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Abstract

We propose and characterize weighted-egalitarian values for cooperative transferable utility games. Each weighted-egalitarian value divides the worth of the grand coalition into two parts and allocates them through equality and proportionality based on exogenous player weights. We characterize the family of all weighted-egalitarian values by employing the standard axioms of efficiency and linearity, in addition to two novel axioms: ω-ratio invariance for symmetric players and symmetry in weights. We then show that relaxing linearity to additivity and adding coalitional monotonicity results in a subfamily of affine combinations of equal division and weighted division values. Furthermore, using an axiom called monotonicity in weights, we characterize the family of convex combinations of equal division and weighted division values.

Keywords: cooperative game, axiomatization, equal division value, weighted division value, equality
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1. Introduction

Cooperative game theory deals with the allocation of a certain worth resulting from the cooperation among players. Equality and proportionality are often regarded as fairness criteria for allocating resources (e.g., Moulin (2004) and Thomson (2019)). In various applications, it is natural to assume that players have exogenous weights that measure differences in their negotiation ability or importance (Kalai and Samet, 1987). This raises a question about studying values (i.e. allocation rules) in cooperative games that consider not only equality but, at the same time, also proportionality principles that account for players' weights.

The construction of values that naturally associate with the exogenous weights of players was pioneered by Shapley (1953) with the introduction of the weighted Shapley value, which allocates the Harsanyi dividends of each coalition among its members proportionally to their positive weights. Similarly, the weighted division (WD) values allocate the total worth among all players proportionally to their weights (e.g., Béal et al. (2016) and Kongo (2019)). Other weighted values based on proportionality principles include the Harsanyi solutions (Derks et al., 2000), weighted coalition structure values (Levy and Mclean, 1989), weighted solidarity values (Calvo and Gutiérrez-López, 2014), and weighted surplus division values (Kongo, 2019). In contrast, the equal division (ED) value is an extreme egalitarian value, that allocates the total worth equally among all players (e.g., van den Brink (2007), van den Brink and Funaki (2009), and Casajus and Huettner (2014)). In this paper, we explore a family of weighted values that offer a compromise between proportionality and equality principles.

We introduce a family of values called *weighted-egalitarian values*, which are constructed as follows. First, inspired by Zou et al. (2021), the worth of the grand coalition is divided into two parts, each of which is determined by a linear function of the worths of all coalitions. Next, one part is allocated among all players proportionally to their importance, as given by exogenous weights, and the other is allocated equally among all players. This allocation process demonstrates that the weighted-egalitarian values strike a balance between egocentrism and egalitarianism. Applying different linear functions can give rise to different values, including the WD and ED values. Our values also allow for the possibility that a player with zero weight may receive a non-zero payoff, which distinguishes them from other weighted division values, except for the weighted surplus division values. This argument is supported by the experimental evidence in De Clippel and Rozen (2022), which clearly shows that a player who adds no value to any coalition (i.e., the null player) may recieve a non-zero payoff.

To explore weighted-egalitarian values, we introduce three axioms that evaluate the effect of the weights of players on the payoffs. The first one is *ω-ratio invariance for symmetric players*, which states that the difference between the payoffs of symmetric players and any other player's payoff should be in the same proportion to the difference between their weights. This axiom is akin to *ω-mutual dependence* in Nowak and Radzik (1995), which states that the payoffs of mutually dependent players (also symmetric players) are in the same proportion to their weights. The second axiom is *monotonicity in weights*, which states that the player with the larger weight among two symmetric players should not receive less profit (or incur less cost) than the other player. This ensures that greater returns or risks are accompanied by greater weights. Note that these two axioms are variations of Shapley's symmetry axiom in the sense that they imply equal payoffs for symmetric players with equal weights. Finally, to deal with the special case where *all* weights are equal, we apply the *symmetry in weights* requirement that all players get an equal payoff in that case.

We identify subfamilies of weighted-egalitarian values using the aforementioned novel and standard axioms. Our first result characterizes the whole family of weighted-egalitarian values by combining ω-ratio invariance for symmetric players and symmetry in weights with the standard axioms of efficiency and linearity. As mentioned before, the ED and WD values are well-known examples of weighted-egalitarian values. We then focus on combinations of these two values. Our second result characterizes a subfamily of affine combinations of the ED and WD values by adding coalitional monotonicity, as introduced in van den Brink (2007) to the axioms mentioned above. This subfamily is identical to the family of convex combinations of the WD and EWD values, where the EWD value assigns to every player the average of the WD values of all other players. Our third result characterizes the family of convex combinations of the ED and WD values by adding monotonicity in weights. In this case, symmetry in weights is redundant. This family considers a fixed weight system but flexible balance coefficients and is closely related to the family of WD values in previous works such as Béal et al. (2016) and Kongo (2019), where weight systems are flexible.

