TI 2023-063/VI
Tinbergen Institute Discussion Paper

Dynamic determinants of optimal global climate policy

Revision: September 2023

Michael Grubb[1]
Rutger-Jan Lange[2]
Nicolas Cerkez[3]
Pablo Salas[4]
Ida Sognnaes[5]

1 University College London
2 Erasmus University Rotterdam and Tinbergen Institute
3 University College London
4 University of Cambridge
5 University of Ohio

Dynamic determinants of optimal global climate policy

Michael Grubb Rutger-Jan Lange Nicolas Cerkez
Pablo Salas Ida Sognnaes*†‡

September 27, 2023

Abstract

We explore how optimal emission abatement trajectories are affected by dynamic characteristics of greenhouse-gas emitting systems, such as inertia, induced innovation, and path-dependency, by formulating a compact and analytically tractable model with stylized damage assumptions to derive the optimal cost-benefit pathway. Our analytic solutions highlight how simple dynamic parameters affect the optimal abatement trajectory (including the optimal current effort and the cost of delay). The conventional cost-benefit result (i.e., an optimal policy with rising marginal costs that reflects discounted climate damages) arises only as a special case in which the dynamic characteristics of emitting systems are assumed to be insignificant. More generally, our model yields useful policy insights for the transition to deep decarbonization, showing that enhanced early action may greatly reduce both damages and abatement costs in the long run.

JEL Codes: C61, O30, Q54

*We thank Simon Dietz and Alex Teytelboym for helpful comments and suggestions. Earlier versions of this paper were presented at the annual conference of the International Association of Energy Economics (IAEE) in Washington, the 6th World Congress of Environmental and Resource Economists (WCERE) in Gothenburg, and the European Association of Energy and Resource Economists (EAERE) in Manchester. We are grateful to participants in these sessions for encouraging and useful feedback.

†Grubb: Institute for Sustainable Resources, UCL (m.grubb@ucl.ac.uk); Lange: School of Economics, Erasmus University of Rotterdam (lange@ese.eur.nl); Cerkez: Department of Economics, UCL (nicolas.cerkez.16@ucl.ac.uk); Salas: International Finance Corperation (psalasbravo@ifc.org); Sognnaes: Center for International Climate Research (ida.sognnas@cicero.oslo.no). Corresponding author: Rutger-Jan Lange.

‡Funding from the Joseph Rowntree Trust (Grubb), the ESRC program "Rebuilding Macroeconomics" (Grubb), the Institute of New Economic thinking (INET) (ID# INO19-00004) (Grubb and Salas), Conicyt (Comisión Nacional de Investigación Cientifica y Tecnológica, Gobierno de Chile) (Salas), the ERSC (ES/N013174/1) (Salas), and Paul and Michelle Gilding (Prince of Wales Global Sustinability Fellowship in Radical Innovation and Disruption (Salas) is gratefully acknowledged.

1 Introduction

As concern over climate change grows, objectives to cut greenhouse gas emissions have become increasingly ambitious. The emphasis on more rapid and radical action is reflected in the joint governmental goals agreed in the COP21 agreement in Paris in 2015, and in national targets to reach “net zero” emissions—required if global temperatures are to be stabilized. These now cover all major economies and a large share of global emissions (IPCC, 2022). Such ambition implies major and potentially rapid sectoral transformations, raising important questions about the economics of deep decarbonization, including cost-benefit, optimal effort, and trajectories given the dynamic characteristics of global emitting systems (Nordhaus, 2019; Stern, 2022).

Integrated assessment models (IAMs) of climate change can broadly be divided into stylized aggregate cost-benefit models and more complex process-based IAMs (Weyant, 2017; Nikas et al., 2019). The former (e.g., Nordhaus, 1991, 1993; Golosov et al., 2014; van der Ploeg and Rezai, 2019) are common in the mainstream economics literature, focusing in particular on optimal responses given assumed climate change damages, but often neglect dynamic aspects of the emitting systems such as capital stock and innovation processes (Acemoglu et al., 2012; Vogt-Schilb et al., 2018). The latter type involve more detailed representations of energy and land use systems, including dynamic aspects—as used in the IPCC Assessments (IPCC, 2007, 2014, 2022). Such IAMs, 17 of which are reviewed by Harmsen et al. (2021), primarily focus on modeling pathways toward fixed goals while drawing on large databases and technology-specific assumptions. The resulting complexity can inhibit transparency, and may not illuminate underlying economic mechanisms or key sensitivities.

This article contributes to a nascent literature seeking to bridge these schools. Building on the intuition in Grubb et al. (1995), we develop a stylized cost-benefit model—both analytically tractable and transparent—to evaluate the optimal balance between emissions-driven changes in temperature (Ricke and Caldeira, 2014; Mattauch et al., 2020) and dynamic features of emitting systems identified in the empirical literature: inertia, induced innovation, and path dependency. Our model allows for an analytic solution that yields insights into how just a few, key, dynamic assumptions affect optimal abatement, with the ultimate goal of informing debates on the optimal effort and timing of abatement, *beyond* assumptions about damages.

First, *inertia* in the system arises most obviously from the duration, construction times, and displacement of long-lived capital stock. The importance of assumptions about the malleability of capital stock in optimal growth models has been known for half a century (Newbery, 1972), but many climate-macroeconomic models which focus on the long run have no inertia, or—implicitly or explicitly—assume a high elasticity of substitution between “green” and “dirty” technologies (e.g., Acemoglu et al., 2012, 2016; Hassler et al., 2020). This becomes problematic in view of the typically long timescales of emitting capital stock (Pottier et al., 2014) and growth rates for clean technology (Wilson and Grubler, 2015). Inertia thus has important implications in the face of higher damage costs (Howard and Sterner, 2017) and in meeting the goals of the Paris Agreement (IPCC, 2022).

This aspect of dynamic costs has recently been represented—at a cost of considerable complexity—in IAMs in terms of capital investment; e.g., in the context of a fixed temperature goal rather than a cost-benefit analysis (Vogt-Schilb et al., 2018), or in a DICE-like cost-benefit model including capital stock (Baldwin et al., 2020). Our treatment is more stylized, parameterizing the scale of such adjustment costs—the resistance to accelerating abatement—in terms of a *characteristic transition time* of the global energy system.

Turning to *induced innovation*, it is established that technological progress is induced by investment and scale, including learning-by-doing and economies of scale (see Grubb et al., 2021, for a systematic review of empirical findings). Induced innovation encompasses endogeneity in innovation between high- and low-carbon technologies (Acemoglu et al., 2012, 2014; Aghion and Jaravel, 2015) and in economic systems more widely (Gillingham et al., 2008; Dietz and Stern, 2015). A growing number of IAMs incorporate some form of induced innovation (e.g., Acemoglu et al., 2012; Baldwin et al., 2020), illustrating themes that similarly emerge from our more stylized analysis. Whereas inertia in itself introduces adjustment costs, investments associated with induced innovation may be associated with cheaper enduring abatement.

Finally, inertia and induced innovation together contribute to the third feature: *path dependency* in emitting systems (Aghion et al., 2016, 2019). Specifically, the enduring impact of greater abatement in one period can be found not only in induced cost reductions in specific targeted technologies, but also in changes to the overall system that yield long-term emissions reductions beyond the directly amortized costs. These may range from lasting low-carbon infrastructure (e.g., in buildings, transport, and electricity networks) and targeted low-carbon innovation to enduring changes in networks, institutions, and policy landscapes and expectations. The optimal effort involved to shift the emissions pathway in part reflects the degree of path dependency of the system.

In pursuit of transparency and tractability, our model seeks to represent the *implications* of these dynamic factors, rather than mimicking the *processes* themselves. The underlying structure can be characterized in terms of Gillingham and Stock's (2018) distinction between static and dynamic costs. Static costs are those for which the cost of a given degree of abatement (relative to a reference emission projection) is predetermined by exogenous modeling assumptions, conventionally represented in terms of marginal abatement cost curves. Dynamic costs are those which are incurred at a given point time in time, but which do not endure and may relate *directly* to abatement cost curves in other periods. Specifically, we introduce a term to represent rate-dependent "transitional costs" comprising two components: a *characteristic transition time* of the emitting system, and a parameter representing the contribution of transitional investments to reducing the "static" component of abatement costs. We characterize the latter as the *pliability* of the system—the extent to which, ultimately, the system can endogenously adjust to abatement.

Our analytic solution illustrates the influence of key parameters, and also reveals qualitatively different behaviors, with three distinct patterns of optimal emission pathways (regimes): the conventional results arising from models with static abatement costs only, one with moderate transitional costs, and one with predominantly transitional costs.

A system with purely static abatement, akin to that in the standard DICE model, implies a sudden drop in annual emissions (i.e., an initial discontinuity in annual emissions). Intuitively, it makes sense to abate up to the value of avoided discounted climate damages.

The system behaves fundamentally differently in the presence of transitional costs. First, it transforms at a steadier rate, as higher inertia (i.e., longer characteristic transition times in our model) smooths the pace of reductions over time. As the initial abatement effort focuses on transforming the system, it is not immediately associated with reduced emissions (as emissions cannot suddenly drop, as in static-cost models). Nevertheless, the initial optimal abatement effort may substantially exceed that in purely static-cost models. Second, to the extent that the transitional costs are associated with reductions in enduring static costs (e.g., through induced innovation, infrastructure, and other path-dependent effects represented in our model by non-zero pliability), there are enhanced benefits to early action. The optimal effort thus exceeds the immediate social cost of carbon emissions. Third, faced with considerable inertia, it may be optimal to "overshoot" into a period of negative emissions which draw peak temperatures downward.

We also derive analytic solutions when faced with a hard temperature constraint, representing an extreme non-linearity in damages. These show that with only static costs, the optimal solution tends to involve "overshoot" followed by steeply negative emissions as the constraint approaches. More realistic dynamic assumptions result in very different strategies, tending to linearly reduce emissions towards zero, though with potential "overshoot" when the constraint is very tight given inertia in the emitting system.

The analysis offers several important contributions to the literature seeking to introduce dynamic characteristics into stylized IAMs. Our introduction of a representative characteristic transition time of the emitting system, indexing a cost associated with *accelerating change* from the reference emissions trajectory, offers an amalgamated proxy for the wide range of specific capital stock and technology growth-rate assumptions required in more complex models. It thus offers a simple approach to evaluating the importance of the resulting inertia.

Similarly, the introduction of a parameter characterizing the pliability of the emitting system reflects the implications of extensive research that a significant fraction of the cost of abatement—classically presented as exogenous assumptions extending far into the future—may in reality be transitional. We thereby capture the broad implications of low-carbon learning-by-doing, scale economies, and infrastructure investments, all of which may be expected to bring down the "static" cost of low carbon options relative to high-carbon ones. Consequently, our model allows us to explore what is implied if the *ex-ante* abatement costs that are typically assumed to be unavoidable in most models are in fact pliable—awaiting the required scale of investment to secure the breakthroughs needed to set energy systems on a different course.

We also provide analytic insights to results found in more complex numerical models. For example, it is well known that cost-benefit results are sensitive to the assumed discount rate r. We derive solutions from our model for the optimal degree of initial effort and find that it is exceptionally sensitive to the discount rate (i.e., proportional to $1/r^6$) in the "static" case, if the reference trajectory is rising significantly and there are no exogenously defined backstop technologies. We further find that any degree of pliability necessarily increases the optimal initial effort. Our model also illustrates that with high enough pliability—as determined by the ratio of climate damages to abatement costs and the (inverse) inertia of the system—the optimal cost-benefit emissions trajectory always includes "overshoot".

Finally, in a cost-benefit setting, if climate damages and abatement costs have equal degrees of convexity and the reference case involves rising emissions, then there will be an optimal, constant positive level of emissions unless abatement costs fall exogenously, or the system is fully pliable. However if climate damages are extremely steep, or (almost equivalently) proxied by a fixed temperature goal, then the behavior of a "static cost only" system changes radically as soon as inertia is introduced—with emissions trending approximately linearly towards zero—and the optimal initial effort is driven entirely by the cost of transforming the system.

