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On Bubbles in Cryptocurrency Prices

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Abstract

This paper investigates how cryptocurrencies relate to concepts such as bubbles, Ponzi-schemes and digital gold in a tractable model for cryptocurrency prices. Investors in the baseline equilibrium hold coins to sell them at a profit to future users if they anticipate in increase in transactional demand per coin. Investors in a bubble equilibrium hold the cryptocurrency because they expect its price to appreciate merely due to future investment inflows. Investors who participate in a bubble equilibrium for a cryptocurrency with non-negative money growth experience Ponzi-scheme equivalent payoffs in the aggregate. The net investment inflows required to sustain a bubble equilibrium are smaller for cryptocurrencies with less new issuance, a lower level of transactional demand and higher growth in transactional demand. Cryptocurrencies with negative issuance (e.g., that burn transaction fees) may generate positive aggregate cash flows to investors even if their price path follows a bubble trajectory.

Keywords: Asset pricing, Bitcoin, crypto-asset, exchange rates, rational bubble

JEL Classification Numbers: E51, F31, G1.

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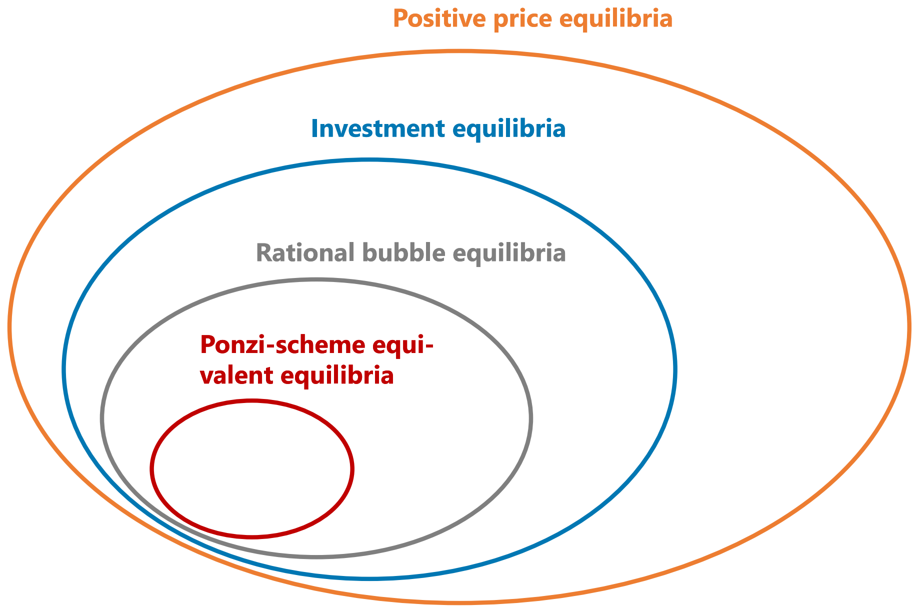
I. Introduction

The terminology employed to describe cryptocurrencies like Bitcoin is notably diverse. Skeptics have called Bitcoin *a Ponzi-scheme* (Welch, 2017; Carstens, 2018) and *the mother of all bubbles* (Roubini, 2018). Enthusiasts have called it *the flagship of a new asset class* (Harvey et al., 2021) and *digital gold* (Popper, 2016; Fink, 2024). These expressions arise from underlying beliefs about the characteristics of cryptocurrencies and their (dis)connection to price trends. The purpose of this paper is to articulate how underlying beliefs regarding the properties of cryptocurrencies and their prices relate to the use of terms such as bubbles, Ponzi-schemes and digital gold. We do so by investigating the various possible price trajectories of cryptocurrencies in a consistent theoretical framework.

The present paper develops a deliberately simple but general model for cryptocurrency prices that allows for classical rational bubble equilibria in the sense of Blanchard (1979) and Blanchard and Watson (1982).[1] Cryptocurrencies in the model differ from securities in that they pay no dividends and that they experience a certain dollar-amount of user demand, for example, to facilitate payments. We use the model to characterize the various possible equilibrium paths for the exchange rates of cryptocurrencies. A snapshot of those equilibrium price paths is provided in Figure 1. Depending on the properties of the cryptocurrency, there may be equilibria where exclusively users choose to hold cryptocurrency as well as investment equilibria where some agents hold the cryptocurrency purely for financial gain. In the investment equilibrium that correspond to the baseline equilibrium, the investors hold tokens to sell them at a profit to future users in anticipation of an increase in user demand.

[1]The concept of rational bubbles in asset prices has been described as fundamentally flawed because such bubbles are said to eventually outgrow all other wealth in the world. This critique is generally not consistent with the theory on rational bubbles. Theory predicts that the expected return of an asset of which the price contains a rational bubble component equals the required return in equilibrium, as is the case for all other financial assets. Suppose all financial assets were to face the same required return and a rational bubble were present in the price of a single asset. Then the value of that asset as a share of total financial wealth would be constant in expectation (abstracting from capital distributions, new issuance, the emergence of new financial assets, etc). If the bubble asset faced a below-average required return, e.g., because investors perceive the asset as an insurance for bad states of the world, then its share in total financial wealth would be diminishing in expectation.

Figure 1: Possible Equilibria for the Exchange Rates of Cryptocurrencies



Note: The figure shows the logical relationships among various sets of equilibrium price trajectories for cryptocurrencies. The areas are not meant to accurately reflect the prevalence of equilibria in practice.

The other investment equilibria are so-called rational bubble equilibria. In a rational bubble equilibrium, investors expect ongoing appreciation driven by future investment inflows rather than user demand. Such bubble equilibria can display a payoff equivalence to Ponzi-schemes in that investors lose in the aggregate by investing in such cryptocurrencies if the bubble persists, but, surprisingly, they do not have to be. Whether investors lose in the aggregate depends on characteristics of a cryptocurrency such as future issuance as well as whether investors perceive it as digital gold.

The analysis begins by determining the equilibrium exchange rate path that we will refer to as the baseline equilibrium. The baseline equilibrium is characterized by an exchange rate that is driven by either user demand or investor demand, or both. Only users hold the cryptocurrency if the expected growth rate in user demand is low or negative. Investors in the baseline equilibrium choose to temporarily hold the cryptocurrency only if they anticipate sufficiently high growth in user demand and sufficiently low money growth. Even though it may be mainly investors who hold the cryptocurrency initially, they expect to sell their coins at a profit to users in the future. The current exchange rate in the baseline equilibrium depends on the peak level of the user demand per coin rather than the user demand over the entire lifetime of the cryptocurrency. Any exchange rate lower than the baseline equilibrium cannot reflect an equilibrium.

The paper also derives bubble equilibrium paths for the exchange rates of cryptocurrencies that are reminiscent of classical rational bubbles in security prices. The exchange rate of a cryptocurrency on a bubble equilibrium path is higher than that on the baseline equilibrium path. A bubble equilibrium path is characterized by investors who hold the cryptocurrency solely due to anticipated exchange rate appreciation, without the exchange rate appreciation being driven by an expected increase in user demand. Rather, investors expect the exchange rate to appreciate due to expected future increases in investment demand.

We analyze the net investment inflows implied by various equilibrium paths for the prices of cryptocurrencies. Sustaining a bubble equilibrium for a cryptocurrency requires an ongoing flow of investment of which the level depends on various aspects. The required inflow of investors' funds increases in the issuance of new cryptocurrency units and in the level of user demand. It will be lower temporarily or even turn negative whenever there is growth in underlying user demand. The share of the coins held by users on a bubble equilibrium path tends to decrease over time, although temporary fluctuations may occur.

