TI 2023-081/VII
Tinbergen Institute Discussion Paper

Aftermarket Welfare and Procurement Auctions

Vladimir A. Karamychev[1]

1 Erasmus University Rotterdam and Tinbergen Institute

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This Version: December 17, 2023

Abstract. Aftermarket social welfare is largely determined by a procurement auction design. Auctions select firms for operating aftermarkets, and auctions may also impose restrictions on aftermarket prices the winner can charge. This paper compares aftermarket social welfare generated by first-price and second-price procurement auctions. It reveals that the social welfare ranking depends on the monotonicity properties of the augmented demand elasticity, defined as a product of the demand elasticity and the firm's relative markup. When the augmented elasticity is price independent, first-price and second-price procurement auctions are welfare-equivalent. When it increases (or decreases) with price, first-price (or second-price) auctions are welfare-superior.

Key Words: Aftermarket, Procurement auctions, Social Welfare, Monopoly.

JEL Classification: D44, H57, L12.

* I thank Emiel Maasland, Maarten Janssen, Bauke Visser, and seminar participants at ESE and EARIE for helpful comments and suggestions.

[a] Erasmus University Rotterdam and Tinbergen Institute, karamychev@ese.eur.nl.

1. Introduction

Auctions are often used in public procurement procedures, in which governments select firms to provide public services, *e.g.*, transportation, to society. The firm that wins a procurement auction is awarded the right to operate the aftermarket. Traditionally, procurement auctions' design takes one of the two forms. In the first design form, the auction outcome determines the amount that the winning firm pays (license fee) or receives (procurement subsidy) but has *no* influence on the aftermarket price that the winner may charge. Procurement license auctions and reverse auctions for subsidies are general examples of this auction design. In the second design form, the auction outcome *is* the aftermarket price that the winning firm may not exceed. Unit-price auctions are examples of this auction design. In unit-price auctions, firms bid prices at which they are willing to sell in the aftermarket, with the lowest bidder winning. Since firms bid differently in different auction formats, the resulting aftermarket prices and the levels of aftermarket social welfare also differ.

This paper compares the levels of aftermarket social welfare generated by first-price, FPA, and second-price, SPA, unit-price procurements auctions, in which firms bid aftermarket prices. These are the two auction formats that naturally arise in a procurement context. If an auction is a sealed-bid auction, it is an FPA. If it is an open auction, such as a reverse English auction, it is outcome-equivalent to an SPA.[1] Sometimes, firms bid multiple prices in unit-price auctions, as in Ewerhart and Fieseler (2003).[2] This paper considers unit-price auctions in which firms bid a single price. A resent example of such auctions is the Italian auction for the Gradual Protection Service (GPS) in 2022. In this auction, firms bid prices at which they were committed to provide GPS to their clients.[3]

Since an FPA coincides with a Bertrand competition game when rivals' costs are unknown, this paper builds on Spulber (1995) and extends it into the analysis of social welfare implications. Spulber (1995) proves the existence and uniqueness of a Bayesian equilibrium in

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[1] For a variety of different auction formats used in practice for procurement and other purposes, see the FCC website at https://www.fcc.gov/auction-formats.

[2] In auctions for unit-price contracts, a bid is multi-dimensional and consists of multiple prices. For each bid, the auctioneer computes the score of the bid, and the lowest score wins the contract.

[3] For the GPS provisions, see annex B on the website of the Italian Regulatory Authority for Energy, Networks and Environment, ARERA, at https://www.arera.it/it/docs/20/491-20.htm and https://www.arera.it/it/docs/22/208-22.htm for the 2021 and 2022 auctions respectively.

an FPA and provides a characterization of this equilibrium. However, analyzing the welfare effects of an FPA using analytical methods is generally not feasible. As a result, the literature on welfare effects of unit-price auctions has mostly relied either on a computational model as in Lunander (2002), or on an experimental study as in Shachat and Wei (2012).

This paper begins with the analysis of a setting where it is possible to rank an FPA versus an SPA, despite the intractability of auctions' Bayesian equilibria. In this setting, firms have heterogeneous fixed costs and equal variable costs. For notational convenience, an augmented elasticity is defined as a product of the elasticity of the aftermarket demand and the monopoly relative markup. Drawing on insights from auction theory and industrial organization, this paper shows that it is the monotonicity property of the augmented elasticity that determines the ranking. If the augmented elasticity is price-independent, an FPA and an SPA are welfare-equivalent; if it increases with price, an FPA is welfare-superior to an SPA; and the reverse holds if it decreases. This result can be explained by the fact that when bidders only differ in fixed costs, the revenue equivalence theorem holds so that an FPA and an SPA result in the same expected monopoly profit. Consequently, the ranking is determined by consumer surplus alone. For consumers, an auction is like a lottery where the outcome is the revenue that all consumers jointly pay to the monopolist. It is well known that an FPA lottery dominates an SPA lottery in the second-order stochastic dominance sense. Therefore, the ranking is determined by the risk attitude of the consumer surplus function with respect to the monopoly revenue. When the augmented elasticity increases with price, consumer surplus is concave in revenue, consumers are risk averse as a whole, and an FPA dominates an SPA.

Next, a general cost case is considered. To obtain tractable results, it is assumed that the degree of cost heterogeneity is small, and the auction games are solved in an approximation. Once again, the only factor that determines the ranking of the auctions is the monotonicity property of the augmented elasticity. Finally, the paper provides an example where both an FPA and an SPA are analytically tractable. In this example, an FPA may yield as much as 7% higher social welfare relative to an SPA.

The economic literature lists several advantages and drawbacks of using an FPA and an SPA. Rothkopf *et al.* (1990) provide seven reasons for why an SPA is rare. On the other hand, an FPA loses their efficiency property once the symmetry of bidders fails. The literature that compares outcomes of an FPA and an SPA is quite versatile and extensive, yet it focuses on either auction efficiency, *i.e.*, whether the highest-valuation bidder or the lowest-cost firm wins, or auction optimality, *i.e.*, on whether the expected revenue is the highest or the expected procurement costs are the lowest. An FPA performs better than an SPA when bidders are risk averse (see Holt 1980, and Maskin and Riley 1984), when the values are "almost" common (see Klemperer 1998), when bidders can collude (see Robinson 1985), and when bidders bid under budget constraints (see Che and Gale 1998). An SPA performs better than an FPA when values are interdependent and signals are affiliated (see Milgrom and Weber 1982), and when the auction allocation results in financial externalities (see Maasland and Onderstal 2007). When bidders are asymmetric (see Maskin and Riley 2000) or in the presence of allocative and informational externalities (see Jehiel *et al.* 1999), either auction can be revenue superior. This paper contributes to this literature by showing that in a procurement framework, even in the symmetric independent private value setting, where the revenue equivalence theorem holds, an FPA and an SPA have different welfare properties once the auction is followed by an aftermarket.

The rest of the paper is organized as follows. Section 2 presents the model, which is analyzed in Section 3. Section 4 discusses conditions under which subsidy or license procurement auctions may affect social welfare and concludes the paper. The Appendix contains all extensive derivations and proofs.

2. The Model

The model is a reformulation of Spulber (1995). We consider a monopoly aftermarket with a downward-sloping market demand $D(p)$, $D' < 0$. There are $n \geq 2$ firms that participate in auction A, where $A \in \{FP, SP\}$, and FP stands for an FPA and SP stands for an SPA. Each firm i is characterized by a cost parameter $\theta_i \in [0,1]$ that is the firm's private information. For all $i \in \{1, \dots, n\}$, parameters θ_i are independent draws from $[0,1]$ according to a CDF $F(x)$ that is differentiable with a positive density $F' > 0$. Firm i of type $\theta_i = \theta$ has a total production cost $c(q, \theta)$, where q is its output. The cost parameter θ positively affects both the total and the marginal cost, so that $c_\theta \geq 0$ and $c_{q\theta} \geq 0$; hereinafter, subscripts p, θ, and q denote partial differentiation.

In auction A, bidder i submits a bid b_i^A. We only consider symmetric Bayesian equilibria in which all firms use the same bidding function $b^A(\theta)$, *i.e.*, $b_i^A = b^A(\theta_i)$. We do so because no asymmetric equilibria in undominated strategies exist. The firm that has submitted the lowest bid wins the auction, becomes a monopolist, charges price p^A in the aftermarket, and gets profit $\pi^A = \pi(p^A, \theta_i)$, where:

$$\pi(p,\theta) \stackrel{\text{def}}{=} pD(p) - c(D(p),\theta).$$

It is assumed that the profit function $\pi(p,\theta)$ is concave in p and is positive at its maximum at $p = p^M(\theta)$; the latter is determined by the first-order condition $\pi_p(p^M(\theta),\theta) = 0$. Thus, it is assumed that $\pi_{pp} < 0$ and $\pi(p^M(\theta),\theta) > 0$. Assumptions made on the cost function $c(q,\theta)$ imply that the cost parameter θ negatively affects the total and the marginal profit, *i.e.*, $\pi_\theta \leq 0$ and $\pi_{p\theta} \leq 0$. It also follows that profit π increases in price, *i.e.*, $\pi_p > 0$, on $p \in [0, p^M)$.