Overall, transitional investments reduce long-term abatement costs, curtailing the overall costs of climate change (i.e., the sum of climate damages and abatement costs). Even with moderate climate-damage assumptions, if emitting systems are substantially path dependent (i.e., have a high degree of pliability), we find that the optimal long-run temperature increase may well lie below 2 degrees Celsius.

The main model is described in section 2. In section 3 we present and discuss the analytic solution, computing it for calibrated parameter values. Section 4 analytically calculates the optimal abatement efforts at time zero, while section 5 addresses the cost of delaying action. Section 6 concludes.

2 The model

2.1 High-level optimization problem

We define cumulative emissions at time t, relative to pre-industrial times, as $E(t)$ measured in gigatonnes of carbon (GtC). We take $t = 0$ to mean today. The historical path of $E(t)$, i.e., for $t \leq 0$, is fixed and cannot be changed. $E_{\mathrm{ref}}(t)$ is a reference trajectory that matches historical cumulative emissions for $t \leq 0$. Cumulative emissions to date are fixed at $E_0 := E_{\mathrm{ref}}(0)$.

Going forward, i.e., for time $t > 0$, $E_{\mathrm{ref}}(t)$ represents a "business as usual" scenario absent any substantial abatement effort. This trajectory is suboptimal in the context of climate change, such that $E(t)$ will optimally diverge from $E_{\mathrm{ref}}(t)$ for $t > 0$. For notational simplicity, annual emissions are denoted by $e(t) := E'(t)$, measured in GtC per year (GtC/yr). The reference trajectory of annual emissions is written as $e_{\mathrm{ref}}(t) = E'_{\mathrm{ref}}(t)$. The constant $e_{\mathrm{ref}}(0) = e_0$ represents current-day emissions.

The stylized high-level problem we are interested in solving is

$$\min_{\{E(t)\}_{t=0}^{t=T}} \int_0^T \exp(-r\,t)\, F[E(t), e(t), e'(t)]\,\mathrm{d}t, \tag{1}$$

$$\text{s.t.} \qquad E(0) = E_0, \tag{2}$$

$$\text{and} \qquad e(0) = e_0, \quad \text{(this restriction is optional)}. \tag{3}$$

Here $\{E(t)\}_{t=0}^{t=T}$ denotes the path of cumulative emissions $E(t)$ from $t = 0$ to $t = T$, where $T > 0$ measured in years (yr) is the time horizon, which may be infinite, $r > 0$ is the discount rate, and $F[\cdot,\cdot,\cdot]$ is a function depending on $E(t)$ and its first two derivatives, denoted $e(t)$ and $e'(t)$. The cumulative emissions path $E(t)$ for $0 \leq t \leq T$ together with the boundary conditions (2) and optionally (3) implies the annual emissions path $e(t)$ for $0 \leq t \leq T$, as well as its rate of change, $e'(t)$. This means that $E(t)$ can—without loss of generality—be used as the control variable.

The function $F[\cdot,\cdot,\cdot]$, measured in USD per year, is the sum of a climate-damage function $D[\cdot]$ and an abatement-cost function $C[\cdot,\cdot]$:

$$F[E(t), e(t), e'(t)] \;:=\; D[E(t)] \,+\, C[e(t), e'(t)]. \tag{4}$$

The damage function reflects the form in the majority of stylized IAMs, which relate climate damages to global temperature change, using the finding that this is closely proportional to cumulative CO_2 emissions (neglecting shortlived gases). Consequently, at any given point in time, climate damages $D[\cdot]$ depend on cumulative emissions up to that point, i.e., $E(t)$, and omitting lags for simplicity (see section 2.2 for details).

Abatement costs $C[\cdot,\cdot]$, on the other hand, depend on both annual emissions $e(t)$ and their rate of change, $e'(t)$. Most stylized IAMs take $C = C[e(t)]$, i.e., without dependence on $e'(t)$, such that the abatement cost at time t depends solely on the annual emissions at time t, relative to the reference level. As indicated, these "static" costs represent the classic structural form of an abatement cost curve. To these we add transitional costs by allowing $C[\cdot,\cdot]$ to depend additionally on $e'(t)$. As outlined, this comprises the elements of inertia and induced innovation (see section 2.3 for details).

The minimization problem (1) is subject to constraint (2), implying that cumulative emissions $E(t)$ must be continuous at $t = 0$: we cannot instantly extract carbon from the atmosphere. Models with static costs, which optimize annual emissions in sequential equilibria, yield discontinuities in annual emissions when climate damages are introduced, and steep reductions if a low-carbon technology suddenly becomes competitive. With transitional dynamics, however, such jumps in global emissions are implausible (and very costly). Constraint (3) implies that $E(t)$ smoothly matches the reference trajectory at $t = 0$, by making annual emissions $e(t)$, too, continuous at $t = 0$. To maintain comparability with standard models without inertia, this constraint is optional.

2.2 Climate-damage function

It is now well established that global temperature change is closely related to cumulative emissions. A central estimate is that global temperatures increase by 1 degree Celsius with each additional 600 GtC in cumulative emissions (IPCC, 2021, Table SPM.2). In line with much of the stylized literature, including the common default assumption in DICE, we assume that global damages increase quadratically with temperature. The climate-damage function, $D[\cdot]$, is thus simply:

$$D[E(t)] \; = \frac{d}{8} E(t)^2, \tag{5}$$

where $d \geq 0$ is a damage parameter with dimensions $\mathrm{USD}/(\mathrm{yr} \times \mathrm{GtC}^2)$, such that damages have a dimension of USD/yr. The numerical factor 1/8 is arbitrary and chosen for later convenience.

Damage function (5) ignores any time lag between emission reductions and their impact on temperature. Contrary to common assumptions, this time lag is relatively small: Ricke and Caldeira (2014) estimate the median time lag (until maximum warming occurs) to be just over 10 years (see also Mattauch et al., 2020). A lag of L years could be introduced with a simple transposition from $d/8E(t)^2$ to $d/8E(t-L)^2$. This would reduce the net present value of the damage by $(1 - \mathrm{e}^{-Lt})$, a minor change with no impact on the structural insight of the paper. We therefore omit this for simplicity.

2.3 Abatement-cost function

We specify the abatement cost function, $C[\cdot,\cdot]$, in terms of abatement $a(t)$, and its rate of change $a'(t)$ as follows:

$$C[e(t), e'(t)] := \; c\left[q\, a(t)^2 \;+\; 2\, p\, \tau^2\, a'(t)^2\right], \tag{6}$$

$$\text{where abatement } a(t) := \; e_{\mathrm{ref}}(t) - e(t). \tag{7}$$

Here $c > 0$ is an overall cost-scaling constant, measured in $\mathrm{USD} \times \mathrm{yr}/\mathrm{GtC}^2$, and the numerical factor of 2 in the second term of equation (6) is arbitrary but included for later convenience.

Abatement at time t, $a(t)$ is measured in GtC/yr relative to baseline, while $a'(t)$, in GtC/yr^2, represents its rate of change. The resulting cost function expresses annual expenditure on emissions abatement in USD/yr.

The first term in equation (6) captures the static costs, the traditional stylized formulation of abatement costs as a nonlinear function of the degree of abatement relative to a baseline projection, scaled by q where $q \in [0, 1]$ is a dimensionless number. In common with several other stylized models, we assume that the static abatement cost at time t increases quadratically with the abatement effort at time t, giving rise to a term that scales with $a(t)^2$.[1]

The second term captures the transitional cost, which is proportional to the square of the rate of change of abatement and measures how rapidly the system is forced to deviate from the reference trajectory.[2] Here we introduce two parameters, τ and p:

1. Parameter $\tau > 0$, measured in years, reflects the intrinsic inertia—i.e., the resistance to change—of the system in terms of a characteristic transition time (Harmsen et al., 2021): the higher τ, the longer it takes to achieve a given level of abatement for a given cost (or the more costly it is to overcome this inertia). The characteristic time τ therefore offers a compact proxy for capital stock retirement and growth rate constraints on new technologies in more complex models.

2. Parameter $p \in [0, 1]$ reflects the "pliability" of the system, with $q = 1 - p$ being its complement. We use p to explore the implications of some portion of costs being transitional that otherwise, in a purely exogenous framework, would have been attributed to static costs. The case $p = 0$ represents the classical exogenous treatment in which all costs are static, while $p = 1$ represents the opposite extreme, in which all costs turn out to be transitional, with innovation and infrastructural developments resulting in low carbon technology systems which are fully competitive with the incumbent fossil-fuel-based industries.

Central to our model is the contention that *ex-ante* estimates of (exogenously defined) marginal cost curves in practice implicitly conflate static and dynamic costs in two aspects. First, much innovation and cost reduction—beyond public R&D—is induced by economic incentives and associated investment, involving private R&D, iterative economies of scale, learning-by-doing, and supply chain developments associated with deployment. Second, an important element of the costs facing many new technologies arises from mismatch with existing physical and institutional infrastructures, as, for example, seen by the needs of electric vehicles for different fuelling and maintenance infrastructures. These and other factors mean that introducing a carbon tax may involve an initial cost which declines as the economy adjusts (Aghion et al., 2016; Shapiro and Metcalf, 2023).

Inertia and induced innovation together create path dependence. The ratio $p\tau^2/q$ can be taken to represent the degree of path dependency of the system: if there is high inertia and significant pliability, then after transitional abatement in one period, the system will have extensive path dependence and will tend to stick to its new trajectory.

[1] Several other stylized models assume quadratic abatement costs. Nordhaus (2013) has $a(t)^{2.8}$. Grubb et al. (2018) show that learning-by-doing tends to reduce not only the scale but the convexity of the marginal cost curve.

[2] Though there is less evidence on the functional form of transitional costs, they are clearly convex (see Grubb et al., 2018). Vogt-Schilb et al. (2018) assume that the cost of capital retirement increase quadratically with the pace and Bauer et al. (2016) assume that costs for renewable deployment increase quadratically with the rate of accelerating renewable expansion in the REMIND model.

Symbol	Meaning	Dimension	Calibration
p, q	pliability and its complement $q = 1 - p$	none	
t, T	time, time horizon	yr	
τ	characteristic transition time	yr	15
r	discount rate	1/yr	0.025
$C[\cdot,\cdot], D[\cdot]$	abatement cost and climate damage	USD/yr	
c	abatement cost parameter	USD $\times$ yr/GtC^2	0.026
d	damage parameter	USD/(yr $\times$ GtC^2)	0.00002
$E(t), E_{\mathrm{ref}}(t)$	cumulative emissions at time t	GtC	
E_0	cumulative emissions at $t = 0$	GtC	665
$e(t), e_{\mathrm{ref}}(t)$	annual emissions at time t	GtC/yr	
$a(t)$	abatement at time t, $a(t) := e_{\mathrm{ref}}(t) - e(t)$	GtC/yr	
e_0	annual emissions at $t = 0$	GtC/yr	10.4
$e'(t), e'_{\mathrm{ref}}(t)$	rate of change of annual emissions at time t	GtC/yr^2	
$a'(t)$	rate of change of abatement at time t	GtC/yr^2	
e_1	reference growth of annual emissions	GtC/yr^2	0.12

Table 1: Overview of symbols, dimensions, and calibration

2.4 Business-as-usual scenario

To complete the model setup, the reference path of cumulative emissions is specified by assuming linear growth in annual emissions as follows:

$$e_{\mathrm{ref}}(t) := e_0 + e_1\, t, \quad t \geq 0, \tag{8}$$

$$E_{\mathrm{ref}}(t) = E_{\mathrm{ref}}(0) + \int_0^t e_{\mathrm{ref}}(s)\,\mathrm{d}s = E_0 + e_0\, t + \frac{e_1}{2}\, t^2, \quad t \geq 0, \tag{9}$$

where $e_0 \geq 0$ is the annual emissions at $t = 0$, while $e_1 \geq 0$, measured in GtC/yr^2, represents a (linear) growth rate of annual emissions assuming "business as usual." We take as our reference scenario a view in which global emissions rise at moderate pace of $e_1 = 120$ MtC/yr^2 (0.12 GtC/yr^2), approximating the average trend since the financial crisis. Problem (1) has now been specified in its entirety. Table 1 contains an overview of all symbols used as well as their dimensions.