To study the equivalence between Ponzi-schemes and cryptocurrencies, we postulate a condition that we refer to as a Ponzi-scheme equivalence condition. This condition requires the equivalence between the cash flows of investors in a cryptocurrency and investors in a Ponzi-scheme. A Ponzi-scheme is an operation where the organizers pay returns to earlier investors from funds put into the scheme by later investors, who are then paid from funds contributed by even later investors, while the organizers skim off the scheme. An individual investor may profit from joining a Ponzi-scheme provided that the scheme continues, but the future cash flows of investors in the aggregate are characterized by a negative present value if the scheme persists. Similarly, the Ponzi-scheme equivalence condition requires the future *aggregate* cash flows of investors from investing in the cryptocurrency to have a negative present value.

The analysis shows that a cryptocurrency does not satisfy the Ponzi-scheme equivalence condition if its exchange rate path follows the baseline equilibrium. The cash outflows that investors experience when they acquire cryptocurrency in the baseline equilibrium will be balanced by expected cash inflows from selling cryptocurrency to users in the future. A cryptocurrency of which the exchange rate follows the baseline equilibrium path does not share the feature with Ponzi-schemes that future cash flows for investors have a negative present value in the aggregate.

Any cryptocurrency with non-negative money growth and nonzero user demand that follows a bubble equilibrium satisfies the Ponzi-scheme equivalence condition whenever its exchange rate path follows a bubble equilibrium. Sustaining a bubble equilibrium for such a cryptocurrency requires a sustained investment inflow that continuously drives up the exchange rate without investors ever profiting *in the aggregate* by cashing out their cryptocurrency holdings. Satisfying the aggregate user demand requires a decreasing number of coins as the exchange rate continuous to increase in the bubble equilibrium. Investors face cash outflows in the aggregate because they continue to accumulate cryptocurrency by purchasing increasingly expensive units held by cryptocurrency users. The present value of future aggregate cash flows to investors is negative even though the expected return of participating in the bubble equilibrium for individual investors satisfies their required return due to a persistent expected appreciation of the exchange rate.

For cryptocurrencies with *negative money growth*, the relationship between bubble equilibrium paths and the Ponzi-scheme equivalance condition is more nuanced. Cryptocurrencies may face negative money growth, for example, because transaction fees that are paid with the cryptocurrency are partly burned (e.g., Ethereum). We show that, depending on the parameters, cryptocurrencies with negative money growth can exhibit bubble equilibrium paths without satisfying the Ponzi-scheme equivalence condition. Investors may benefit both individually and in the aggregate from investing in such a cryptocurrency, even if the cryptocurrency's exchange rate follows a bubble equilibrium path. The net cash inflow for investors in both the baseline equilibrium and a bubble equilibrium come from purchases made by users who replenish their balances to desired levels. A cryptocurrency with negative money growth on a bubble path is less likely to satisfy the Ponzi-scheme equivalence condition if it experiences higher user growth or if the money growth is more negative. Moreover, it is less likely to satisfy the Ponzi-scheme equivalence condition if investors have a lower required return for holding the cryptocurrency. A lower required return could be the consequence of investors perceiving a cryptocurrency as a digital gold that acts as an insurance for particularly bad states of the world.

The remainder of this paper is organized as follows. Section II discusses related literature. Section III introduces the model. Sections IV and V and characterize, respectively, the baseline equilibrium and the bubble equilibria. Section VI analyzes how investor flows in the different equilibria relate to the Ponzi-scheme equivalence condition. Section VII generalizes the results regarding investment flows in bubble equilibria and their relationship to investors' payoffs in Ponzi-schemes to a model where user demand responds to the expected return on holding the cryptocurrency. Section VIII provides concluding remarks. Proofs are in the appendix.

II. Related Literature

The present paper builds upon a fast-growing theoretical literature on the exchange rates of cryptocurrencies (Athey et al., 2016; Bakos and Hałaburda, 2022; Biais et al., 2023; Chiu and Koeppl, 2022; Cong et al., 2021, 2022; Garratt and Van Oordt, 2022, 2023; Gryglewicz et al., 2021; Sockin and Xiong, 2023a,b). User demand in those models is driven by users who derive either transactional or utility benefits from holding the cryptocurrency. Some models also model explicitly the demand by forward-looking investors on the dynamics of cryptocurrency prices (Bolt and Van Oordt, 2020; Garratt and Van Oordt, 2024; Karau and Moench, 2023; Prat et al., 2024; Wei and Dukes, 2021). The theoretical environments in those papers, with one exception, do not allow for the classical rational bubble equilibria driven by expectations of investor as explored by, among others, Blanchard (1979) and Blanchard and Watson (1982). The aforementioned exception is the valuable study by Wei and Dukes (2021). They focus on the impact of rational bubble equilibria on user adoption in an environment where the required return equals zero. They find that bubble equilibria may accelerate adoption of a cryptocurrency by regular users. Differently, we focus on the investor inflows that are necessary to a sustain rational bubble equilibria in an environment that permits for nonzero required returns. Allowing for a nonzero required return allows us to compare the relationship between the present value of investors net inflows for various potential price equilibria and Ponzi-schemes. The comparison reveals among others that bubble equilibria can have payoffs to investors that are in the aggregate equivalent to Ponzi-schemes, but not all.

The volatile and explosive price trajectories of cryptocurrencies also have been a popular subject in the empirical literature (Yermack, 2015). The search query for "Bitcoin AND Bubble" on Google Scholar returns no less than 27,000 results. Cheah and Fry (2015) and Cheung et al. (2015) apply the methodology of Phillips et al. (2015) to detect explosive paths on Bitcoin prices. Later studies vary in terms of statistical methods and data. Chaim and Laurini (2019), Geuder et al. (2019) and Cretarola and Figà-Talamanca (2020) apply alternative statistical methods on the price series of Bitcoin and Ethereum. Hafner (2020) considers a larger set of cryptocurrencies using a method that allows for time-varying volatility. Li et al. (2022) find that media attention is associated with higher future returns when bitcoin prices follow a bubbly price trajectory. Enoksen et al. (2020) relate trading and transaction volume to explosive price trajectories when analyzing a variety of cryptocurrencies. Corbet et al. (2018) assesses the explosiveness of nonprice series such as the Bitcoin block size and mining power. The theoretical results in the present paper offer valuable insights into factors that may render cryptocurrencies more or less susceptible to explosive price paths. These factors are discussed in greater detail in the concluding remarks.

III. Model

The model is in the tradition of classical rational bubble models (Blanchard, 1979; Blanchard and Watson, 1982). Classical rational bubble models build off from two main attributes: A rational expectations asset market model and a market-clearing condition. The rational expectations asset market model defines when investors are willing to hold an asset. We build a tractable model that applies this setup to cryptocurrencies.

Time is discrete and denoted by $t = 0, 1, 2, ...$ We denote the exchange rate of the cryptocurrency in terms of dollars at time t by S_t. The exogenous number of units of the cryptocurrency at time t is denoted by $M_t > 0$. The demand for a cryptocurrency consists of investment demand and utility demand.

The aggregate number of cryptocurrency units held purely for investment purposes is denoted by $Z_t \geq 0$. We impose a non-negativity constraint on the investment position reflecting the fact that investors in the aggregate cannot bring additional units into existence, even though individual investors could maintain long or short positions. Investment demand is determined by what is known as a simple rational expectations asset market model. Let $r > 0$ denote the required return on capital for investment holdings in the cryptocurrency. Investor behaviour is governed by the following assumption.