In an FPA, the winning firm sets the aftermarket price p^{FP} equal to its winning bid b^{FP}, which only depends on the winner's type θ:

$$p^{FP}(\theta) \stackrel{\text{def}}{=} b^{FP}(\theta).$$

In an SPA, the aftermarket price p^{SP} is determined by the lowest losing bid, which is $\min\{b^{SP}(\theta_{-i})\}$. If it happens that this lowest losing bid exceeds the (optimal) monopoly price $p^M(\theta)$ of firm i, *i.e.*, if $\min\{b^{SP}(\theta_{-i})\} > p^M(\theta)$, the winner has an incentive to lower its price to the monopoly price $p^M(\theta)$. This increases not only its own profit, but also sales, consumer surplus, and, eventually, social welfare. Not allowing the monopolist to reduce its price in this case leads to a lower level of social welfare which an SPA generates. That is why we assume that the aftermarket price p^{SP} equals either the lowest losing bid $\min\{b^{SP}(\theta_{-i})\}$ or the monopoly price $p^M(\theta)$, whichever is the lowest:[4]

$$p^{SP}(\theta,\theta_{-i}) \stackrel{\text{def}}{=} \min\{b^{SP}(\theta_{-i}), p^M(\theta)\}.$$

Aftermarket consumer surplus CS^A is determined by the aftermarket price alone so that $CS^A = CS(p^A)$, where:

$$CS(p) \stackrel{\text{def}}{=} \int_{x \geq p} D(x)\, dx.$$

Social welfare is defined as a sum of the monopoly profit and consumer surplus:

$$SW^A \stackrel{\text{def}}{=} \pi^A + CS^A.$$

In an SPA, price p^{SP} is a random variable for a given winning type θ, and so are the levels of profit π^{SP}, consumer surplus CS^{SP}, and social welfare SW^{SP}. In an FPA, to the contrary, price p^{FP}, profit π^{FP}, consumer surplus CS^{FP}, and the social welfare SW^{FP} are deterministic. Conditional on θ, the expected aftermarket price $\bar{p}^A(\theta)$, the expected monopoly profit $\bar{\pi}^A(\theta)$, the expected consumer surplus $\overline{CS}^A(\theta)$, and the expected social welfare $\overline{SW}^A(\theta)$ are defined as the following conditional expectations of p^A, π^A, CS^A, and SW^A:

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[4] *Cf.* Spulber 1995, section IV on franchise competition.

$$\bar{p}^A(\theta) \stackrel{\text{def}}{=} \mathbb{E}[p^A|\theta \text{ wins } A],$$
$$\bar{\pi}^A(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\pi^A|\theta \text{ wins } A] = \mathbb{E}[\pi(p^A,\theta)|\theta \text{ wins } A],$$
$$\overline{CS}^A(\theta) \stackrel{\text{def}}{=} \mathbb{E}[CS^A|\theta \text{ wins } A] = \mathbb{E}[CS(p^A)|\theta \text{ wins } A],$$
$$\overline{SW}^A(\theta) \stackrel{\text{def}}{=} \mathbb{E}[SW^A|\theta \text{ wins } A] = \bar{\pi}^A(\theta) + \overline{CS}^A(\theta).$$

The analysis below focuses on conditions under which it is an FPA that is welfare superior, or an SPA, or they are welfare equivalent.

3. Analysis

Let us begin with an SPA. Since profit $\pi(p,\theta)$ is monotonically increasing on $p \in [0, p^M]$, the only undominated strategy of a firm of type θ is to bid its break-even price. Thus, the equilibrium bidding function $b^{SP}(\theta)$ is implicitly defined by:

$$\pi(b^{SP}(\theta),\theta) = 0.$$

Bidding function $b^{SP}(\theta)$ is well-defined, differentiable, and increasing.

An FPA is analyzed in Spulber (1995), where it is shown that there exists a unique Bayesian equilibrium of an FPA, it is symmetric, and the bidding function $b^{FP}(\theta)$ satisfies the following ordinary differential equation, ODE in short:

$$b^{FP'}(\theta) = (n-1)\frac{F'(\theta)}{1-F(\theta)} \cdot \frac{\pi(b^{FP},\theta)}{\pi_p(b^{FP},\theta)},$$

with the boundary condition $\pi(b^{FP}(1),1) = 0$. It implies that $b^{FP}(1) = b^{SP}(1)$ so that the highest cost firm bids its break-even price in an FPA. The bidding function $b^{FP}(\theta)$ is increasing so that other, lower-cost firms bid lower and if they win, they get strictly positive monopoly profits, *i.e.*, $\pi(b^{FP}(\theta),\theta) > 0$ for $\theta \in [0,1)$.

Since a closed-form solution $b^{FP}(\theta)$ to the above ODE is not feasible, as well as the explicit expression for $b^{SP}(\theta)$, the social welfare analysis proceeds in the following three steps. First, in Section 3.1, a case where $c_{q\theta} = 0$ is considered. In this case, θ only affects firms' fixed costs; firms' variable costs are equal and publicly known. The main result of the paper is obtained in this setting, and without having to solve intractable equations. Using insight from auction theory and IO, the social welfare levels generated by an FPA and an SPA can be compared indirectly. Then, Section 3.2, considers a general case under the assumption that the cost parameter θ has a small influence on firms' costs. Both FPA and SPA are solved in approximations. The solutions confirm that the main result continues to hold in a general setting, provided the degree of firms' cost heterogeneity is small. To get insights into the effect of a large cost heterogeneity on the auctions' welfare ranking, Section 3.3 presents a tractable example in which firms have constant private marginal costs. This example illustrates how a large heterogeneity in variable cost favors the first-price auction.

3.1. Fixed Cost Heterogeneity Analysis

Suppose $c_{q\theta} = 0$ so that the cost function can be written as a sum of the fixed and the variable costs: $c(q,\theta) = FC(\theta) + VC(q)$. Without loss of generality, $FC(\theta) = \theta$ is assumed so that:

$$c(q,\theta) = \theta + VC(q).$$

As a result, monopoly profit π becomes additively separable and can be written as follows:

$$\pi(p,\theta) = r(p) - \theta,$$

where $r(p)$ is defined as the firm's revenue net of variable cost, net revenue in short:

$$r(p) \stackrel{\text{def}}{=} pD(p) - VC\big(D(p)\big).$$

Since $r' = \pi_p > 0$ for $p < b^{FP}(1)$, the net revenue function $r(p)$ has an inverse, which we denote by $\tilde{p}(x)$:

$$\tilde{p}(x) \stackrel{\text{def}}{=} r^{-1}(x),$$

and which is implicitly defined by:

$$\pi(\tilde{p}(x), x) = 0.$$

Thus, $\tilde{p}(\theta_i)$ is the break-even price of type θ_i monopolist: by charging price $\tilde{p}(\theta_i)$, the monopolist gets net revenue $r\big(\tilde{p}(\theta_i)\big) = \theta_i$ and zero profit. It follows that $b^{SP}(\theta) = \tilde{p}(\theta)$.

It can be seen now that bidding an aftermarket price p in an auction is equivalent to bidding the aftermarket net revenue r, provided the aftermarket price is set to $\tilde{p}(r)$. The winner's profit, *i.e.*, its auction payoff, equals $(r - \theta)$, which is additively separable so that the standard theory of auctions applies. In particular, the revenue equivalence theorem, RET in short, holds. One of the implications of RET is that the winner of type θ gets equal expected payoffs in an FPA and an SPA:

$$\bar{\pi}^{FP}(\theta) = \bar{\pi}^{SP}(\theta).$$

Since

$$\bar{\pi}^{A}(\theta) = \mathbb{E}[r(p^A)|\theta \text{ wins } A] - \theta,$$

it follows that the expected net revenues are also equal:

$$\mathbb{E}\big[r\big(p^{FP}(\theta)\big)\big|\theta \text{ wins } FP\big] = \mathbb{E}\big[r\big(p^{SP}(\theta,\theta_{-i})\big)\big|\theta \text{ wins } SP\big].$$

However, in an FPA, $r(p^{FP}(\theta))$ is deterministic, whereas in an SPA, $r(p^{SP}(\theta,\theta_{-i}))$ is a random variable, *i.e.*, a lottery, with the same mean $r(p^{FP}(\theta))$ but positive variance. In other words, conditional on θ, lottery $r(p^{SP}(\theta,\theta_{-i}))$ is a mean-preserving spread of (a degenerate) lottery $r(p^{FP}(\theta))$. Therefore, the FPA aftermarket revenue $r(p^{FP}(\theta))$ dominates the SPA aftermarket revenue $r(p^{SP}(\theta,\theta_{-i}))$ in the second-order stochastic dominance sense.

