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Degree Centrality, von Neumann-Morgenstern Expected Utility and Externalities in Networks*

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Abstract. This paper aims to connect the social network literature on centrality measures with the economic literature on von Neumann-Morgenstern expected utility functions using cooperative game theory. The social network literature studies various concepts of network centrality, such as degree, betweenness, connectedness, and so on. This resulted in a great number of network centrality measures, each measuring centrality in a different way. In this paper, we aim to explore which centrality measures can be supported as von Neumann-Morgenstern expected utility functions, reflecting preferences over different network positions in different networks. Besides standard axioms on lotteries and preference relations, we consider *neutrality to ordinary risk*. We show that this leads to a class of centrality measures that is fully determined by the degrees (i.e. the numbers of neighbours) of the positions in a network. Although this allows for externalities, in the sense that the preferences of a position might depend on the way how other positions are connected, these externalities can be taken into account only by considering the degrees of the network positions. Besides bilateral networks, we extend our result to general cooperative TU-games to give a utility foundation of a class of TU-game solutions containing the Shapley value.

JEL Classification: D85, D81, C02

Keywords: weighted network, degree, centrality measure, externalities, neutrality to ordinary risk, expected utility function

1 Introduction

This paper aims to connect the social network literature on centrality measures with the economic literature on von Neumann-Morgenstersn expected utility functions. The social network literature studies various concepts of network centrality, such as degree, betweenness, connectedness, and so on. This resulted in a great number of network centrality measures, each measuring centrality in a different way. Examples can be found in, e.g. (1), (29), (24), (12; 13), (4), (14), (5), (17), (23), (3), (11), to name a few. A large part

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of decision making in the economic literature is based on von Neumann-Morgenstern expected utility functions (22). Specifically, properties of preference relations allowing them to be represented by a von Neumann-Morgenstern expected utility function are studied.

Building on (16), (27) gives a von-Neumann-Morgenstern expected utility foundation of solutions for cooperative transferable utility games; see also (28). Specifically, adding axioms with respect to worst and better elements and two risk neutrality axioms (neutrality to ordinary risk and neutrality to strategic risk) to a set of usual axioms of preference relations, the Shapley value (31) stands out as such a utility function expressing preferences over different player positions in different games. In this paper, we generalize the work of (7) who apply (27)'s theory to bilateral networks obtaining a utility foundation of network centrality measures. The importance of developing a utility foundation for network centrality measures is also stressed in, e.g. (9) who introduce a network formation game where the building of relations (in their case authority relations) is driven by utility maximizing individuals.

Bilateral networks can be represented by a special class of cooperative games, the so-called 2-games of (10), where worth is generated by two-player coalitions, being the links in the network. The Shapley value of these games boils down to the degree measure of the associated networks. In (7), (27)'s theory is applied to undirected networks (and then extended to directed networks), where they show that without requiring neutrality to strategic risk, one obtains a characterization of the degree measure as a von Neumann-Morgenstern expected utility function over network positions. The degree is an important characteristic of a position in a network, and is useful in the explanation of many network phenomena, for example in explaining the level of cooperation in threshold games on a network as in (26), (19).

As (27), (7) also use axioms specifying a worst position and identify at least one position that is better than this. They assume that the worst is being isolated, i.e. having no neighbours. This implies that being isolated is as good as being in the empty network that has no relations, and thus implies that it ignores externalities with respect to the connectedness of other nodes as long as one is isolated. Besides, a main disadvantage of this axiom is that it is not related to risk attitudes.

Neutrality to ordinary risk requires that one is indifferent between a position in a convex combination of two networks and playing a lottery between these two networks with the probabilities corresponding to the weights in this convex combination. In the underlying paper, we want to explore which centrality measures can be supported as von Neumann-Morgenstern expected utility functions by, besides some standard regularity axioms and anonymity, only assuming neutrality to ordinary risk. Specifically, we make no assumption about what is the worst possible network position. It turns out that the only centrality measures that satisfy these axioms are based on the degrees of the network positions, specifically are a combination of the degree of a position and the average degree over all positions in the network. This implies that it allows for externalities, but only in terms of the average degree. In this way, we give a foundation of these network centrality measures as utility functions over network positions, expressing preferences over positions in networks, bringing in externalities of connections between other positions. We obtain this main result by first characterizing this class of centrality measures that is determined by the degree of the positions in the network.