3 Analytic solutions

3.1 Statement of Theorem 1

Optimization problem (1) permits an analytic solution as described here.

Theorem 1 *Consider optimization problem* (1) *with damage parameter $d > 0$ and infinite time horizon ($T = \infty$). The optimal path is*

$$E(t) \;=\; E_\star \,+\, e_\star \cdot t \,+\, \sum_{j=1}^{2} Z_j \,\exp\left(z_j\, t/2\right), \quad t \geq 0, \tag{10}$$

where constants $E_\star$, with dimension GtC, and $e_\star$, with dimension GtC/yr, are

$$E_\star = \frac{8\, c\, q\,(e_0\, r - e_1)}{d} + 64\, e_1\, r^2 \left(\frac{c\, p\, \tau^2}{4\, d} - \left(\frac{c\, q}{d}\right)^2 \right), \quad e_\star = 8 \frac{c\, q\, e_1\, r}{d}. \tag{11}$$

The form of the exponential constants z_j, with dimension 1/yr, and Z_j, with dimension GtC, for $j = 1, 2$ depend on the pliability p of the system relative to a critical threshold $p^\star$, defined as

$$p^\star := 1 - \frac{\sqrt{1+4x} - 1}{2x} \in (0, 1),$$

where $x := c/(d\tau^2) \in (0, \infty)$ is a dimensionless characteristic of the system. The solution then comprises three mathematically distinct regimes:

1. **No pliability.** *Assume $p = 0$ and impose constraint (2). Then*

$$z_1 = r - \sqrt{r^2 + \frac{d}{2\,c\,q}}, \quad Z_1 = E_0 - E_\star, \qquad z_2 = 0, \qquad Z_2 = 0. \tag{12}$$

2. **Medium pliability.** *Assume $0 < p \leq p^\star$ and impose constraints (2) and (3). Then $z_1 = z_+$ and $z_2 = z_-$, where*

$$z_\pm = r - \sqrt{v \pm \sqrt{u}} \; < \; 0, \tag{13}$$

where $u := \frac{q^2}{p^2\tau^4} - \frac{d}{cp\tau^2}$ and $v := r^2 + \frac{q}{p\,\tau^2}$. Both z_1 and z_2 are real valued and strictly negative. The exponential constants Z_j for $j = 1, 2$ are given by

$$Z_1 = \frac{2(e_0 - e_\star) + z_-(E_\star - E_0)}{z_+ - z_-} \qquad\qquad Z_2 = \frac{2(e_0 - e_\star) + z_+(E_\star - E_0)}{z_- - z_+}. \tag{14}$$

3. **High pliability.** *Assume $p > p^\star$ and impose constraints (2) and (3). Then $z_1 = z_+$ and $z_2 = z_-$, where*

$$z_\pm \; = \; r \; - \; w \; \pm \; \mathrm{i}\, \frac{\sqrt{|u|}}{2w}, \tag{15}$$

where $\mathrm{i} = \sqrt{-1}$, $w := \frac{\sqrt{v+\sqrt{v^2+|u|}}}{\sqrt{2}}$, while u, v remain as under point 2. Both z_1 and z_1 are complex valued, with real parts that are strictly negative. Constants Z_1, Z_2 remain as in equation (14), but with $z_\pm$ as in equation (15).

Proof: A standard application of the calculus of variations (e.g., Goldstein et al., 2013) gives a fourth-order differential equation for the solution $E(t)$. Conjecture (10) yields expressions for $E_\star$ and $e_\star$, as well as a fourth-order polynomial equation for the constants z_j for $j = 1, 2$. Two roots can be discarded because of the (implicit) boundary condition at $T = \infty$. The two remaining roots can be found analytically, giving z_1 and z_2. Depending on the regime, these are either real (no-pliability and medium-pliability regimes) or complex (high-pliability regime). The constants Z_j for $j = 1, 2$ follow from the boundary conditions at $t = 0$ and are expressable in terms of z_1 and z_2. In all cases, the cumulative emissions path $E(t)$ remains real and implies the marginal emissions path $e(t)$ by taking the first derivative. Details of the proof can be found in the online Appendix A.

3.2 Discussion of Theorem 1

Equation (10) in Theorem 1 gives the optimal solution $E(t)$ for $t \geq 0$. As a sanity check, it can be verified that the limits (i) $d \to 0$ or (ii) $c \to \infty$ imply $E(t) \to E_{\text{ref}}(t)$. That is, when (i) damages are zero or (ii) the abatement cost approaches infinity, the optimal path is equal to the reference path. For non-zero damages ($d > 0$) and finite abatement cost ($c < \infty$), the path of $E(t)$ lies below that of $E_{\text{ref}}(t)$ for $t \geq 0$. In particular, equation (10) gives $E(t)$ as the sum of a constant $E_\star$, a linear function of time with slope $e_\star$, and a sum of two exponential functions. The (real parts of the) exponential parameters z_j for $j = 1, 2$ are negative, such that, as $t \to \infty$, these terms vanish.

In all cases, the optimal marginal emissions path $e(t)$ is implied by the cumulative emissions path $E(t)$ via a straightforward differentiation with respect to time. Further, in all cases an analytic solution remains possible even for a finite optimization horizon T. The resulting, somewhat more involved, expressions are available from the authors upon request.

Theorem 1 implies that optimal long-run annual emissions $e(t) = E'(t)$ are constant at the level $e_\star$ given in equation (11). The positive emissions are determined by the balance of damages and abatement costs, both of which increase quadratically in our model if reference emissions are rising. As might have been anticipated, $e_\star$ is an increasing function of the abatement-cost parameter c, the business-as-usual emissions parameter e_1, and the discount rate r, and a decreasing function of the damage parameter d.

For a fully pliable, path-dependent system ($p = 1, q = 0$), or stable reference emissions ($e_1 = 0$), we have $e_\star = 0$, i.e., it is optimal in the long run to decarbonize entirely—as would be the case for more convex damages, as touched on in Theorem 2 below.[3]

The three regimes differ in how this optimal asymptotic emissions level $e_\star$ is reached. The no-pliability solution, where $p = 0$, is akin to the standard DICE solution, and implies an initial drop in emission (prompt jump in abatement).[4] Whenever $p > 0$, i.e., for a system with any positive degree of inertia, such an immediate response is impossibly costly. Hence, in regimes 2 and 3, the path of $e(t)$ remains continuous at $t = 0$, avoiding a discontinuity in annual emissions. In the medium-pliability regime, the optimal long-run emissions level is reached by more steadily cutting emissions to this level. The abatement effort (cost) at time zero typically exceeds that in regime 1, because part of this effort is related to the transformation of the emitting systems, the results of which are not immediately visible in the marginal emissions path.

In the high-pliability regime, the exponential parameters z_j for $j = 1, 2$ are imaginary. Naturally, the cumulative emissions profile $E(t)$ remains real valued. The corresponding response can be equivalently written in terms of trigonometric functions as follows:

$$\begin{aligned} E(t) &= E_\star + e_\star \cdot t \\ &\quad + \exp\left(\frac{\widehat{z}\,t}{2}\right)\left[\frac{2(e_0 - e_\star) + \widehat{z}(E_\star - E_0)}{\widetilde{z}} \sin\left(\frac{\widetilde{z}\,t}{2}\right) + (E_0 - E_\star)\cos\left(\frac{\widetilde{z}\,t}{2}\right)\right], \end{aligned} \tag{16}$$

where $e_\star, E_\star$ remain as in Theorem 1, while $\widehat{z}, \widetilde{z}$ are real numbers defined as $\widehat{z} := \text{Re}(z_+)$ and

[3] This reflects a point which is intuitively obvious but rarely made explicit: whether or not a cost-benefit model converges to zero emissions depends on the relative convexity of assumed damage and abatement costs, unless a "backstop" technology kicks in which also in effect reduces the convexity of the abatement cost curve.

[4] In practice, plots from DICE do not show this because emissions at time $t = 0$ are set equal to the actual emissions and the discontinuity occurs in the first unconstrained five-year period, shown as $t + 5$.

$\widetilde{z} := \mathrm{Im}(z_+)$.

The intuition for the third regime is that, when accelerating abatement is expensive (but damages are significant), this constrains the pace of abatement, leading to some degree of "overshoot" of concentrations and temperature before later correcting (steering back, with negative emissions). This explains the appearance of trigonometric functions: emissions oscillate toward the long-term optimum. We note that "overshoot" has become of feature of scenarios to meet more ambitious temperature goals in many complex IAMs of the IPCC (IPCC, 2022), but has not generally been observed in stylized cost-benefit models.

3.3 Statement of Theorem 2

A key structural assumption in the preceding analysis concerns the functional form of damages, in particular that damages for a given temperature increase with the same (quadratic) form as abatement costs. This results in the specific form of the solution involving the convergence on a constant emissions rate, which reflects the optimal trade-off between the damages and abatement costs. It is thus natural to question the sensitivity of Theorem 1 to this assumption.

Science has largely framed the problem of climate change in terms of planetary risks, and emphasized that the risk of major climate damages arising from earth systems (or indeed, some vulnerable social systems) crossing various thresholds increase sharply with the degree of temperature change.[5]

While solutions involving other functional forms of damages in general are not analytically tractable, we here investigate the implications of highly non-linear damages by the proxy of a limit on temperature, which can represented by a limit on cumulative emissions. This results in our second theorem, which confirms the main takeaways of Theorem 1.

Theorem 2 *Consider optimization problem* (1) *without damages (i.e., $d = 0$) but a constraint on temperature change, represented by limit on cumulative emissions E_T within a finite time horizon $T < \infty$. Let $\Delta_T := E(T) - E_{\mathrm{ref}}(T)$ and $\delta_T := e(T) - e_{\mathrm{ref}}(T)$ be deviations from the reference trajectory at the terminal time point (both negative).*

1. **No inertia/pliability.** *Assume $p = 0$. Impose initial constraint* (2) *and terminal constraint $E(T) = E_T$. Then the optimal path is*

$$E(t) = E_{\mathrm{ref}}(t) + \Delta_T \, \frac{\exp(rt) - 1}{\exp(rT) - 1}, \qquad 0 \le t \le T. \tag{17}$$

2. **Positive inertia/pliability.** *Assume $p < 0 < 1$. Impose initial constraints* (2)-(3) *and terminal constraints $E(T) = E_T$ and $e(T) = e_T$. Then the optimal path is*

$$E(t) \;=\; E_{\mathrm{ref}}(t) \;+\; \sum_{i=1}^{2} \left[Z_i \, \frac{\mathrm{e}^{z_i\, t} - 1}{z_i} \right] - (Z_1 + Z_2) \frac{\mathrm{e}^{r\, t} - 1}{r}, \qquad 0 \le t \le T, \tag{18}$$

where constants z_i for $i = 1, 2$ with dimensions yr^{-1} are $z_1 = (r - \sqrt{r^2 + 2q/(p\tau^2)})/2$

[5] Weitzman (2009) underlined the importance of highly non-linear risks, which he explored in terms of damages rising with temperatures.

and $z_2 = (r + \sqrt{r^2 + 2q/(p\tau^2)})/2$. *Constants* Z_i *for* $i = 1, 2$ *measured in GtC/yr are*

$$Z_1 = \frac{\left(e^{z_2 T} - e^{rT}\right)\Delta_T - \left(\frac{e^{z_2 T}-1}{z_2} - \frac{e^{rT}-1}{r}\right)\delta_T}{\left(e^{z_2 T} - e^{rT}\right)\left(\frac{e^{z_1 T}-1}{z_1} - \frac{e^{rT}-1}{r}\right) - \left(e^{z_1 T} - e^{rT}\right)\left(\frac{e^{z_2 T}-1}{z_2} - \frac{e^{rT}-1}{r}\right)}, \tag{19}$$

$$Z_2 = \frac{-\left(e^{z_1 T} - e^{rT}\right)\Delta_T + \left(\frac{e^{z_1 T}-1}{z_1} - \frac{e^{rT}-1}{r}\right)\delta_T}{\left(e^{z_2 T} - e^{rT}\right)\left(\frac{e^{z_1 T}-1}{z_1} - \frac{e^{rT}-1}{r}\right) - \left(e^{z_1 T} - e^{rT}\right)\left(\frac{e^{z_2 T}-1}{z_2} - \frac{e^{rT}-1}{r}\right)}. \tag{20}$$