This allows us to compare the expected social welfare levels that an FPA and an SPA generate by looking at the degree of risk aversion of the social welfare function written as a function of the aftermarket revenue r:

$$\widetilde{SW}(r) \stackrel{\text{def}}{=} \pi(\tilde{p}(r),\theta) + CS(\tilde{p}(r)) = (r-\theta) + \int\limits_{p\geq\tilde{p}(r)} D(p)\,dp.$$

Notice that since the monopoly profit $\pi(\tilde{p}(r),\theta) = (r-\theta)$ is linear in r, the degree of the social welfare risk aversion equals the degree of aversion of consumer surplus $CS(\tilde{p}(r))$.

When $CS(\tilde{p}(r))$ is linear in r, consumers are jointly risk neutral, $\widetilde{SW}(r)$ is linear as well, and both auctions are welfare equivalent. When $CS(\tilde{p}(r))$ is concave, consumers are jointly risk averse, $\widetilde{SW}(r)$ is also concave, and an FPA is welfare superior. When $CS(\tilde{p}(r))$ is convex, consumers are jointly risk seeking, $\widetilde{SW}(r)$ is convex, and an SPA is welfare superior. The first-order derivative of $\widetilde{SW}(r)$ is:

$$\widetilde{SW}'(r) = 1 - D(\tilde{p}(r))\tilde{p}'(r) = 1 - \frac{D(\tilde{p}(r))}{r'(\tilde{p}(r))} = 1 - \frac{1}{\left(1 - \frac{(p-MC)}{p}\varepsilon\right)} = 1 - \frac{1}{(1-\tilde{\varepsilon})}.$$

where $MC(q)$ is the marginal cost, $\varepsilon(p)$ is the aftermarket demand elasticity, and $\tilde{\varepsilon}(p)$ is the augmented elasticity:

$$MC(q) \stackrel{\text{def}}{=} VC'(q),$$

$$\varepsilon(p) \stackrel{\text{def}}{=} \frac{pD'(p)}{D(p)},$$

$$\tilde{\varepsilon}(p) \stackrel{\text{def}}{=} \frac{p-MC}{p}\varepsilon = \frac{(p-MC)D'}{D}.$$

All these functions are evaluated at output $q = D(p)$ and price $p = \tilde{p}(r)$.

The augmented elasticity $\tilde{\varepsilon}$ has the following interpretation. Raising a monopoly price marginally has two opposite effects on monopoly profit:

$$\pi_p = D - D\tilde{\varepsilon}.$$

The positive effect is D, it is direct, and it comes from getting one extra marginal unit of revenue from each current consumer. The negative effect is $D\tilde{\varepsilon}$, it is indirect, and it comes from a decrease in demand and the resulting from it change in the total production cost. Distributing this negative effect over all current consumers results in $\tilde{\varepsilon}$ per consumer. Thus, $\tilde{\varepsilon}(p)$ is a per-consumer marginal profit loss due to the indirect effect of a marginal price increase. The first-order condition for the monopoly profit maximization problem, which is $\tilde{\varepsilon}(p^M) = 1$, simply states that the benefit of raising price marginally, which is 1, must be equal the cost of doing so, which is $\tilde{\varepsilon}$. Since $\pi_p > 0$ for $p < p^M$, it follows that $\tilde{\varepsilon}(p) < 1$ in this price range.

The second-order derivative of $\widetilde{SW}(r)$ is:

$$\widetilde{SW}''(r) = -\frac{\tilde{\varepsilon}'\tilde{p}'}{(1-\tilde{\varepsilon})^2} = -\frac{\tilde{\varepsilon}'}{(1-\tilde{\varepsilon})^2 r'\big(\tilde{p}(r)\big)} = -\frac{\tilde{\varepsilon}'}{(1-\tilde{\varepsilon})^3 D}.$$

The following proposition states the result.

Proposition 1.

When firms are heterogeneous in fixed cost only, the monotonicity of the augmented elasticity $\tilde{\varepsilon}'$ determines whether an FPA dominates an SPA in terms of social welfare.

1. If $\tilde{\varepsilon}'(p) = 0$, an FPA and an SPA are equivalent.
2. If $\tilde{\varepsilon}'(p) > 0$, an FPA dominates an SPA.
3. If $\tilde{\varepsilon}'(p) < 0$, an SPA dominates an FPA.

Due to RET, the expected profits of the monopolist are the same in an FPA and an SPA. The difference in social welfare comes from the difference in consumer surplus only. This difference is determined by the price derivative of the augmented elasticity. Whether the augmented elasticity is increasing in price is an empirical question for a given market. However, for any bounded market demand function, the augmented elasticity is necessarily increasing when price approaches marginal cost. Therefore, part 3 of the proposition never holds for a practical demand function for the whole price range.[5]

From a theoretical modeling perspective, often used in the literature specifications with non-decreasing marginal cost functions and linear, power, and constantly elastic demand functions, *e.g.*, $D = (1-p)^\gamma$, $D = 1 - p^\gamma$, and $D = p^{-\gamma}$, are all characterized by $\tilde{\varepsilon}' > 0$. Moreover, since for non-decreasing marginal cost, the relative markup $\frac{p-MC}{p}$ is increasing in price, even in cases where demand elasticity is decreasing, the augmented elasticity might well

[5] For unbounded demand functions, the augmented elasticity can be decreasing, as in case of zero marginal cost and demand function $D(p) = 1 + \frac{1}{p}$. In such a case, $\tilde{\varepsilon} = \frac{1}{p+1}$ which is decreasing in price.

be increasing. The assumption $\pi_{pp} < 0$ also puts some restriction on how negative $\tilde{\varepsilon}'$ can be: $\pi_{pp} = D(1 - \tilde{\varepsilon}) < 0$ implies $\tilde{\varepsilon}' > \frac{D'}{D}(1 - \tilde{\varepsilon})$. These arguments suggest that in theoretical and applied literature, it is more likely that an FPA is welfare superior to an SPA. Yet, the following is an exception.

Corollary 1.
With constant, perfectly inelastic demand, an FPA and an SPA are welfare equivalent.

When aftermarket demand is constant, the total monopoly cost is a fixed cost, and $\tilde{\varepsilon} = 0$. Then, Proposition 1 applies and the equivalence follows from its Part 1. Non-trivial social welfare implications only arise when aftermarket demand is not constant and responds to price changes.

Proposition 1 ranks an FPA and an SPA only. Nevertheless, the welfare superiority of an FPA is easy to establish.

Corollary 2.
If $\tilde{\varepsilon}'(p) > 0$, an FPA maximizes the aftermarket social welfare over all efficient equilibria of all possible auction formats.

An efficient Bayesian equilibrium is necessarily monotone, symmetric, in pure strategies, and with full participation. In this setting, all auctions result in the same expected aftermarket revenue, according to RET. Conditional on the type of the winner, the revenue after an FPA is deterministic. Consequently, when the social welfare function $\widetilde{SW}(r)$ is concave, which is the case when $\tilde{\varepsilon}'(p) > 0$, social welfare generated by an FPA is strictly higher than social welfare generated by any other auction format in which the revenue is stochastic.

3.2. Small Variations in Cost Analysis

When $c_{q\theta} > 0$ so that the cost parameter θ positively affects firms' variable cost, the standard theory of auctions is not applicable. Approximation techniques must be used instead. Let us assume that θ has only small effect on firms' profits and on all their derivatives. This is as if all firms were almost identical and θ measured small differences between them. To simplify the notation, and without loss of generality, we assume that θ is uniformly distributed over the $[1 - \alpha, 1]$ interval. This assumption does not reduce the generality because for any CDF $F(x)$ of θ, random variable $(1 - \alpha) + \alpha F(\theta)$ is distributed uniformly on the $[1 - \alpha, 1]$ interval.

Taking $(1-\alpha)+\alpha F(\theta)$ to be the new cost parameter results in the desired distribution. The assumption of a small support $[1-\alpha, 1]$ allows us to use approximation techniques and find solutions to an FPA and an SPA as power series of $(\theta-1)$.