The rest of this paper is organized as follows. Section 2 covers basic definitions and notation. In Section 3, we introduce the concept of weighted-egalitarian values and three new axioms. Section 4 presents the axiomatization of three subfamilies of weighted-egalitarian values. Finally, Section 5 concludes. All proofs are provided in Appendix.

2. Basic definitions and notation

A *cooperative game with transferable utility*, or simply a game, is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns a real number $v(S)$ to each coalition $S \subseteq N$, satisfying $v(\emptyset) = 0$. For each coalition $S \subseteq N$, its cardinality will be denoted by $|S|$ or s. We assume $|N| \geq 3$. As we consider the set of players N to be fixed, we represent a game by its characteristic function v. The class of games with player set N is denoted by $\mathcal{G}^N$.

For any $\emptyset \neq T \subseteq N$, the *unanimity game* u_T is given by $u_T(S) = 1$ if $S \supseteq T$, and $u_T(S) = 0$ otherwise; the *standard game* e_T is given by $e_T(S) = 1$ if $S = T$, and $e_T(S) = 0$ otherwise. Players $i, j \in N$ are *symmetric* in $v \in \mathcal{G}^N$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. For $v, w \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, the game $av + bw$ is defined by $(av + bw)(S) = av(S) + bw(S)$ for all $S \subseteq N$.

A *weight system* ω on N is a vector of real numbers $\omega = (\omega_i)_{i \in N}$ such that $\sum_{k\in N}\omega_k = 1$ and $\omega_i \geq 0$ for all $i \in N$. The set of all weight systems on N is denoted by Ω.

A *value* on $\mathcal{G}^N$ is a function ψ that assigns a single payoff vector $\psi(v) \in \mathbb{R}^N$ to every game $v \in \mathcal{G}^N$. The number $\psi_i(v)$ represents the payoff of player i in game v.

The *equal division (ED) value* on $\mathcal{G}^N$ is defined by

$$ED_i(v) = \frac{1}{n}v(N),$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

The *weighted division value* on $\mathcal{G}^N$ associated with weight system ω (WD^ω) is defined by

$$WD_i^\omega(v) = \omega_i v(N),$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

Note that $ED = WD^\omega$ for $\omega_i = \frac{1}{n}$ for all $i \in N$. Abusing notation slightly, we often use ψ instead of a weighted value, ψ^ω, associated with weight system ω.

3. Weighted egalitarian values and axioms

In this paper, we consider the following weighted values.

Definition 1. Suppose that

$$\alpha(v) = \sum_{S\subseteq N} \alpha_S v(S) \tag{1}$$

is a linear function on $\mathcal{G}^N$ where every α_S, $S \subseteq N$, is a parameter in $\mathbb{R}$. Given a weight system ω, the *weighted-egalitarian value* on $\mathcal{G}^N$ associated with ω is defined by

$$\psi_i^{\omega,\alpha}(v) = \omega_i \alpha(v) + \frac{1}{n}\big(v(N) - \alpha(v)\big), \tag{2}$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

We refer to the class of values defined by (2) for all linear functions $\alpha(v)$ as the family of weighted-egalitarian values. The ED value is obtained when $\alpha(v) = 0$, the WD value is obtained when $\alpha(v) = v(N)$, and their affine combination $\beta WD_i + (1-\beta)ED_i = \beta\omega_i v(N) + (1-\beta)\frac{v(N)}{n}$ is obtained when $\alpha(v) = \beta v(N)$. In our axiomatizations below, we will use one parametrized axiom that depends on the weight system ω, but the function α does not explicitly appear in the axioms.

First, we recall some known and desirable axioms of values.

- **Efficiency**. For all $v \in \mathcal{G}^N$, it holds that $\sum_{i\in N} \psi_i(v) = v(N)$.

- **Linearity**. For all $v, w \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, it holds that $\psi(av + bw) = a\psi(v) + b\psi(w)$.

- **Additivity**. For all $v, w \in \mathcal{G}^N$, it holds that $\psi(v + w) = \psi(v) + \psi(w)$.

- **Symmetry**. For all $v \in \mathcal{G}^N$ and $i, j \in N$ such that i and j are symmetric in v, it holds that $\psi_i(v) = \psi_j(v)$.

- **Coalitional monotonicity**. For all $v, w \in \mathcal{G}^N$ and all $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, it holds that $\psi_i(v) \geq \psi_i(w)$.

Efficiency, linearity, additivity, and symmetry are standard. Coalitional monotonicity, introduced in van den Brink (2007), states that the payoff of a player does not decrease if the worths of all coalitions containing this player do not decrease.

In what follows, we introduce three additional novel axioms that formalize how players' payoffs are affected by their weights.