3. **Full pliability.** *Assume* $p = 1$. *Impose initial constraints* (2)-(3) *and terminal constraints* $E(T) = E_T$ *and* $e(T) = e_T$. *Then the optimal path is*

$$E(t) \;=\; E_{\text{ref}}(t) \;+\; Z_1\, t \;+\; Z_2\, t\, e^{r\,t} \;-\; (Z_1 + Z_2)\, \frac{e^{r\,t} - 1}{r}, \qquad 0 \le t \le T, \tag{21}$$

where constants Z_i *for* $i = 1, 2$ *with dimensions GtC/yr are*

$$Z_1 = \frac{(\delta_T - r\Delta_T) rT e^{rT} - \delta_T (e^{rT} - 1)}{(e^{rT} - 1)^2 - r^2 T^2 e^{rT}}, \quad Z_2 = \frac{(\delta_T - r\Delta_T)(e^{rT} - 1) - \delta_T rT}{(e^{rT} - 1)^2 - r^2 T^2 e^{rT}}. \tag{22}$$

Proof: The proof is similar to that of Theorem 1, requiring us to solve a fourth-order differential equation for $E(t)$. The key difference is that terminal emissions conditions are enforced at $T < \infty$. Details are available on request.

3.4 Discussion of Theorem 2

With no inertia, i.e., $\tau = 0$, initial and terminal constraints on emission rates are irrelevant since emissions can jump with no transitional cost. Behavior is then very different from the other cases. Note that with $p = 0$

$$e(t) = e_{\text{ref}} + \Delta_T \frac{r \exp(rt)}{\exp(rT) - 1}$$

When T, the time horizon, is moderately close relative to the impact of discounting, the result is a large discontinuous initial jump of magnitude $\Delta_T \frac{r \exp(rt)}{\exp(rT)-1}$ at time 0. With a larger T, the denominator increases and the emissions trajectory can defer subsequent action, with "overshoot" followed by accelerating abatement towards large negative emissions in the final stages, at high but short-lived annual costs, extensively discounted.

With positive pliability, i.e. $p > 0$, inertia prevents the sudden jump and the dynamics are different. Time derivatives of all the terms in (18) involving Z_1 and Z_2 have the property of linear dependence on t when t is small.

The general equation is greatly simplified for the case of full pliability ($p = 1, q = 0$). Note that in this case we have

$$e(t) \;=\; e_{\text{ref}}(t) \;+\; Z_1\,(1 - e^{rt}) \;+\; Z_2\, r\, t\, e^{rt}$$

The emissions trajectory is invariant with respect to τ also. The consequence is that whereas in the cost-benefit case higher inertia leads to (slightly) less effort, with more convex damage the opposite is the case. With a temperature constraint, the optimal initial effort increases directly in proportion to τ^2 , as well as (of course) the severity of the constraint.

Summarizing, with a fixed binding goal for temperature (and hence cumulative emissions), the behavior of the system has parallels with the cost-benefit case when the system has no learning or inertia —an initial jump, followed by deferral of stronger action, until the constraint approaches (or as damages accumulate). With $p > 0$, however, the emissions trend inclines towards linear reductions in the earlier stages, to minimize transitional costs and exploit the possibilities of lowered abatement costs.

3.5 Numerical results

3.5.1 Calibration

This subsection describes how we calibrate the parameters of our model. An overview of the calibrated parameter values is given in Table 1.

Emissions. We define $t = 0$ to be 2019 and take values for E_0, e_0, and e_1 from the most recent IPCC report (IPCC, 2021). Specifically we set $E_0 = 665\text{GtC}$, $e_0 = 10.4\text{GtC/yr}$, and $e_1 = 0.120\text{GtC/yr}^2$.

Discount rate. We assume the real discount rate to be 2.5% per year, based on the expert elicitation survey by Drupp et al. (2018). This is a compromise between "prescriptive" and "descriptive" rates, leaning more towards the latter in that after a few decades it leads to significant discounting of costs and damages.

Climate damages. Our climate-damage estimates draw upon Nordhaus (2013) and Howard and Sterner (2017), both of which present damage estimates as proportional to the square of global temperature change, as in our model. Howard and Sterner's (2017) "preferred damage specification" is almost four times the Nordhaus (2013) value. We take a central benchmark value midway between these, resulting in $0.00002\,\text{USD}/(\text{yr} \times \text{GtC}^2)$.

Abatement costs. The vast majority of literature specifies abatement costs in terms of marginal abatement costs, some derived for specific projected years. Based on the extensive review by Harmsen et al. (2021, Figure 1) of a dozen different complex IAMs, we take an average benchmark abatement cost parameter $c = 0.026\,\text{USD} \times \text{yr}/\ \text{GtC}^2$, equivalent to a marginal abatement cost of $370\,\text{USD/tC} = 100\,\text{USD/tCO}_2$ for 50% emissions reduction from reference (7GtC/yr), in the middle of their reported range.

Characteristic time. There is little empirical literature on transition timescales. The review by Harmsen et al. (2021) introduces this metric for the first time, documenting a median value of 13.5 years across all the complex-process IAMs they review. We take our benchmark value as $\tau = 15$ years.

3.5.2 Results for Theorem 1

We present our main results for five different scenarios. Figure 1 displays annual emissions (in GtC per year), global mean temperature increases with respect to pre-industrial times (in degrees Celsius), along with annual damages from climate change and annual abatement costs (in trillion USD per year) for a system with no pliability (i.e., $p = 0$), a system with full pliability (i.e., $p = 1$), and three scenarios that relate to regions in between (i.e., $p \in \{0.25, 0.5, 0.75\}$).

No pliability, $p = 0$. The system with purely static costs resembles that used in DICE and other classical IAMs (hereafter, "classical"), as discussed in section 2. There is a prompt reduction in annual emissions (by about one third), but after this initial drop, emissions continue to rise steadily throughout the century. This is because abatement in this scenario cannot keep up with the rising emissions from the business-as-usual scenario. As can be seen, behavior in the classical case with exogenous and static costs ($p = 0$) is similar to the climate-policy ramp observed in DICE. Annual abatement investment increases from below 500 billion USD per year to above 1.5 trillion USD per year in 2100. As cumulative emissions continue to rise, global mean temperatures rise above 2.5 degrees Celsius by 2100 and continue rising beyond. Damages increase correspondingly over time, reaching more than 4 trillion USD per year by the end of the century.

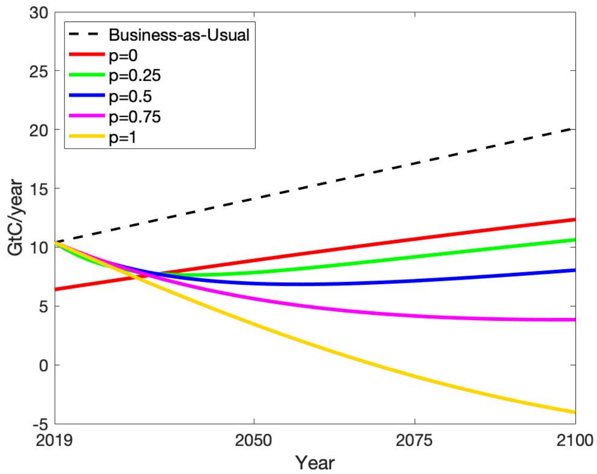
As soon as $p > 0$, an immediate (discontinuous) emissions reduction is no longer possible. As indicated above, emissions in these positive-pliability scenarios initially decline (approximately) linearly. Once they cross below the classical $p = 0$ case, which happens after roughly $\tau = 15$ years, the behavior varies widely across the different cases.

Full pliability, $p = 1$. At the opposite extreme, the scenario with a fully pliable system is one with solely transitional costs and no static costs. Emissions decline steadily and reach net zero emissions around 2065. Afterwards, net annual emissions become negative, implying that cumulative emissions will decrease (as indicated for the high-pliability regime). With temperature increases proportional to cumulative emissions, the global mean temperature increases to about 1.6 degrees Celsius from pre-industrial levels at the time of net zero, and decreases slightly thereafter.

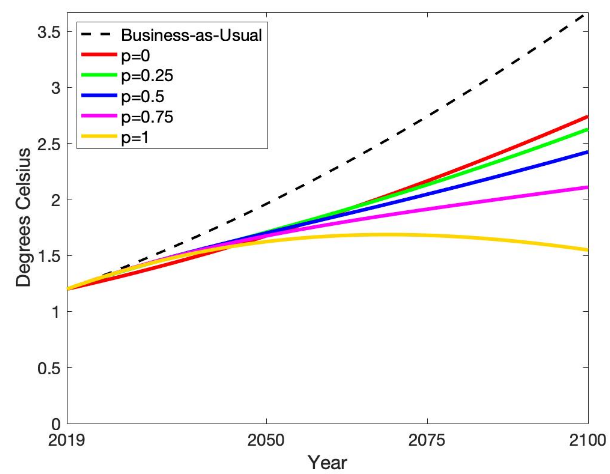
With full pliability, the optimal policy involves substantially higher initial expenditure than in the other scenarios. Initial annual abatement investment in the fully pliable system is, at over 1 trillion USD per year, almost three times greater than in the non-pliable system. The trend is also reversed: optimal effort decreases rather than increases over time, reaching less than 500 billion USD per year by 2100. Damages in this scenario remain much lower than in the other two cases: between one and two trillion USD per year between 2018 and 2100.

Intermediate pliability, $p = 0.25$, $p = 0.5$, $p = 0.75$. In all intermediate cases, emissions after the crossing point stay below those in the classical ($p = 0$) case, but do not reach zero; as explained above, they asymptote towards a constant level. Given the absence of a "backstop" technology in our model, global temperatures, damages, and abatement costs all keep rising, though the $p = 0.75$ case only reaches 2 degrees Celsius towards the end of the century, reflecting an initial doubling of the effort, which remains above the classical cost for the first half of the century, reaping the rewards in the second half with lower abatement costs as well as lower damages.