In the first-order approximation, FOA in short, firms' bids, aftermarket prices, and monopoly profits are all written as linear functions of $(\theta-1)$, plus a reminder $o(\theta-1)$ in the Peano's form. Thus, both equilibrium profit and price functions are linear in $(\theta-1)$, so that profit π is linear in price p in the FOA. This is equivalent to the market demand being constant. According to Corollary 1, an FPA and an SPA are welfare equivalent in the FOA. Thus, the social welfare difference between an FPA and an SPA is of the second or a higher order. That is why we begin with the second-order approximation, SOA in short. We omit the reminders in the expressions that follow.

In the SOA, we write $b^A(\theta)$ as follows:

$$b^A(\theta) = b_0^A + b_1^A(\theta-1) + \frac{1}{2}b_2^A(\theta-1)^2,$$

where b_k^A are to be found k^{th}-order terms, that are independent of θ. When $\theta=1$, condition $\pi(b^A(\theta),\theta)=0$ implies $b_0^A=b_0$ where b_0 is the bid of the highest cost firm in both an FPA and an SPA, determined by $\pi(b_0,1)=0$. In the same vein, we write $\pi(b^A,\theta)$ and $\pi_p(b^A,\theta)$ as their second-order Tailor series:

$$\pi(b^A(\theta),\theta) = \pi_0^A + \pi_1^A(\theta-1) + \frac{1}{2}\pi_2^A(\theta-1)^2,$$

$$\pi_p(b^A(\theta),\theta) = \pi_{p0}^A + \pi_{p1}^A(\theta-1) + \frac{1}{2}\pi_{p2}^A(\theta-1)^2,$$

where π_k^A and π_{pk}^A are the corresponding k^{th}-order terms of the profit function $\pi(b^A(\theta),\theta)$ and its derivative $\pi_p(b^A(\theta),\theta)$. Since $\pi(b^A(1),1)=0$, it follows that $\pi_0^A=0$.

In an SPA, the condition that determines $b^{SP}(\theta)$ is $\pi(b^{SP}(\theta),\theta)=0$. In an FPA, the condition that determines $b^{FP}(\theta)$ is the ODE, which for the uniform on the $[1-\alpha,1]$ interval distribution of θ is:

$$(1-\theta)\pi_p(b^{FP}(\theta),\theta)b^{FP\prime}(\theta) = (n-1)\pi(b^{FP}(\theta),\theta).$$

By plugging the above expansions for $b^A(\theta)$, $\pi(b^A,\theta)$, and $\pi_p(b^A,\theta)$ into the two equilibrium conditions and discarding terms that are of a higher than the second order, we get algebraical equations that determine b_1^A, and b_2^A. Then, we compute the expected conditional on θ aftermarket prices $\bar{p}^A(\theta)$, monopoly profits $\bar{\pi}^A(\theta)$, consumer surpluses $\overline{CS}^A(\theta)$, and the social welfare levels $\overline{SW}^A(\theta)$, and compare them between an FPA with an SPA. The augmented elasticity becomes a function of the price and the cost parameter so that we write it as $\tilde{\varepsilon}(p,\theta)$. The following proposition states the result.

Proposition 2.

In the second-order approximation, conditional on θ:

1. The expected profits in an FPA and in an SPA are equal: $\bar{\pi}^{FP}(\theta) = \bar{\pi}^{SP}(\theta)$.
2. The expected price in an FPA is lower than in an SPA:

$$\bar{p}^{FP}(\theta) - \bar{p}^{SP}(\theta) = \frac{(n-1)}{2(n+1)n^2}\frac{\pi_{pp}\pi_\theta^2}{\pi_p^3}(1-\theta)^2 < 0.$$

3. If $\tilde{\varepsilon}_p(b_0,1) > 0$, the expected consumer surplus in an FPA is higher than that in an SPA:

$$\overline{CS}^{FP}(\theta) - \overline{CS}^{SP}(\theta) = \frac{(n-1)D_0^2}{2(n+1)n^2}\frac{\pi_\theta^2}{\pi_p^3}(1-\theta)^2\tilde{\varepsilon}_p(b_0,1) > 0.$$

Part 1 of the proposition is RET in the second-order approximation. This can be interpreted as follows. When $\pi_{p\theta} = 0$ so that θ only affects fixed costs, RET holds. When $\pi_{p\theta} < 0$ so that θ also affects variable costs, RET can only hold if the effect of $\pi_{p\theta}$ on the profit of the monopolist in SOA is identical in an FPA and an SPA. Part 1 delivers exactly this result: in SOA, the effect of θ on marginal cost influences expected profits in an FPA and an SPA in an identical manner so that RET continues to hold.

Since monopoly profit is concave in price, and, conditional on θ, the variance of the aftermarket price in an FPA, which is zero, is smaller than that in an SPA, RET can only hold if the expected aftermarket price is lower in an FPA than in an SPA. This is Part 2 of the proposition. Part 3 of the proposition provides the same ranking as Proposition 1 does. To get a deeper understanding of why the welfare ranking in SOA is exactly the same as in the fixed cost heterogeneity case of Proposition 1, we note that:

$$\tilde{\varepsilon}_p(p,\theta) = \frac{1}{D}\left(\frac{D'}{D}\pi_p - \pi_{pp}\right).$$

This implies that in SOA, the welfare ranking is only determined by price-partials π_p and π_{pp} of the profit function. The cross-partial derivative $\pi_{p\theta} = -c_{q\theta}D'$, which captures the effect of θ on marginal cost $c_q(q,\theta)$ and which enters the expressions for $\bar{\pi}^A(\theta)$ and $\overline{CS}^A(\theta)$ (see the proof of Proposition 2), cancels out in the comparison. Thus, the effect of $\pi_{p\theta}$ on consumer surplus in SOA is also identical in an FPA and an SPA, just like the effect of $\pi_{p\theta}$ on the profit of the monopolist. As a result, the marginal cost heterogeneity does not contribute to the ranking. The only effect that remains is the effect of θ on fixed cost, *i.e.*, π_θ. That is why Proposition 2 delivers the same result as Proposition 1 does.

When $\tilde{\varepsilon}_p(b_0, 1) = 0$, an FPA and an SPA are welfare equivalent in the SOA. This means that the social welfare ranking is determined by the third-order effects. The following proposition states this result.

Proposition 3.

Let $\tilde{\varepsilon}_p(b_0, 1) = 0$. Then:

1. If $\tilde{\varepsilon}_{p\theta}(b_0, 1) < 0$ then an FPA results in higher levels of profit, consumer surplus, and social welfare.
2. If $\tilde{\varepsilon}_{p\theta}(b_0, 1) > 0$ then an SPA results in higher levels of profit, consumer surplus, and social welfare.

Appendix provides expressions for the profit and consumer surplus differences; both are proportional to $\tilde{\varepsilon}_{p\theta}(b_0, 1)$. This result can be interpreted as follows. If the highest-cost firm with $\theta = 1$ wins the auction, it charges price $p = b_0$ in the aftermarket and faces the augmented elasticity $\tilde{\varepsilon}(b_0, 1)$. If $\tilde{\varepsilon}_p(b_0, 1) > 0$, according to Proposition 2, an FPA is welfare-superior to an SPA. Suppose $\tilde{\varepsilon}_p(b_0, 1) = 0$ and $\tilde{\varepsilon}_{p\theta}(b_0, 1) < 0$. Then, in some neighborhood where $\theta < 1$, $\tilde{\varepsilon}_p(b_0, 1) > 0$ holds. This implies that in some neighborhood where $p < b_0$, $\tilde{\varepsilon}(p, \theta)$ increases in price. According to Proposition 3, also in this case, an FPA is welfare-superior to an SPA. Therefore, Proposition 3 confirms that the monotonicity of the augmented elasticity is the only determinant of the welfare ranking of an FPA and an SPA.

The above approximation results are obtained under the assumption that θ is distributed over a small interval $[1 - \alpha, 1]$. This is equivalent to the assumption that relative effects of θ on the profit function $\pi(p, \theta)$ and on all its partial derivatives are small. Hence, the approximation is applicable when θ is uniformly distributed over the whole unit interval [0,1] provided $\left|\frac{\pi_\theta}{\pi}\right| \ll 1$, $\left|\frac{\pi_{p\theta}}{\pi_p}\right| \ll 1$, $\left|\frac{\pi_{pp\theta}}{\pi_{pp}}\right| \ll 1$, $\left|\frac{\pi_{p\theta\theta}}{\pi_{p\theta}}\right| \ll 1$, *etc.*

3.3. Marginal Cost Heterogeneity Example

Here is an analytically tractable example with constant marginal costs that take values from zero and up to the maximal consumers' willingness to pay. Let the aftermarket demand be $D(p) = (1 - p)^\gamma$, with $\gamma > 0$, and firms' cost function be $c(q, \theta) = q\theta$. Parameter θ

represents firms' (constant) unit production cost. The following proposition provides the closed-form solutions for an FPA and an SPA, and the welfare ranking.