Assumption 1 (Rational expectations market model) *Investors adjust their investment holdings such that, for any t where $Z_t > 0$,*

$$\mathbb{E}_t \left(S_{t+1}\right) = (1 + r)S_t. \tag{1}$$

Moreover, investors do not hold the asset, so that $Z_t = 0$, whenever $\mathbb{E}_t \left(S_{t+1}\right) \leq (1 + r)S_t$.

This assumption reflects the idea that risk-neutral investors would purchase more of a cryptocurrency if the expected return were higher than the required return. Purchases by investors would drive up the current exchange rate until (1) holds true, after which there are no incentives for investors to further adjust their holdings. If the expected return were less than the required return, then selling pressure by rational investors would put downward pressure on the current exchange rate. The investors would continue to sell until either (1) holds true or $Z_t = 0$.

The user demand in terms of dollars is exogenous and denoted by $X_t^{\$} \geq 0$.[2] The model is agnostic regarding the source of the user demand. The primary interpretation of the user demand is the transactional demand for a cryptocurrency that is used as a means of payments but not as a unit of account (Bolt and Van Oordt, 2020; Prat et al., 2024). If individuals use a cryptocurrency to make payments for a total dollar-amount of $T_t^{\$}$ dollars, and if the cryptocurrency units used to making payments have an average velocity of V_t^*, then this implies a transactional demand of $X_t^{\$} = T_t^{\$}/V_t^*$ dollars.[3] In certain cases, a cryptocurrency may be used as the exclusive means of payment to transact on a particular platform (Cong et al., 2021; Gryglewicz et al., 2021). Other tokens are used to offer discounts to loyal customers if customers use them to pay for services provided by their issuers (Garratt and Van Oordt, 2024). An alternative interpretation of the user demand is the utility demand that originates from benefits associated with tokens other than making payments. Tokens may serve as a membership credential that grants access to a particular platform (Bakos and Hałaburda, 2022).

The market-clearing condition requires the total value of all units to reflect the combined value of units held to satisfy user demand and the units held purely for investment purposes.

Assumption 2 (Market-clearing condition) *For any t, we have that*

$$M_t S_t = X_t^{\$} + Z_t S_t. \tag{2}$$

[2]We allow the level of user demand to respond to the expected rate of appreciation in Section VII.

[3]See Bolt and Van Oordt (2020) for more details; here, we intentionally use the same notation.

The final assumption reflects the idea that users can always dispose of cryptocurrency without incrurring any costs, a process that is sometimes referred to as *burning* cryptocurrency.

Assumption 3 (Free disposal) *Individuals can dispose of cryptocurrency units at no cost so that* $S_t \geq 0$ *for any* t*.*

The model setup is closed with the following equilibrium definition.

Definition 1 *An equilibrium path for the exchange rate is defined as any path* $\mathbb{E}_0(S_0), \mathbb{E}_0(S_1), \mathbb{E}_0(S_2), ...$ *that satisfies Assumptions 1-3 for given sequences* $(M_0, M_1, M_2, ...)$ *and* $(X_0^{\$}, X_1^{\$}, X_2^{\$}, ...)$*.*

IV. Baseline Equilibrium

It will be convenient to rewrite the market clearing condition in the form of an equation for the exchange rate.

Lemma 1 *At any time* t *where* $Z_t < M_t$*,*

$$S_t = \frac{X_t^{\$}}{M_t - Z_t}. \tag{3}$$

The equation reflects the intuitive notion that a higher exchange rate may follow from higher user demand (i.e., transactional demand or utility demand), a lower number of issued tokens, or a higher investment demand. The lemma also implies a hypothetical reference level for the exchange rate in the absence of any investment demand defined as $Q_t =: X_t^{\$}/M_t$. This reference level does not need to correspond to an equilibrium level for the exchange rate. Investors may have incentives to purchase units of the cryptocurrency if they anticipate a sufficiently strong appreciation the exchange rate. Nonzero investment demand would raise the exchange rate above the hypothetical reference level.

Traditional rational bubble models for asset prices typically derive an equilibrium with a fundamental value that reflects the present value of expected cash flows. The fundamental value in such models reflects the floor for the equilibrium value of the asset if negative price bubbles are ruled out (in our model, they are ruled out because of Assumption 3). A fundamental value in the sense of the present value of discounted cash flows does not apply directly to cryptocurrencies that do not pay dividends. However, we can derive a similar floor for the equilibrium exchange rate path of cryptocurrencies. We refer to this path as the baseline equilibrium path.

To find the floor for the equilibrium exchange rate of cryptocurrencies, we first define $\tau(1)$ as the time at which the discounted value of the hypothetical reference level without investors, Q_t, is maximized, i.e.,

$$\tau(1) =: \inf \operatorname*{argmax}_{t \in \mathbb{N}} \frac{Q_t}{(1+r)^t}. \tag{4}$$

The $\mathbb{N}$ corresponds to all positive integers including zero. The derivation of the baseline equilibrium requires $\tau(1)$ to be finite which is true as long as the growth rate of user demand per coin stabilizes at a level that is less than the required return at some point in the future. If there are multiple $t \in \mathbb{N}$ that maximize the argument, then the infimum-operator ensures that $\tau(1)$ corresponds to the time at which the maximum occurs first. At $t = \tau(1)$, we repeat the same procedure to define $\tau(2)$ as the next point in time that maximizes the discounted value of the reference level for the exchange rate. Repeating this procedure yields the sequence $(\tau(1), \tau(2), \tau(3), ...)$ that corresponds to the points in time where the future discounted value of Q_t is maximized from the perspective of, respectively, time $t = (\tau(0), \tau(1), \tau(2), ...)$ where $\tau(0) = -1$ by convention. Formally, the values of $\tau(n)$ for $n \in \mathbb{N}$ are defined as

$$\tau(n) =: \inf \operatorname*{argmax}_{t \in \mathbb{N}^{>\tau(n-1)}} \frac{Q_t}{(1+r)^t}. \tag{5}$$

Given the sequence $\tau(n)$, we obtain the path for the exchange rate under the baseline equilibrium in the following proposition.

Proposition 1 (Baseline equilibrium) *The lowest possible level of the exchange rate on an equilibrium path for any t such that $\tau(n-1) < t \leq \tau(n)$ is*