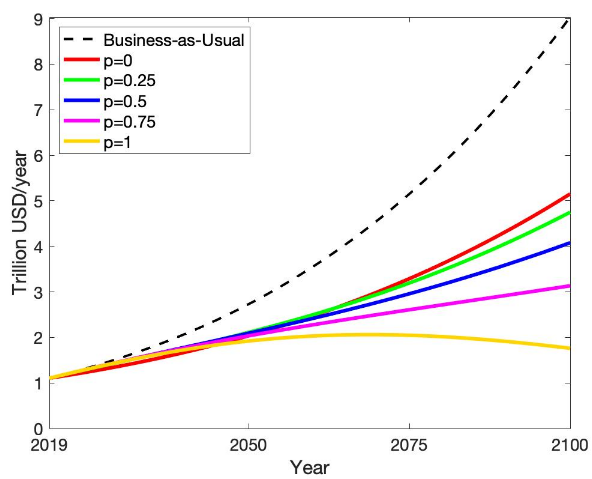
While damages are lower for higher p in panel (c), abatement costs in panel (d) for $p > 0$ (but $p \neq 1$) are (mostly) above the classical case with $p = 0$. The intuition is that the larger



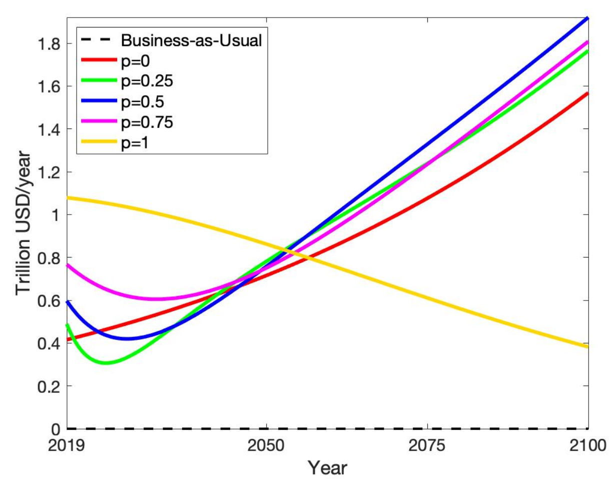
(a) Annual emissions



(b) Global mean temperature increase

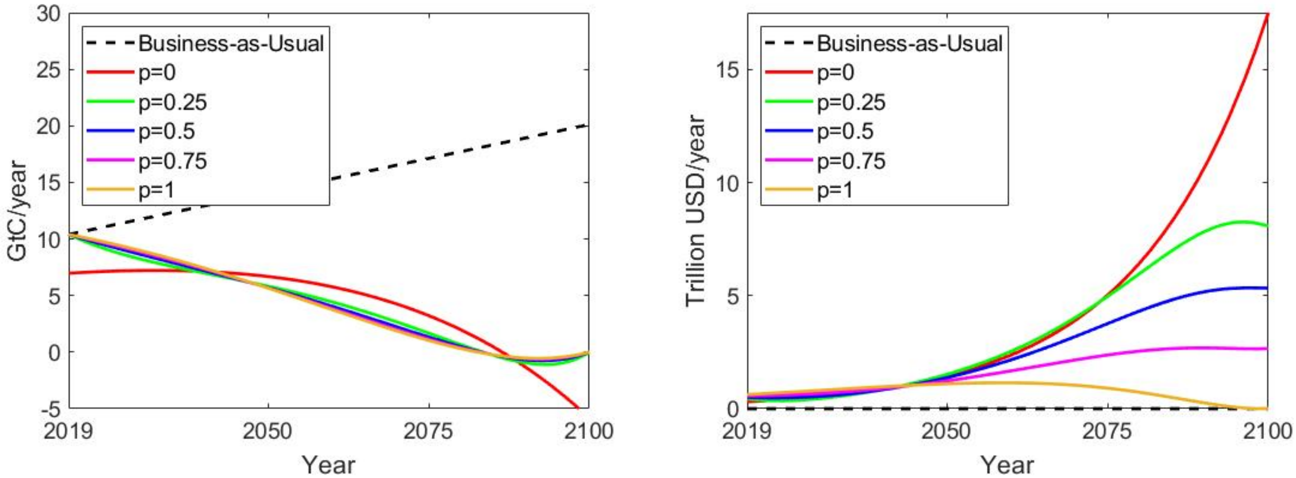


(c) Annual damages



(d) Annual abatement costs

Figure 1: Optimal policy and implications for $p \in \{0, 0.25, 0.5, 0.75, 1\}$



(a) Annual emissions

(b) Annual abatement costs

Figure 2: Optimal policy and implications with limit on cumulative emissions

initial abatement effort in the intermediate-pliability cases leads to reduced damages later on. Given that damages are approximately three times higher than abatement costs for any given p, the reduction in damages is larger than the increase in abatement (compared to the business-as-usual scenario).

Hence, in terms of overall discounted costs, any $p > 0$ ultimately reduces the NPV associated with responding to climate change, ie. a lower value of the (optimized) objective function (1). However as we explore in section 4, it has the opposite implication for the optimal initial effort.

3.5.3 Results for Theorem 2

Figure 2 plots annual emissions and annual abatement costs resulting from a temperature constraint (limit on cumulative emissions), i.e., Theorem 2. We set $T = 81$ and $e(T) = 0$, so that annual emissions are constrained to reach zero at the end of the century. Furthermore, $E(T) = 1000$ GtC, which equates to *remaining* cumulative emissions of 335GtC (1230GtCO$_2$), giving a near two-thirds chance of staying "below 2 degrees Celsius" IPCC (2021).

For the classical case of $p = \tau = 0$, we see the expected immediate jump in annual emissions. Thereafter, emissions, however, do not start to decline further until the middle of the century. After about 2085 they go below zero, into a period of steeply negative emissions in order to bring the global temperature back down.

With positive inertia and pliability, the figure shows that emissions decline almost linearly irrespective of the degree of inertia or pliability across the range shown. They dip slightly below zero in the last decade.

Comparing Theorem 2 broadly confirms the findings of Theorem 1 and shows that our cost-benefit approach leads to similar results than an approach with a fixed temperature target. For the remainder of the paper—where we calculate the optimal effort at time zero and the cost of delay—we therefore only present results using the set-up from our main cost-benefit model. The results using the other approach are available upon request.

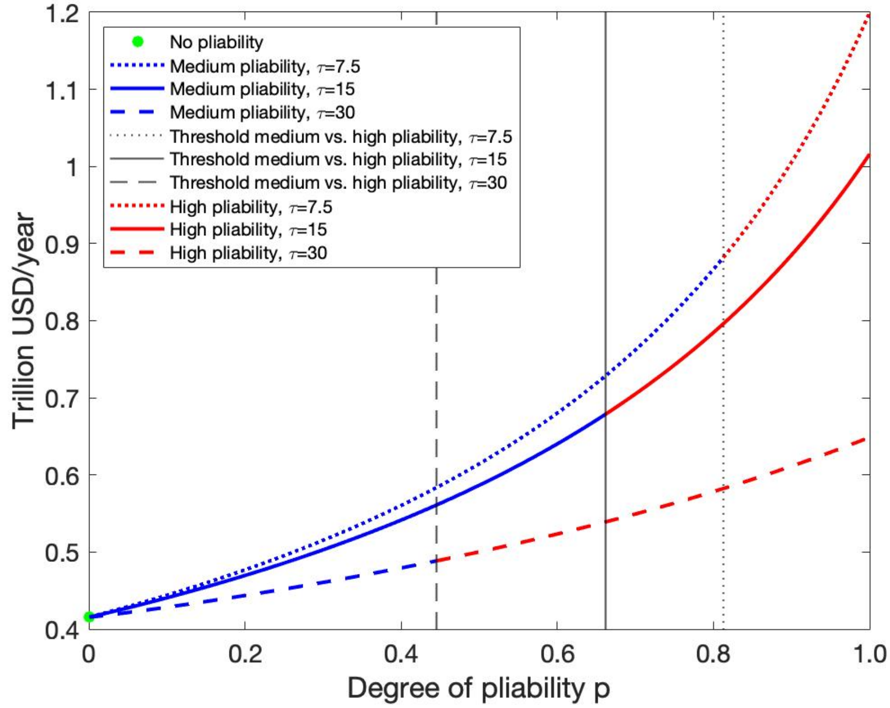


Figure 3: Optimal abatement effort at time zero from equation (24)

4 Abatement effort at time zero

Having obtained the optimal path of cumulative emissions $E(t)$ in three regimes in Theorem 1, we can directly compute the optimal degree of initial abatement effort (expenditure at time t=0), and explore key sensitivities.

4.1 No inertia, no pliability regime

For $p = 0$, substituting the exact solution (10) into the cost function $C[\cdot, \cdot]$ in equation (6) and evaluating the result at $t = 0$ yields the optimal current abatement effort measured in USD/yr as follows:

$$C[e(0), e'(0)]\Big|_{p=0} = c\left[e_0 - e_\star - \frac{z_1}{2}(E_0 - E_\star)\right]^2\Big|_{p=0,q=1}, \qquad (23)$$
$$= \left[\frac{e_1^2}{r^6} + \frac{2\,e_0\,e_1}{r^5} + \frac{e_0^2 + 2\,e_1\,E_0}{r^4} + \frac{2\,e_0\,E_0}{r^3} + \frac{E_0^2}{r^2}\right] \cdot \frac{d^2}{64\,c} + O(d^3),$$

where $E_\star$ and z_1 are given in equation (11) and (12), respectively. The second line is a straightforward first-order Taylor expansion in the square of the damage parameter. From this expression, it is clear that the optimal level of effort today is extremely sensitive to the discount rate r, which appears to the power of six in the denominator whenever $e_1 \neq 0$. The ratio d^2/c confirms that effort tends to increase nonlinearly with d, while higher abatement cost c suppresses effort because it reduces the benefit/cost ratio (and with discounting it is cheaper to defer the effort required). In terms of initial emission conditions, e_0, e_1, and E_0 all also increase the optimal effort.

Numerous studies with DICE have underlined sensitivity to the discount rate, but to our knowledge none have identified it analytically to such a remarkable degree. Note that the first two terms in the expansion are driven by e_1, while the inverse quartic and cubic dependencies involve e_0. We interpret this as follows: without inertia, the solution suggests that in the presence of climate damages, optimal emissions today are much lower than actual emissions. At low discount rates, the solution suggests an immediate, large reduction in the starting level, which is amplified further to counteract the rising trend of future emissions. In published results from DICE and similar numeric models, this immediate reduction in annual emissions is somewhat obscured by the five-year time steps typically used, but the underlying logic is one of a sudden, potentially dramatic jump so as to "start from somewhere else." In isolation of any consideration of dynamic constraints, it is unclear how useful this is as a policy-relevant insight, since the global energy system clearly cannot make overnight jumps in its emission levels and trajectories, as acknowledged in DICE itself (Nordhaus, 2019).

4.2 Inertia and positive-pliability regimes

For $p \neq 0$, substituting the exact solution (10) into the cost function $C[\cdot,\cdot]$ in equation (6) and evaluating the result at $t = 0$ yields the optimal current abatement effort as follows:

$$C[e(0), e'(0)] = \; 2\,c\,p\,\tau^2 \left[e_1 - (e_0 - e_\star) \cdot \frac{z_1 + z_2}{2} - (E_\star - E_0) \cdot \frac{z_1 \times z_2}{4} \right]^2, \tag{24}$$

For a small degree of pliability (low p), the dependencies can be clarified as:

$$C[e(0), e'(0)] = C[e(0), e'(0)]\Big|_{p=0} + V \times \; c\,\tau\,\sqrt{2p} + O(p), \tag{25}$$

where V is a constant,[6] $e_\star$ and $E_\star$ are as in equation (11), while z_1 and z_2 are as in equation (13) (medium-pliability regime). Equation (25) is a straightforward Taylor expansion in powers of p around the point $p = 0$, where the first term is given in equation (23). In equation (25), the fact that the second term scales with τ reflects the fact that with more inertia, greater effort is required to change the emissions trajectory, in proportion to the characteristic timescale of the emitting system. The dependence on $\sqrt{p}$ shows that effort is very sensitive to p as p approaches 0. As soon as there is any transitional cost, i.e,, for any $p > 0$, the system cannot be moved to a different starting point as in the no-pliability regime; hence, the high sensitivity to p can be explained by this qualitatively different nature of the solution. In general, the optimal initial effort increases with p, as more effort is exerted into transforming the system.[7]

[6] V is defined as follows

$$V = \left[e_1 - (e_0 - e_\star)\frac{r + z_-}{2} + (E_0 - E_\star)\frac{r\,z_-}{4} \right] \cdot \left[2(e_0 - e_\star) - z_-(E_0 - E_\star) \right]\Bigg|_{p=0,q=1},$$

$$z_\pm = r \pm \sqrt{r^2 + \frac{d}{2c}},$$

[7] A similar equation as equation (25) but for $p > p^\star$ can be obtained by plugging $p = 1$ into equation (24). As this yields no new insights, we do not display this formula. It is available upon request.