The first axiom, called *ω-ratio invariance for symmetric players*, depends on the weights ω and states that if two players in a game have equal productivity, then the ratio of their payoffs to other players is proportional to their difference in weight. This imposes that players with equal productivity receive the same payoff if and only if their weights are the same. In this sense, this axiom is a weighted variant of Shapley's symmetry axiom. Let $\omega \in \Omega$.

- **ω-ratio invariance for symmetric players**. For all $v \in \mathcal{G}^N$ and all $i, j, k \in N$ such that i and j are symmetric in v, it holds that

$$[\psi_i(v) - \psi_k(v)](\omega_j - \omega_k) = [\psi_j(v) - \psi_k(v)](\omega_i - \omega_k).$$

The second axiom, called *monotonicity in weights*, states that if two players in a game have equal productivity, then the player with a larger weight should not receive less payoff or incur less cost (The terms "profit" and "cost" refer to $v(N) \geq 0$ and $v(N) \leq 0$, respectively). When all players have the same weight, this axiom boils down to symmetry.

- **Monotonicity in weights**. For all $v \in \mathcal{G}^N$, all $\omega \in \Omega$, and all $i, j \in N$ such that $\omega_i \geq \omega_j$ and i, j are symmetric in v, it holds that $\psi_i(v) \geq \psi_j(v)$ if $v(N) \geq 0$, and $\psi_i(v) \leq \psi_j(v)$ if $v(N) \leq 0$.

Notice that ω-ratio invariance for symmetric players does not have any bite when all weights are equal. Therefore, for that case we require a *symmetry in weights* axiom which states that if *all* players have the same weight, then all players get the same payoff. Let $\omega \in \Omega$.

- **Symmetry in weights**. If $\omega_h = \frac{1}{n}$ for all $h \in N$, then for all $v \in \mathcal{G}^N$, it holds that $\psi_i(v) = \psi_j(v)$ for all $i, j \in N$.

We remark that we use symmetry in weights only in the first two of our three characterizations.

4. Main results

We first provide an axiomatization of the family of weighted-egalitarian values (for a given ω) involving ω-ratio invariance for symmetric players and symmetry in weights.

Theorem 1. *Let ω be a weight system. A value ψ on $\mathcal{G}^N$ satisfies efficiency, linearity, ω-ratio invariance for symmetric players, and symmetry in weights if and only if there exists a linear function $\alpha : \mathcal{G}^N \to \mathbb{R}$ of the form (1) such that $\psi = \psi^{\omega,\alpha}$.*

The next result shows that adding coalitional monotonicity to Theorem 1 leads to a subfamily of affine combinations of the ED and WD values. Notably, linearity can be weakened to additivity.

Theorem 2. *Let ω be a weight system. A value ψ on $\mathcal{G}^N$ satisfies efficiency, additivity, ω-ratio invariance for symmetric players, symmetry in weights, and coalitional monotonicity if and only if there exists $\beta \in [-\frac{1}{n-1}, 1]$ such that*

$$\psi_i(v) = \beta\omega_i v(N) + (1-\beta)\frac{v(N)}{n}, \tag{3}$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

Remark 1. Define the following modification of the WD value:

$$EWD_i(v) = \frac{1}{n-1}\sum_{j\in N\setminus\{i\}} \omega_j v(N).$$

The family of values characterized in Theorem 2 can also be expressed as $\{\gamma WD + (1-\gamma)EWD \mid \gamma \in [0,1]\}$ with $\gamma = \frac{n-1}{n}\beta + \frac{1}{n}$. This formulation clarifies that each member of the resulting family makes a trade-off between egocentrism (γ) and altruism ($1-\gamma$). Specifically, the payoff $\omega_i v(N)$ represents an egocentric allocation for player i, while the payoff $\frac{1}{n-1}\sum_{j\in N\setminus i}\omega_j v(N)$ can be interpreted as an altruistic allocation for player i.

Remark 2. If symmetry is required in Theorem 2, then ω-ratio invariance for symmetric players, symmetry in weights and additivity become redundant, as follows from van den Brink (2007, Theorem 3.3) which shows that an even weaker symmetry[1] together with efficiency and coalitional monotonicity axiomatize the ED value. Consequently, the weights ω_i in (3) have no effect, which implies $\beta = 0$ and thus $\psi(v) = \frac{v(N)}{n}$.

[1]For all $v \in \mathcal{G}^N$ such that all players in N are symmetric in v, there exists a $c^* \in \mathbb{R}$ such that $\psi_i(v) = c^*$ for all $i \in N$.

Remark 3. As can be seen from the proofs of Theorems 1 and 2 in the appendix, symmetry in weights is only used for the case when all weights are equal. If we only consider weight vectors where not all weights are equal, then the characterization results in Theorems 1 and 2 hold without symmetry in weights.

