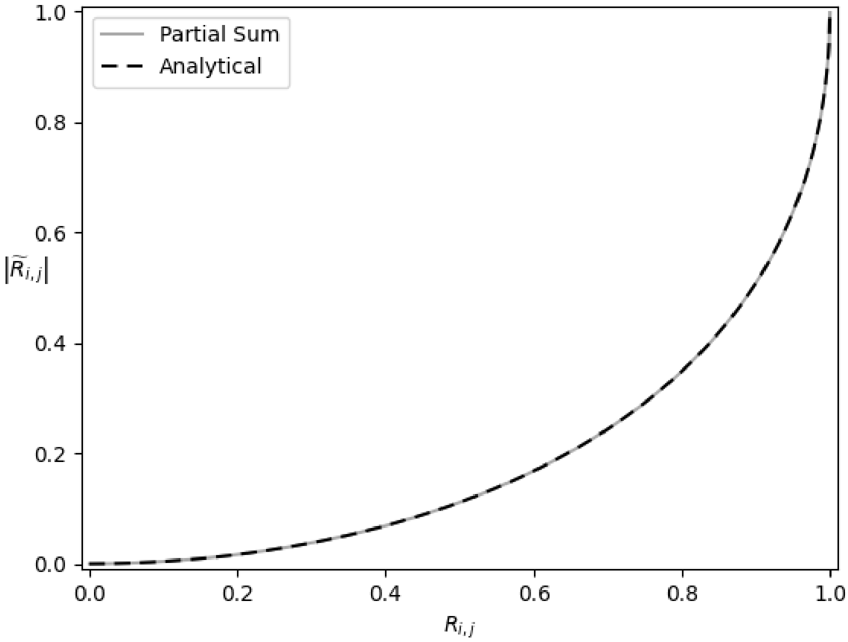

Figure 1: Absolute correlation $|\widetilde{R}_{i,j}|$ as a function of $R_{i,j}$, computed using the partial sum approximation based on (3) with 10^5 terms and analytically using the result in (4).

Proof. Using the result in (4), we have

$$|\widetilde{R}_{i,j}| = \frac{4}{\pi^2}\big[\arcsin(R_{i,j})\big]^2 = \left[\frac{2}{\pi}\arcsin(R_{i,j})\right]^2 = \left\{\mathbb{E}\left[\operatorname{sgn}(\varepsilon^y_{i,t}\varepsilon^y_{j,t})\right]\right\}^2,$$

where the final equality follows from Theorem 1 in Pelagatti and Sbrana (2024), which shows that $\mathbb{E}\left[\operatorname{sgn}(XY)\right] = \frac{2}{\pi}\arcsin(\rho)$ if X and Y are jointly normal with zero means and correlation ρ.

■

3 Inverse and asymptotic impact

Another advantage of the closed-form expression in (4) is that its inverse can be obtained analytically.

Corollary 1.

$$|R_{i,j}| = \sin\left(\frac{\pi}{2}\sqrt{|\widetilde{R}_{i,j}|}\right). \tag{9}$$

The above result allows for transforming estimates of $\widetilde{R}_{i,j}$ into estimates of $R_{i,j}$, where the sign of $R_{i,j}$ can be estimated separately, for example, based on the signs of the products $y_{i,j}y_{j,t}$ as in Harvey et al. (1994). Consequently, we find that the moment-based estimator of Ahsan and Dufour (2024), which relies on the linearization in (2), retains its closed-form property as an estimator for the actual MSV model. Furthermore, the expression in (9) provides insight into the asymptotic impact of estimating $|R_{i,j}|$

using $|\widetilde{R}_{i,j}|$. The following result applies to estimators based on the linearized form that are asymptotically normal under suitable regularity conditions (e.g., Harvey et al. 1994; Ahsan and Dufour 2024).

Lemma 1. *Let $\{y_t\}_{t=1}^{T}$ be a sample of observations generated by the MSV process in (1) with $|R_{i,j}| \in (0,1)$ for $i,j \in \{1,\ldots,d\}$. Suppose $\widetilde{r}_{i,j}$ is a corresponding estimator of $|\widetilde{R}_{i,j}|$ such that $\sqrt{T}(\widetilde{r}_{i,j} - |\widetilde{R}_{i,j}|) \xrightarrow{\mathrm{d}} \mathrm{N}(0, \widetilde{V}_{i,j})$ as $T \to \infty$. Consider the estimator $r_{i,j} = g(\widetilde{r}_{i,j})$, with $g(x) = \sin\left(\frac{\pi}{2}\sqrt{x}\right)$, defined through (9). Then,*

$$\sqrt{T}\big(r_{i,j} - |R_{i,j}|\big) \xrightarrow{\mathrm{d}} \mathrm{N}(0, V_{i,j}) \qquad \textit{as} \qquad T \to \infty,$$

with asymptotic variance

$$V_{i,j} = \frac{\pi^2}{16|\widetilde{R}_{i,j}|} \cos^2\left(\frac{\pi\sqrt{|\widetilde{R}_{i,j}|}}{2}\right) \widetilde{V}_{i,j}. \tag{10}$$

Proof. By the chain rule, we have

$$g'(x) = \frac{\pi}{4\sqrt{x}} \cos\left(\frac{\pi\sqrt{x}}{2}\right),$$

which shows that g' is continuous and is defined for $x > 0$. Since $|R_{i,j}| \in (0,1)$ was assumed, (4) implies that $|\widetilde{R}_{i,j}| \in (0,1)$, ensuring that $g(x)$ is continuously-differentiable at $x = |\widetilde{R}_{i,j}|$. Therefore, the delta method (e.g., Van der Vaart, 2000, Theorem 3.1) can be applied to obtain

$$\sqrt{T}\big(r_{i,j} - |R_{i,j}|\big) = \sqrt{T}\left[g(\widetilde{r}_{i,j}) - g\big(|\widetilde{R}_{i,j}|\big)\right] \xrightarrow{\mathrm{d}} \mathrm{N}\left(0, \left[g'\big(|\widetilde{R}_{i,j}|\big)\right]^2 \widetilde{V}_{i,j}\right) \qquad \text{as} \qquad T \to \infty,$$

which establishes the result.

