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Abstract

This paper disentangles the added value of using high-frequency-based (realized) covariance measures on multivariate volatility forecasting into two pillars: the realized variances and realized correlations and quantifies the corresponding economic gains using a broad set of portfolio performance metrics. Using state-of-the-art models based on daily returns and realized (co)variances, we predict the conditional covariance matrix on a daily, weekly, biweekly, and monthly frequency, both for dimensions 30 and 50. We evaluate the forecasts statistically using various loss functions and economically by constructing Global Minimum Variance (GMV) portfolios. Using a data set of 50 liquid U.S. stocks from 2001 to 2019, we find that the inclusion of realized variances largely accounts for the improvement in statistical forecast performance (between 65% and at least 78%). The results on the GMV portfolios show that realized covariance models exhibit lower ex-post volatility but tend to produce substantially lower ex-post mean returns compared to models with realized variances and daily returns. Consequently, Sharpe Ratios increase roughly by 35%, leading to significant utility gains, equivalent to up to 500 basis points per year. Combined, our results indicate that there is no economic gain by modeling correlations dynamically, either using daily returns or realized correlations.

Key words: multivariate volatility, high-frequency data, realized variances, realized correlations

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1 Introduction

Modeling and forecasting the volatility of financial asset returns is a crucial element in quantitative portfolio management. Since the development of the GARCH model (Bollerslev, 1986), a substantial body of literature has emerged on modeling volatility using daily returns (see Bauwens et al. (2006) for a survey). This body of work has been further enhanced by novel approaches to estimating covariances more precisely through the use of high-frequency (HF) data, as demonstrated by the work of Andersen et al. (2003); Barndorff-Nielsen and Shephard (2004); Barndorff-Nielsen et al. (2011), among others. These measures of covariances have been used to build the so-called 'realized' covariance models, such as the CAW (Conditional Autoregressive Wishart) model of Gourieroux et al. (2009), the multivariate HEAVY model of Noureldin et al. (2012), the HAR-DRD model of Oh and Patton (2016) and the multivariate volatility model of Chiriac and Voev (2011).

Typically, the statistical and economic gains of the aforementioned realized covariance models have been tested using relatively small portfolios -ranging from 9 to 15 assets-, as in Gorgi et al. (2018), Noureldin et al. (2012) and Archakov et al. (2024). However, their relative performance in higher-dimension portfolios has received less attention. Notably, Hautsch et al. (2015) address this gap by using a large portfolio (i.e. more than 50 assets) to demonstrate the economic potential of incorporating HF-based covariances. The authors show that Global Minimum Variance Portfolios (GMVPs) constructed with HF-based covariance forecasts achieve significantly lower volatility than those built using models based solely on daily returns. Additionally, they find that utility gains from using these HF-based models are particularly substantial for highly risk-averse investors.

This paper builds on the work of Hautsch et al. (2015) by further exploring the sources of economic and statistical gains from high-frequency (HF)-based covariance models on medium-sized portfolios (30 to 50 assets). To quantify the benefits of HF data in multivariate volatility forecasting, we employ several state-of-the-art models that include 1) only daily returns, 2) daily returns and realized variances, and 3) full realized covariance matrices. We then disentangle the contributions of realized variances and realized correlations and introduce a broader set of economic performance metrics to evaluate the forecasts. This comprehensive approach allows us to quantify better the specific benefits of incorporating

HF data into multivariate volatility forecasting models.

Using daily returns and realized (co)variances based on 5-minute data of up to 50 highly liquid U.S. stocks during the period 2001 - 2019, we estimate three types of multivariate volatility models to forecast the conditional covariance matrix. Our first group of models uses solely returns, such as the BEKK model of Engle and Kroner (1995) and the CCC and (corrected) DCC-GARCH models of Bollerslev (1990) and Engle (2002); Aielli (2013). The second group consists of models that incorporate both daily returns and realized variances, specifically, the univariate HEAVY-GAS model of Opschoor et al. (2018), combined with the CCC and DCC models mentioned above. Lastly, we include three different models that focus directly on realized covariance matrices: the CAW model of Gourieroux et al. (2009), the multivariate HEAVY-GAS-tF model by Opschoor et al. (2018) and the HAR-DRD model of Oh and Patton (2016). Following Hautsch et al. (2015), we construct daily, weekly, biweekly, and monthly (cumulative) forecasts of the covariance matrix for one set of 30 assets and one set with all 50 assets in our sample. We evaluate our forecasts statistically, using the Root Mean Standard Error or Frobenius norm (RMSE), the QLIK, and the STEIN loss functions, and economically by constructing GMVPs.

Our main result consists of two parts. First, we examine the forecasts' statistical performance, and we find that most of the improvement in loss functions when incorporating HF data into the models comes from its use in modeling the variances rather than the correlations. Around 70% of the decrease in RMSE and 91% of the STEIN function can be attributed to using realized variances. These percentages are even higher for daily forecasts. Notably, we observe that the value of the STEIN loss function calculated for this forecast horizon is the lowest in the case of models that combine realized variances and daily returns.

The second part of our results focuses on economic performance metrics. Our analysis reveals that while models incorporating the full realized covariance matrix achieve the lowest ex-post portfolio volatility, this benefit often comes with a significant reduction in ex-post portfolio returns compared to models using a mixed approach (combining daily returns with realized variances only). The later set of models produces the highest Sharpe ratios, even when accounting for various levels of transaction costs. Notably, the Sharpe ratios increase by an average of 35% relative to the best-performing realized covariance model. These findings remain consistent across the analyzed time horizons and hold true for portfolios with 30 and 50 dimensions.

We also used the utility framework proposed by Fleming et al. (2001, 2003) to examine the potential utility gains associated with including HF data in volatility forecasting across different levels of risk aversion. We find that the utility gains derived from using a model that incorporates only realized variances relative to one that uses the full covariance matrix (such as the HAR-DRD model) could be substantial, equivalent in monetary terms to an increase of around 300 up to 500 annual basis points. Finally, we observe that modeling correlation dynamics does not add significant value to an investor interested in minimizing risk, regardless of their level of risk aversion. We often observe that it is even more favorable for an investor to use a model with constant correlations that are regularly updated (for example, by using the moving annual window) than a model that includes correlation dynamics.

Our results are robust across three key aspects: First, we consider different portfolio dimensions, specifically 30 and 50 assets. Second, we confirmed the robustness of our results by varying the in-sample period length, and third, by changing the re-estimation window length, with monthly parameter updates yielding results consistent with those obtained from yearly updates.

Our work is closely related to the work by Halbleib and Voev (2016), who study the performance of mixed-approach models with a focus on squared errors. Using 30 highly liquid stocks traded on the NYSE, the authors show that a mixed specification outperforms models using only daily returns regarding forecast precision (lower squared forecast errors). Moreover, such specification performs at par with more computationally intensive models based on HF data. We extend their approach by considering a larger portfolio and by evaluating not only the forecasts' statistical performance but also the *economic* impact of using mixed frequencies by constructing GMVPs as in Hautsch et al. (2015).

The remainder of this paper is organized as follows. Section 2 discusses briefly the data used in the paper. Section 3 provides an overview of the models included in our comparison exercise. Section 4 lists the statistical loss functions and explains how we construct and evaluate the GMVPs' outcomes. Section 5 discusses the results of our empirical application. Section 7 presents our main conclusions.

2 Data

Our data set consists of daily realized (co)variances and daily open-to-close log returns of 50 randomly chosen highly liquid U.S. equities from various sectors from the S&P 500 index over the period January 2, 2001, until December 6, 2019. This results in a sample of $4,696$ observations. For each stock, we observe consolidated trades (transaction prices) extracted from the Trade and Quote (TAQ) database with a time-stamp precision of one second before 2014 and one millisecond after 2014. We first clean the high-frequency data following the guidelines of Brownlees and Gallo (2006) and Barndorff-Nielsen et al. (2009). We then construct Realized Covariance matrices using fixed 5-minute intervals in the returns series.

Table 1 provides an overview of 50 Tickers and their GICS sector. As Panel A indicates, the stocks come from nine different sectors, most of them belonging to the financial industry. Panel B shows a few abnormal observations in the constructed realized (co)variances. These values are winsorized at a very high level (99.95 level for AIG, F, and WMB, 99.93 in the case of MS). Most of these outliers were observed around the peak of the Global Financial Crisis in 2008.

3 The modeling framework

A departing point for our modeling framework is the following return equation for a $K \times 1$ vector of daily open-to-close log-returns $\boldsymbol{r}_t = (r_{1t}, \ldots, r_{Kt})^\top$, with $t = 1, \ldots, T$

$$\boldsymbol{r}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t = \boldsymbol{\mu} + \boldsymbol{V}_t^{1/2} \boldsymbol{z}_t, \quad \boldsymbol{z}_t | \mathcal{F}_{t-1} \sim N(0, \boldsymbol{I}_K), \tag{1}$$

where $\boldsymbol{\mu}$, $\boldsymbol{\epsilon}_t$, and $\boldsymbol{V}_t$ denote a vector with constants, demeaned returns, and the conditional covariance matrix of the return vector, respectively. Furthermore, $\boldsymbol{z}_t$ is a multivariate Normal distributed $k \times 1$ innovation vector with unit covariance matrix $\boldsymbol{I}_k$. Since we work with daily data, we set $\boldsymbol{\mu}$ equal to 0.

We present three different types of models for $\boldsymbol{V}_t$. The first type includes only daily returns. The second one uses a combination of daily returns and realized variances, and the third models $\boldsymbol{V}_t$ directly by using realized covariance matrices.

We consider two frequently used models for $\boldsymbol{V}_t$ in equation (1) based on daily returns,

Table 1: Tickers, GICS sectors, and Outliers

This table provides an overview of the Tickers and corresponding sector of 50 U.S. stocks from the S&P 500 index. Panel A lists the sector, the Ticker, and the number of companies within each sector. Panel B shows abnormal values of the daily realized variance of our sample. We list the Ticker, the date, the abnormal, and the winsorized realized variance using a level of 99.95 (AIG, F, and WMB) and 99.93 (MS).