The lower bound of the parameter β in Theorem 2 (i.e., $-\frac{1}{n-1}$) may seem innocuous[2] as shown in Remark 1, but it leads to a counterintuitive result: the player with the smaller weight among two symmetric players may recieve a larger payoff than the other player. To avoid this, the next result uses monotonicity in weights instead of symmetry in weights in Theorem 2 to identify the family of convex combinations of the ED and WD values.

Theorem 3. *Let ω be a weight system. A value ψ on $\mathcal{G}^N$ satisfies efficiency, additivity, ω-ratio invariance for symmetric players, monotonicity in weights, and coalitional monotonicity if and only if there exists $\beta \in [0,1]$ such that*

$$\psi_i(v) = \beta\omega_i v(N) + (1-\beta)\frac{v(N)}{n},$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

Remark 4. Theorem 3 still holds when monotonicity in weights is limited to the cases where the worth of the grand coalition is non-negative. This can be replaced by a comparable principle that ensures each player's payoff to be weakly increasing with her weight. Specifically, for all $v \in \mathcal{G}^N$, $i \in N$, and $\omega^1, \omega^2 \in \Omega$ such that $v(N) \geq 0$ and $\omega_i^1 \geq \omega_i^2$, it holds that $\psi_i^{\omega^1}(v) \geq \psi_i^{\omega^2}(v)$.

Remark 5. We notice that Theorem 3 does not use symmetry in weights anymore.

Remark 6. We show the independence of the axioms in Theorem 1.

[2]This lower bound is also present in some axiomatic results of parameter values, such as the GUC-subfamily of rules in Bergantiños and Moreno-Ternero (2022, Theorem 2).

(i) Not *efficiency*: The *null value* defined by $\psi_i(v) = 0$ for all $v \in \mathcal{G}^N$ and all $i \in N$.

(ii) Not *linearity*: The value defined by

$$\psi_i(v) = \omega_i + \frac{1}{n}\big[v(N) - 1\big], \tag{4}$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

(iii) Not *ω-ratio invariance for symmetric players*: The value defined by

$$\psi_i(v) = \frac{(\omega_i)^2}{\sum_{j\in N}(\omega_j)^2} v(N), \tag{5}$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

(iv) Not *symmetry in weights*: The value defined by

$$\psi_i(v) = \begin{cases} \frac{i}{\sum_{j\in N} j} v(N), & \text{if } \ \omega_k = \omega_\ell \ \text{ for all } k,\ell \in N; \\ \frac{1}{n} \text{ for all } k \in N, & \text{otherwise.} \end{cases} \tag{6}$$

for all $v \in \mathcal{G}^N$ and all $i \in N$.

Remark 7. We show the independence of the axioms in Theorems 2 and 3.

(i) Not *efficiency*: The null value.

(ii) Not *additivity*: The value defined by (4).

(iii) Not *ω-ratio invariance for symmetric players*: The value defined by (5).

(iv) Not *symmetry in weights*: The value defined by (6).

(v) Not *monotonicity in weights* in Theorem 3: The value defined by $\psi_i(v) = \frac{v(N)}{n-1} - \frac{\omega_i v(N)}{n-1}$ for all $v \in \mathcal{G}^N$ and all $i \in N$.

(vi) Not *coalitional monotonicity*: The value defined by $\psi_i(v) = 2\omega_i v(N) - \frac{v(N)}{n}$ for all $v \in \mathcal{G}^N$ and all $i \in N$.

We note that Theorems 2 and 3 remain valid when coalitional monotonicity is replaced with either non-negativity or grand coalition monotonicity, the latter introduced in Casajus and Huettner (2014). Non-negativity requires that for all $v \in \mathcal{G}^N$ with $v(N) \geq 0$, it holds that $\psi_i(v) \geq 0$ for all $i \in N$, while grand coalition monotonicity requires that for all $v, w \in \mathcal{G}^N$ such that $v(N) \geq w(N)$, it holds that $\psi_i(v) \geq \psi_i(w)$ for all $i \in N$.

5. Conclusion

In this paper, we have introduced weighted-egalitarian values to account for both weight proportionality and equality principles in the allocation of the total worth. We have characterized the family of weighted-egalitarian values, as well as two subfamilies: one consisting of some affine combinations of the ED and WD values, and the other consisting of convex combinations of the ED and WD values. These results offer valuable insights into weighted values in cooperative games, which balance the principles of proportionality and equality in resource allocation.

While some justifications have been proposed for additivity in the value theory of cooperative games, it is often viewed as a technical requirement. Consequently, a significant amount of literature is dedicated to axiomatic results that attempt to dispense with additivity. Although additivity is crucial to our results, it would be interesting to explore alternative characterizations that do not depend on it.

Appendix: proofs

Prior to presenting the proofs of our main results, we introduce a lemma that characterizes the values of unanimity games which satisfy efficiency, ω-ratio invariance for symmetric players, and symmetry in weights.