$$S_t^* = \frac{Q_{\tau(n)}}{(1+r)^{\tau(n)-t}}.$$

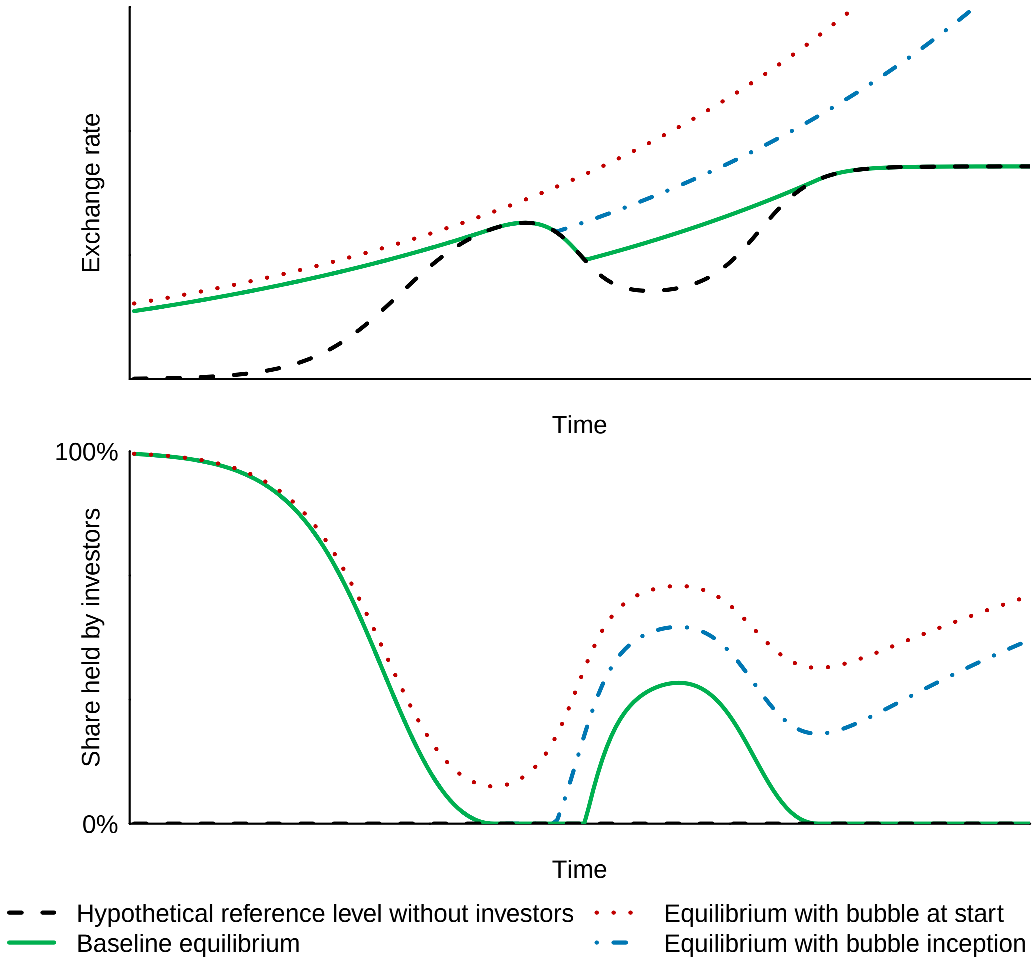
Proof. See Appendix A. ■

The upper panel of Figure 2 provides an illustration of the exchange rate path under the baseline equilibrium. The black dashed line reflects the evolution of the hypothetical reference level without investors.[4] The green line reflects the exchange rate path under the baseline equilibrium. The initial exchange rate under the baseline equilibrium is higher than the reference level, because investors hold the tokens in expectation of higher future demand by users. This is reflected by the share of coins held by investors in the bottom panel of the figure. The share of coins held by investors can be calculated as $Z_t/M_t = 1 - Q_t/S_t^*$. Initially, investors hold almost all the tokens in circulation. The share of coins held by investors approaches zero the closer one gets to the peak in the discounted level of the user demand per coin (i.e., when t approaches the next value in the sequence of $\tau(n)$).

The exchange rate on the baseline equilibrium path equals to the reference rate $Q_{\tau(n)}$ at any $t = \tau(n)$. At this point in time, all cryptocurrency units are held by users and no units are held by investors. If the peak value of the discounted user demand per coin occurs at $t = 0$ (i.e., $\tau(1) = 0$), then we have that the baseline equilibrium is characterized by zero initial investor demand so that the equilibrium corresponds to a positive price equilibrium but not an investment equilibrium in Figure 1. Otherwise, some cryptocurrency units will be held by investors so that the baseline equilibrium corresponds to an investment equilibrium.

[4]The double-hump shaped path for the reference level in the upper panel of Figure 2 illustrates a scenario with strong initial growth and then a temporary decline in user demand. The path is arbitrary and was generated using a combination of three logistic functions.

Figure 2: Exchange Rate and Investment Share in Equilibrium



The exchange rate of a cryptocurrency in the baseline equilibrium and the fundamental price of a security have crucially different relationships to their underlying values. For a security, the material underlying values are the discounted cash flows over the entire lifetime of the security. The fundamental price is calculated as the *sum* of the discounted cash flows of the security. For a cryptocurrency, the material underlying value is the discounted user demand per coin. The current exchange rate under the baseline equilibrium is determined by the *peak value* of the discounted user demand per coin. This peak value occurs at $t = \tau(1)$ and, hence, the initial exchange rate under the baseline equilibrium equals $Q_{\tau(1)}/(1+r)^{\tau(1)}$.

V. Bubble Equilibria

The exchange rate under the baseline equilibrium in Proposition 1 is based on one particular solution of the difference equation in Assumption 1. Also other solutions to the difference equation that reflect equilibria exist. In particular, at $t = 0$, the exchange rate may be higher due to the presence of a rational bubble component that is stacked on top of the baseline equilibrium. Such equilibria with a bubble component from the outset are characterized in the following proposition.

Proposition 2 (Rational bubble equilibrium) *Any path for the expected exchange rate that satisfies*

$$\mathbb{E}_0 S_t = (S_0^* + B)(1 + r)^t$$

for $B \geq 0$ *and any* $t \in \mathbb{N}$ *constitutes an equilibrium path.*

The initial exchange rate in a bubble equilibrium characterized by the proposition consists of two components: The exchange rate under the baseline equilibrium in Proposition 1 plus a non-negative rational bubble component, $B \geq 0$.

An illustration of an equilibrium with a bubble component in the initial exchange rate is provided by the blue dotted line in the upper panel of Figure 2. The exchange rate and the share of cryptocurrency held by investors under this equilibrium are larger than in the baseline equilibrium, and the divergence tends to increase over time. Investors have incentives to hold the cryptocurrency despite of the bubble component because the cryptocurrency is expected to continue to appreciate at a rate that equals the required return on capital. The share of coins held by investors in a bubble equilibrium slowly converges to all coins in circulation as shown by the blue dotted line in the lower panel of Figure 2, even though temporary growth in user demand can lead to a short-term reduction in the share held by investors.

The expected exchange rate of a cryptocurrency on a bubble equilibrium path in Proposition 2 depends on the required return on capital and the initial level of the exchange rate. Proposition 2 permits bubble exchange rate paths that contain random elements – it describes the exchange rate path in terms of expectations. Blanchard and Watson (1982) suggest an intuitive example of a bubble equilibrium where the exchange rate grows at a faster speed than the required return as long as the bubble persists, but where there is a positive probability that the bubble will burst. The expected return on such a stochastic bubble path still equals the required return because the higher rate of appreciation is counterbalanced by the downside risk. Alternative stochastic bubble paths with more complex dynamics are possible too.

The previous proposition covers the case in which there is a rational bubble component from the outset. If there was no rational bubble component from the outset, then a rational bubble component to the price could still commence in the future if the expected return in the baseline equilibrium is sufficiently low. The following proposition characterizes the equilibria that involve the inception of a bubble in cryptocurrency prices.

Proposition 3 (Rational bubble inception) *Suppose $S_t = S_t^*$ for $t = \tau(n)$ given some value $n \in \mathbb{N}$. Then any path for the expected exchange rate that satisfies*

$$\mathbb{E}_{\tau(n)} S_t = (S^*_{\tau(n)+1} + B)(1+r)^{t-\tau(n)-1}$$

for B s.t. $0 \leq B \leq S^*_{\tau(n)}(1+r) - S^*_{\tau(n)+1}$ and all $t > \tau(n)$ constitutes an equilibrium path.