Panel A: Sectors and Tickers

GICS nr	GICS sector	# Comp	Tickers
1	Capital Goods	6	AA, BA, CAT, HON, F, NOC
2	Financials	14	AXP, JPM, AIG, BAC, C, KEY, MTB
			COF,USB,BBT,STI,WFC,GS,MS
3	Energy	4	GE,XOM,MUR,SLB
4	Consumer Services	5	HD, MCD, WMT, TGT, BXP
5	Consumer Non-Durables	3	KO,MO,SYY
6	Health Care	7	PFE, ABT, BAX, JNJ, LLY, THC, MMM
7	Public Utilities	5	AEP, AEE, DUK, SO, WMB
8	Technology	3	IBM,DOV,HPQ
9	Basic industries	2	PG,FLR
10	Transportation	1	LUV

Panel B: Abnormal values

Ticker	date	value	ws value
AIG	20080915	2839	1533
	20080916	5396	1533
F	20081008	481	452
	20081120	1661	452
WMB	20020723	939	896
	20020726	2025	896
MS	20080917	1242	720
	20080918	1658	720
	20081010	1407	720

namely the (scalar) BEKK-GARCH model of Engle and Kroner (1995) and the (corrected) DCC-GARCH model of Engle (2002); Aielli (2013). The first model reads

$$\boldsymbol{V}_{t+1} = (1 - A - B)\boldsymbol{\Omega} + A\,(\boldsymbol{r}_t - \boldsymbol{\mu})(\boldsymbol{r}_t - \boldsymbol{\mu})^\top + B\,\boldsymbol{V}_t, \tag{2}$$

with scalar parameters A and B and a $k \times k$ matrix-valued intercept parameter $\boldsymbol{\Omega}$. We apply covariance targeting to estimate the intercept matrix: $\hat{\boldsymbol{\Omega}} = \sum_{t=1}^{T} \hat{\boldsymbol{\epsilon}}_t \hat{\boldsymbol{\epsilon}}_t^\top$.

The (corrected) cDCC model (Aielli, 2013; Engle, 2002) disentangles $\boldsymbol{V}_t$ into conditional variances and correlations, which are then modeled separately. That is, $\boldsymbol{V}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t$ with $\boldsymbol{D}_t = \text{diag}(\sigma_{1t}, \ldots, \sigma_{Kt})\top$ a diagonal matrix of volatilities. In a first step, we model the conditional variances $\sigma^2_{i,t}$ individually using a GARCH(1,1) model (Bollerslev, 1986),

$$r_{it} = \mu_i + \sigma_{it} u_{it}, \quad u_{it} \sim N(0,1) \tag{3}$$

$$\sigma^2_{i,t+1} = \omega_i + \alpha_i \epsilon^2_{i,t} + \beta_i \sigma^2_{i,t}, \tag{4}$$

where $\epsilon_{i,t}$ is the i-th entry of the vector $\boldsymbol{\epsilon}_t$. Similar to the BEKK model, we estimate ω_i by a variance targeting approach. The second step consists of modeling the (quasi) correlation matrices $\boldsymbol{Q}_t$, which are driven by the de-volatised returns $u_{it} = \epsilon_{it}/\sigma_{it}$. The model can be summarized as follows:

$$\boldsymbol{u}_t = \boldsymbol{D}_t^{-1} \boldsymbol{\epsilon}_t \tag{5}$$

$$\boldsymbol{Q}_{t+1} = (1 - \delta - \lambda)\bar{\boldsymbol{Q}} + \delta \left(\boldsymbol{Q}_t^{*1/2} \boldsymbol{u}_t \boldsymbol{u}_t^\top \boldsymbol{Q}_t^{*1/2} \right) + \lambda \boldsymbol{Q}_t \tag{6}$$

$$\boldsymbol{R}_t = \boldsymbol{Q}_t^{*-1/2} \boldsymbol{Q}_t \boldsymbol{Q}_t^{*-1/2} \tag{7}$$

where $\boldsymbol{u}_t$ is a vector of de-volatized returns and $\boldsymbol{Q}_t^* = \mathrm{diag}(q_{ii,t}), i = 1, \ldots, K$.Again, we use correlation targeting to estimate $\bar{\boldsymbol{Q}}$. This model is labeled as DCC-GARCH. As a special case of the cDCC model, we also consider the Constant Conditional Correlation (CCC-GARCH) model of Bollerslev (1990) by setting δ and λ to zero. In this model, the correlation dynamics are not explicitly modeled. Instead, the correlations are recalculated regularly according to a moving window.

The second class of models introduces HF-based realized volatility measures into the modeling framework. We considered the univariate version of the HEAVY-GAS-tF model proposed by Opschoor et al. (2018). This score-driven model assumes fat-tailed distributions for the returns and the realized variances and innovates the latent conditional variance by the (sum of the) score of these distributions. This results in an updating mechanism that is robust against incidental large returns and large values of the realized variance. The univariate HEAVY-GAS-tF model reads

$$r_{it} = \mu_i + \sigma_{it} u_{it}, \quad u_{it}|\mathcal{F}_{t-1} \sim t(0, 1, \nu_{0,i}) \tag{8}$$

$$RV_{i,t}|\mathcal{F}_{t-1} \sim F(\sigma^2_{i,t}, \nu_{1,i}, \nu_{2,i}) \tag{9}$$

$$\sigma^2_{i,t+1} = (1 - \beta_i)\omega_i + \alpha_i\, s_{i,t} + \beta_i\, \sigma^2_{i,t} \tag{10}$$

$$\begin{aligned} s_{it} &= \frac{1}{\nu_{1,i} + 1} \sigma^2_{i,t} \nabla_{i,t} \sigma^2_{i,t} \\ &= \frac{w_{it}(r_{it} - \mu_i)^2 - \sigma^2_{i,t}}{\nu_{1,i} + 1} \\ &+ \frac{\nu_{1,i}}{\nu_{1,i} + 1} \left[\frac{\nu_{1,i} + \nu_{2,i}}{\nu_{2,i} - 2} \frac{RV_{i,t}}{\left(1 + \frac{\nu_{1,i}\ RV_{i,t}}{(\nu_{2,i}-2)\sigma^2_{i,t}}\right)} - \sigma^2_{i,t} \right], \end{aligned} \tag{11}$$

with $\nabla = \frac{\partial \log t(\cdot)}{\partial \sigma^2_{i,t}} + \frac{\partial \log F(\cdot)}{\partial \sigma^2_{i,t}}$ and $w_{it} = (\nu_{0,i} + 1)/\left(\nu_{0,i} - 2 + \frac{(r_{it}-\mu_i)^2}{\sigma^2_{i,t}}\right)$. Note that the (conditional) correlations are again modeled by the CCC or the cDCC models. The complete models are then labeled as CCC(DCC)-HEAVY-GAS.

The third class of models relies on directly modeling the realized covariance matrices $\boldsymbol{RC}_t$. The literature on this type of models has expanded rapidly in recent years (see Archakov et al. (2024) for an overview). Our selection of models is designed to capture the diverse approaches present in this growing body of research on high-frequency multivariate variance forecasting. We will use the CAW model of Gourieroux et al. (2009), the multivariate version of the HEAVY-GAS-tF model by Opschoor et al. (2018) and the HAR-DRD model of Oh and Patton (2016) as classical benchmarks. The CAW model assumes a conditional Wishart distribution for $\boldsymbol{RC}_t$ with a latent time varying mean $\boldsymbol{V}_t$:

$$\boldsymbol{RC}_t \sim \mathcal{W}(\boldsymbol{V}_t, \nu_W) \tag{12}$$

$$\boldsymbol{V}_{t+1} = (1 - A - B)\boldsymbol{\Omega} + A\, \boldsymbol{RC}_t + B\, \boldsymbol{V}_t, \tag{13}$$

with scalar parameters A and B and $\boldsymbol{\Omega}$ an intercept matrix, estimated by the covariance targeting approach using the sample mean of all realized covariance matrices.

The multivariate HEAVY-GAS-tF model by Opschoor et al. (2018) uses the score-driven framework by Creal et al. (2013). It is based on two measurement equations that depend on $\boldsymbol{V}_t$: one for $\boldsymbol{r}_t$ and another for $\boldsymbol{RC}_t$. Given these two measurement equations, the innovation of $\boldsymbol{V}_t$ consists of the (scaled) score: the partial derivative of the log-density w.r.t. $\boldsymbol{V}_t$. In sum, the model reads

$$\boldsymbol{r}_t|\mathcal{F}_{t-1} \sim t(\boldsymbol{\mu}, \boldsymbol{V}_t, \nu_0) \quad \boldsymbol{RC}_t|\mathcal{F}_{t-1} \sim F(\boldsymbol{V}_t, \nu_1, \nu_2) \tag{14}$$

$$\boldsymbol{V}_{t+1} = (1-B)\Omega + A\,\boldsymbol{S}_t + B\,\boldsymbol{V}_t \tag{15}$$

$$\begin{aligned}\boldsymbol{S}_t &= \frac{1}{\nu_1+1}\boldsymbol{V}_t \nabla \boldsymbol{V}_t \\ &= \frac{w_t(\boldsymbol{r}_t-\boldsymbol{\mu})(\boldsymbol{r}_t-\boldsymbol{\mu})' - \boldsymbol{V}_t}{\nu_1+1} \\ &+ \frac{\nu_1}{\nu_1+1}\left[\frac{\nu_1+\nu_2}{\nu_2-k-1}\,\boldsymbol{RC}_t\left(\mathbf{I}_k + \frac{\nu_1\,\boldsymbol{V}_t^{-1}\,\boldsymbol{RC}_t}{\nu_2-k-1}\right)^{-1} - \boldsymbol{V}_t\right],\end{aligned} \tag{16}$$

with $\nabla = \frac{\partial \log t(\cdot)}{\partial \boldsymbol{V}_t} + \frac{\partial \log F(\cdot)}{\partial \boldsymbol{V}_t}$ and $w_t = (\nu_0+k)/(\nu_0 - 2 + (\boldsymbol{r}_t-\boldsymbol{\mu})\top \boldsymbol{V}_t^{-1}(\boldsymbol{r}_t-\boldsymbol{\mu}))$.