Lemma 1. *Let ω be a weight system. A value ψ on the class of scaled unanimity games $\{au_T\}_{\emptyset \neq T \subseteq N}$, satisfies efficiency, ω-ratio invariance for symmetric players and symmetry in weights if and only if there exists a function*

$g\colon \mathcal{U}^N \to \mathbb{R}$ *such that*

$$\psi_i(au_T) = \frac{a}{n} + g(au_T)(\omega_i - \frac{1}{n}), \tag{7}$$

for all u_T*,* $a \in \mathbb{R}$*,* $a \neq 0$*, and all* $i \in N$*, where* $\mathcal{U}^N$ *is the class of all scaled unanimity games on* N*.*

Proof. It is straightforward to show that the value defined by (7) satisfies the three axioms. To show the "only if" part, let ψ be a value that satisfies the three axioms. We will derive the formula for $\psi_i(au_T)$ for any $au_T \in \mathcal{U}^N$.

First, notice that, if $\omega_i = \frac{1}{n}$ for all $i \in N$, then by symmetry in weights we immediately obtain that $\psi_i(au_T) = \frac{a}{n}$ for all $i \in N$, which coincides with (7) for any function $g(au_T)$.

Therefore, in the rest of the proof we assume that there exist $i, j \in N$ with $\omega_i \neq \omega_j$. We consider the following two cases.

(a) Suppose that players i and j are symmetric in au_T, i.e., $i, j \in T$ or $i, j \in N \backslash T$. Recall that $|N| \geq 3$. For every $k \in N \backslash \{i, j\}$, by ω-ratio invariance for symmetric players,

$$[\psi_i(au_T) - \psi_k(au_T)](\omega_j - \omega_k) = [\psi_j(au_T) - \psi_k(au_T)](\omega_i - \omega_k).$$

Then,

$$\psi_k(au_T)(\omega_i - \omega_j) = \psi_j(au_T)(\omega_i - \omega_k) - \psi_i(au_T)(\omega_j - \omega_k). \tag{8}$$

Subtracting $\psi_i(au_T)(\omega_i - \omega_j)$ from both sides of (8) yields:

$$[\psi_k(au_T) - \psi_i(au_T)](\omega_i - \omega_j) = [\psi_j(au_T) - \psi_i(au_T)](\omega_i - \omega_k).$$

Since $\omega_i \neq \omega_j$, it follows that

$$\psi_i(au_T) - \psi_k(au_T) = \frac{\psi_i(au_T) - \psi_j(au_T)}{\omega_i - \omega_j}(\omega_i - \omega_k). \tag{9}$$

Denote $g_i^j(au_T) = \frac{\psi_i(au_T)-\psi_j(au_T)}{\omega_i-\omega_j}$. Since (9) also holds for $k \in \{i,j\}$ (due to the fact that $k=i$ implies $\omega_i - \omega_k = 0$, and $k=j$ implies $\frac{\omega_i-\omega_k}{\omega_i-\omega_j} = 1$), summing (9) over all $k \in N$ yields:

$$n\psi_i(au_T) - \sum_{k\in N} \psi_k(au_T) = g_i^j(au_T)\big(n\omega_i - \sum_{k\in N}\omega_k\big).$$

Since $\sum_{k\in N}\psi_k(au_T) = a$ (by efficiency), and $\sum_{k\in N}\omega_k = 1$ (by definition), it follows that $n\psi_i(au_T) - 1 = g_i^j(au_T)(n\omega_i - 1)$. Hence,

$$\psi_i(au_T) = \frac{1}{n} + g_i^j(au_T)(\omega_i - \frac{1}{n}). \tag{10}$$

Similarly,

$$\psi_j(au_T) = \frac{1}{n} + g_j^i(au_T)(\omega_j - \frac{1}{n}). \tag{11}$$

By substituting (10) and (11) into (9) and using $g_i^j(au_T) = g_j^i(au_T)$, we obtain that, for all $k \in N \setminus \{i,j\}$,

$$\begin{aligned}\psi_k(au_T) &= \psi_i(au_T) - g_j^i(au_T)(\omega_i - \omega_k)\\ &= \frac{1}{n} + g_i^j(au_T)(\omega_i - \frac{1}{n}) - g_i^j(au_T)(\omega_i - \omega_k)\\ &= \frac{1}{n} + g_i^j(au_T)(\omega_k - \frac{1}{n}).\end{aligned} \tag{12}$$

Denote $g(au_T) = g_i^j(au_T)$. Take $i,j,k \in N$. Suppose without loss of generality that $\omega_i \neq \omega_k$. (Notice that we assumed that $|N| \geq 3$ and $\omega_i \neq \omega_j$.) Then

$$\begin{aligned}g_i^k(au_T) &= \frac{\psi_i(au_T) - \psi_k(au_T)}{\omega_i - \omega_k}\\ &= \frac{\psi_i(au_T) - \psi_j(au_T)}{\omega_i - \omega_j} = g_i^j(au_T) = g(au_T),\end{aligned} \tag{13}$$

where the second equality follows from ω-ratio invariance for symmetric players.