An illustration of an bubble equilibrium path that involves the inception of a bubble in the exchange rate is given by the red dash-dotted line in Figure 2. The exchange rate under the baseline equilibrium is on a downward trajectory when the inception of the bubble occurs, and, hence, expected returns in the baseline equilibrium are less than the required return. This is relevant because the size of the bubble at inception as measured by B is limited in magnitude since the expected return on the cryptocurrency's exchange rate at the moment of inception cannot exceed the required return in equilibrium. The expected return in the baseline equilibrium must be less than the required return in order to allow for the possible inception of a bubble in investor's expectations to exist. Otherwise, the possible inception of the bubble with a nonzero probability would raise the expected return above the required return, which cannot occur in equilibrium.[5]

VI. Sustaining a Bubble Equilibrium

The previous sections explored the possible equilibrium price paths given Assumptions 1–3. The present section explores the net investment flows that are implied by the equilibrium price paths and their relationship to Ponzi-schemes. The first result is that the following condition must hold true for the expected net investment inflows to sustain the rational bubble equilibria characterized in Propositions 2–3.

Proposition 4 *A rational bubble equilibrium for a cryptocurrency can persist if and only if*

$$\underbrace{\mathbb{E}_t \Delta Z_{t+1} S_{t+1}}_{\substack{\text{Net investment} \\ \text{inflow}}} = \underbrace{\mathbb{E}_t \Delta M_{t+1} S_{t+1}}_{\substack{\text{Value of} \\ \text{newly issued} \\ \text{units}}} - \underbrace{\Delta X^{\$}_{t+1}}_{\substack{\text{New user} \\ \text{demand}}} + \underbrace{r X^{\$}_{t}}_{\substack{\text{Appreciation} \\ \text{of units held} \\ \text{by users}}}. \tag{6}$$

Proof. See Appendix B. ■

The expression for the net inflow of investors' funds required to sustain a rational bubble consists of three components.[6] The first component is the value of newly issued cryptocurrency units. Newly issued units need to be absorbed by investors to sustain a bubble equilibrium, and, hence, the larger the new issuance the larger the net inflow of investors' funds nec-

[5] Our result is related to but different from that of Diba and Grossman (1987). They argue against the inception of rational bubbles in security prices, because the nonzero probability of a switch to a bubble equilibrium would raise the expected return of a security above the required return. This limits the probability of the inception of new bubbles in their to zero. Differently, we analyze a cryptocurrency with an expected return that may be lower than the required return, in which case the coins are held by users alone. The expected return in the baseline equilibrium is by definition equal to or less than the required return whenever one moves from $t = \tau(n)$ to $t = \tau(n) + 1$.

[6] The Δ-operator applies to the first symbol by convention; e.g., $\Delta Z_{t+1} S_{t+1} = (Z_{t+1} - Z_t) S_{t+1}$.

essary to sustain a rational bubble. The second component is the change in user demand. An increase in user demand reduces the number of units that need to be acquired by investors. A smaller net investment inflow is required to sustain a bubble equilibrium if the user demand expands. The required net inflow may turn negative if the increase in user demand is sufficiently large. The third component stems from the existing user demand for a cryptocurrency. Sustaining a bubble equilibrium path implies a continuous appreciation of the coins held by users resulting in a reduction in the number of coins they need to hold for transactional or utility purposes. Investors need to acquire the coins that users sell to sustain the equilibrium, which number depends on the rate of appreciation which depends in equilibrium on the required return. The required net investment inflow in (6) will be lower if investors require a lower return on investments in the cryptocurrency because they perceive it as a digital gold that provides insurance for bad states of the world.

The condition in Proposition 4 hints that sustaining a bubble exchange rate paths may require a continuous inflow of investors funds like a Ponzi-scheme. A Ponzi-scheme is distinct from a cryptocurrency in the literal sense in that investors can choose to buy or sell them at the prevailing market price, but it could display similarities in that the prolongation of a Ponzi-scheme requires a sustained inflow of aggregate investors' funds like certain equilibria for cryptocurrency prices. We explore the relationship between bubble exchange rate paths and Ponzi-schemes by postulating the following Ponzi-scheme equivalence condition regarding the cash flows for investors.

Condition 1 (Ponzi-scheme equivalance) *The remaining cash flows for investors in the aggregate have or will have a negative present value at some point in time in the future, that is,*

$$-\sum_{i=1}^{\infty} \frac{\mathbb{E}_T \Delta Z_{T+i} S_{T+i}}{(1+r)^i} < 0 \quad \textit{for some} \quad T \geq 0.$$

The condition sums all the expected aggregate cash flows to investors at some point in the future and requires the present value to be negative. The minus-sign in front of the sum accounts for the fact that net sales by investors – cash inflows from the perspective of the investors – correspond to negative values of ΔZ_t. Net purchases – cash outflows for investors – correspond to positive values of ΔZ_t. A cryptocurrency satisfies the Ponzi-scheme equivalence condition for example if its equilibrium exchange rate path requires a continuous inflow of investors funds: This would imply $\Delta Z_t > 0$ for all $t > T$, and, hence, violate Condition 1. The condition requires that there will be some point in time where the remaining discounted cash flows from coin sales by investors will be less than the discounted cost of the remaining coin purchases by investors.

A. Non-negative Money Growth

We first consider aggregate payoffs of investors in bubble equilibria for cryptocurrencies that exhibit non-negative money growth ($\Delta M_t \geq 0$ for any $t > T$). Denote the growth rate of user demand by $g_t = (X_{t+1} - X_t)/X_t$. Consider a cryptocurrency for which the growth stabilizes at some point in the – potentially distant – future such that the user growth rate will be lower than the required return on capital, i.e., $g_t < r$ for any $t > T$ given some T. The following proposition summarizes our result for such cryptocurrencies.

Proposition 5 *Consider a cryptocurrency with non-negative issuance* $\Delta M_t \geq 0$ *and for which the user demand stabilizes at some distant point in the future such that the growth rate* $g_t < r$ *for any* $t > T$ *given some* $T \geq 0$*. The cryptocurrency satisfies the Ponzi-scheme condition if its exchange rate follows a bubble equilibrium path.*

Proof. See Appendix C. ■

The intuition for this result is that an ongoing appreciation of the exchange rate is only sustainable if investors purchase cryptocurrency from users, who need fewer and fewer coins as the exchange rate continues to increase. Sustained user growth can help to alleviate the need for investors to purchase units from users, but this reduction is not sufficient if the growth rate is less than the required return of capital ($g_t < r$). The ongoing purchases of cryptocurrency by investors from users ad infinitum results in a negative present value of the remaining aggregate cash flows from the perspective of investors. Despite of the negative cash flows for investors in the aggregate, every individual investor who acquires cryptocurrency in the bubble equilibrium and sells it in the future is expected to earn the required return.

B. *Negative Money Growth*

Some cryptocurrencies exhibit negative money growth. Negative money growth can be a design feature. For example, Ethereum has shown a period of negative money growth after its switch to a mechanism where part of the tokens paid as transaction fees are burned. A hidden source of negative money growth is the coins to which users have lost access permanently.

To illustrate the relationship between the cash flows for investors in Ponzi-schemes and bubble equilibria for cryptocurrencies with negative money growth, we extend the model to allow for a scenario where every period a proportion $f > 0$ of the user demand $X_t^{\$}$ is burnt as transaction fees (or, alternatively, lost) such that $\Delta M_{t+1} S_{t+1} = -f X_t^{\$}$.[7] The following proposition states the Ponzi-scheme equivalence result for such a cryptocurrency.