Finally, the HAR-DRD model of Oh and Patton (2016) accounts for slowly declining autocorrelations in realized variances and correlations - i.e. the so-called long-memory behavior - by assuming a HAR model (Corsi, 2009) for both realized quantities. More specifically, let us first decompose the realized covariance matrix into realized variances and realized correlations:

$$\boldsymbol{RC}_t = \sqrt{\boldsymbol{RV}_t}\,\boldsymbol{RCORR}_t\,\sqrt{\boldsymbol{RV}_t}, \tag{17}$$

where $\boldsymbol{RV}_t = \mathrm{diag}(\boldsymbol{RC}_t)$ is a diagonal matrix with the realized variances on the diagonal and $\boldsymbol{RCORR}_t$ the matrix with realized correlations, obtained via $\boldsymbol{RV}_t^{-1/2}\boldsymbol{RC}_t\boldsymbol{RV}_t^{-1/2}$.

Then, in a first step, the (logarithm of the) individual realized variances is modeled by the HAR model as:

$$\begin{aligned}\log \boldsymbol{RV}_{ii,t+1} = \beta_{0,i} + \beta_{1,i}\log \boldsymbol{RV}_{ii,t} + \beta_{2,i}\frac{1}{5}\sum_{k=1}^{5}\log \boldsymbol{RV}_{ii,t-k+1} + \\ \beta_{3,i}\frac{1}{22}\sum_{k=1}^{22}\log \boldsymbol{RV}_{ii,t-k+1} + \boldsymbol{\eta}_{ii,t+1},\end{aligned} \tag{18}$$

with coefficients $\beta_{0,i}, \ldots, \beta_{3,i}$ $(i = 1, \ldots, K)$ estimated by OLS. The realized correlations are modeled in a second step by the following HAR model

$$\text{vech}(\boldsymbol{RCORR}_{t+1}) = (1-a-b-c)\text{vech}(\boldsymbol{RC\bar{O}RR}) + a\,\text{vech}(\boldsymbol{RCORR}_t) + \\ b\,\frac{1}{5}\sum_{k=1}^{5}\text{vech}(\boldsymbol{RCORR}_{t-k+1}) + c\,\frac{1}{22}\sum_{k=1}^{22}\text{vech}(\boldsymbol{RCORR}_{t-k+1}) + \boldsymbol{\xi}_{t+1}, \tag{19}$$

where the coefficients (a, b, c) are again estimated by OLS.

3.1 Estimation

We apply our models to two portfolios: one with 30 randomly selected assets and the other including all 50 assets from our sample. All model parameters are estimated by quasi Maximum Likelihood or by the Composite Likelihood method, combined with a (co)variance targeting method, with the exception of the parameters of the HAR-DRD model, which are estimated by OLS. The usage of the Composite Likelihood (CL) method stems from Pakel et al. (2021), who show that the estimated BEKK and DCC (GARCH or HEAVY-GAS) parameters could suffer from bias. In particular, the innovation parameter a may be biased downwards, while the persistence parameter b may tend towards one as the portfolio dimension grows. We use the CL method to solve this issue. This method consists of dividing the data set into bivariate pairs and maximizing the sum of the log-likelihood of these bivariate pairs.

Table 2 lists all models and their conditional distributions and indicates which models will be estimated via the CL method. The table shows that we use the Gaussian distribution for univariate and multivariate returns. One could interpret this as a 'quasi Maximum Likelihood' approach. In the case of the cDCC model, we follow Engle (2002) and decompose the Gaussian likelihood into the volatility and correlation parts, which are estimated separately. In the case of the CCC(DCC)-HEAVY-GAS models, we maximize the sum of the log-likelihood of the Student's t distribution and the F distribution.

Table 2: Overview of used models and their conditional distribution
This table reports the multivariate volatility models considered in this paper. We list the model name and its conditional distribution and indicate the cases where we use the Composite Likelihood method of Pakel et al. (2021) instead of the maximum likelihood method.

Models	Conditional Distribution	Composite Likelihood
Models based on returns		
BEKK	Gaussian	yes
CCC-GARCH	Gaussian	no
DCC-GARCH	Gaussian	yes
Models based on returns and realized variances		
CCC-HEAVY-GAS	t, F	no
DCC-HEAVY-GAS	t, F, Gaussian (DCC part)	yes
Models based on realized covariances		
CAW	Wishart	no
HEAVY-GAS	t, F	no
HAR-DRD		no

4 Forecasting and evaluation

Once we obtain parameter estimates for all the models described above, we compute (cumulative) h-step ahead forecasts of the conditional covariance matrix $\boldsymbol{V}_t$. We follow Hautsch et al. (2015) and set h equal to 1, 5, 10 and 22. This corresponds to daily, weekly, biweekly, and monthly forecasts.

We then evaluate the statistical performance of the forecasted covariance matrices based on proper statistical loss functions (Laurent et al., 2013), such as the RMSE, as well as the QLIK and STEIN loss functions. All losses functions depend on the predicted $\boldsymbol{V}_{t+1}$ and the true realized covariance matrix $\boldsymbol{RC}_{t+1}$:

$$RMSE_{t+1} = \sqrt{\text{trace}\left((\boldsymbol{RC}_{t+1} - \boldsymbol{V}_{t+1})^\top(\boldsymbol{RC}_{t+1} - \boldsymbol{V}_{t+1})\right)}, \tag{20}$$

$$QLIK_{t+1} = \log|\boldsymbol{V}_{t+1}| + \text{trace}(\boldsymbol{V}_{t+1}^{-1}\boldsymbol{RC}_{t+1}), \tag{21}$$

$$STEIN_{t+1} = \text{trace}(\boldsymbol{RC}_{t+1}\boldsymbol{V}_{t+1}^{-1}) - \log\left|\boldsymbol{RC}_{t+1}\boldsymbol{V}_{t+1}^{-1}\right| - K \tag{22}$$

We use the Model Confidence Set (MCS) proposed by Hansen et al. (2011) to test on the lowest average loss using a significance level of 5%. The MCS automatically accounts for the dependence between model outcomes, given that all models are based on the same data.

Motivated by the mean-variance optimization setting of Markowitz (1952), we then evaluate our forecasts economically by examining the GMVPs' performance. This portfolio can be obtained by solving an optimization problem where an investor seeks to minimize the 1-step-ahead portfolio volatility at time t, subject to a fully invested portfolio. The resulting quadratic problem can be written as

$$\min \boldsymbol{w}_{t+1|t}^{\top} \boldsymbol{V}_{t+1} \boldsymbol{w}_{t+1|t}, \qquad \text{s.t.} \qquad \boldsymbol{w}_{t+1|t}^{\top} \iota = 1, \tag{23}$$

with solution

$$w_{t+1|t} = \frac{\boldsymbol{V}_{t+1|t}^{-1} \iota}{\iota^{\top} \boldsymbol{V}_{t+1|t}^{-1} \iota}. \tag{24}$$

We assess the predictive ability of the different models by comparing the implied ex-post portfolio volatility, $\sigma_{p,t+1} = \sqrt{\boldsymbol{w}'_{t+1|t} \boldsymbol{RC}_{t+1} \boldsymbol{w}_{t+1|t}}$, using the MCS.

Beyond the GMVP's volatility, we also calculate several other relevant portfolio performance metrics, such as turnover (TO_t), concentration (CO_t), and the total short position (SP_t) for each competing model at time t. The turnover measures the portfolio's value bought/sold when rebalancing it to its new optimal position from t to $t+1$. A model that produces more stable covariance matrix forecasts will typically produce GMVPs with less turnover and, hence, lower transaction costs. This effect would lead to a gain in trading strategies. The total turnover at time t is defined as

$$TO_t = \sum_{i=1}^{N} \left| w_{i,t+1|t} - w_{i,t|t-1} \frac{1 + r_{i,t}}{1 + \boldsymbol{w}_{t|t-1}^{\top} \boldsymbol{r}_t} \right|, \tag{25}$$

where $w_{i,t|t-1}$ is the i-th element of the weight vector $\boldsymbol{w}_{t|t-1}$ and $r_{i,t}$ the return of asset i at time t. Following Bollerslev et al. (2018), we assume a fixed transaction cost c proportional to the turnover rate in the portfolio. The portfolio excess return net of transaction costs for given proportional transaction costs cTO_t now reads

$$r_{p,t} = \boldsymbol{w}_{t|t-1}^{\top} \boldsymbol{r}_t - cTO_t, \tag{26}$$

with c equal to 0, 1 and 2% respectively.

Portfolio concentration and total portfolio short position are both measures of the amount of extreme portfolio allocations. Less concentrated portfolios and those with fewer short positions could be easier and cheaper to implement, as rebalancing over time implies lower transaction costs. Again, more stable forecasts of $\boldsymbol{V}_{t+1}$ should result in less extreme portfolio weights. The portfolio concentration is defined as

$$CO_t = \left(\sum_{i=1}^{N} w_{i,t|t-1}^2 \right)^{1/2}, \tag{27}$$

while the total portfolio short position SP_t is given by

$$SP_t = \sum_{i=1}^{N} w_{i,t|t-1} \cdot I[w_{i,t|t-1} < 0], \tag{28}$$

with $I[\cdot]$ an indicator function that takes the value one if the i-the element of the weight vector is lower than zero.

Finally, we use the utility-based framework of Fleming et al. (2001, 2003) to assess the relative economic advantages of using different forecasting models. This approach has been used by Bollerslev et al. (2018), among others, and is based on the assumption that an investor has quadratic utility with a risk aversion parameter γ. The realized utility of the portfolio return based on the forecasted covariances from model j equals

$$U(r_{p,t}^j, \gamma) = (1 + r_{p,t}^j) - \frac{\gamma}{2(1+\gamma)}(1 + r_{p,t}^j)^2. \tag{29}$$

Given two different models j and l, the return Δ_γ that the investor with risk aversion γ would like to give up to switch from model j to l can be obtained by solving

$$\sum_{t=1}^{P} U(r_{p,t}^j, \gamma) = \sum_{t=1}^{P} U(r_{p,t}^l - \Delta_\gamma, \gamma). \tag{30}$$

We test the null-hypothesis $\Delta_\gamma = 0$ using the Reality Check of White (2000), based on the stationary bootstrap of Politis and Romano (1994), using 999 bootstrap samples with an average block length of 22 days.