■

Figure 2 shows the ratio of the asymptotic variances $V_{i,j}/\widetilde{V}_{i,j}$, defined through (10), for different values of $|\widetilde{R}_{i,j}|$. The ratio decreases with the absolute value of the correlation, the variances being equal at $|\widetilde{R}_{i,j}| \approx 0.28$. Beyond this point, estimators of $|R_{i,j}|$ defined via (9) become asymptotically more precise than the corresponding estimators of $|\widetilde{R}_{i,j}|$.

The continuous mapping theorem ensures that the boundary values $|R_{i,j}| = 0$ and $|R_{i,j}| = 1$ can be consistently estimated using (9). However, estimates of $|R_{i,j}|$ obtained from small values of $|\widetilde{R}_{i,j}|$ should be treated with caution, as the ratio of the variances increases without bound when $|\widetilde{R}_{i,j}|$ approaches zero, potentially leading to imprecise estimates of $|R_{i,j}|$. In practice, applications involving low correlations between shocks in the log returns may therefore present challenges for estimators based on the linearized form of the MSV model.

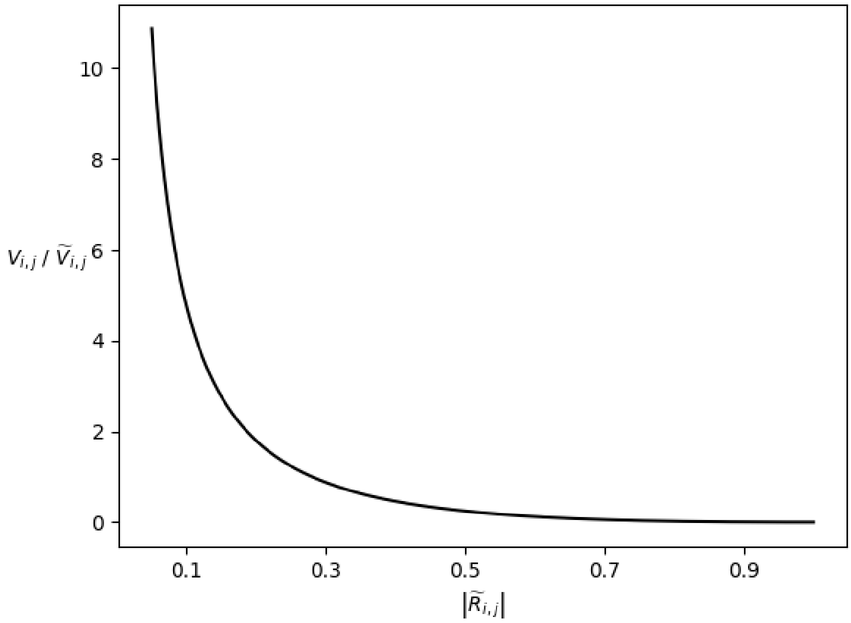

Figure 2: Ratio of asymptotic variances $V_{i,j}/\widetilde{V}_{i,j}$ defined through (10).

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Appendix A Alternative proof of Theorem 1 for the special cases $R_{i,j} = \pm 1$

Proof. Since $\arcsin^2(\pm 1) = \pi^2/4$, the result in (4) for $R_{i,j} = \pm 1$ is equivalent to

$$|R_{i,j}| = 1 \implies \left|\widetilde{R}_{i,j}\right| = 1. \tag{11}$$

To see why the implication in (11) holds, we first note that $R_{i,j}$ represents the correlation between the jointly normal random variables $\varepsilon^y_{i,t}$ and $\varepsilon^y_{j,t}$, such that

$$\begin{bmatrix} \varepsilon^y_{i,t} \\ \varepsilon^y_{j,t} \end{bmatrix} \sim \mathrm{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & R_{i,j} \\ R_{i,j} & 1 \end{bmatrix}\right).$$

Now suppose (without loss of generality) that $R_{i,j} = 1$, so that $\varepsilon^y_{i,t} = \varepsilon^y_{j,t}$ holds in mean square. This can be seen, for example, by letting $w = (1, -1)$ and noting that

$$\mathbb{E}(\varepsilon^y_{i,t} - \varepsilon^y_{j,t})^2 = \mathbb{V}\text{ar}\left[\varepsilon^y_{i,t} - \varepsilon^y_{j,t}\right] = \mathbb{V}\text{ar}\left[w(\varepsilon^y_{i,t}, \varepsilon^y_{j,t})'\right] = w \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} w' = 0.$$

The equality in mean square implies that $\varepsilon^y_{i,t} = \varepsilon^y_{j,t}$ also holds almost surely. Similarly, we have that $\widetilde{\varepsilon}^y_{i,t} = \log[(\varepsilon^y_{i,t})^2] = \log[(\varepsilon^y_{j,t})^2] = \widetilde{\varepsilon}^y_{j,t}$ almost surely. Since $\widetilde{\varepsilon}^y_{i,t}$ has bounded moments of any finite order, the correlation between $\widetilde{\varepsilon}^y_{i,t}$ and $\widetilde{\varepsilon}^y_{j,t}$ exists, hence it must be equal to one. This establishes the implication in (11) for $R_{i,j} = 1$; the case of $R_{i,j} = -1$ follows by analogy.

■