Proposition 6 *Consider a cryptocurrency where every period a fraction of the user demand $f > 0$ is burnt and for which the growth rate of user demand g is such that $g < r$ for any $t > 0$.*

1. *If $f < r - g$ and the exchange rate follows a bubble equilibrium path, then the cryptocurrency satisfies the Ponzi-scheme equivalence condition.*

2. *If $f > r - g$ and the exchange rate follows a bubble equilibrium path, then the cryptocurrency does not satisfy the Ponzi-scheme equivalence condition.*

Proof. See Appendix D. ■

[7] The expression assumes $\Delta M_{t+1} = -f X_t^{\$} / S_{t+1}$. Alternatively, one could assume $\Delta M_{t+1} = -f X_t^{\$} / S_t$. This has no impact on Propositions 6 and 7, except that the f in any expression should be replaced by $f/(1 + r)$.

All bubble equilibria satisfy the Ponzi-scheme equivalence condition if the proportion of the user demand that is burnt every period is less than the difference between the required return and the growth in user demand ($f < r-g$). We find a different result for the situation where the proportion of user demand paid as fees exceeds the difference between the required return and the growth in user demand ($f > r - g$). For such cryptocurrencies, bubble equilibria do not satisfy the Ponzi-scheme equivalence condition. This is why there is a non-empty set of bubble equilibria with payoffs that are not equivalent to Ponzi-schemes in Figure 1.

A cryptocurrency for which $f > r - g$ is an interesting scenario from an economic point of view in that the user demand per coin and, hence, the reference level would grow forever at a rate $f + g$ which is greater than the required return r if there weren't any investors. The presence of investors ensures that the exchange rate does not appreciate at a rate higher than the required return in equilibrium. The mechanism functions as follows. Investors who acquire the cryptocurrency drive up the initial exchange rate. The higher initial exchange rate implies that users how fewer coins, and, hence, burn a smaller number of coins as transaction fees. The number of coins declines at a slower speed resulting in a slower increase in the user demand per coin than in the absence of investors. In equilibrium, the user demand per coin will grow at precisely the same rate as the required return. The remaining cash flows to investors will be positive in such an equilibrium. Users have to purchase tokens from investors to replenish their cryptocurrency balances after burning the transaction fees which generates a positive aggregate cash flow from users to investors.

The following proposition illustrates the equilibrium exchange rate path and the evolution in the share of coins held by investors.

Proposition 7 *Consider a cryptocurrency with no new issuance, with a constant growth rate of user demand $g < r$, and with users who burn a proportion $f > r - g$ of their coins as transaction fees at any t. The following exchange rate path constitutes an equilibrium:*

$$S_t = (1+r)^t \frac{f}{r-g} \frac{X_0^{\$}}{M_0 - U} \quad \textit{for any} \quad t, \tag{7}$$

where $U = 0$ corresponds to the baseline equilibrium, and where any U such that $0 < U < M_0$ corresponds to a bubble equilibrium. The equilibrium share of coins held by investors equals

$$\frac{Z_t}{M_t} = \frac{\frac{f-(r-g)}{f}(M_0 - U)\left(\frac{1+g}{1+r}\right)^t + U}{(M_0 - U)\left(\frac{1+g}{1+r}\right)^t + U}.$$

Proof. See Appendix E. ■

The share of coins held by investors slowly converges to all coins in a bubble equilibrium ($0 < U < M_0$) as was the case for the earlier bubble equilibria where the money issuance was exogenous. The evolution of the share of coins held by investors with the endogenous burning of coins is different for the baseline equilibrium ($U = 0$). Rather than converging to zero as was the case with the exogenous money issuance, the share of coins held by investors is constant: Investors in the baseline equilibrium sell precisely the amount to restore the balance between the shares held by users and investors.

The finding that bubble equilibria do not satisfy the Ponzi-scheme condition if $f > r - g$ does not mean that the net present value of the cash flows for investors of investing in the cryptocurrency will be non-negative in the aggregate. The Ponzi-scheme equivalence condition tests whether the future cash flows for investors have a negative present value in the aggregate. The condition does not consider the initial cost for investors to acquire the coins. If investors were to pay the prevailing market price for their initial position in cryptocurrency at $t = 0$, then the cost of their position would be $S_0 Z_0$. From the proposition, we can then derive the following corollary.

Corollary 1 *Consider the equilibria characterized in Proposition 7. The baseline equilibrium (i.e., $U = 0$) is the only equilibrium with a non-negative net present value for investors in the aggregate, i.e.,*

$$-S_0 Z_0 - \sum_{t=1}^{\infty} \frac{\mathbb{E}_0 \Delta Z_t S_t}{(1+r)^t} \geq 0.$$

Investors experience positive cash flows in the aggregate in bubble equilibria where $f > r - g$, so the Ponzi-scheme equivalence condition is not satisfied, but the present value of the aggregate cash flows is less than the initial cost of acquiring cryptocurrency in the first period. Despite of the negative present value in the aggregate, every individual investor who acquires cryptocurrency and sells it in the future is still expected to earn the required return.

VII. Discussion

A. *Endogenous user demand*

The user demand in the model is assumed to be inelastic with respect to the expected return on the cryptocurrency. Although this assumption simplifies the model considerably, it ignores how the demand for a means of payment depends on the opportunity cost of holding it. Alternatively, one could consider a more general model where the user demand at time t equals $X_t^{\$}(\mathbb{E}_t R_{t+1}) = A(\mathbb{E}_t R_{t+1})\widehat{X}_t^{\$}$, where the $\widehat{X}_t^{\$}$ is exogenous, and where $A(\cdot)$ is a non-negative finite scaling factor that increases in the expected return $\mathbb{E}_t R_{t+1} = \mathbb{E}_t(S_{t+1}/S_t)$ for $\mathbb{E}_t R_{t+1} \geq 0$. The original model can be considered as a special case where $A(\mathbb{E}_t R_{t+1}) = 1$. The more general model allows user demand to depend on time-specific fundamental factors as measured by $\widehat{X}_t^{\$}$ as well as the expected rate of appreciation.

The main results regarding the investment flows in bubble equilibria and their relationship to investors' payoffs in Ponzi-schemes extend immediately to the more general model.[8] Proposition 4, 5, 6 and 7 as well as Corollary 1 all hold true in the more general model.[9] The reason why these results also hold true in the more general model is straightforward. Investors hold the cryptocurrency in any equilibrium where the exchange rate contains a bubble component, so the expected return must equal the required return in bubble equilibria (Assumption 1). Hence, we must have $A(\mathbb{E}_t R_{t+1}) = A(1 + r)$ whenever the cryptocurrency price contains a bubble component in equilibrium, so that the user demand $X_t^{\$}$ becomes

[8]The solution for the baseline equilibrium is more challenging in the general model. The path of the user demand per coin does not need to be unique in the baseline equilibrium for a given sequence of $\widehat{X}_t^{\$}$.

[9]The growth rates referred to in those propositions should in the more general model be understood as growth rates of the underlying fundamentals, i.e., $g_t = (\widehat{X}_{t+1} - \widehat{X}_t)/\widehat{X}_t$.

a re-scaled version of the underlying fundamental factor $\widehat{X}_t^{\$}$. The user demand takes the maximum possible value in a bubble equilibrium since the expected return cannot exceed the required return in any equilibrium.