5 Results

We use a moving estimation window of 1000 observations (corresponding to roughly four calendar years), leaving P = 3,696 observations for the out-of-sample period, starting on December 28th, 2004. We re-estimate the models' parameters after 250 days (one year) and construct h-step ahead forecasts at each day t. As previously noted, we consider one set of $K = 30$ assets and a set of all 50 assets in our sample. Section 5.1 explains the results on the average loss functions, and it is followed by a discussion on the GMVP results in Section 5.2.

5.1 Statistical loss functions

Tables 6 - 7 show the average RMSE, QLIK, and STEIN loss functions across our set of models estimated for the two portfolios. In addition, we report the p-value of the MCS using a significance level of 5%. The models are listed in the following order: first, we present the models using daily returns only (BEKK, CCC(DCC)-GARCH), then, the models combining realized variances and daily returns (CCC(DCC)-HEAVY-GAS), and finally, models using the full realized covariance matrix (CAW, HEAVY-GAS-tF, and HAR-DRD).

Both tables present a consistent pattern of results across all forecast horizons. All the statistical loss functions exhibit a general decline when realized variances are incorporated into the model, achieving their lowest values with models that fully integrate the realized covariance matrix. Specifically, the HEAVY-GAS-tF model, and occasionally the HAR-DRD model, yield the lowest values among all the models considered. This outcome aligns with theoretical expectations, as these models utilize the most comprehensive information available to model and forecast the realized covariance matrix.

To indicate which part of this decrease can be attributed to the inclusion of realized variances, we first take the average values of the loss functions of models using daily returns only, CCC(DCC)-GARCH, and that of the models incorporating the realized variances only (CCC(DCC)-HEAVY-GAS). We then measure the contribution of the realized variances as the reduction in the statistical loss functions when transitioning from the CCC(DCC)-GARCH models to the CCC(DCC)-HEAVY-GAS models, relative to the difference between the CCC(DCC)-GARCH models and the average loss function of the realized multivariate models (CAW, HEAVY-GAS and HAR-DRD).

Table 5 presents our first main result: a substantial portion of the reduction in statistical loss functions associated with incorporating HF data into the models is attributable to the introduction of realized variances. The impact is particularly significant for the RMSE and STEIN loss functions, with reductions ranging between 61% and 78% for the first and of at least 88% for the second. We observe that in the case of the STEIN loss function, the value is lowest for the models that use realized covariances only (CCC(DCC)-HEAVY-GAS model) in the daily horizon for both the 30- and 50-asset portfolios. In contrast, when measured by the change in the QLIK loss function, the contribution of realized variances is smaller, accounting for 55% of the decrease in the 30-asset portfolio and 49% in the 50-asset portfolio, on average, across various forecast horizons.

5.2 Global Minimum Variance portfolio

Apart from evaluating our (cumulative) h-step ahead forecasts statistically, we also evaluate them economically by constructing GMVPs and examining a set of economic performance measures based on these portfolios. Tables 6 and 7 present the main results of the portfolio exercise for dimensions 30 and 50, respectively. Each panel (A, B, C, D) considers a different forecast horizon, ranging from daily to monthly. For each horizon, we calculated the annualized mean portfolio return and volatility (measured by the ex-post realized portfolio standard deviation and the standard deviation of the ex-post returns), the turnover (TO), concentration (CO), and total short position (SP), as well as the Sharpe ratio and economic gains in case of different transaction costs (0, 1 and 2 percent). Specifically, we report the economic gain, expressed in annual basis points, that an investor with quadratic utility and a particular risk aversion is willing to give up to switch from the HAR-DRD model to the model specified in each column. Negative (positive) bold values indicate that the investors would be significantly inclined (disinclined) to use the particular model in the column instead of the HAR-DRD model.

Both Tables show us three important take-aways. First of all, models that use the full realized covariance matrix produce the lowest ex-post realized portfolio standard deviation.[1]

[1]The superior performance of this type of model has been evidenced by recent work by Archakov et al. (2024) and Gorgi et al. (2018) in portfolios with smaller dimensions.

Table 3: Out-of-sample point forecasts (dimension 30)

This table shows the mean of the RMSE, QLIK, and STEIN loss functions based on daily, weekly, and biweekly predictions of the covariance matrix of returns of 30 stocks according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,000 observations and re-estimated after 250 observations. The lowest value of all loss functions across the models is marked in bold. In addition, we report the mcs p-values based on a 5% significance level. The p-values of the models within the model confidence set are marked in bold. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2005 until December 2019 and contains 3,696 observations.

	BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts								
RMSE	35.13	34.36	33.93	29.54	29.09	27.86	28.37	26.64
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**
QLIK	53.51	34.02	35.13	31.76	32.79	30.81	30.80	**30.63**
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.27)**	**(0.55)**	**(1.00)**
STEIN	43.86	39.83	39.05	37.53	**37.04**	42.74	38.51	39.25
mcs p-value	(0.00)	(0.00)	(0.00)	(0.06)	**(1.00)**	(0.00)	(0.01)	(0.00)
Panel B: Weekly forecasts								
RMSE	151.84	153.65	151.85	125.63	123.05	112.61	114.02	**111.47**
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	**(0.61)**	**(0.61)**	**(1.00)**
QLIK	101.01	82.82	83.42	81.15	81.85	80.25	**79.94**	80.05
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.12)**	**(1.00)**	**(0.66)**
STEIN	15.01	13.14	13.14	8.75	8.98	9.32	**7.54**	8.86
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel C: Biweekly forecasts								
RMSE	296.68	306.43	303.68	250.04	245.01	221.57	**219.95**	220.75
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.98)**	**(1.00)**	**(0.98)**
QLIK	120.84	104.37	104.73	102.62	103.10	101.96	**101.42**	101.46
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.02)	**(1.00)**	**(0.86)**
STEIN	12.41	11.98	11.95	6.69	6.91	6.88	**5.24**	6.49
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel D: Monthly forecasts								
RMSE	659.41	690.92	687.47	584.37	573.95	518.62	**488.68**	518.58
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.23)**	**(1.00)**	**(0.23)**
QLIK	142.91	129.30	129.41	127.67	127.88	127.33	126.28	**126.13**
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.69)**	**(1.00)**
STEIN	10.59	12.37	12.24	6.00	6.13	6.01	**4.28**	5.36
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)

Table 4: Out-of-sample point forecasts (dimension 50)

This table shows the mean of the RMSE, QLIK, and STEIN loss functions based on daily, weekly, and biweekly predictions of the covariance matrix of returns of 50 stocks according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,000 observations and re-estimated after 250 observations. The lowest value of all loss functions across the models is marked in bold. In addition, we report the mcs p-values based on a 5% significance level. The p-values of the models within the model confidence set are marked in bold. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2005 until December 2019 and contains 3,696 observations.

	BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts								
RMSE	62.01	59.53	58.98	51.73	50.91	49.42	51.09	**46.81**
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**
QLIK	121.55	61.77	64.38	57.99	60.62	56.16	**56.08**	56.56
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.87)**	**(1.00)**	**(0.21)**
STEIN	191.47	185.58	180.22	177.71	**173.69**	210.62	185.70	188.67
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)	(0.00)	(0.00)
Panel B: Weekly forecasts								
RMSE	265.41	260.30	257.69	216.20	211.26	195.25	202.83	**191.49**
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	**(0.27)**	**(0.16)**	**(1.00)**
QLIK	197.93	142.87	144.61	140.18	142.25	138.44	**137.83**	138.63
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.21)**	**(1.00)**	**(0.21)**
STEIN	35.83	27.86	27.91	20.03	20.68	23.17	**17.99**	21.61
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel C: Biweekly forecasts								
RMSE	517.20	517.80	513.67	427.70	418.09	378.91	388.41	**376.85**
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.82)**	**(0.62)**	**(1.00)**
QLIK	228.08	178.56	179.79	175.87	177.44	174.52	**173.51**	174.04
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.05)**	**(1.00)**	**(0.37)**
STEIN	28.64	22.73	22.84	14.00	14.71	15.64	**11.31**	14.61
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel D: Monthly forecasts								
RMSE	1149.1	1177.2	1172.0	1003.8	985.60	888.52	**853.70**	890.57
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.56)**	**(1.00)**	**(0.56)**
QLIK	260.02	219.87	220.45	217.54	218.41	216.56	**214.72**	214.82
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	**(0.88)**
STEIN	23.70	21.70	21.61	11.58	12.09	12.59	**8.28**	11.11
mcs p-value	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)

Table 5: Percentage improvement in loss functions

This table shows the percentage decrease in statistical loss functions attributed to the inclusion of realized variances. It is calculated as the reduction in the RMSE, QLIK, and STEIN average loss functions obtained when transitioning from the CCC(DCC)-GARCH models to the CCC(DCC)-HEAVY models, relative to the difference between the average loss function of the CCC /DCC-GARCH models and that of the realized multivariate models, CAW, HEAVY-GAS and HAR-DRD. This analysis is conducted for daily, weekly, biweekly, and monthly predictions of the covariance matrix of 30 and 50 stock returns. Parameters are estimated with a moving window of 1,000 observations and re-estimated after 250 observations. An *na* indicates that the average loss function of the CCC(DCC)-HEAVY-GAS models is lower than that of the realized multivariate models. The out-of-sample period goes from January 2005 until December 2019 and contains 3,696 observations.

	RMSE		STEIN		QLIK	
h/dim	30	50	30	50	30	50
Daily	74%	78%	*na*	*na*	60.0%	55%
Weekly	71%	72%	94%	*na*	51.0%	46%
Biweekly	68%	69%	90%	94%	57.6%	49%
Monthly	61%	61%	88%	89%	51.4%	46%
Average	69%	70%	90%	92%	55%	49%

In particular, the HEAVY-GAS-tF model achieves a 9.2% reduction in ex-post realized portfolio standard deviation compared to the CCC-HEAVY-GAS model, which relies solely on realized variances and daily returns. In addition, it is the only model in the MCS in almost every horizon for dimensions 30 and 50. However, when focusing on the standard deviation of the ex-post portfolio returns, the results are somewhat less strong, with a percentage decrease of 2.8% on average relative to the CCC-HEAVY-GAS model. Notably, in the case of monthly forecasts, the CCC-HEAVY-GAS model delivers the lowest ex-post portfolio volatility.