From this, together with (10)-(12), we can conclude that (7) holds for any $k \in N$.

(b) Suppose that players i and j are asymmetric in u_T. Suppose without loss of generality that $i \in T$ and $j \in N\backslash T$. For every $k \in T\backslash\{i\}$, by ω-ratio invariance for symmetric players,

$$\psi_k(au_T) - \psi_j(au_T) = \frac{\psi_i(au_T) - \psi_j(au_T)}{\omega_i - \omega_j}(\omega_k - \omega_j). \tag{14}$$

Obviously, this equality also holds for $k = i$.

For every $k \in (N\backslash T)\backslash\{j\}$, by ω-ratio invariance for symmetric players,

$$\psi_k(au_T) - \psi_i(au_T) = \frac{\psi_j(au_T) - \psi_i(au_T)}{\omega_j - \omega_i}(\omega_k - \omega_i). \tag{15}$$

Obviously, this equality also holds for $k = j$.

Summing (14) over all $k \in T$, summing (15) over all $k \in N \setminus T$, adding this all together and applying ω-ratio invariance for symmetric players, gives

$$\sum_{k\in T}(\psi_k(au_T) - \psi_j(au_T)) + \sum_{k\in N\backslash T}(\psi_k(au_T) - \psi_i(au_T))$$

$$= \sum_{k\in T}\frac{\psi_i(au_T) - \psi_j(au_T)}{\omega_i - \omega_j}(\omega_k - \omega_j) + \sum_{k\in N\backslash T}\frac{\psi_j(au_T) - \psi_i(au_T)}{\omega_j - \omega_i}(\omega_k - \omega_i).$$

Using $\sum_{k\in N}\psi_k(au_T) = a$, $\sum_{k\in N}\omega_k = 1$ and $g^i_j(au_T) = \frac{\psi_i(au_T)-\psi_j(au_T)}{\omega_i-\omega_j}$, this is equivalent to

$$a - t\psi_j(au_T) - (n-t)\psi_i(au_T) = g^i_j(au_T) - tg^i_j(au_T)\omega_j - (n-t)g^i_j(au_T)\omega_i$$

$$\Leftrightarrow$$

$$a - t\psi_j(au_T) - (n-t)\psi_i(au_T) = g^i_j(au_T)(1 - t\omega_j - (n-t)\omega_i)$$

$$\Leftrightarrow$$

$$a - t(\psi_j(au_T) - \psi_i(au_T)) - n\psi_i(au_T) = g_j^i(au_T)(1 - t(\omega_j - \omega_i) - n\omega_i)$$

$$\Leftrightarrow \text{ by definition of } g_j^i(au_T)$$

$$a - tg_j^i(au_T)(\omega_j - \omega_i) - n\psi_i(au_T) = g_j^i(au_T)(1 - t(\omega_j - \omega_i) - n\omega_i)$$

$$\Leftrightarrow$$

$$a - n\psi_i(au_T) = g_j^i(au_T)(1 - n\omega_i)$$

$$\Leftrightarrow$$

$$\psi_i(au_T) = \frac{1}{n} + g_j^i(au_T)(\omega_i - \frac{1}{n})$$

Since, as in Case (a), we can show that $g_j^i(au_T) = g_j^k(au_T)$ for all $i, j, k \in N$, we obtain that (7) holds also in this case.

□

Proof of Theorem 1. It is easy to check that any value defined by (2) satisfies the four axioms. To show the "only if" part, let ψ be a value that satisfies the four axioms. By Lemma 1, there exists a function g such that, for all $a \in \mathbb{R}$, $i \in N$, and $\emptyset \neq T \subseteq N$,

$$\psi_i(au_T) = \frac{a}{n} + g(au_T)(\omega_i - \frac{1}{n}). \tag{16}$$

If $\omega_i = \frac{1}{n}$ for all $i \in N$ then the result immediatley follows from symmetry in weights.

Therefore, suppose that there exist $i, j \in N$ with $\omega_i \neq \omega_j$.