VIII. Concluding Remarks

This paper focused on the question how cryptocurrency price paths relate to concepts such as new asset classes, bubbles, Ponzi schemes and digital gold. The analysis reveals how those terms relate to differences in underlying beliefs regarding future peak values of the user demand per coin and discount rates.

High crypto prices can be justified by a high expected peak value in terms of user demand. Describing a cryptocurrency as equivalent to a Ponzi-scheme is not justified if the current price reflects the discounted value of the expected peak value in user demand. This holds true even if the current price action seems to be driven mostly by investors' actions rather than user dynamics. Even though investors experience outflows when they acquire cryptocurrency, they expect to profit from selling crypto to users in the future. An observer could easily mistake a high price observed in a baseline equilibrium for a bubble when underestimating the expected peak value in user demand. Moreover, higher prices could be justified by a lower discount rate resulting from investors perceiving cryptocurrency as a digital gold that provides insurance against bad states of the world.

The picture looks less rosy if high current prices are the consequence of investors expecting price increases solely due to higher future investor demand, and not due to the expected peak level of future user demand. Such bubble price paths are possible in equilibrium and are associated with a gradually increasing share of coins held purely for investment purposes. The analysis reveals that, for cryptocurrencies with nonnegative money growth, such price paths are associated with Ponzi-scheme equivalent payoffs to investors in the aggregate. Even though individual investors are expected to earn their required return if the bubble persists, they do experience negative cash flows in the aggregate. Finally, investors do not necessarily experience Ponzi-scheme equivalent payoffs in the aggregate if they invest in a bubble equilibrium for a cryptocurrency with negative money growth.

For the empiricist or experimenter, our results provide a theoretical foundation for potential factors that make cryptocurrencies more or less susceptible to explosive price paths. For reasons outside the model, it may seem less plausible that an equilibrium can persist if it requires a high rather than a low net investment inflow over a sustained period of time. Combining such an a posteriori statement with Proposition 4 implies the following predictions: A bubble equilibrium for cryptocurrencies is *less plausible* with (1) *higher new issuance*, (2) *lower growth in user demand*, and (3) *higher existing user demand*. It is noteworthy that user demand acts as a double-edged sword: Growth in user demand reduces the required investment inflow to sustain a bubbly exchange rate path initially, but the existing level of user demand increases the required investment inflow. Finally, the required investment inflow for bubbles is smaller when the required return is lower which typically would be the case if investors perceive a cryptocurrency as a digital gold that acts as an insurance for bad states of the world.

For the theorist, the condition in Corollary 1 may prove useful to rule out rational bubble equilibria if the baseline equilibrium is the object of study. A theorist may be tempted to ruling out bubble equilibria by imposing an alternative condition that limits the asymptotic growth of the exchange rate so that the present value of the future exchange rate converges to zero, akin to a transversality condition. The disadvangage of such a transversality-like condition is that it may rule out both the bubble equilibria *and* the baseline equilibrium as we have seen in the case studied in Proposition 7.[10]

[10]From Eq. (7), it is immediate that the baseline equilibrium condition in Proposition 7 violates the condition $\lim_{t\to\infty} S_t/(1+r)^t = 0$ since $\lim_{t\to\infty} S_t/(1+r)^t > 0$ for $U = 0$ provided that $X_0^{\$} > 0$.

Appendix: Proofs

A. *Proof of Proposition 1*

We first consider t such that $0 < t \leq \tau(1)$.

Suppose $X^{\$}_{\tau(1)} > 0$. The definition of Q_t implies $S_{\tau(1)}$ cannot be less than $Q_{\tau(1)}$ if Assumption 2 holds true (Lemma 1). This proves the lower bound for $t = \tau(1)$. Moreover, if $S_{\tau(1)} = Q_{\tau(1)}$, then any level of the exchange rate $S_t(1+r)^{\tau(1)-t} < Q_{\tau(1)}$ for $0 < t < \tau(1)$ would violate Assumption 1. This proves the lower bound for any $0 < t < \tau(1)$. Assumption 3 also holds true on the path since $X^{\$}_{\tau(1)} > 0$ implies $S^*_t > 0$ for $0 < t \leq \tau(1)$, so the path S^*_t in the proposition constitutes an equilibrium for any t such that $0 < t \leq \tau(1)$.

Repeating the argument for $n = 2, 3, ...$ provides the proof for any t such that $\tau(n-1) < t \leq \tau(n)$.

The special case where $X^{\$}_{\tau(1)} = 0$ implies $X^{\$}_t = 0$ for any $t \in \mathbb{N}$, so that $\tau(n) = n$. In this case, the path described in the proposition implies $S_t = Q_t = 0$ for any t, which is the lowest possible level of the exchange rate that does not violate Assumption 3.

B. *Proof of Proposition 4*

From Lemma 1, we have

$$\mathbb{E}_t(\Delta S_{t+1}) = \mathbb{E}_t\left(\frac{X^{\$}_{t+1}}{M_{t+1} - Z_{t+1}} - \frac{X^{\$}_t}{M_t - Z_t}\right).$$

Moreover, since investors are willing to hold cryptocurrency on a bubble equilibrium path, we have $\mathbb{E}_t(\Delta S_{t+1}) = rS_t$ from Assumption 1. Combining both expressions gives

$$rS_t = \mathbb{E}_t\left(\frac{X^{\$}_{t+1}(M_t - Z_t) - X^{\$}_t(M_{t+1} - Z_{t+1})}{(M_{t+1} - Z_{t+1})(M_t - Z_t)}\right),$$

and, since $S_t(M_t - Z_t) = X_t^{\$}$,

$$
\begin{aligned}
rX_t^{\$} &= \mathbb{E}_t\left(\frac{X_{t+1}^{\$}(M_t - Z_t) - X_t^{\$}(M_{t+1} - Z_{t+1})}{(M_{t+1} - Z_{t+1})}\right), \\
&= \mathbb{E}_t\left(\frac{X_{t+1}^{\$}(M_t - Z_t) - X_{t+1}^{\$}(M_{t+1} - Z_{t+1}) + \Delta X_{t+1}^{\$}(M_{t+1} - Z_{t+1})}{M_{t+1} - Z_{t+1}}\right), \\
&= \mathbb{E}_t\left(S_{t+1}(M_t - Z_t) - S_{t+1}(M_{t+1} - Z_{t+1}) + \Delta X_{t+1}^{\$}\right), \\
&= -\mathbb{E}_t \Delta M_{t+1} S_{t+1} + \mathbb{E}_t \Delta Z_{t+1} S_{t+1} + \Delta X_{t+1}^{\$}.
\end{aligned}
$$

Rearranging the last expression gives the expression for $\mathbb{E}_t \Delta Z_{t+1} S_{t+1}$ in the proposition.

C. Proof of Proposition 5

Aggregating the relationship for the net inflows from investors in Proposition 4 for all $t > T$ with discounting and iterating expectations back to $t = T$ gives

$$
\sum_{i=1}^{\infty} \frac{\mathbb{E}_T \Delta Z_{T+i} S_{T+i}}{(1+r)^i} = \sum_{i=1}^{\infty} \frac{\mathbb{E}_T \Delta M_{T+i} S_{T+i}}{(1+r)^i} + \sum_{i=1}^{\infty} \frac{-\Delta X_{T+i}^{\$} + rX_{T+i-1}^{\$}}{(1+r)^i}. \tag{8}
$$

Using $M_{T+i} \geq 0$ (non-negative money growth) and $\Delta X_{T+i} = (1 + g_{T+i})X_{T+i-1}^{\$} - X_{T+i-1}^{\$} = g_{T+i}X_{T+i-1}^{\$}$ gives

$$
\begin{aligned}
\sum_{i=1}^{\infty} \frac{\mathbb{E}_T \Delta Z_{T+i} S_{T+i}}{(1+r)^i} &\geq \sum_{i=1}^{\infty} \frac{rX_{T+i-1}^{\$} - g_{T+i}X_{T+i-1}^{\$}}{(1+r)^i}, \qquad (9) \\
&\geq 0, \qquad (10)
\end{aligned}
$$

where the last inequality holds because $g_t < r$ for all $t \geq T$. The present value of the net cash inflows from investors is larger than zero which is the same as saying that that present value of the remaining cash flows is negative from the perspective of investors.