Proposition 4.

When $D(p) = (1-p)^\gamma$, $c(q,\theta) = q\theta$, and θ is uniformly distributed over the $[0,1]$ interval:

1. FPA and SPA bidding functions are $b^{FP}(\theta) = \theta + \frac{1}{n+\gamma}(1-\theta)$ and $p^{SP}(\theta) = \theta$.
2. An FPA results in a lower conditional expected aftermarket price:
$$\bar{p}^{FP}(\theta) - \bar{p}^{SP}(\theta) = -\frac{\gamma}{n(n+\gamma)}\left(1 - \frac{n+\gamma}{1+\gamma}\left(\frac{\gamma}{1+\gamma}\right)^{n-1}\right)(1-\theta) < 0.$$
3. An FPA results in higher both conditional expected aftermarket profit and consumer surplus:
$$\frac{\bar{\pi}^{FP}(\theta)}{\bar{\pi}^{SP}(\theta)} = \frac{\overline{CS}^{FP}(\theta)}{\overline{CS}^{SP}(\theta)} = 1 + R(\gamma, n),$$

where $R(0,n) = R(\gamma,1) = R(\gamma,\infty) = R(\infty,n) = 0$ and $R(\gamma,n) > 0$ otherwise.

Auction bidding functions turn out to be linear, and the computation of expected prices, profits, and consumer surpluses is straightforward, although algebraically involved. As one can see, an FPA outperforms an SPA in this example in all respects: the expected market price is lower, and the expected profit and consumer surplus are higher. An FPA Pareto-dominates an SPA.

Figure 1 shows graphs of $R(\gamma,n)$ as a function of the demand parameter γ, for different numbers of firms, $n \in \{2,\ 4,\ 10\}$. The relative social welfare gain of using an FPA for $n = 2$ firms and linear demand with $\gamma = 1$, is 6.67%. When $n = 1$ or $n \to \infty$, function $R(\gamma,n)$ converges to zero, illustrating the fact that an FPA and an SPA perform equally well in case of one firm only, and in case of very many firms. When the demand parameter γ converges either to zero or to infinity, similarly, $R(\gamma,n)$ converges to zero because demand becomes inelastic in these limits, and an FPA and an SPA perform equally well in this case. Note that consumers and firms both benefit from an FPA, and their relative benefits are the same.

4. Discussion and Conclusion

In this paper, we have observed that first-price and second-price procurement auctions result in different levels of social welfare in the aftermarket. The monotonicity property of the augmented elasticity plays a crucial role in determining the auction ranking. When firms differ

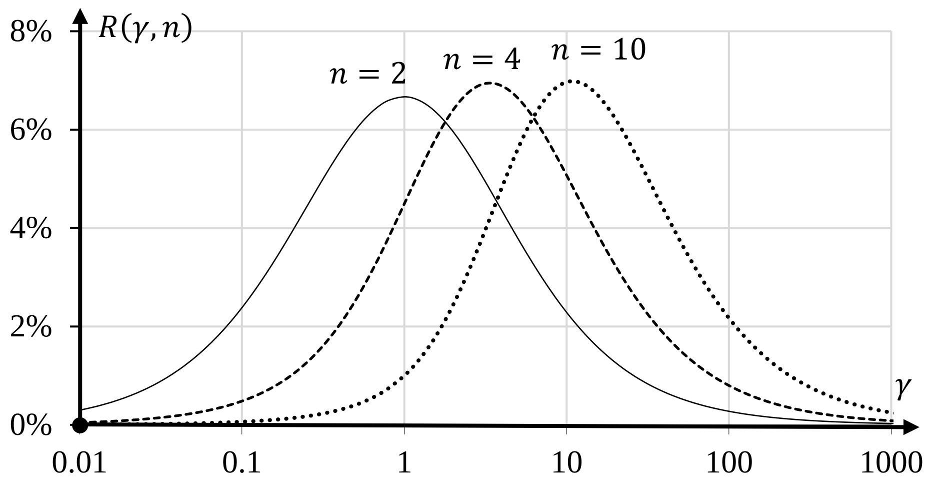

Figure 1. Function $R(\gamma, n)$, in %%, as a function of γ (log scale), for $n = 2$ (solid curve), $n = 4$ (dashed curve), and $n = 10$ (dotted curve).

only in their fixed costs or when the degree of cost heterogeneity is small, this property is the only factor that determines the ranking. We have also seen an example demonstrating that an FPA may yield almost 7% more social welfare than an SPA.

When a procurement auction design takes the form of a license or a subsidy auction, economic literature often assumes that the auction winning bid of the monopolist affects neither the aftermarket demand nor consumer surplus. Under this assumption, the auction format has indeed no effect on social welfare. However, when consumer surplus depends on the total cost structure of the monopolist, this assumption fails, and the auction format does have an effect on social welfare. This happens when two conditions hold.

The first condition is that the total cost of the monopolist has an effect on its operational decisions. This can occur if, *e.g.*, the monopolist is liquidity constrained. Kamphorst *et al.* (2020) explain how fixed costs can affect firms' pricing decisions. In our framework, the license fee or the received subsidy is a part of the monopoly fixed cost.

The second condition is that the aftermarket consumer surplus depends on the monopoly operational decisions. Let us consider a procurement of a public transportation contract. It is reasonable to assume that, in addition to the transportation fare, other service characteristics determine the aftermarket demand for transportation. Some of these characteristics, such as service routes and frequencies, can be included in the procurement contract and regulated. However, there are many others that are difficult, if not impossible, to contract upon, such as the cleanness of buses, the friendliness of bus drivers, onboard comfort, pleasure, and safety, the functionality and performance of on-board facilities such as USB chargers, Wi-Fi, airconditioning, *etc*. These are all elements of a multi-dimensional service quality that are often set by the operating monopoly and are not fully regulated by the contract. We can aggregate all these elements by referring to them as the 'effort' that the monopoly management puts into the service. This effort has a positive effect on consumer demand and surplus but is costly for the monopolist. The auction winner maximizes its profit by setting this effort optimally.

When these two conditions hold, the subsidy that the monopolist receives or the license fee that it pays affect its operational decisions, including the choice of the optimal effort level. The latter, in turn, affects utility levels that consumers derive from using public transportation and, consequently, the demand for it. As a result, auction format affects aftermarket social welfare. The resulting auction ranking will likely depend on the specification details of the market demand and profit functions.

References

Che, Y-K. and I.L. Gale, 1998, Standard Auctions with Financially Constrained Bidders, *Review of Economic Studies*, 65(1): 1-21, https://www.jstor.org/stable/2567001.

Ewerhart, C. and K. Fieseler, 2003, Procurement Auctions and Unit-Price Contracts, *The RAND Journal of Economics* 34(3): 569-581, https://www.jstor.org/stable/1593747.

Holt, C.A. Jr., 1980, Competitive Bidding for Contracts under Alternative Auction Procedures, *Journal of Political Economy* 88(3): 433-445, https://www.jstor.org/stable/1831924.

Jehiel, P., B. Moldovanu, and E. Stacchetti, 1999, Multidimensional Mechanism Design for Auctions with Externalities, *Journal of Economic Theory*, 85(2): 258-293, doi: https://doi.org/10.1006/jeth.1998.2501.

Kamphorst, J., E. Mendys-Kamphorst, and B. Westbrock, 2020, Fixed costs matter even when the costs are sunk, *Economics Letters* 195:109428, doi: https://doi.org/10.1016/j.econlet.2020.109428

Klemperer, P.D., 1998, Auctions with Almost Common Values: the "Wallet Game" and its Applications, *European Economic Review*, 42(3-5): 757-769, doi: https://doi.org/10.1016/S0014-2921(97)00123-2.

Lunander, A., 2002, Procurement Bidding in First-Price and Second-Price, Sealed-Bid Auctions within the Common-Value Paradigm, *Computational Economics* 19: 227–244, doi: https://doi.org/10.1023/A:1015008130780.

Maasland, E. and S. Onderstal, 2007, Auctions with Financial Externalities, *Economic Theory*, 32(3): 551-574, https://www.jstor.org/stable/27822573.

Maskin, E.S. and J.G. Riley, 1984, Optimal Auctions with Risk Averse Buyers, *Econometrica*, 52(6): 1473-1518, doi: https://doi.org/10.2307/1913516.