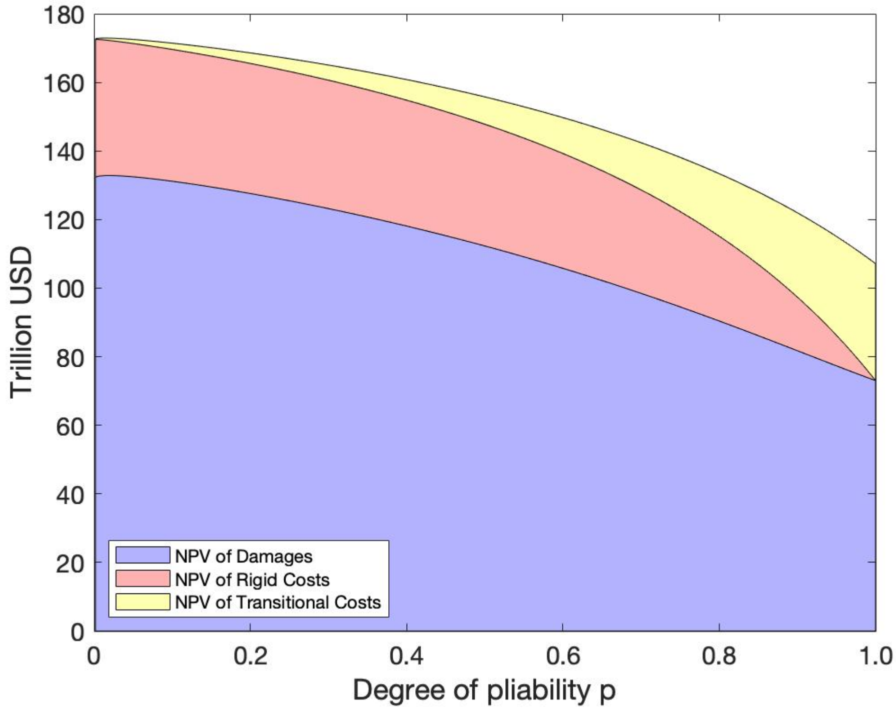


Figure 4: Decomposition of optimal value of objective function (1)

4.3 Numerical illustration

Figure 3 displays the optimal abatement effort at time zero for our calibration from above, but with three values of the characteristics transition time τ, i.e., $\tau \in \{7.5, 15, 30\}$.

Optimal initial effort is increasing in p and decreasing in τ. The gains of induced innovation can easily be reaped in a flexible system with low inertia. If, however, inertial timescales put a serious brake on the optimal pace of abatement achieved, this dampens the response, and hence the benefits available, in the entire system. Policies to remove obstacles to faster transitions—many of which may be political and distributional—enhance the gains, and consequently, the justified effort.[8]

5 Cost of delay

We end by exploring the influence of dynamic factors on the cost of delay, which we define as the sensitivity of objective function (1) to an infinitesimal delay $\mathrm{d}t$ in implementing the optimal solution given in Theorem 1. During this short period of delay, the emissions profile is assumed to equal its reference trajectory, after which the re-optimized policy is implemented. Optimization problem (1) is formally unchanged after a delay $\mathrm{d}t$ if we recognize that the initial conditions E_0 and e_0 have shifted to $E_0 + e_0\mathrm{d}t$ and $e_0 + e_1\mathrm{d}t$, respectively.[9]

[8]Note that the impact of characteristic transition time τ is reversed if the climate damages are more convex: with a fixed temperature constraint, the constancy of initial pace of emissions reduction implies the optimal initial effort required is proportional to τ^2.

[9]For analytic tractability we choose an infinitesimal delay $\mathrm{d}t$; the results can be generalized to allow for any (non-infinitesimal) delay $\Delta t > 0$.

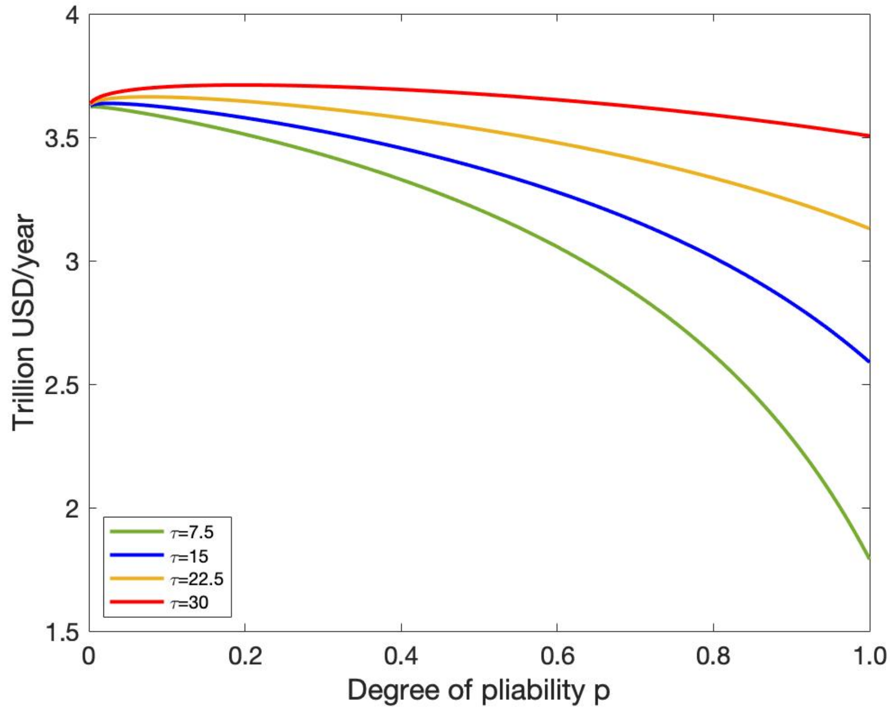


Figure 5: Cost of delay for $\tau \in \{7.5, 15, 22.5, 30\}$

First, we analytically compute (see the online Appendix B) the value of the objective function (1) under the optimal policy given in Theorem 1. We refer to this quantity as the optimal net present value (NPV) associated with problem (1). Figure 4 shows how this optimal NPV, including its three components (damages, static cost, and transitional cost), vary with p. The optimal NPV is decreasing in p, while climate damages make up around two thirds of the total across the range.

Second, we analytically compute (again, see the online Appendix B) the sensitivity of the optimal NPV with respect to a short delay $\mathrm{d}t$. The results shown in Figure 5 demonstrate that the cost of delay is decreasing in p from around $\sim$3.6 trillion USD per year (for $p = 0$) to around $\sim$2.6 trillion USD per year for our benchmark value $\tau = 15$. The fact that the cost of delay is a large multiple of the optimal effort at time zero suggests that the optimized objective function is highly sensitive to the initial conditions. Even as the optimal effort at time zero is relatively modest, a short delay of this optimal effort may be exceedingly costly; indeed, much more costly than the optimal effort itself.

With low pliability, climate change is more costly overall to deal with and climate damages are substantially higher. However, the system faces no inertial barrier. The ability to drop emissions immediately is valuable in terms of the large immediate marginal impact on $E(t)$, and every year that passes without such action squanders this potential, substantially increasing long-run damages. At higher pliability, abatement effort shifts towards transitional investments with enduring benefits, but the scale of (marginal) reduced climate damages is lower because the overall scale of long-run climate change is curtailed. Higher inertia, by impeding rapid response, reduces the pace at which the system can exploit lower static costs, but increases the marginal value of the achievable emission reductions. At a characteristic transition time of $\tau = 30$ years, these two effects roughly cancel each other out and the overall cost of delay is almost independent of the degree of system pliability.

6 Conclusion

We have constructed a stylized model that focuses on dynamic features of abatement costs, splitting the latter into static costs and elements of transitional costs. Our analytic results illuminate key sensitivities, and we demonstrated that the relative degree (as well as the absolute scale) of these cost components has important implications for optimal trajectories, initial effort, and long-run economics of the system.

Capital stock and other factors create aggregate inertia, represented in our model in terms of a *characteristic transition time*. Technology deployment is typically associated with cost reductions, and that investments in low carbon infrastructure (e.g., buildings, networks) may last many decades, we argue that *ex-ante* estimates of marginal cost curves in practice conflate static and transitional cost elements. The extent to which technologies and systems may adjust to transitional abatement-related effort is characterized in terms of the *pliability* of the system.

Many models assume marginal abatement costs to be exogenously definable, positive and enduring. Compared to this classical formulation, the optimal response in systems with high pliability tend to start with optimally linear reductions in emissions, driven by higher initial effort, and result in lower long-run temperature change and damages. In a cost-benefit setting, higher (lower) inertia slightly dampens (amplifies) these benefits in our model. In a setting with highly convex damages—represented in extreme by a fixed temperature goal—inertia is critical, with the optimal effort in our model increasing with τ^2.

The extent to which *ex-ante* assumed (static) abatement costs are actually transitional is of course uncertain. However, we note that, for example, numerous modelling studies from a decade ago assumed *ex-ante* that solar energy and electric vehicles would be enduringly expensive ways of reducing emissions. Initially, they were, but with hindsight it is clear that much of the early investment in deployment drove innovation and scale economies. Hence, we contend that many technology cost projections in reality conflate assumed static costs with the need for transitional investment, or adjustment costs, at technology or system levels.

Moreover, whilst zero technology cost is impossible, abatement is not a technology but a difference between the cost of low and high carbon technologies. Over the past decade, following large investments, several low carbon technologies have indeed become as cheap as fossil fuels, suggesting that exploring a full range of $0 \leq p \leq 1$ is not unreasonable as a research inquiry. We have shown how much it matters.

Our aim is that this model will inspire further research on the dynamic features of emitting systems. Gaining a deeper understanding of different approaches to dealing with inertial timescales, induced innovation, and path dependency is crucial to help inform policymaking on one of the most important threats facing humanity this century.

Bibliography

Acemoglu, Daron, Philippe Aghion, Leonardo Bursztyn, and David Hemous. 2012. The environment and directed technical change. *American Economic Review.* 102(1):131–66.

Acemoglu, Daron, Philippe Aghion, and David Hémous. 2014. The environment and directed technical change in a north–south model. *Oxford Review of Economic Policy.* 30(3):513–530.

Acemoglu, Daron, Ufuk Akcigit, Douglas Hanley, and William Kerr. 2016. Transition to clean technology. *Journal of Political Economy.* 124(1):52–104.

Aghion, Philippe and Xavier Jaravel. 2015. Knowledge spillovers, innovation and growth. *The Economic Journal.* 125(583):533–573.

Aghion, Philippe, Antoine Dechezleprêtre, David Hemous, Ralf Martin, and John Van Reenen. 2016. Carbon taxes, path dependency, and directed technical change: Evidence from the auto industry. *Journal of Political Economy.* 124(1):1–51.

Aghion, Philippe, Cameron Hepburn, Alexander Teytelboym, and Dimitri Zenghelis. 2019. Path dependence, innovation and the economics of climate change. In *Handbook on Green Growth.* Edward Elgar Publishing.

Baldwin, Elizabeth, Yongyang Cai, and Karlygash Kuralbayeva. 2020. To build or not to build? Capital stocks and climate policy. *Journal of Environmental Economics and Management.* 100:102235.

Bauer, Nico, Ioanna Mouratiadou, Gunnar Luderer, Lavinia Baumstark, Robert J Brecha, Ottmar Edenhofer, and Elmar Kriegler. 2016. Global fossil energy markets and climate change mitigation–An analysis with REMIND. *Climatic Change.* 136(1):69–82.

Dietz, Simon and Nicholas Stern. 2015. Endogenous growth, convexity of damage and climate risk: How Nordhaus' framework supports deep cuts in carbon emissions. *The Economic Journal.* 125(583):574–620.

Drupp, Moritz A, Mark C Freeman, Ben Groom, and Frikk Nesje. 2018. Discounting disentangled. *American Economic Journal: Economic Policy.* 10(4):109–34.

Gillingham, Kenneth and James H Stock. 2018. The cost of reducing greenhouse gas emissions. *Journal of Economic Perspectives.* 32(4):53–72.

Gillingham, Kenneth, Richard G Newell, and William A Pizer. 2008. Modeling endogenous technological change for climate policy analysis. *Energy Economics.* 30(6):2734–2753.

Goldstein, H., C.P. Poole, and J. Safko. 2013. *Classical mechanics.* Pearson.