D. *Proof of Proposition 6*

If a fraction f of the coins paid as transaction fees by users are burned, and if there is no further issuance of coins, then $\mathbb{E}_T \Delta M_{T+i} S_{T+i} = -f X^{\$}_{T+i-1}$. Then, from (8), we have

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{\mathbb{E}_T \Delta Z_{T+i} S_{T+i}}{(1+r)^i} &= \sum_{i=1}^{\infty} \frac{-f X^{\$}_{T+i-1} - g X^{\$}_{T+i-1} + r X^{\$}_{T+i-1}}{(1+r)^i}, \\ &= \frac{r-f-g}{1+g} \sum_{i=1}^{\infty} \frac{(1+g)^i}{(1+r)^i} X^{\$}_T, \\ &= \frac{r-f-g}{r-g} X^{\$}_T.\end{aligned}$$

This value can be both positive and negative, depending on the level of the transaction fees. If $f < r-g$, then the present value of the future cash flows to investors is negative, so that the Ponzi-scheme equivalence condition is satisfied for any equilibrium where coins continually appreciate at an expected rate r (i.e., bubble equilibria in this case). If the fees $f > r-g$, then the present value of the remaining cash flows to investors would be positive, so that the Ponzi-scheme equivalence condition is not satisfied in an equilibrium where coins continuously appreciate at an expected rate r (i.e., the equilibria described by Proposition 7 in this case).

E. *Proof of Proposition 7*

Market clearing (Assumption 2) requires that, for any $t \geq T$,

$$S_{t+1} = \frac{X^{\$}_{t+1}}{M_{t+1} - Z_{t+1}} = \frac{(1+g)X^{\$}_t}{M_t - f X^{\$}_t / S_{t+1} - Z_{t+1}} = \frac{(1+f+g)X_t}{M_t - Z_{t+1}}.$$

The rational expectation market model (Assumption 1) requires

$$\mathbb{E}_t \frac{S_{t+1}}{S_t} = (1+f+g)\mathbb{E}_t \frac{M_t - Z_t}{M_t - Z_{t+1}} = (1+r).$$

For non-stochastic Z_{t+1}, one can rewrite this condition into the following difference equation for the speculative position

$$Z_{t+1} = M_t - \frac{1+f+g}{1+r}(M_t - Z_t) = \frac{1+f+g}{1+r} Z_t - \frac{f+g-r}{1+r} M_t.$$

Similarly, we derive the number of cryptocurrency units as

$$M_{t+1} = M_t - fX_t/S_{t+1} = M_t - \frac{f}{1+f+g}(M_t - Z_{t+1}) = \frac{1+g}{1+f+g} M_t + \frac{f}{1+f+g} Z_{t+1}.$$

Plugging in the difference equation for Z_{t+1} yields the difference equation for the existing number of currency units as

$$M_{t+1} = \frac{f}{1+r} Z_t + \frac{1+r-f}{1+r} M_t.$$

Thus, we have the system of difference equations

$$\begin{pmatrix} Z_{t+1} \\ M_{t+1} \end{pmatrix} = \begin{bmatrix} \frac{1+f+g}{1+r} & -\frac{f+g-r}{1+r} \\ \frac{f}{1+r} & \frac{1+r-f}{1+r} \end{bmatrix} \begin{pmatrix} Z_t \\ M_t \end{pmatrix}, \tag{11}$$

with distinct and real eigenvalues $(\lambda_1, \lambda_2) = \left(\frac{1+g}{1+r}, 1\right)$, and eigenvectors $v_1 = \left(\frac{f+g-r}{f}, 1\right)$ and $v_2 = (1, 1)$. The system has the following solution

$$\begin{pmatrix} Z_{t+i} \\ M_{t+i} \end{pmatrix} = C \begin{bmatrix} \frac{f+g-r}{f} \\ 1 \end{bmatrix} \left(\frac{1+g}{1+r}\right)^i + U \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1^i. \tag{12}$$

From the solution of M_{t+i} for $i = 0$, we solve $C = M_t - U$, so that we find the generic solution

$$Z_{t+i} = \frac{f+g-r}{f}(M_t - U)\left(\frac{1+g}{1+r}\right)^i + U,$$
$$M_{t+i} = (M_t - U)\left(\frac{1+g}{1+r}\right)^i + U.$$

What values of U correspond to valid equilibria? Note that $\lim_{i\to\infty}(Z_{t+i}, M_{t+i}) = (U, U)$. Hence, a valid equilibrium requires $U \geq 0$: Otherwise, Z_{t+i} and M_{t+i} would converge to negative numbers. Similarly, a valid equilibrium also requires $M_t > U$. Any solution with a value of U such that $0 \leq U < M_t$ corresponds to a valid equilibrium.

The floor for the equilibrium exchange rate corresponds to the case where $U = 0$, so that $Z_t = M_t(f + g - r)/f$, and $S_t = (X_t/M_t)(f/(r-g))$. Any values of U such that $0 < U < M_t$ corresponds to equilibrium exchange rate paths with higher levels. The share of coins held by speculators can be calculated for all equilibria as

$$\frac{Z_{t+i}}{M_{t+i}} = \frac{\frac{f+g-r}{f}(M_t - U)\left(\frac{1+g}{1+r}\right)^i + U}{(M_t - U)\left(\frac{1+g}{1+r}\right)^i + U}.$$

This number converges to one for all equilibria corresponding to $U > 0$. Hence, these equilibria exhibit a similar pattern in investment holdings as bubble equilibria. If $U = 0$, then the share held by speculators will be constant at $(f + g - r)/f$.

F. *Proof of Corollary 1*

From the proof of Proposition 6, we have

$$-\sum_{i=1}^{\infty} \frac{\mathbb{E}_0 \Delta Z_{t+i} S_{t+i}}{(1+r)^i} = \frac{f+g-r}{r-g} X_0^{\$}. \tag{13}$$

Moreover, from Proposition 7 and its proof, we have

$$\begin{aligned} -S_0 Z_0 &= -\left(\frac{f}{r-g}\frac{X_0^{\$}}{M_0-U}\right)\left(\frac{f+g-r}{f}(M_0-U)+U\right), \\ &= -\frac{f+g-r}{f}X_0^{\$} - U\left(\frac{f}{r-g}\frac{X_0^{\$}}{M_0-U}\right). \end{aligned} \tag{14}$$

Summing (13) and (14) gives

$$-S_0 Z_0 - \sum_{i=1}^{\infty}\frac{\mathbb{E}_0 \Delta Z_{t+i} S_{t+i}}{(1+r)^i} = -U\left(\frac{f}{r-g}\frac{X_0^{\$}}{M_0-U}\right).$$

The only value of U such that $0 \leq U < M_0$ for which this expression for the net present value is non-negative is $U = 0$. This value of U corresponds to the baseline equilibrium.

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