Maskin, E.S. and J.G. Riley, 2000, Asymmetric Auctions, *Review of Economic Studies*, 67(3): 413-438, https://www.jstor.org/stable/2566960.

Milgrom, P.R. and R.J. Weber, 1982, A Theory of Auctions and Competitive Bidding, *Econometrica*, 50(5): 1089-1122, doi: https://doi.org/10.2307/1911865.

Robinson, M.S., 1985, Collusion and the Choice of Auction, *RAND Journal of Economics*, 16(1): 141-145, doi: https://doi.org/10.2307/2555595.

Rothkopf, M.H, T.J. Teisberg, and E.P. Kahn, 1990, Why Are Vickrey Auctions Rare? *Journal of Political Economy* 98(1): 94-109, https://www.jstor.org/stable/2937643.

Shachat, J., and, L. Wei, 2012, Procuring Commodities: First-Price Sealed-Bid or English Auctions, *Marketing Science* 31(2): 317-333, https://www.jstor.org/stable/41488025.

Spulber, D.F., 1995, Bertrand Competition when Rivals' Costs are Unknown, *The Journal of Industrial Economics* 43(1): 1-11, https://doi.org/10.2307/2950422.

Appendix

Proof of Proposition 1 is in the main text.

Proof of Proposition 2.

First, we evaluate π_k^A and π_{pk}^A:

$$\pi_1^A = \frac{d}{d\theta}\pi(b^A(\theta),\theta) = \pi_p b_1^A + \pi_\theta,$$

$$\pi_2^A = \frac{d^2}{d\theta^2}\pi(b^A(\theta),\theta) = \pi_{pp}(b_1^A)^2 + 2\pi_{p\theta}b_1^A + \pi_p b_2^A + \pi_{\theta\theta},$$

$$\pi_{p1}^A = \frac{d}{d\theta}\pi_p(b^A(\theta),\theta) = \pi_{pp}b_1^A + \pi_{p\theta}.$$

The term π_{p2}^A won't be necessary because π_p enters the ODE as a term $(1-\theta)\pi_p$, and the term π_{p2}^A is discarded because it becomes the third-order term. Then, $\pi(b^A,\theta)$ (in the SOA) and $\pi_p(b^A,\theta)$ (in the FOA) become:

$$\pi(b^A,\theta) = \left(\pi_p b_1^A + \pi_\theta\right)(\theta-1) + \frac{1}{2}\left(\pi_{pp}(b_1^A)^2 + 2\pi_{p\theta}b_1^A + \pi_p b_2^A + \pi_{\theta\theta}\right)(\theta-1)^2,$$

$$\pi_p(b^A,\theta) = \pi_p + \left(\pi_{pp}b_1^A + \pi_{p\theta}\right)(\theta-1).$$

In an SPA, the equilibrium condition $\pi(b^{SP},\theta)=0$ implies the following two equations:

$$0 = \pi_p b_1^{SP} + \pi_\theta,$$

$$0 = \pi_{pp}(b_1^{SP})^2 + 2\pi_{p\theta}b_1^{SP} + \pi_p b_2^{SP} + \pi_{\theta\theta}.$$

These equations determine b_1^{SP} and b_2^{SP} so that the SPA bidding function is:

$$b^{SP}(\theta) = b_0 + \frac{\pi_\theta}{\pi_p}(1-\theta) - \frac{1}{2}\left(\frac{\pi_{pp}\pi_\theta^2}{\pi_p^3} - \frac{2\pi_{p\theta}\pi_\theta}{\pi_p^2} + \frac{\pi_{\theta\theta}}{\pi_p}\right)(1-\theta)^2.$$

In an FPA, the ODE $(1-\theta)\pi_p(b^{FP},\theta)b^{FP\prime}(\theta) = (n-1)\pi(b^{FP},\theta)$ becomes:

$$\left(\pi_p + \left(\pi_{pp}b_1^{FP} + \pi_{p\theta}\right)(\theta-1)\right)\left(b_1^{FP} + b_2^{FP}(\theta-1)\right) =$$

$$= -(n-1)\left(\pi_p b_1^{FP} + \pi_\theta + \frac{1}{2}\left(\pi_{pp}(b_1^{FP})^2 + 2\pi_{p\theta}b_1^{FP} + \pi_p b_2^{FP} + \pi_{\theta\theta}\right)(\theta-1)\right).$$

Dropping the term of the order $(\theta-1)^2$ results in the following two equations:

$$\pi_p b_1^{FP} = -(n-1)\left(\pi_p b_1^{FP} + \pi_\theta\right),$$

$$\pi_p b_2^{FP} + \left(\pi_{pp}b_1^{FP} + \pi_{p\theta}\right)b_1^{FP} = -\frac{(n-1)}{2}\left(\pi_{pp}(b_1^{FP})^2 + 2\pi_{p\theta}b_1^{FP} + \pi_p b_2^{FP} + \pi_{\theta\theta}\right).$$

These equations determine b_1^{FP} and b_2^{FP} so that the SPA bidding function is:

$$b^{FP}(\theta) = b_0 + \frac{(n-1)}{n}\frac{\pi_\theta}{\pi_p}(1-\theta)$$

$$-\frac{(n-1)}{2(n+1)}\left(\frac{n^2-1}{n^2}\frac{\pi_{pp}\pi_\theta^2}{\pi_p^3} - 2\frac{\pi_{p\theta}\pi_\theta}{\pi_p^2} + \frac{\pi_{\theta\theta}}{\pi_p}\right)(1-\theta)^2.$$

Next, we compute the expected conditional on θ aftermarket prices:

$$\bar{p}^{FP}(\theta) = b^{FP}(\theta),$$

$$\bar{p}^{SP}(\theta) = \mathbb{E}[p^{SP}(\theta,\theta_{-i})|\theta \text{ wins } SP] = \mathbb{E}\left[b^{SP}(z)\middle|\theta, z = \min\{\theta_j \middle| j \neq i\}\right].$$

In the latter computation, we omit the monopoly price in the evaluation of $p^{SP}(\theta,\theta_{-i})$ because $b^{SP}(\theta_j) \leq b^{SP}(1) = b_0 < p^M(1)$. Using the CDF $F(x) = \frac{x-1+\alpha}{\alpha}$ of θ, we denote the CDF of z, the lowest order statistics of $(n-1)$ types of all losing bidder, by $G(x)$:

$$G(x) = 1 - \big(1 - F(x)\big)^{n-1} = 1 - \left(\tfrac{1-x}{\alpha}\right)^{n-1}.$$

Computing $\bar{p}^{SP}(\theta)$ results in:

$$\bar{p}^{SP}(\theta) = \frac{1}{1 - G(\theta)} \int_{\theta}^{1} b^{SP}(z)\, dG(z) = \frac{(n-1)}{(1-\theta)^{n-1}} \int_{\theta}^{1} b^{SP}(z)(1-z)^{n-2}\, dz$$

$$= b_0 + \frac{(n-1)}{n} \frac{\pi_\theta}{\pi_p}(1-\theta) - \frac{(n-1)}{2(n+1)} \left(\frac{\pi_{pp}\pi_\theta^2}{\pi_p^3} - 2\frac{\pi_{p\theta}\pi_\theta}{\pi_p^2} + \frac{\pi_{\theta\theta}}{\pi_p} \right)(1-\theta)^2.$$

Then,

$$\bar{p}^{FP}(\theta) - \bar{p}^{SP}(\theta) = \frac{(n-1)}{2(n+1)n^2} \frac{\pi_{pp}\pi_\theta^2}{\pi_p^3}(1-\theta)^2 < 0$$

for all $\theta < 1$, due to $\pi_{pp} < 0$ and $\pi_p > 0$, and part 2 of the proposition follows.