Since linearity implies $\psi_i(au_T) = a\psi_i(u_T)$, we obtain from (7) that

$$\frac{a}{n} + g(au_T)(\omega_i - \frac{1}{n}) = \frac{a}{n} + ag(u_T)(\omega_i - \frac{1}{n}) \text{ for all } i \in N.$$

Since there is at least one $i \in N$ with $\omega_i \neq \frac{1}{n}$, this implies that

$$g(au_T) = ag(u_T). \tag{17}$$

Then,

$$\begin{aligned}
\psi_i(v) &= \psi_i\left(\sum_{T\subseteq N, T\neq\emptyset} \Delta_v(T)u_T\right) \\
&= \sum_{T\subseteq N, T\neq\emptyset} \psi_i\left(\Delta_v(T)u_T\right) \\
&= \sum_{T\subseteq N, T\neq\emptyset} \left(\frac{\Delta_v(T)}{n} + g\left(\Delta_v(T)u_T\right)\right)\left(\omega_i - \frac{1}{n}\right) \\
&= \frac{v(N)}{n} + \left(\sum_{T\subseteq N, T\neq\emptyset} \Delta_v(T)g(u_T)\right)\left(\omega_i - \frac{1}{n}\right), \qquad (18)
\end{aligned}$$

where the first equality holds since every $v \in \mathcal{G}^N$ can be expressed as $v = \sum_{T\subseteq N, T\neq\emptyset} \Delta_v(T)u_T$ where $\Delta_v(T) = \sum_{S\subseteq T}(-1)^{|T|-|S|}v(S)$, the second equality follows from linearity, the third equality follows from (16), and the last equality follows from (17).

Denote $h(v) = \sum_{T\subseteq N, T\neq\emptyset} \Delta_v(T)g(u_T)$. Comparing (2) with (18), we only need to show that there exist $\alpha_S,\ S \subseteq N$ such that $h(v) = \sum_{S\subseteq N} \alpha_S v(S)$.

Let $v, w \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$. Since $\Delta_{av+bw}(T) = a\Delta_v(T) + b\Delta_v(T)$ for all $T \subseteq N, T \neq \emptyset$, we have $h(av + bw) = ah(v) + bh(w)$. This shows that h is a linear function on $\mathcal{G}^N$. Hence, $h(v)$ is a linear function with respect to $(2^n - 1)$ parameters $v(S)$, $\forall S \subseteq N$ and $S \neq \emptyset$. Therefore, $h(v) = \sum_{S\subseteq N} \alpha_S v(S)$, where every α_S is a parameter in $\mathbb{R}$.

□

Proof of Theorem 2. The "if" part is obvious. To show the "only if" part, let ψ be a value that satisfies the five axioms. Similar to Lemma 1, the combination of efficiency, ω-ratio invariance for symmetric players, and symmetry in weights determines that there exists a function $f\colon \mathcal{E}^N \to \mathbb{R}$ such that

$$\psi_i(ce_T) = \begin{cases} f(ce_T)(\omega_i - \frac{1}{n}), & \text{if } \emptyset \neq T \subsetneq N; \\ \frac{c}{n} + f(ce_N)(\omega_i - \frac{1}{n}), & \text{if } T = N, \end{cases} \tag{19}$$

for all e_T, $c \in \mathbb{R}$, $c \neq 0$, and all $i \in N$, where $\mathcal{E}^N$ is the class of all scaled standard games on N.

Since every $v \in \mathcal{G}^N$ can be expressed as $v = \sum_{S \subseteq N, S \neq \emptyset} v(S)e_S$, we have

$$\begin{aligned} \psi_i(v) &= \psi_i\left(\sum_{S \subseteq N, S \neq \emptyset} v(S)e_S\right) \\ &= \sum_{S \subseteq N, S \neq \emptyset} \psi_i\left(v(S)e_S\right) \\ &= \frac{v(N)}{n} + \left(\sum_{S \subseteq N, S \neq \emptyset} f\left(v(S)e_S\right)\right)\left(\omega_i - \frac{1}{n}\right), \end{aligned} \tag{20}$$

where the second equality follows from additivity, and the third equality holds by using (19).

Next, we will derive the formula for $f(ce_T)$ by distinguishing two cases with respect to T.

(i) Consider $f(ce_T)$ for $\emptyset \neq T \subsetneq N$. For the zero game v_0 defined by $v_0(S) = 0$ for all $S \subseteq N$, efficiency and additivity imply $\psi_i(v_0) = 0$ for all $i \in N$. Without loss of generality, we assume $c \geq 0$. By coalitional monotonicity, we have that $\psi_i(ce_T) = \psi_i(v_0) = 0$ for all $i \in N \backslash T$, and $\psi_i(ce_T) \geq \psi_i(v_0) = 0$ for all $i \in T$. Efficiency then implies that $0 = \sum_{i \in N} \psi_i(ce_T) = \sum_{i \in N \backslash T} \psi_i(ce_T) + \sum_{i \in T} \psi_i(ce_T) = \sum_{i \in T} \psi_i(ce_T) \geq 0$, yielding $\psi_i(ce_T) = 0$ for all $i \in T$. Hence, $\psi_i(ce_T) = 0$ for all $i \in N$, which, with

(19) implies $f(ce_T)(\omega_i - \frac{1}{n}) = 0$. This equality holds regardless of the sign of $(\omega_i - \frac{1}{n})$, and therefore we have $f(ce_T) = 0$.