Second, when considering economic performance metrics such as the Sharpe ratio, we observe a distinct pattern across different portfolio sizes, forecast horizons, and transaction costs. The CCC-HEAVY-GAS model consistently achieves the highest Sharpe ratios, while the CAW and HEAVY-GAS-tF models tend to produce the lowest. This difference is primarily driven by the higher ex-post returns obtained with the former model. Although turnover, concentration ratios, and short positions suggest that the CCC-HEAVY-GAS model is more expensive to implement than any of the realized multivariate models across all horizons (particularly the CAW and HEAVY-GAS-tF models), the difference in mean ex-post returns from the former model is large enough to offset the additional transaction costs, even at the highest level included in our analysis (2%).

The utility gains obtained following Fleming et al. (2003) approach offer additional insights into the economic trade-offs between the models included in our comparison exercise. Tables 6 and 7 show that the utility gains for an investor using the CCC-HEAVY-GAS model instead of the HAR-DRD model (which typically produces the highest Sharpe Ratios among realized multivariate models) are substantial. Specifically, an investor would be willing to pay between 270 and 500 basis points per year to switch between these models. This advantage remains significant even when accounting for different levels of transaction costs. Notably, we also observe that for horizons longer than one day, an investor would be better off using the HAR-DRD model than using a model that produces lower ex-post portfolio volatility, such as the multivariate HEAVY-GAS model, which should be less expensive to implement (due to substantially lower turnover). Again, this result is driven by the differences in ex-post mean returns. Put differently, our results indicate that pursuing a GMV strategy based on a model with full realized covariance matrix does not necessarily produce the best portfolio performance metrics.

A second finding that points in this direction is that the economic performance of the forecasts obtained with the CCC-HEAVY-GAS models surpasses the one obtained with the DCC-HEAVY-GAS model. The apparent irrelevance of the correlations *dynamics* in these models arises due to two reasons. First, the Sharpe ratio is considerably higher for the first model due to higher annualized returns combined with relatively similar portfolio volatilities. Additionally, the turnover of the CCC-HEAVY-GAS models is lower than that of the HAR-DRD model.

In addition, we note three small side results, which hold for both portfolio sizes. The first one concerns the models using daily returns only. We notice that modeling volatilities and correlations separately (as in the CCC-GARCH model) is better than modeling returns jointly (as in the BEKK model). Second, despite the low turnover of the CAW and the HEAVY-GAS-tF model relative to the HAR-DRD model, the latter model is preferred due to higher ex-post annualized returns.[2]

[2] In an additional analysis available upon request, we observe that the CCC(DCC)-HEAVY model (including the realized variance into the model equation of the conditional variance as in Shephard and Sheppard (2010), while leaving the correlation models intact) performs roughly at par with the CCC-HEAVY-GAS model. The former has a lower turnover, while the latter has higher annualized returns. However, in the case of dimension 50, the gain in utility to switch from the HAR-DRD model to the CCC-HEAVY-GAS model is statistically significant. At the same time, this is not significant for the CCC-HEAVY model in the case of daily and weekly forecasts.

In sum, we find that GMVPs built using realized covariance models exhibit the lowest ex-post portfolio volatilities. However, the advantage of these models relative to those using realized variances and daily returns becomes smaller when we examine other popular volatility measures, such as the standard deviation of ex-post portfolio returns. In addition, we observe that the latter class of models produces higher Sharpe ratios, even in the presence of transaction costs, and generates the highest utility gains for investors with varying levels of risk aversion. Moreover, these potential utility gains can be substantial -up to 500 basis points a year- for an investor who wants to switch from the best-performing model using the full realized covariance matrix (HAR-DRD) to a model that uses daily returns and realized variances. These results are robust across different portfolio sizes. Lastly, the similarity in ex-post portfolio volatilities between the CCC and DCC versions of the GARCH and HEAVY-GAS models and substantially lower ex-post returns for the models that include correlations dynamics, suggests that there are no significant economic gains from modeling correlations dynamically, either using returns or realized correlations.

6 Robustness checks

We conducted several robustness checks to validate the reliability and consistency of our empirical results. In line with the methodology of Hautsch et al. (2015), we focused on three different dimensions. We have already shown in our results section that our main results hold for different portfolio sizes($k = 30$ and $k = 50$). In this section, we focus on assessing the impact of varying the length of the updating and estimation windows.

Tables A.1 and A.2 in Appendix A show the average statistical loss functions and the results on the GMVPs when re-estimating the parameters of each model on a monthly basis ($T_w = 22$). Again, we observe that nearly 75% of the improvement in the RMSE and at least 93% of the STEIN loss functions can be attributed to the use of realized variances (Table A.5).

Consistent with our main findings, we observe that although the HEAVY-GAS-tF model has significantly the lowest ex-post realized portfolio volatility, the CCC-HEAVY-GAS model delivers again the highest Sharpe ratio due to the higher mean ex-post returns and the lower turnover. Moreover, we find that an investor with quadratic utility is willing to

Table 6: The GMV portfolio, dimension 30

This table shows portfolio statistics of the Global Minimum Variance portfolio, based on daily, weekly, biweekly, and monthly predictions of the 30 × 30 covariance matrix, according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,000 observations and re-estimated after 250 observations. We report the average annualized return and standard deviation (both using the true realized covariance matrix and daily returns), as well as the turnover (TO), portfolio concentration (CO), and short positions (SP). The lowest portfolio volatilities are marked in bold. We also list the p-value that belongs to the model confidence set approach (mcs) of the lowest ex-post daily volatility using the true realized covariance matrix based on a 5% significance level. Bold p-values correspond to models that belong to the model confidence set. Finally, the table reports the economic gains of switching from each model listed in the column to the HAR-DRD model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. A bold Δ_γ significantly differs from zero at the 5% level. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2005 until December 2019 and contains 3696 observations.

		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts									
Mean return		7.301	8.592	6.931	11.198	9.18	6.001	5.750	7.177
$\hat{\sigma}_p$		10.524	9.611	9.603	9.175	9.15	8.438	**8.352**	8.527
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.732	10.775	10.837	10.590	10.60	**9.913**	9.945	10.032
TO		0.406	0.450	0.463	0.664	0.71	0.300	0.238	0.821
CO		0.568	0.475	0.490	0.441	0.46	0.378	0.360	0.399
SP		-0.651	-0.393	-0.426	-0.360	-0.40	-0.239	-0.205	-0.276
$c = 0\%$	Sharpe	0.622	0.797	0.640	1.057	0.87	0.605	0.578	0.715
	Δ_1	6	-134	33	**-396**	-195	116	142	
	Δ_{10}	173	-64	109	**-344**	-141	106	134	
$c = 1\%$	Sharpe	0.535	0.692	0.532	0.899	0.70	0.529	0.518	0.509
	Δ_1	-98	-227	-57	**-436**	-223	-15	-5	
	Δ_{10}	69	-158	18	**-384**	-170	-26	-13	
$c = 2\%$	Sharpe	0.448	0.587	0.424	0.741	0.53	0.453	0.457	0.303
	Δ_1	-203	-321	-148	**-475**	-252	-146	-152	
	Δ_{10}	-36	-251	-72	**-423**	-198	-157	-160	
Panel B: Weekly forecasts									
Mean return		7.959	8.524	7.085	10.478	8.627	5.632	5.379	7.512
$\hat{\sigma}_p$		10.650	9.918	9.896	9.432	9.391	8.738	**8.636**	8.820
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.548	10.274	10.364	9.917	10.002	9.720	**9.525**	9.628
TO		0.394	0.344	0.355	0.588	0.623	0.282	0.231	0.632
CO		0.558	0.468	0.481	0.429	0.441	0.376	0.358	0.386
SP		-0.627	-0.400	-0.429	-0.342	-0.370	-0.233	-0.202	-0.250
$c = 0\%$	Sharpe	0.689	0.830	0.684	1.057	0.862	0.579	0.565	0.780
	Δ_1	-24	-95	50	**-294**	-108	**189**	**212**	
	Δ_{10}	161	-36	117	**-268**	-74	**197**	**203**	
$c = 1\%$	Sharpe	0.672	0.813	0.666	1.027	0.831	0.565	0.552	0.747
	Δ_1	-36	-109	36	**-296**	-108	**171**	**192**	
	Δ_{10}	149	-50	103	**-270**	-75	179	**183**	
$c = 2\%$	Sharpe	0.655	0.796	0.649	0.997	0.800	0.550	0.540	0.714
	Δ_1	-48	-124	22	**-298**	-109	154	**172**	
	Δ_{10}	137	-65	89	**-272**	-75	162	163	

(continued from previous page)