Next, we compute the conditional on θ expected aftermarket profits $\bar{\pi}^{A}(\theta)$. In an FPA:

$$\bar{\pi}^{FP}(\theta) = \pi(b^{FP}(\theta), \theta)$$

$$= \big(\pi_p b_1^{FP} + \pi_\theta\big)(\theta - 1) + \frac{1}{2}\big(\pi_{pp}(b_1^{FP})^2 + 2\pi_{p\theta} b_1^{FP} + \pi_p b_2^{FP} + \pi_{\theta\theta}\big)(\theta-1)^2$$

$$= \frac{1}{n}\pi_\theta(\theta - 1) + \frac{1}{(n+1)}\left(\pi_{\theta\theta} - \frac{(n-1)}{n}\frac{\pi_{p\theta}\pi_\theta}{\pi_p}\right)(\theta-1)^2.$$

To compute $\bar{\pi}^{SP}(\theta)$, we first write $\pi(b^{SP}(z), \theta)$ in the SOA:

$$\pi(b^{SP}(z), \theta) = \tfrac{d\pi}{dz}(z-1) + \tfrac{d\pi}{d\theta}(\theta - 1)$$
$$+ \tfrac{1}{2}\left(\tfrac{d^2\pi}{dz^2}(z-1)^2 + 2\tfrac{d^2\pi}{dz d\theta}(z-1)(\theta-1) + \tfrac{d^2\pi}{d\theta^2}(\theta-1)^2\right).$$

where all derivatives are evaluated at $z = \theta = 1$:

$$\tfrac{d\pi}{dz} = \pi_p b_1^{SP},\ \ \tfrac{d\pi}{d\theta} = \pi_\theta,\ \ \tfrac{d^2\pi}{dz^2} = \pi_{pp}(b_1^{SP})^2 + \pi_p b_2^{SP},\ \ \tfrac{d^2\pi}{dz d\theta} = \pi_{p\theta} b_1^{SP},\ \ \tfrac{d^2\pi}{d\theta^2} = \pi_{\theta\theta},$$

so that

$$\pi(b^{SP}(z), \theta) = \pi_\theta(\theta - z) + \frac{1}{2}\left(\frac{2\pi_{p\theta}\pi_\theta}{\pi_p} - \pi_{\theta\theta}\right)(z-1)^2 - \frac{\pi_{p\theta}\pi_\theta}{\pi_p}(z-1)(\theta-1)$$
$$+ \frac{1}{2}\pi_{\theta\theta}(\theta-1)^2.$$

Then:

$$\bar{\pi}^{SP}(\theta) = \mathbb{E}\big[\pi(b^{SP}(z), \theta)\big|\theta, z = \min\{\theta_j \big| j \neq i\}\big] = \frac{1}{1-G(\theta)} \int_{\theta}^{1} \pi(b^{SP}(z), \theta)\, dG(z)$$

$$= \frac{1}{n}\pi_\theta(\theta-1) + \frac{1}{n+1}\left(\pi_{\theta\theta} - \frac{(n-1)}{n}\frac{\pi_{p\theta}\pi_\theta}{\pi_p}\right)(1-\theta)^2 = \bar{\pi}^{FP}(\theta),$$

and part 1 of the proposition follows.

Consumer surplus in the SOA is:

$$CS\big(b^A(\theta)\big) = CS_0 - D_0 b_1^A(\theta - 1) - \frac{1}{2}(D_0'(b_1^A)^2 + D_0 b_2^A)(\theta - 1)^2,$$

where $CS_0 = CS(b_0)$, $D_0 = D(b_0)$, and $D_0' = D'(b_0)$. Therefore, for an FPA:

$$\overline{CS}^{FP}(\theta) = CS^{FP}\big(b^{FP}(\theta)\big) = CS_0 - D_0 b_1^{FP}(\theta - 1) - \frac{1}{2}(D_0'(b_1^{FP})^2 + D_0 b_2^{FP})(\theta - 1)^2$$

$$= CS_0 - D_0 \frac{(n-1)}{n}\frac{\pi_\theta}{\pi_p}(1-\theta)$$

$$- \frac{(n-1)D_0}{2(n+1)}\left(\frac{n^2-1}{n^2}\frac{\pi_\theta^2}{\pi_p^2}\left(\frac{D_0'}{D_0} - \frac{\pi_{pp}}{\pi_p}\right) + 2\frac{\pi_{p\theta}\pi_\theta}{\pi_p^2} - \frac{\pi_{\theta\theta}}{\pi_p}\right)(1-\theta)^2,$$

and for an SPA:

$$\overline{CS}^{SP}(\theta) = \mathbb{E}\big[CS\big(b^{SP}(z)\big)\big|\theta, z = \min\{\theta_j \big| j \neq i\}\big] = \frac{1}{1-G(\theta)}\int_\theta^1 CS\big(b^{SP}(z)\big)\, dG(z)$$

$$= CS_0 - D_0 \frac{(n-1)}{n}\frac{\pi_\theta}{\pi_p}(1-\theta)$$

$$- \frac{(n-1)D_0}{2(n+1)}\left(\frac{D_0'}{D_0}\frac{\pi_\theta^2}{\pi_p^2} - \frac{\pi_{pp}\pi_\theta^2}{\pi_p^3} + \frac{2\pi_{p\theta}\pi_\theta}{\pi_p^2} - \frac{\pi_{\theta\theta}}{\pi_p}\right)(1-\theta)^2.$$

Then:

$$\overline{CS}^{FP}(\theta) - \overline{CS}^{SP}(\theta) = \frac{(n-1)D_0}{2(n+1)n^2}\frac{\pi_\theta^2}{\pi_p^3}\left(\frac{D_0'}{D_0}\pi_p - \pi_{pp}\right)(1-\theta)^2.$$

To evaluate the last expression, we use $\pi_p(p,\theta) = D(p)(1-\tilde{\varepsilon})$ to obtain $\pi_{pp} = D_0'(1-\tilde{\varepsilon}) - D_0\tilde{\varepsilon}_p$. Therefore, $\frac{D_0'}{D_0}\pi_p - \pi_{pp} = D_0\tilde{\varepsilon}_p$. This ends the proof. ■

Proof of Proposition 3.

The proof is identical to the proof of Proposition 2, we only provide expressions for the third-order terms. For the bidding functions:

$$b_3^{FP} = \frac{(n-1)^3}{n^3}\frac{\pi_{ppp}\pi_\theta^3}{\pi_p^4} - \frac{(n-1)}{(n+2)}\frac{\pi_{\theta\theta\theta}}{\pi_p} + 3\frac{(n-1)}{(n+2)}\left(\pi_{p\theta\theta} - \frac{(n^2-1)}{n^2}\frac{\pi_{pp\theta}\pi_\theta}{\pi_p}\right)\frac{\pi_\theta}{\pi_p^2}$$

$$+ 3\frac{(n-1)}{(n+2)}\left(\frac{(n-1)(3n^2+6n+1)}{n^2(n+1)}\frac{\pi_{pp}\pi_\theta^2}{\pi_p^2} - 2\frac{\pi_{p\theta}\pi_\theta}{\pi_p} + \pi_{\theta\theta}\right)\frac{\pi_{p\theta}}{\pi_p^2}$$

$$- 3\frac{(n-1)^2}{n(n+1)}\left(\pi_{pp}\pi_{\theta\theta} + \frac{(n^2-1)}{n^2}\frac{\pi_{pp}^2\pi_\theta^2}{\pi_p^2}\right)\frac{\pi_\theta}{\pi_p^3},$$

$$b_3^{SP} = \frac{\pi_{ppp}\pi_\theta^3}{\pi_p^4} - 3\frac{\pi_{pp\theta}\pi_\theta^2}{\pi_p^3} + 3\frac{\pi_{p\theta\theta}\pi_\theta}{\pi_p^2} - \frac{\pi_{\theta\theta\theta}}{\pi_p} - 3\frac{\pi_{pp}^2\pi_\theta^3}{\pi_p^5} + 9\frac{\pi_{pp}\pi_{p\theta}\pi_\theta^2}{\pi_p^4} - 3\frac{\pi_{pp}\pi_{\theta\theta}\pi_\theta}{\pi_p^3}$$
$$- 6\frac{\pi_{p\theta}^2\pi_\theta}{\pi_p^3} + 3\frac{\pi_{p\theta}\pi_{\theta\theta}}{\pi_p^2}.$$

For the conditional expected profits:

$$\bar{\pi}_3^{FP} = \frac{3}{(n+2)}\left(\pi_{\theta\theta\theta} + \frac{n^2-1}{n^2}\frac{\pi_{pp\theta}\pi_\theta^2}{\pi_p^2} - \frac{2(n-1)}{n}\frac{\pi_{p\theta\theta}\pi_\theta}{\pi_p}\right.$$
$$\left. + \frac{(n-1)}{(n+1)}\left(\frac{1-n^2}{n^2}\frac{\pi_{pp}\pi_\theta^2}{\pi_p^2} + 2\frac{\pi_{p\theta}\pi_\theta}{\pi_p} - \pi_{\theta\theta}\right)\frac{\pi_{p\theta}}{\pi_p}\right),$$

$$\bar{\pi}_3^{SP} = \frac{3}{(n+2)}\left(\pi_{\theta\theta\theta} + \frac{(n-1)}{(n+1)}\frac{\pi_{pp\theta}\pi_\theta^2}{\pi_p^2} - \frac{2(n-1)}{n}\frac{\pi_{p\theta\theta}\pi_\theta}{\pi_p}\right.$$
$$\left. - \frac{(n-1)}{(n+1)}\left(\frac{\pi_{pp}\pi_\theta^2}{\pi_p^2} - 2\frac{\pi_{p\theta}\pi_\theta}{\pi_p} + \pi_{\theta\theta}\right)\frac{\pi_{p\theta}}{\pi_p}\right).$$