Golosov, Mikhail, John Hassler, Per Krusell, and Aleh Tsyvinski. 2014. Optimal taxes on fossil fuel in general equilibrium. *Econometrica.* 82(1):41–88.

Grubb, Michael, Thierry Chapuis, and Minh Ha Duong. 1995. The economics of changing course: Implications of adaptability and inertia for optimal climate policy. *Energy Policy.* 23(4-5):417–431.

Grubb, Michael, J Mercure, Pablo Salas, R Lange, and Ida Sognnaes. 2018. Systems innovation, inertia and pliability: A mathematical exploration with implications for climate change abatement. Cambridge Working Paper Economics 1819.

Grubb, Michael, Paul Drummond, Alexandra Poncia, Will McDowall, David Popp, Sascha Samadi, Cristina Penasco, Kenneth Gillingham, Sjak Smulders, Matthieu Glachant, et al. 2021. Induced innovation in energy technologies and systems: A review of evidence and potential implications for CO_2 mitigation. *Environmental Research Letters*. 16(4):043007.

Harmsen, Mathijs, Elmar Kriegler, Detlef P Van Vuuren, Kaj-Ivar van der Wijst, Gunnar Luderer, Ryna Cui, Olivier Dessens, Laurent Drouet, Johannes Emmerling, Jennifer Faye Morris, et al. 2021. Integrated assessment model diagnostics: Key indicators and model evolution. *Environmental Research Letters*. 16(5):054046.

Hassler, John, Per Krusell, Conny Olovsson, and Michael Reiter. 2020. On the effectiveness of climate policies. Working Paper. IIES Stockholm University.

Howard, Peter H and Thomas Sterner. 2017. Few and not so far between: A meta-analysis of climate damage estimates. *Environmental and Resource Economics*. 68(1):197–225.

IPCC. 2007. Climate Change 2007: Mitigation of Climate Change. Contribution of Working Group III to the Fourth Assessment Report of the Intergovernmental Panel on Climate Change.

IPCC. 2014. Climate Change 2014: Mitigation of Climate Change. Contribution of Working Group III to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change.

IPCC. 2021. Climate Change 2021: The Physical Science Basis. Contribution of Working Group I to the Sixth Assessment Report of the Intergovernmental Panel on Climate Chang.

IPCC. 2022. Climate Change 2022: Mitigation of Climate Change. Contribution of Working Group III to the Sixth Assessment Report of the Intergovernmental Panel on Climate Change.

Mattauch, Linus, H Damon Matthews, Richard Millar, Armon Rezai, Susan Solomon, and Frank Venmans. 2020. Steering the climate system: Using inertia to lower the cost of policy: Comment. *American Economic Review*. 110(4):1231–37.

Newbery, David. 1972. The importance of malleable capital in optimal growth models. *Journal of Economic Theory*. 5(1):21–41.

Nikas, Alexandros, Haris Doukas, and Andreas Papandreou. 2019. A detailed overview and consistent classification of climate-economy models. In Doukas, H., A. Flamos, and J. Lieu, editors, *Understanding risks and uncertainties in energy and climate policy*. pages 1–54. Springer, Cham.

Nordhaus, William. 2019. Climate change: The ultimate challenge for economics. *American Economic Review*. 109(6):1991–2014.

Nordhaus, William D. 1991. To slow or not to slow: The economics of the greenhouse effect. *The Economic Journal*. 101(407):920–937.

Nordhaus, William D. 1993. Rolling the "DICE": An optimal transition path for controlling greenhouse gases. *Resource and Energy Economics*. 15:27–50.

Nordhaus, William D. 2013. *The climate casino.* Yale University Press.

Pottier, Antonin, Jean-Charles Hourcade, and Etienne Espagne. 2014. Modelling the redirection of technical change: The pitfalls of incorporeal visions of the economy. *Energy Economics*. 42:213–218.

Ricke, Katharine L and Ken Caldeira. 2014. Maximum warming occurs about one decade after a carbon dioxide emission. *Environmental Research Letters*. 9(12):124002.

Shapiro, Alan Finkelstein and Gilbert E Metcalf. 2023. The macroeconomic effects of a carbon tax to meet the us paris agreement target: The role of firm creation and technology adoption. *Journal of Public Economics*. 218:104800.

Stern, Nicholas. 2022. A time for action on climate change and a time for change in economics. *The Economic Journal*. 132(644):1259–1289.

van der Ploeg, Frederick and Armon Rezai. 2019. Simple rules for climate policy and integrated assessment. *Environmental and Resource Economics*. 72(1):77–108.

Vogt-Schilb, Adrien, Guy Meunier, and Stéphane Hallegatte. 2018. When starting with the most expensive option makes sense: Optimal timing, cost and sectoral allocation of abatement investment. *Journal of Environmental Economics and Management*. 88:210–233.

Weitzman, Martin L. 2009. On modeling and interpreting the economics of catastrophic climate change. *The Review of Economics and Statistics*. 91(1):1–19.

Weyant, John. 2017. Some contributions of integrated assessment models of global climate change. *Review of Environmental Economics and Policy*. 11(1):115–137.

Wilson, Charlie and Arnulf Grubler. 2015. Historical characteristics and scenario analysis of technological change in the energy system. In Vos, Rob and Diana Alarcón, editors, *Technology and innovation for sustainable development*. pages 45–80. Bloomsbury Publishing New York, NY.

A Proof of Theorem 1

Euler-Lagrange equations. The optimization problem described in (1) can be solved using standard Euler-Lagrange (EL) methods. The only non-standard feature is that the control variable $E(t)$ appears alongside both of its first and second derivative in the integrand F. For this reason, the standard EL equation is adjusted to include a third term as follows

$$0 = \frac{\mathrm{d}(\mathrm{e}^{-rt}F)}{\mathrm{d}E} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}(\mathrm{e}^{-rt}F)}{\mathrm{d}E'} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\mathrm{d}(\mathrm{e}^{-rt}F)}{\mathrm{d}E''}. \tag{A.1}$$

where $F := F(E, E', E'')$ as in equation (4), where primes denote derivatives. Explicitly computing all derivatives we obtain

$$\left[-4p\tau^2 e_1 r^2 - 2q(e_0 r + e_1(rt-1)), \tfrac{d}{4c}, 2qr, 4p\tau^2 r^2 - 2q, -8rp\tau^2, 4p\tau^2\right] \begin{bmatrix} 1 \\ E(t) \\ E^{(1)}(t) \\ E^{(2)}(t) \\ E^{(3)}(t) \\ E^{(4)}(t) \end{bmatrix} = 0,$$

which we here express as an inner product involving $E(t)$ and its four derivatives. This expression makes clear that in general we are faced with an inhomogenous linear ordinary differential equation (ODE) of fourth order. As is standard, the solution can be written as the sum of two solutions: one that solves the homogenous ODE and one that solves the inhomogenous ODE.

Solution to inhomogenous ODE. The inhomogenous ODE can be solved by a linear function of time, which we write as

$$E(t) = E_\star + e_\star t, \tag{A.2}$$

where $E_\star$ and $e_\star$ are constants to be found. For this candidate solution $E(t)$, the second, third, and fourth derivatives are zero. Solving the resulting simplified ODE for B and b, we obtain

$$e_\star = 8\frac{c\,q\,e_1\,r}{d} \qquad E_\star = \frac{c\,q\,(e_0\,r - e_1)}{d/8} + 64\,e_1\,r^2\left(\frac{c\,p\,\tau^2}{4\,d} - \left(\frac{c\,q}{d}\right)^2\right). \tag{A.3}$$

This simple solution already yields one important insight into the long-term behavior of our solution: in the long run, optimal cumulative emissions are linear in time, such that annual emissions are optimally constant. Specifically, the optimal long-run constant level of emissions is given by the parameter $e_\star$ above. As can be seen, it decreases with the damage parameter d, but increases with the static-cost component q, the discount rate r, and the increase of marginal emissions in the reference scenario, given by e_1.

While the inhomogenous ODE determines the optimal long-term emissions path, the particular solutions to the homogeneous ODE determine the optimal course of action in the short-term. We discuss these next.

Solution to homogenous ODE. To solve the homogenous ODE, we look for solutions that are exponential in time. Indeed, the homogenous ODE of fourth order allows for four independent solutions taking the form

$$E(t) = \sum_{j=1}^{4} Z_j \exp\left(\frac{z_j\, t}{2}\right), \tag{A.4}$$

where the parameters Z_j and z_j remain to be determined for $j = 1, 2, 3, 4$. Substituting this candidate solution into the ODE and simplifying, we find that the constants z_j for each $j = 1, 2, 3, 4$ must solve the following fourth-order polynomial equation

$$\left[d/c, 4\, q\, r, 4\, r^2\, p\, \tau^2 - 2\, q, -4\, r\, p, p\, \tau^2\right] \begin{bmatrix} 1 \\ z_j \\ z_j^2 \\ z_j^3 \\ z_j^4 \end{bmatrix} = 0, \qquad j = 1, 2, 3, 4. \tag{A.5}$$

Generally, this equation is of fourth order, unless $p = 0$, in which case it is only of second order (note the last two entries of the row vector).

Full solution. The full solution is obtained by summing the solutions to the homogeneous and inhomogeneous ODEs, i.e.,

$$E(t) = E_\star + e_\star\, t + \sum_{j=1}^{4} Z_j \exp\left(\frac{z_j t}{2}\right), \tag{A.6}$$

where the parameters $E_\star$ and $e_\star$ are given by (A.3), the constants z_j for $j = 1, 2, 3, 4$ are the roots of the fourth-order polynomial equation given in (A.5), and the four constants Z_j for $j = 1, 2, 3, 4$ remain to be determined by four boundary conditions, as discussed below. These boundary conditions will need to ensure that $E(0) = E_\star + \sum_{j=1}^{4} Z_j = E_0$, thereby putting a constraint on the Z_j's.

Boundary conditions. In general, the four constants Z_j are determined by a total of four boundary conditions to be specified at either $t = 0$ or $t = T$. At $t = 0$, we impose $E(0) = E_0$, reflecting the fact that cumulative emissions (relative to pre-industrial times) at time zero are fixed. For systems with any positive transitional cost ($p > 0$), we also impose $E'(0) = E'_{\text{ref}}(0) = e_{\text{ref}}(0) = e_0$, because sudden jumps in marginal emissions would incur an infinite cost. By imposing both boundary conditions, we ensure that the path of cumulative emissions $E(t)$ smoothly matches that of the reference trajectory of cumulative emissions $E_{\text{ref}}(t)$.

At $t = T$, we are faced with two free boundary conditions, as endpoint $E(T)$ and its derivative $E'(T)$ are left to be determined by the optimizer. However, in the limit as $T \to \infty$, which we consider below, two of the four homogenous solutions can be discarded (set to zero), as they blow up exponentially, thereby causing infinite damages. As such, only two constants Z_j, for $j = 1, 2$ remain, which can be determined by the two boundary conditions at $t = 0$.

If $p = 0$, the ODE and polynomial equation are of second order. In this case, only a single boundary condition at $t = 0$ is required, which we take to be $E(0) = E_0$. In this case, a jump in marginal (but not cumulative) emissions at time zero is permitted.

Solutions under three regimes. The optimal solution behaves differently, qualitatively, depending on the numerical values of the parameters. Specifically, three regimes can be identified. We present the solution in each of three mutually exclusive and collectively exhaustive regimes:

1. No pliability: $p = 0$,
2. Medium pliability: $0 < p \leq p_\star$, which implies $c\,q^2 \geq p\,\tau^2\,d$,
3. High pliability: $p > p_\star$, which implies $c\,q^2 < p\,\tau^2\,d$.