		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel C: Biweekly forecasts									
Mean return		8.136	8.838	7.463	9.827	8.150	5.108	4.891	6.780
$\hat{\sigma}_p$		10.663	10.165	10.145	9.530	9.490	8.900	**8.789**	8.973
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		12.359	10.476	10.671	9.891	10.177	9.937	9.739	**9.664**
TO		0.381	0.303	0.315	0.518	0.545	0.266	0.224	0.542
CO		0.547	0.470	0.483	0.419	0.429	0.374	0.357	0.378
SP		-0.602	-0.414	-0.441	-0.325	-0.348	-0.228	-0.198	-0.232
$c = 0\%$	Sharpe	0.658	0.844	0.699	0.994	0.801	0.514	0.502	0.702
	Δ_1	-106	-198	-58	**-302**	-132	170	**190**	
	Δ_{10}	169	-121	37	**-282**	-84	195	**196**	
$c = 1\%$	Sharpe	0.651	0.836	0.692	0.980	0.787	0.507	0.496	0.688
	Δ_1	-110	-204	-64	**-303**	-132	163	**182**	
	Δ_{10}	165	-127	31	**-282**	-84	188	**188**	
$c = 2\%$	Sharpe	0.643	0.829	0.684	0.967	0.774	0.501	0.491	0.673
	Δ_1	-114	-210	-69	**-304**	-132	156	**174**	
	Δ_{10}	161	-133	26	**-283**	-84	181	180	
Panel D: Monthly forecasts									
Mean return		7.581	8.590	7.211	9.150	7.549	4.679	4.529	6.263
$\hat{\sigma}_p$		10.669	10.521	10.513	9.655	9.626	9.136	**9.014**	9.171
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	**(0.23)**
StDev ret		13.395	10.754	10.952	**9.723**	10.051	10.268	10.211	9.818
TO		0.359	0.261	0.273	0.409	0.428	0.245	0.211	0.442
CO		0.527	0.478	0.489	0.405	0.412	0.372	0.355	0.367
SP		-0.556	-0.430	-0.451	-0.301	-0.315	-0.223	-0.192	-0.208
$c = 0\%$	Sharpe	0.566	0.799	0.658	0.941	0.751	0.456	0.444	0.638
	Δ_1	-90	-223	-83	**-290**	-126	163	**177**	
	Δ_{10}	304	-129	30	**-299**	-104	205	**214**	
$c = 1\%$	Sharpe	0.563	0.796	0.656	0.936	0.746	0.453	0.441	0.633
	Δ_1	-91	-225	-85	**-290**	-126	161	**175**	
	Δ_{10}	302	-131	28	**-299**	-104	203	**212**	
$c = 2\%$	Sharpe	0.560	0.793	0.653	0.931	0.741	0.450	0.439	0.628
	Δ_1	-92	-227	-87	**-290**	-127	158	**172**	
	Δ_{10}	301	-133	26	**-299**	-104	201	**209**	

Table 7: The GMV portfolio, dimension 50

This table shows portfolio statistics of the Global Minimum Variance portfolio, based on daily, weekly, biweekly, and monthly predictions of the 50 × 50 covariance matrix, according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,000 observations and re-estimated after 250 observations. We report the average annualized return and standard deviation (both using the true realized covariance matrix and daily returns), as well as the turnover (TO), portfolio concentration (CO), and short positions (SP). The lowest portfolio volatilities are marked in bold. We also list the p-value that belongs to the model confidence set approach (mcs) of the lowest ex-post daily volatility using the true realized covariance matrix based on a 5% significance level. Bold p-values correspond to models that belong to the model confidence set. Finally, the table reports the economic gains of switching from each model listed in the column to the HAR-DRD model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. A bold Δ_γ significantly differs from zero at the 5% level. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2005 until December 2019 and contains 3696 observations.

		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts									
Mean return		4.816	9.450	7.583	12.831	9.90	6.851	6.628	9.049
$\hat{\sigma}_p$		10.641	9.509	9.542	9.073	9.08	8.129	**8.017**	8.283
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.793	10.600	10.537	10.386	10.29	**9.569**	9.667	9.658
TO		0.475	0.591	0.608	0.902	0.97	0.331	0.255	1.164
CO		0.577	0.466	0.488	0.442	0.46	0.366	0.346	0.402
SP		-0.982	-0.608	-0.662	-0.593	-0.66	-0.397	-0.348	-0.489
$c = 0\%$	Sharpe	0.408	0.892	0.720	1.235	0.96	0.716	0.686	0.937
	Δ_1	**446**	-31	155	**-371**	-79	**219**	**242**	
	Δ_{10}	**653**	56	236	**-305**	-22	**211**	**243**	
$c = 1\%$	Sharpe	0.307	0.751	0.574	1.017	0.72	0.629	0.619	0.633
	Δ_1	272	-175	15	**-437**	-128	9	13	
	Δ_{10}	**479**	-89	95	**-371**	-71	1	14	
$c = 2\%$	Sharpe	0.205	0.610	0.429	0.798	0.49	0.541	0.553	0.329
	Δ_1	99	-319	-125	**-503**	-177	**-201**	**-216**	
	Δ_{10}	305	-233	-45	**-437**	-120	**-208**	**-215**	
Panel B: Weekly forecasts									
Mean return		4.838	8.847	7.353	11.168	8.639	5.981	5.881	8.038
$\hat{\sigma}_p$		10.773	9.812	9.813	9.345	9.345	8.434	**8.302**	8.599
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.662	10.218	10.336	9.667	9.734	9.483	**9.299**	9.436
TO		0.467	0.461	0.467	0.806	0.860	0.318	0.251	0.910
CO		0.567	0.467	0.487	0.430	0.449	0.364	0.345	0.387
SP		-0.954	-0.632	-0.682	-0.576	-0.632	-0.391	-0.345	-0.452
$c = 0\%$	Sharpe	0.415	0.866	0.711	1.155	0.888	0.631	0.632	0.852
	Δ_1	343	-73	77	**-311**	-57	**206**	**214**	
	Δ_{10}	**556**	-3	159	**-290**	-31	**210**	**203**	
$c = 1\%$	Sharpe	0.395	0.843	0.689	1.113	0.843	0.614	0.619	0.803
	Δ_1	321	-96	55	**-316**	-60	176	**181**	
	Δ_{10}	**534**	-26	136	**-296**	-34	180	170	
$c = 2\%$	Sharpe	0.375	0.820	0.666	1.071	0.798	0.597	0.605	0.755
	Δ_1	299	-119	33	**-321**	-62	147	148	
	Δ_{10}	**512**	-48	114	**-301**	-36	151	136	

(continued from previous page)

		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel C: Biweekly forecasts									
Mean return		4.890	8.945	7.697	10.533	8.350	5.318	5.472	7.263
$\hat{\sigma}_p$		10.772	10.038	10.034	9.454	9.448	8.600	**8.456**	8.755
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.593	10.531	10.810	9.768	10.016	9.773	9.601	**9.560**
TO		0.458	0.407	0.416	0.713	0.756	0.306	0.247	0.783
CO		0.557	0.469	0.488	0.421	0.437	0.362	0.344	0.376
SP		-0.924	-0.655	-0.702	-0.560	-0.607	-0.386	-0.341	-0.424
$c = 0\%$	Sharpe	0.422	0.849	0.712	1.078	0.834	0.544	0.570	0.760
	Δ_1	259	-158	-31	**-325**	-104	**197**	**180**	
	Δ_{10}	**456**	-67	88	**-306**	-63	216	183	
$c = 1\%$	Sharpe	0.412	0.840	0.702	1.060	0.815	0.536	0.563	0.739
	Δ_1	251	-168	-40	**-327**	-105	185	166	
	Δ_{10}	**447**	-77	78	**-308**	-63	204	170	
$c = 2\%$	Sharpe	0.402	0.830	0.693	1.041	0.796	0.528	0.557	0.719
	Δ_1	242	-177	-49	**-329**	-106	173	153	
	Δ_{10}	**439**	-86	69	**-309**	-64	192	156	
Panel D: Monthly forecasts									
Mean return		4.763	8.528	7.415	10.045	8.211	4.633	4.992	6.726
$\hat{\sigma}_p$		10.721	10.379	10.377	9.592	9.581	8.852	**8.692**	8.958
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.04)
StDev ret		12.630	10.601	11.094	**9.558**	10.010	10.608	10.173	9.812
TO		0.445	0.353	0.363	0.564	0.596	0.288	0.240	0.641
CO		0.538	0.472	0.487	0.409	0.421	0.360	0.342	0.362
SP		-0.870	-0.676	-0.712	-0.535	-0.570	-0.379	-0.335	-0.385
$c = 0\%$	Sharpe	0.377	0.805	0.668	1.051	0.820	0.437	0.491	0.686
	Δ_1	228	-172	-55	**-334**	-146	217	177	
	Δ_{10}	**521**	-94	73	**-359**	-127	294	211	
$c = 1\%$	Sharpe	0.373	0.801	0.665	1.044	0.814	0.434	0.488	0.678
	Δ_1	226	-175	-59	**-335**	-147	213	172	
	Δ_{10}	**519**	-97	70	**-360**	-128	290	207	
$c = 2\%$	Sharpe	0.369	0.797	0.661	1.037	0.807	0.431	0.485	0.671
	Δ_1	224	-179	-62	**-336**	-148	209	168	
	Δ_{10}	**517**	-100	67	**-361**	-128	286	202	

pay a fee ranging from 235 to 452 basis points per year to switch from the HAR-DRD model to the CCC-HEAVY-GAS model.

Finally, we increase the estimation window from 1,000 to 1,500 days (roughly equivalent to six years). Tables A.3 and A.4 show the corresponding results. The percentage of improvement in the QLIK and RMSE attributed to the use of realized variances is similar to the one observed in our original specification. However, in the case of the STEIN function, this percentage decrease is lower (75% on average vs. 91%). The results regarding the economic performance of GMVPs are broadly similar to those in our main specification. The CCC-HEAVY-GAS model continues to outperform models using the full realized covariance matrix, this time with an even greater Sharpe ratio difference and larger utility gains, ranging from 360 to 581 basis points per year.

Overall, we observe that our main results are robust against changes in the portfolio size and the length of the estimation and updating windows. Consistent with our main specification, we find that using HF data to model variances contributes the most to the statistical performance improvements (roughly 69%). Moreover, Sharpe ratios and utility gains can become even higher when using a larger estimation window.

7 Conclusions

This study provides new insights into the advantages of using high-frequency data to model financial volatilities in medium-sized portfolios. We build upon the work of Hautsch et al. (2015) - who find that incorporating realized measures of covariances does matter for portfolio selection - by evaluating both the statistical accuracy and the economic performance of forecasts obtained using a variety of approaches in multivariate covariance forecasting, and investigating the sources of these gains. We do so by disentangling the realized covariances into realized variances and realized correlations. Our results indicate that a significant portion of the improvement in forecasts' statistical performance derived from incorporating HF data can be obtained using models that combine daily returns and realized variances, particularly on the daily horizon (over 76%). Furthermore, although our results confirm that the models that use the full realized covariance matrix offer superior ex-post portfolio volatilities, we observe significant economic advantages of simpler models that combine realized variances and daily returns, particularly the CCC-HEAVY-GAS, as evidenced by its higher Sharpe ratios and substantial utility gains. These findings are robust across different horizons, portfolio sizes (30 and 50 assets), and various estimation and updating window lengths. Finally, our results indicate that dynamically modeling correlations, as done in DCC models, do not yield significant economic benefits over simpler approaches for portfolios of this size.