For the conditional expected consumer surpluses:

$$\overline{CS}_3^{FP} = 3\frac{(n-1)}{(n+2)}\left(\frac{n^2-1}{n^2}\frac{\pi_{pp\theta}\pi_\theta^2}{\pi_p^3} - \frac{\pi_{p\theta\theta}\pi_\theta}{\pi_p^2} + \frac{\pi_{\theta\theta\theta}}{3\pi_p}\right.$$
$$\left. + \left(2\frac{\pi_{p\theta}\pi_\theta}{\pi_p^3} - \frac{n^2-1}{n^2}\frac{\pi_{pp}\pi_\theta^2}{\pi_p^4} - \frac{\pi_{\theta\theta}}{\pi_p^2}\right)\pi_{p\theta}\right)D_0,$$

$$\overline{CS}_3^{SP} = 3\frac{(n-1)}{(n+2)}\left(\frac{\pi_{pp\theta}\pi_\theta^2}{\pi_p^3} - \frac{\pi_{p\theta\theta}\pi_\theta}{\pi_p^2} + \frac{\pi_{\theta\theta\theta}}{3\pi_p} + \left(2\frac{\pi_{p\theta}\pi_\theta}{\pi_p^3} - \frac{\pi_{pp}\pi_\theta^2}{\pi_p^4} - \frac{\pi_{\theta\theta}}{\pi_p^2}\right)\pi_{p\theta}\right)D_0.$$

Since the first and second-order terms cancel in an FPA and an SPA comparison, the profit difference is:

$$\bar{\pi}^{FP}(\theta) - \bar{\pi}^{SP}(\theta) = \frac{3(n-1)(1-\theta)^3}{n^2(n+1)(n+2)}\frac{\pi_\theta^2}{\pi_p}\left(\frac{\partial^2}{\partial p\partial\theta}\ln\pi_p\right).$$

Using $\pi_p = D(1-\tilde{\varepsilon})$ we obtain $\frac{\partial}{\partial\theta}\ln\pi_p = -\frac{\tilde{\varepsilon}_\theta}{(1-\tilde{\varepsilon})}$ and:

$$\frac{\partial^2}{\partial p\partial\theta}\ln\pi_p = -\frac{\tilde{\varepsilon}_{p\theta}}{(1-\tilde{\varepsilon})} = -\frac{D_0}{\pi_p}\tilde{\varepsilon}_{p\theta},$$

due to $\tilde{\varepsilon}_p = 0$. Hence,

$$\bar{\pi}^{FP}(\theta) - \bar{\pi}^{SP}(\theta) = -\frac{3(n-1)(1-\theta)^3 D_0}{n^2(n+1)(n+2)}\frac{\pi_\theta^2}{\pi_p^2}\tilde{\varepsilon}_{p\theta}.$$

Similarly, the consumer surplus difference is

$$\overline{CS}^{FP}(\theta) - \overline{CS}^{SP}(\theta) = \frac{3(n-1)}{n^2(n+2)}\frac{\pi_\theta^2}{\pi_p^3}\left(\frac{\pi_{pp}\pi_{p\theta}}{\pi_p} - \pi_{pp\theta}\right)D_0(\theta-1)^3$$

$$= -\frac{3(n-1)(1-\theta)^3 D_0^2}{n^2(n+2)} \frac{\pi_\theta^2}{\pi_p^3} \tilde{\varepsilon}_{p\theta}.$$

This ends the proof. ■

Proof of Proposition 4.

In an SPA, firms bid break-even prices so that $b^{SP}(\theta) = \theta$. For an FPA, we use $\pi(p,\theta) = (p-\theta)(1-p)^\gamma$ and $\pi_p(p,\theta) = (1-(1+\gamma)p + \gamma\theta)(1-p)^{\gamma-1}$ to write the ODE as follows:

$$(1-\theta)\left(1-\gamma\frac{b^{FP}-\theta}{1-b^{FP}}\right)b^{FP'}(\theta) = (n-1)(b^{FP}-\theta).$$

Using the substitution $b^{FP}(\theta) = 1-(1-\theta)y(\theta)$, we rewrite this ODE as follows:

$$(1-\theta)\big((1+\gamma)y-\gamma\big)y' = y\big((n+\gamma)y-(n+\gamma-1)\big).$$

It is a separable ODE, it has a unique solution that does not explode at $\theta = 1$, and the solution is $y' = 0$, *i.e.*, $y = \frac{n+\gamma-1}{n+\gamma}$. The corresponding bidding function is:

$$b^{FP}(\theta) = 1 - y(1-\theta) = 1 - \frac{n+\gamma-1}{n+\gamma}(1-\theta) = \theta + \frac{1}{n+\gamma}(1-\theta).$$

This ends the proof of part 1 of the proposition.

Computing the conditional on θ expected aftermarket prices results in:

$$\bar{p}^{FP}(\theta) = b^{FP}(\theta) = \theta + \frac{1}{n+\gamma}(1-\theta),$$

$$\bar{p}^{SP}(\theta) = \mathbb{E}[p^{SP}(\theta,\theta_{-i})|\theta \text{ wins } SP] = \mathbb{E}[z|\theta, z = \min\{\theta_{-i}, p^M\}].$$

Using $p^M(\theta) = \theta + \frac{1-\theta}{1+\gamma}$ and $G(\theta) = 1-(1-\theta)^{n-1}$ we, finally, get:

$$\bar{p}^{SP}(\theta) = \frac{\int_\theta^{p^M(\theta)} z\, dG(z) + p^M(\theta)\big(1-p^M(\theta)\big)^{n-1}}{1-G(\theta)} = \theta + \frac{1}{n}\left(1-\left(\frac{\gamma}{1+\gamma}\right)^n\right)(1-\theta).$$

Therefore,

$$\bar{p}^{FP}(\theta) - \bar{p}^{SP}(\theta) = -\frac{\gamma(1-\theta)}{n(n+\gamma)}\left[1-\frac{n+\gamma}{1+\gamma}\left(\frac{\gamma}{1+\gamma}\right)^{n-1}\right] < 0.$$

The inequality follows from the fact that the bracketed expression equals zero at $n = 1$ and increases in n. This ends the proof of` part 2 of the proposition.

Computing the conditional on θ expected profits and consumer surpluses results in:

$$\bar{\pi}^{FP}(\theta) = \pi(b^{FP}(\theta),\theta) = \frac{(n+\gamma-1)^\gamma}{(n+\gamma)^{\gamma+1}}(1-\theta)^{1+\gamma},$$

$$\overline{CS}^{FP}(\theta) = CS\big(b^{FP}(\theta)\big) = \int_{b^{FP}(\theta)}^{1} D(x)\,dx = \frac{(n+\gamma-1)^{1+\gamma}}{(1+\gamma)(n+\gamma)^{1+\gamma}}(1-\theta)^{1+\gamma},$$

and

$$\bar{\pi}^{SP}(\theta) = \frac{(1-\theta)^{1+\gamma}}{(n-1+\gamma)(n+\gamma)}\left((n-1)+\gamma\left(\frac{\gamma}{1+\gamma}\right)^{n-1+\gamma}\right),$$

$$\overline{CS}^{SP}(\theta) = \frac{1}{(1+\gamma)(n+\gamma)}\left((n-1)+\gamma\left(\frac{\gamma}{1+\gamma}\right)^{n+\gamma-1}\right)(1-\theta)^{1+\gamma},$$

so that:

$$\frac{\bar{\pi}^{FP}(\theta)}{\bar{\pi}^{SP}(\theta)} = \frac{\overline{CS}^{FP}(\theta)}{\overline{CS}^{SP}(\theta)} = 1 + R(\gamma, n),$$

where:

$$R(\gamma, n) \stackrel{\text{def}}{=} \frac{(n-1+\gamma)\left(1-\frac{1}{n+\gamma}\right)^{\gamma}}{(n-1)+\gamma\left(\frac{\gamma}{1+\gamma}\right)^{n-1+\gamma}} - 1.$$

It can be seen (numerically) that $R(\gamma, n) > 0$ for all $n > 1$ and $\gamma > 0$. The limiting properties of $R(\gamma, n)$ follow from its analytical form. This ends the proof. ■