The critical boundary between the medium- and high-pliability regimes is denoted $p^\star$ and is determined by setting p equal to $p^\star$, q equal to $1-p^\star$ and solving for $p^\star$ the equality $c\,q^2 = p\,\tau^2\,d$, i.e., we must solve

$$c\,(1-p^\star)^2 \;=\; p^\star\tau^2 d.$$

This is a quadratic equation in $p^\star$ with two potential solutions. Only one of these potential solutions falls in the range $(0,1)$, which reads

$$p^\star := 1 - \frac{\sqrt{1+4x}-1}{2x} \in (0,1),$$

where $x := c/(d\tau^2) \in (0,\infty)$ is a dimensionless characteristic of the system. For $0 < p \leq p^\star$, it can be verified that $c\,q^2 \geq p\,\tau^2\,d$, such that we are in the medium-pliability regime. For $p > p^\star$, we can be verified that $c\,q^2 < p\,\tau^2\,d$, such that we are in the high-pliability regime.

In each case, an analytic solution is possible, which can be found by (i) solving the (in general) fourth-order polynomial equation, (ii) discarding two of the four solutions to the homogeneous ODE that correspond to the explosive solutions, and (iii) imposing the relevant boundary condition(s) at $t = 0$. We here only report the analytic solution in the case where $T = \infty$, which is economically the most relevant, and for which the solution takes the simplest possible form.

Zero pliability: If $p = 0$, such that the system contains no pliability, the fourth-order ODE simplifies to a second-order ODE. The corresponding second-order polynomial equation allows for two unique roots, one positive and one negative. The positive root can be discarded as it corresponds to an explosive solution, such that we can set $Z_2 = Z_3 = Z_4 = 0$, leaving only Z_1 to be determined. The negative root is given by

$$z_1 = r - \sqrt{r^2 + \frac{d}{2\,c\,q}}. \tag{A.7}$$

Note that $z_1 < 0$; the other root contains a plus instead of a minus in front of the square root and is economically irrelevant. This confirms the first part of equation (12) in Theorem 1. Imposing the boundary conditions $E(0) = E_0$, the constant Z_1 can be determined as

$$Z_1 = E_0 - E_\star, \tag{A.8}$$

where the value of $E_\star$ is given by (A.3) when p is set to zero. This confirms the second part of equation (12) in Theorem 1. For the zero pliability regime, we do not impose $E'(0) = e_0$ such that the optimal level of today's emissions, $E'(0)$, will generally differ from the reference level, e_0. For pliable systems in the two regimes below, a jump in marginal emissions is impossible.

Medium pliability: If $p \neq 0$, $c\,q^2 \geq p\,\tau^2\,d$, such that pliability is non-zero but small in relative terms (i.e., $0 < p \leq p^\star$), the fourth-order polynomial allows for four distinct roots. Two roots are positive and can be discarded from economic arguments, i.e., we set $Z_3 = Z_4 = 0$. The two remaining (negative) roots are given by

$$z_1 = r - \sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\left(\frac{q}{p\,\tau^2}\right)^2 - \frac{d}{c\,p}}}, \tag{A.9}$$

$$z_2 = r - \sqrt{r^2 + \frac{q}{p\,\tau^2} - \sqrt{\left(\frac{q}{p\,\tau^2}\right)^2 - \frac{d}{c\,p\,\tau^2}}}, \tag{A.10}$$

where each displayed square root is a real number because $cq^2 \geq p\,\tau^2\,d$ by assumption in the current regime, which implies $(q/p\tau^2)^2 \geq d/(cp)$. These equations confirm equations (13) in Theorem 1.

Imposing the boundary conditions $E(0) = E_0$ and $E'(0) = e_0$, we find the two constants Z_1 and Z_2 as follows

$$Z_1 = \frac{2(e_0 - e_\star) + z_2(E_\star - E_0)}{z_1 - z_2} \qquad Z_2 = \frac{2(e_0 - e_\star) + z_1(E_\star - E_0)}{z_2 - e_1}. \tag{A.11}$$

These equations confirm equations (14) in Theorem 1.

High pliability: If $p \neq 0$, $c\,q^2 < p\,\tau^2\,d$, such that transitional costs are large in relative terms (i.e., $p > p^\star$), the fourth-order polynomial equation allows for four distinct, complex-valued, roots. To avoid the emissions path exploding as $t \to \infty$, we pick the two roots with negative real parts. Hence, we may set $Z_3 = Z_4 = 0$. The two negative roots z_1 and z_2 differ by only a single sign, such that we can denote them by $z_1 = z_+$ and $z_2 = z_-$, where $z_\pm$ is defined as

$$\begin{aligned} z_\pm \equiv r &- \frac{1}{\sqrt{2}}\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}} \\ &\pm \frac{\mathrm{i}}{\sqrt{2}} \frac{\sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2}}{\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}}}, \end{aligned} \tag{A.12}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, and every displayed square root is a real (positive) number, because $c\,q^2 < p\tau^2 d$ in the current regime. It is clear that both $z_\pm$ have negative real parts as desired. This confirms equation (15) in Theorem 1.

Imposing the boundary conditions $E(0) = E_0$ and $E'(0) = e_0$, we find that the constants Z_1 and Z_2 are identical in form to those in the medium-pliability regime, namely

$$Z_1 = \frac{2(e_0 - e_\star) + z_2(E_\star - E_0)}{z_1 - z_2} \qquad Z_2 = \frac{2(e_0 - e_\star) + z_1(E_\star - E_0)}{z_2 - z_1}. \tag{A.13}$$

However, the numerical values of these constants differ from those in the medium pliability regime, because the two roots z_1 and z_2, which appear in the numerator and denominator, are now complex values. Hence, Z_1 and Z_2 are also complex valued. Naturally, the cumulative emissions path $E(t)$ for all time t remains real valued. After some tedious but straightforward trigonometric algebra, the optimal cumulative emissions trajectory $E(t)$ can be rewritten in trigonometric terms as

$$E(t) = E_\star + e_\star t + \exp\left(\frac{\widehat{z}t}{2}\right)\left[\frac{2(e_0 - e_\star) + \widehat{z}(E_\star - E_0)}{\widetilde{z}}\sin\left(\frac{\widetilde{z}t}{2}\right) + (E_0 - E_\star)\cos\left(\frac{\widetilde{z}t}{2}\right)\right],$$

where $e_\star$ and $E_\star$ are as in (A.3), while $\widehat{z}$ and $\widetilde{z}$ are real numbers coming from the real and imaginary parts of z_1 above. Explicitly, we have

$$\widehat{z} = r - \frac{1}{\sqrt{2}}\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}} \tag{A.14}$$

and

$$\widetilde{z} = \frac{1}{\sqrt{2}}\frac{\sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2}}{\sqrt{r^2 + \frac{q}{p\,\tau^2} + \sqrt{\frac{d}{c\,p\,\tau^2} - \left(\frac{q}{p\,\tau^2}\right)^2 + \left(r^2 + \frac{q}{p\,\tau^2}\right)^2}}}. \tag{A.15}$$

The intuition for the high pliability regime is that, when "steering" is expensive, it might be beneficial to "oversteer" before correcting (steering back) later, which explains the appearance of trigonometric functions in the solution: emissions oscillate towards the long-term optimum. For a fully pliable system in which case $q = 0$, it is optimal to decarbonize the economy completely at some finite time, and even go into negative marginal emissions (capturing carbon dioxide from the atmosphere), also at some finite time, while oscillating (with exponentially decreasing amplitudes) towards a fully decarbonized limit.

In all three regimes, the optimal marginal emissions path $E'(t)$ is implied by the optimal cumulative emissions path $E(t)$ via a straightforward differentiation with respect to time. Further, in all cases an analytic solution remains possible even for a finite optimization horizon T, but the resulting expressions are more involved, because it no longer holds that two out of four roots from the fourth-order polynomial can be discarded (all four roots are relevant in this case). The resulting expressions are available from the authors upon request.

B Analytic solution for NPV and cost of delay

Assume $0 < p \le p^\star$, such that the medium-pliability regime applies; below we extend the results to all $p \in [0, 1]$. Assume $T = \infty$, i.e., an infinite time horizon. Assume the optimal cumulative emissions path $E(t)$ given in equation (10) in Theorem 1. Then the net present value (NPV) of damages can be computed analytically as

$$\int_0^\infty \exp(-r\,t)\,\frac{d}{8}E(t)^2\,\mathrm{d}t = \tag{B.1}$$
$$\frac{d}{8}\left[\frac{2e_\star^2 + 2e_\star E_\star r + E_\star^2 r^2}{r^3} + \frac{4Z_1Z_2}{2r - z_1 - z_2} + \sum_{i=1}^{2}\left\{\frac{Z_i^2}{r - z_i} + \frac{8Z_i(e_\star + E_\star r)}{(2r - z_i)^2} - \frac{4z_iZ_iE_\star}{(2r - z_i)^2}\right\}\right].$$

Second, the NPV of the static-cost component can be computed in closed form as

$$\int_0^\infty \exp(-r\,t)\,c\,q\,[e_{\mathrm{ref}}(t)-e(t)]^2\,\mathrm{d}t = \tag{B.2}$$
$$\frac{c\,q}{4}\Bigg[\frac{8e_1^2}{r^3}+\frac{4(e_\star-e_0)^2}{r}+8e_1\frac{e_0-e_\star}{r^2}+\frac{4z_1z_2Z_1Z_2}{2r-z_1-z_2}$$
$$+\sum_{i=1}^{2}\left\{8(e_\star-e_0)\frac{z_iZ_i}{2r-z_i}+\frac{z_i^2Z_i^2}{r-z_i}-8e_1\frac{2z_iZ_i}{(2r-z_i)^2}\right\}\Bigg].$$

Third, the NPV of the transitional-cost component reads

$$\int_0^\infty \exp(-r\,t)\,2\,c\,p\,\tau^2\,[e'_{\mathrm{ref}}(t)-e'(t)]^2\,\mathrm{d}t = \tag{B.3}$$
$$\frac{cp\tau^2}{8}\left[\frac{16e_1^2}{r}+\frac{z_1^4Z_1^2}{r-z_1}+\frac{z_2^4Z_2^2}{r-z_2}+\frac{4z_1^2z_2^2Z_1Z_2}{2r-z_1-z_2}-16e_1\left(\frac{z_1^2\,Z_1}{2r-z_1}+\frac{z_2^2\,Z_2}{2r-z_2}\right)\right].$$

In equations (B.1), (B.2) and (B.3), the quantities $e_\star$, $E_\star$ are given in equation (11), while z_i for $i=1,2$ are given in equation (13), and Z_i for $i=1,2$ are given in equation (14).

By adding the right-hand side (RHS) of equations (B.1), (B.2) and (B.3), we obtain the optimal NPV of the entire minimization problem (1), i.e.,

$$\mathrm{NPV} = \text{RHS of equations (B.1), (B.2) and (B.3).} \tag{B.4}$$

This optimal NPV remains valid in the limit where p approaches zero, such that the NPV in the no-pliability regime can be obtained as a special case. Moreover, all expressions technically remain valid in the high-pliability regime; while some quantities turn complex, the imaginary parts cancel out and the result is a real-valued number that equals the desired NPV in the high-pliability regime.

The cost of delay discussed in the main text is obtained by comparing the NPV as computed above with the NPV evaluated a small time $\mathrm{d}t$ later, assuming no action is taken in the meanwhile. Our solution remains valid after some delay if we recognize that the initial conditions have shifted. In particular, cumulative emissions have increased from E_0 to $E(0+\mathrm{d}t)=E_0+e_0\mathrm{d}t$, while annual emissions have increased from e_0 to $e(0+\mathrm{d}t)=e_0+e_1\mathrm{d}t$. Hence, with obvious notation,

$$\text{cost of delay} \;=\; \frac{\mathrm{d}\,\mathrm{NPV}}{\mathrm{d}\,E_0}\,e_0 \;+\; \frac{\mathrm{d}\,\mathrm{NPV}}{\mathrm{d}\,e_0}\,e_1, \tag{B.5}$$

where the NPV is given in equation (B.4). The cost of delay is measured in units of currency per units of time. The required derivatives can be computed in closed form by using equations (B.1), (B.2) and (B.3), which depend explicitly on E_0 and e_0. Moreover, the chain rule must be employed to account for the implicit dependence of $E_\star$, Z_1 and Z_2 on the initial conditions E_0 and e_0; the resulting (lengthy) expression for equation (B.5) is available from the authors on request.