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A Additional results

Table A.1: Statistical loss functions with different updating windows (dimension 50)

This table shows the mean of the RMSE, QLIK, and STEIN loss functions based on daily, weekly, and biweekly predictions of the covariance matrix of returns of 50 stocks according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,000 observations and re-estimated after 22 observations. The lowest value of all loss functions across the models is marked in bold. In addition, we report the mcs p-values based on a 5% significance level. The p-values of the models within the model confidence set are marked in bold. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2005 until December 2019 and contains 3,696 observations.

	BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts								
RMSE	64.91	60.27	60.64	51.52	51.32	50.45	50.56	**46.66**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(1.00)
QLIK	125.44	61.21	64.65	57.60	61.13	55.89	**55.78**	56.51
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.75)**	**(1.00)**	(0.04)
STEIN	189.74	182.03	176.88	175.29	**171.77**	208.02	186.21	188.61
	(0.00)	(0.00)	(0.01)	(0.01)	**(1.00)**	(0.00)	(0.00)	(0.00)
Panel B: Weekly forecasts								
RMSE	279.70	265.81	267.94	214.88	213.86	201.42	**200.08**	**191.30**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.05)	(0.12)	(1.00)
QLIK	202.01	142.41	144.90	139.77	142.71	**137.89**	**137.42**	138.43
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.14)	(1.00)	(0.02)
STEIN	35.38	27.49	27.91	19.43	20.53	22.30	**17.93**	21.53
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel C: Biweekly forecasts								
RMSE	545.28	532.28	536.90	424.97	422.95	392.77	382.10	**379.27**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.31)**	**(0.79)**	**(1.00)**
QLIK	232.41	178.17	180.10	175.34	177.74	173.60	**172.98**	173.69
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.11)**	**(1.00)**	**(0.11)**
STEIN	28.20	22.70	23.14	13.42	14.52	14.74	**11.18**	14.44
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel D: Monthly forecasts								
RMSE	1210.4	1224.2	1234.7	989.95	984.15	897.05	**839.84**	901.96
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.21)**	**(1.00)**	**(0.21)**
QLIK	264.27	219.53	220.72	216.54	218.14	214.86	**213.89**	213.99
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.02)	**(1.00)**	**(0.85)**
STEIN	23.15	22.02	22.19	10.81	11.61	11.40	**8.01**	10.78
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)

Table A.2: GMV portfolio analysis with updating window length $T_w = 22$ (dimension 50)

This table shows portfolio statistics of the Global Minimum Variance portfolio, based on daily, weekly, biweekly and monthly predictions of the 50×50 covariance matrix, according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,000 observations and re-estimated after 22 observations. We report the average annualized return and standard deviation (both using the true realized covariance matrix and daily returns), as well as the turnover (TO), portfolio concentration (CO), and short positions (SP). The lowest portfolio volatilities are marked in bold. We also list the p-value that belongs to the model confidence set approach (mcs) of the lowest ex-post daily volatility using the true realized covariance matrix based on a 5% significance level. Bold p-values correspond to models that belong to the model confidence set. Finally, the table reports the economic gains of switching from each model listed in the column to the HAR-DRD model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. A bold Δ_γ is significantly different from zero at the 5% level. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2005 until December 2019 and contains 3696 observations.

		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts									
Mean return		4.437	7.980	5.956	12.181	9.394	7.068	6.603	8.855
$\hat{\sigma}_p$		10.823	9.431	9.487	8.978	9.086	8.119	**8.017**	8.256
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		12.286	10.440	10.367	10.238	10.291	**9.606**	9.680	9.688
TO		0.484	0.601	0.612	0.918	0.986	0.314	0.252	1.166
CO		0.582	0.473	0.495	0.443	0.467	0.364	0.346	0.403
SP		-1.000	-0.617	-0.677	-0.592	-0.672	-0.393	-0.348	-0.490
$c = 0\%$	Sharpe	0.361	0.764	0.575	1.190	0.913	0.736	0.682	0.914
	Δ_1	**470**	95	297	**-327**	-48	**178**	**225**	
	Δ_{10}	**728**	163	358	**-277**	7	**171**	**224**	
$c = 1\%$	Sharpe	0.262	0.619	0.426	0.964	0.671	0.653	0.616	0.611
	Δ_1	**298**	-47	157	**-390**	-93	-37	-5	
	Δ_{10}	**556**	21	218	**-340**	-39	-44	-6	
$c = 2\%$	Sharpe	0.163	0.474	0.277	0.738	0.430	0.571	0.551	0.307
	Δ_1	126	-190	17	**-452**	-139	**-251**	**-236**	
	Δ_{10}	384	-122	79	**-403**	-85	**-259**	**-237**	
Panel B: Weekly forecasts									
Mean return		5.470	7.433	5.849	10.319	7.948	6.202	5.856	7.795
$\hat{\sigma}_p$		10.952	9.778	9.837	9.241	9.342	8.420	**8.301**	8.540
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.984	10.254	10.397	9.539	9.731	9.523	**9.317**	9.514
TO		0.476	0.468	0.473	0.818	0.872	0.301	0.248	0.907
CO		0.572	0.479	0.501	0.431	0.453	0.362	0.345	0.387
SP		-0.972	-0.643	-0.702	-0.574	-0.645	-0.387	-0.345	-0.452
$c = 0\%$	Sharpe	0.456	0.725	0.563	1.082	0.817	0.651	0.629	0.819
$c = 0\%$	Sharpe	0.46	0.72	0.56	1.08	0.82	0.65	0.63	0.82
	Δ_1	259	43	203	**-252**	-13	**159**	**192**	
	Δ_{10}	**500**	110	283	**-250**	6	**160**	**175**	
$c = 1\%$	Sharpe	0.436	0.702	0.540	1.038	0.772	0.635	0.615	0.771
	Δ_1	237	21	182	**-257**	-15	129	159	
	Δ_{10}	**478**	88	262	**-254**	4	130	142	
$c = 2\%$	Sharpe	0.416	0.679	0.517	0.995	0.726	0.619	0.602	0.723
	Δ_1	216	-1	160	**-261**	-17	98	126	
	Δ_{10}	**456**	66	240	**-259**	2	99	109	

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		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel C: Biweekly forecasts									
Mean return		5.847	7.649	6.362	9.585	7.551	5.607	5.455	7.036
$\hat{\sigma}_p$		10.960	10.011	10.070	9.340	9.428	8.583	**8.455**	8.675
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		11.560	10.677	10.905	9.667	9.976	9.787	9.605	**9.582**
TO		0.468	0.413	0.421	0.721	0.766	0.288	0.244	0.778
CO		0.561	0.482	0.503	0.421	0.440	0.361	0.344	0.376
SP		-0.943	-0.665	-0.719	-0.557	-0.619	-0.381	-0.342	-0.423
$c = 0\%$	Sharpe	0.506	0.716	0.583	0.992	0.757	0.573	0.568	0.734
	Δ_1	140	-50	81	**-254**	-48	145	158	
	Δ_{10}	332	53	206	**-246**	-12	163	160	
$c = 1\%$	Sharpe	0.496	0.707	0.574	0.973	0.738	0.565	0.562	0.714
	Δ_1	132	-59	72	**-256**	-48	133	145	
	Δ_{10}	324	44	197	**-248**	-12	151	147	
$c = 2\%$	Sharpe	0.485	0.697	0.564	0.954	0.718	0.558	0.555	0.693
	Δ_1	124	-69	63	**-257**	-48	120	131	
	Δ_{10}	316	34	188	**-249**	-12	139	134	
Panel D: Monthly forecasts									
Mean return		5.548	7.514	6.367	9.069	7.307	4.994	4.943	6.726
$\hat{\sigma}_p$		10.919	10.355	10.408	9.463	9.530	8.828	**8.689**	8.846
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		12.885	10.942	11.433	**9.610**	10.244	10.594	10.176	9.718
TO		0.455	0.357	0.368	0.567	0.600	0.270	0.237	0.632
CO		0.542	0.485	0.502	0.408	0.423	0.358	0.342	0.362
SP		-0.888	-0.685	-0.728	-0.531	-0.578	-0.374	-0.336	-0.383
$c = 0\%$	Sharpe	0.431	0.687	0.557	0.944	0.713	0.471	0.486	0.692
	Δ_1	154	-66	54	**-235**	-53	182	183	
	Δ_{10}	**488**	56	226	**-246**	-2	266	226	
$c = 1\%$	Sharpe	0.426	0.683	0.553	0.937	0.707	0.468	0.483	0.685
	Δ_1	152	-69	51	**-236**	-53	178	178	
	Δ_{10}	**485**	52	223	**-246**	-3	262	221	
$c = 2\%$	Sharpe	0.422	0.679	0.550	0.930	0.700	0.465	0.480	0.677
	Δ_1	150	-72	48	**-237**	-54	174	174	
	Δ_{10}	**483**	49	220	**-247**	-3	258	217	

Table A.3: Statistical loss functions with estimation window length $T_s = 1500$ (dimension 50)

This table shows the mean of the RMSE, QLIK, and STEIN loss functions based on daily, weekly, and biweekly predictions of the covariance matrix of returns of 50 stocks according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,500 observations and re-estimated after 250 observations. The lowest value of all loss functions across the models is marked in bold. In addition, we report the mcs p-values based on a 5% significance level. The p-values of the models within the model confidence set are marked in bold. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2007 until December 2019 and contains 3,196 observations.

	BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts								
RMSE	71.50	66.03	66.55	56.66	56.15	55.41	56.35	**51.25**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**
QLIK	135.62	61.80	65.14	58.64	62.04	57.87	**57.50**	58.22
	(0.00)	(0.00)	(0.00)	(0.03)	(0.00)	**(0.41)**	**(1.00)**	**(0.25)**
STEIN	197.37	200.02	189.33	188.38	**181.15**	202.86	185.96	194.41
	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)	**(0.14)**	(0.00)
Panel B: Weekly forecasts								
RMSE	306.11	281.92	284.29	233.85	230.90	216.86	224.06	**206.82**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(0.06)**	**(0.06)**	**(1.00)**
QLIK	212.63	143.36	146.09	140.96	143.78	140.21	**139.26**	140.38
	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.05)	**(1.00)**	(0.05)
STEIN	39.09	25.56	25.32	21.35	21.92	22.14	**18.06**	22.76
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel C: Biweekly forecasts								
RMSE	593.64	548.06	552.79	457.17	451.61	415.59	427.13	**402.98**
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.25)	(0.24)	(1.00)
QLIK	242.89	178.90	181.14	176.66	178.93	176.16	**174.91**	175.77
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.02)	**(1.00)**	**(0.07)**
STEIN	32.01	18.80	18.92	14.87	15.64	14.85	**11.37**	15.52
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
Panel D: Monthly forecasts								
RMSE	1300.4	1207.0	1217.3	1050.3	1039.0	934.88	**929.36**	935.40
	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	**(0.99)**	**(1.00)**	**(0.99)**
QLIK	274.00	219.94	221.49	218.03	219.48	217.68	**215.94**	216.44
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	**(0.38)**
STEIN	27.19	16.03	16.07	12.15	12.73	11.62	**8.24**	11.79
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)

Table A.4: GMV portfolio analysis for estimation and updating window lengths $T_s = 1500$, $T_w = 250$ (dimension 50)

This table shows portfolio statistics of the Global Minimum Variance portfolio, based on daily, weekly, biweekly, and monthly predictions of the 50 × 50 covariance matrix, according to the following models: BEKK-GARCH, CCC(DCC)-GARCH (CCC(DCC)-GAR), CCC(DCC)-HEAVY-GAS (CCC(DCC)-HGAS), multivariate HEAVY-GAS, CAW, and HAR-DRD (HDRD). Parameters are estimated with a moving window of 1,500 observations and re-estimated after 250 observations. We report the average annualized return and standard deviation (both using the true realized covariance matrix and daily returns), as well as the turnover (TO), portfolio concentration (CO), and short positions (SP). The lowest portfolio volatilities are marked in bold. We also list the p-value that belongs to the model confidence set approach (mcs) of the lowest ex-post daily volatility using the true realized covariance matrix based on a 5% significance level. Bold p-values correspond to models that belong to the model confidence set. Finally, the table reports the economic gains of switching from each model listed in the column to the HAR-DRD model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. A bold Δ_γ is significantly different from zero at the 5% level. Panel A, B, C, and D show results of daily, weekly, biweekly, and monthly forecasts, respectively. The out-of-sample period goes from January 2007 until December 2019 and contains 3,196 observations.

		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel A: Daily forecasts									
Mean return		5.153	9.548	7.238	13.659	9.837	6.981	6.965	9.429
$\hat{\sigma}_p$		11.410	9.632	9.645	9.288	9.295	8.353	**8.237**	8.513
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		12.577	10.915	10.761	10.796	10.620	**9.936**	9.996	9.989
TO		0.526	0.538	0.571	0.868	0.961	0.362	0.256	1.199
CO		0.625	0.448	0.477	0.426	0.454	0.376	0.351	0.406
SP		-1.111	-0.566	-0.647	-0.553	-0.649	-0.433	-0.361	-0.505
$c = 0\%$	Sharpe	0.410	0.875	0.673	1.265	0.926	0.703	0.697	0.944
	Δ_1	**457**	30	227	**-415**	-34	**244**	**246**	
	Δ_{10}	**720**	85	299	**-339**	24	**240**	**247**	
$c = 1\%$	Sharpe	0.304	0.751	0.539	1.062	0.698	0.611	0.632	0.641
	Δ_1	287	-169	69	**-498**	-94	33	9	
	Δ_{10}	**550**	-81	141	**-422**	-36	29	9	
$c = 2\%$	Sharpe	0.199	0.626	0.405	0.860	0.470	0.519	0.568	0.339
	Δ_1	118	-335	-89	**-581**	-154	-178	**-229**	
	Δ_{10}	381	-248	-17	**-505**	-96	-182	**-229**	
Panel B: Weekly forecasts									
Mean return		5.657	9.823	7.647	12.111	8.689	5.808	6.171	8.451
$\hat{\sigma}_p$		11.564	9.901	9.911	9.575	9.572	8.680	**8.538**	8.827
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		12.285	10.280	10.497	9.940	9.979	9.880	**9.603**	9.889
TO		0.522	0.476	0.495	0.795	0.872	0.358	0.254	0.949
CO		0.615	0.444	0.473	0.415	0.440	0.375	0.351	0.391
SP		-1.082	-0.570	-0.645	-0.538	-0.623	-0.430	-0.359	-0.470
$c = 0\%$	Sharpe	0.461	0.956	0.728	1.218	0.871	0.588	0.643	0.855
	Δ_1	306	-133	87	**-366**	-23	**264**	**225**	
	Δ_{10}	**547**	-97	143	**-361**	-15	**263**	**200**	
$c = 1\%$	Sharpe	0.439	0.932	0.705	1.178	0.827	0.570	0.629	0.806
	Δ_1	284	-157	64	**-373**	-27	**234**	**190**	
	Δ_{10}	**526**	-121	120	**-368**	-18	**234**	165	
$c = 2\%$	Sharpe	0.418	0.909	0.681	1.138	0.783	0.551	0.616	0.758
	Δ_1	263	-181	41	**-381**	-31	**205**	155	
	Δ_{10}	**504**	-145	98	**-376**	-22	204	130	

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		BEKK	CCC GAR	DCC GAR	CCC HGAS	DCC HGAS	CAW	HEAVY GAS	HDRD
Panel C: Biweekly forecasts									
Mean return		5.879	10.216	8.347	11.375	8.326	4.941	5.707	7.419
$\hat{\sigma}_p$		11.580	10.067	10.082	9.691	9.677	8.855	**8.704**	8.987
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		12.072	10.543	10.956	10.116	10.371	10.299	**9.984**	10.128
TO		0.517	0.443	0.460	0.720	0.783	0.355	0.252	0.826
CO		0.604	0.445	0.473	0.405	0.428	0.374	0.350	0.380
SP		-1.052	-0.585	-0.657	-0.521	-0.597	-0.427	-0.357	-0.442
$c = 0\%$	Sharpe	0.487	0.969	0.762	1.124	0.803	0.480	0.572	0.733
	Δ_1	176	-275	-84	**-396**	-88	**250**	170	
	Δ_{10}	374	-235	-3	**-397**	-65	**266**	156	
$c = 1\%$	Sharpe	0.476	0.958	0.751	1.106	0.784	0.471	0.565	0.712
	Δ_1	168	-285	-93	**-398**	-89	**238**	155	
	Δ_{10}	366	-245	-12	**-399**	-66	254	142	
$c = 2\%$	Sharpe	0.465	0.948	0.741	1.088	0.765	0.462	0.559	0.691
	Δ_1	160	**-295**	-103	**-401**	-90	**226**	141	
	Δ_{10}	358	-254	-21	**-402**	-67	242	128	
Panel D: Weekly forecasts									
Mean return		5.978	10.073	8.277	10.810	7.960	4.054	5.117	6.822
$\hat{\sigma}_p$		11.544	10.343	10.359	9.828	9.801	9.121	**8.959**	9.206
mcs p-value		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	**(1.00)**	(0.00)
StDev ret		13.344	10.756	11.591	**10.012**	10.612	11.122	10.548	10.302
TO		0.511	0.403	0.419	0.591	0.636	0.353	0.251	0.687
CO		0.584	0.448	0.474	0.391	0.409	0.373	0.349	0.365
SP		-0.993	-0.613	-0.677	-0.492	-0.552	-0.422	-0.352	-0.404
$c = 0\%$	Sharpe	0.448	0.937	0.714	1.080	0.750	0.364	0.485	0.662
	Δ_1	121	**-320**	-131	**-402**	-110	**286**	173	
	Δ_{10}	457	**-273**	5	**-431**	-79	**367**	197	
$c = 1\%$	Sharpe	0.444	0.932	0.710	1.073	0.743	0.361	0.482	0.655
	Δ_1	119	**-324**	-134	**-403**	-111	**282**	168	
	Δ_{10}	455	**-276**	2	**-432**	-80	364	192	
$c = 2\%$	Sharpe	0.439	0.928	0.706	1.066	0.736	0.357	0.480	0.647
	Δ_1	117	**-327**	-137	**-404**	-112	**278**	163	
	Δ_{10}	453	**-279**	-1	**-433**	-80	**360**	187	

Table A.5: Percentage improvement in loss functions

This table shows the percentage decrease in statistical loss functions attributed to the inclusion of realized variances. It is calculated as the reduction in the RMSE, QLIK, and STEIN average loss functions obtained when transitioning from the CCC(DCC)-GARCH models to the CCC(DCC)-HEAVY-GAS models, relative to the difference between the average loss function of the CCC(DCC)-GARCH models and that of the realized multivariate models, CAW, HEAVY-GAS-tF and HAR-DRD. This analysis is conducted for daily, weekly, biweekly, and monthly predictions of the covariance matrix of 50 stock returns, with two alternative specifications: i) parameters are estimated with a moving window of 1,000 observations and re-estimated after 22 observations, and ii) parameters are estimated with a moving window of 1,500 observations and re-estimated after 250 observations. An *na* indicates that the average loss function of the CCC(DCC)-HEAVY-GAS model is lower than that of the realized multivariate models.

	RMSE		STEIN		QLIK	
T_{IS}/T_w	1000/22	1500/250	1000/22	1500/250	1000/22	1500/250
Daily	80%	83%	*na*	*na*	52%	56%
Weekly	76%	75%	*na*	85%	42%	49%
Biweekly	74%	71%	95%	73%	45%	50%
Monthly	69%	60%	90%	66%	47%	49%
Average	75%	72%	93%	75%	47%	51%