(ii) Consider $f(ce_N)$. Let $x_1, x_2 \in \mathbb{R}$. We have that $\frac{x_1+x_2}{n} + f((x_1 + x_2)e_N)\left(\omega_i - \frac{1}{n}\right) = \psi_i((x_1 + x_2)e_N) = \psi_i(x_1 e_N) + \psi_i(x_2 e_N) = \frac{x_1+x_2}{n} + [f(x_1 e_N) + f(x_2 e_N)]\left(\omega_i - \frac{1}{n}\right)$, where both the first and last equalities follow from (20), and the second equality follows from additivity of ψ. This implies that $f((x_1 + x_2)e_N) = f(x_1 e_N) + f(x_2 e_N)$, showing that $f(xe_N)$ is additive with respect to x.

We will show that $f(xe_N)$ is a linear function with respect to x. The theory of additive functions (e.g. Theorem 5.2.1 in Kuczma (2009)) implies that $f(ce_N) = cf(e_N)$ when c is rational. It remains to show that $f(ce_N) = cf(e_N)$ when c is irrational.

Let $\{s_m\}$ be a sequence of rationals which converge to c from below. By coalitional monotonicity, $\psi_i(s_m e_N) \leq \psi_i(ce_N)$ for all $i \in N$, implying $\frac{s_m}{n} + f(s_m e_N)(\omega_i - \frac{1}{n}) \leq \frac{c}{n} + f(ce_N)(\omega_i - \frac{1}{n})$. Hence,

$$\frac{c - s_m}{n} + [f(ce_N) - f(s_m e_N)]\,(\omega_i - \frac{1}{n}) \geq 0.$$

Then, $\lim_{m\to\infty}\left[\frac{c-s_m}{n} + [f(ce_N) - f(s_m e_N)]\,(\omega_i - \frac{1}{n})\right] = \lim_{m\to\infty}[f(ce_N) - f(s_m e_N)](\omega_i - \frac{1}{n}) = [f(ce_N) - \lim_{m\to\infty} s_m f(e_N)](\omega_i - \frac{1}{n}) = [f(ce_N) - cf(e_N)](\omega_i - \frac{1}{n}) \geq 0$. This inequality holds regardless of whether $\omega_i - \frac{1}{n}$ is negative or positive. It must be the case that $f(ce_N) = cf(e_N)$.

Therefore, $f(xe_N)$ is a linear function with respect to x, and thus $f(ce_N) = \alpha_N c$ where $\alpha_N := f(e_N) \in \mathbb{R}$.

Subsequently, it follows from cases (i), (ii) and (20) that

$$\psi_i(v) = \frac{v(N)}{n} + \alpha_N v(N)(\omega_i - \frac{1}{n}).$$

Next, we show that α_N must be a real number in $[-\frac{1}{n-1}, 1]$. Consider any $v \in \mathcal{G}^N$ such that $v(N) \geq 0$. We have $\psi_i(v) = \frac{v(N)}{n} + \alpha_N v(N)(\omega_i - \frac{1}{n}) \geq$

$\psi_i(v_0) = 0$, where the inequality follows from coalitional monotonicity. This implies that $\frac{1}{n} + \alpha_N(\omega_i - \frac{1}{n}) \geq 0$. Since $\omega_i \in [0,1]$, we obtain that the upper bound of α_N is 1 (obtained when $\omega_i = 0$), and the lower bound of α_N is $-\frac{1}{n-1}$ (obtained when $\omega_i = 1$).

Taking $\beta = \alpha_N$ yields the desired assertion. □

Proof of Theorem 3. The "if" part is obvious. To show the "only if" part, let ψ be a value that satisfies the five axioms. First, we show that ψ has the form of (3). Note that in the proof of Theorem 2, symmetry in weights was used only for the case where all weights are equal. The other axioms imply that ψ has the form of (3) when there exist $i, j \in N$ with $\omega_i \neq \omega_j$. When $\omega_i = \omega_j$ for all $i, j \in N$, monotonicity in weights boils down to symmetry. As mentioned in Remark 2, symmetry, efficiency, and coalitional monotonicity uniquely chareacterize $\psi = ED$, which also coincides with (3).

To show that $\beta \geq 0$, we assume that $v \in \mathcal{G}^N$ and $i, j \in N$ are symmetric players in v such that $\omega_i \geq \omega_j$. By (3), we have $\psi_i(v) - \psi_j(v) = \beta(\omega_i - \omega_j)v(N)$. Then, by monotonicity in weights, we have $\beta \geq 0$ whenever $v(N) \geq 0$ or $v(N) \leq 0$. □

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