Other nodes being more connected can have positive as well as negative externalities. Our class of measures contains two parameters, in particular, an externality parameter. Depending on these parameters, we can recover some specific measures, e.g., the degree measure if the externality parameter is equal to zero. By considering some additional axioms on the preferences, we can specify the range for the parameters in the utility functions. Negative (respectively, positive) values of the externality parameter express negative (respectively, positive) externalities. Negative externalities of other nodes being connected to each other always occur when the sum of the values assigned to all nodes is a constant. This is the case, for instance, for the new average degree externality measure which assigns to every node in a network its degree minus the average degree over all nodes, and therefore leads to the sum of the values equal to zero.

As mentioned above, cooperative games, also known as weighted hypergraphs, are a generalization of bilateral networks. We extend our main result to the class of all cooperative games, and by doing so also generalize the result of (27) characterizing the Shapley value. However, since we do not have the 'worst element' axiom, we strengthen his strategic risk neutrality axiom being an axiom that deals with uncertainty regarding the unanimity coalition. In a unanimity game a positive worth is earned by any coalition containing the unanimity coalition and the worth is zero otherwise. (27)'s neutrality to strategic risk requires that a player is indifferent between being in two scaled unanimity games, where it belongs to the unanimity coalition in both games and the worth that is generated in the game is proportional to the number of players in the unanimity coalition. We strengthen this axiom by also requiring this for null players. This leads to a class of solutions that is a combination of the Shapley value and a term that is the same for all players. In this way, we also generalize the result of (27).

This paper is organized as follows. In Section 2, we discuss preliminaries on networks and (16)'s expected utility theory over mixture sets. In Section 3, we characterize a class of centrality measures that are combinations of the degree and average degree of a network. In Section 4, we present our main result showing that these measures are the only ones that represent regular preferences that are neutral to ordinary risk. In Section 5, we extend the main result to cooperative games. Finally, Section 6 contains concluding remarks.

2 Preliminaries

First, we present some basic concepts and notation that will be used in the paper.

Weighted (undirected) networks A *weighted undirected network*, shortly called a *network* in this paper, is a pair (N,ω) consisting of a finite set of *nodes* $N \subset \mathbb{N}$ and a *weight function* $\omega\colon L^c \to \mathbb{R}_+$, where $L^c = \{\{i,j\} \mid i,j \in N,\ i \neq j\}$ denotes the *complete undirected network* on N. An element $\{i,j\} \in L^c$ is a subset of N of size two and is called a *link*. A link $\{i,j\}$ represents a certain bilateral relationship between nodes i and j. A weight function gives a nonnegative weight $\omega(\{i,j\})$ to every link that can be interpreted as the 'importance' or 'strength' of the relationship. By $\mathcal{WG}^N$ we denote the collection of all weight functions on N. We refer to a weighted undirected network simply as a network. Since N is assumed to be fixed, we represent a network (N,ω) by the weight function ω.

A centrality measure for undirected networks is a function $f\colon \mathcal{WG}^N \to \mathbb{R}^N$ that assigns a real number to every node in every undirected network that reflects the 'centrality' of the node in the network. The *degree* of node $i \in N$ in network ω is defined as the sum of the weights of all links containing i, and thus is given by

$$d_i(\omega) = \sum_{j \in N \setminus \{i\}} \omega(\{i,j\}) \tag{1}$$

The *degree measure* is the centrality measure that assigns to any node i in any network ω its degree $d_i(\omega)$.

Let $\Pi(N)$ be the collection of all permutations $\pi\colon N \to N$ on N, i.e. for every rank number $k \in N$ there is a unique $i \in N$ with $\pi(i) = k$. For a network $\omega \in \mathcal{WG}^N$ and a permutation $\pi \in \Pi(N)$, the permuted network $\pi\omega \in \mathcal{WG}^N$ is given by $\pi\omega(\{i,j\}) = \omega(\{\pi^{-1}(i), \pi^{-1}(j)\})$ for every $\{i,j\} \in L^c$, i.e. only the labels of the positions are switched. We say that node $i \in N$ is *isolated* in $\omega \in \mathcal{WG}^N$ if $\omega(\{i,j\}) = 0$ for all $j \in N$. We denote the set of weighted undirected networks where i is an isolated node by $\mathcal{WG}^N_i$. We denote by $\omega^0 \in \mathcal{WG}^N$ the empty network given by $\omega^0(\{i,j\}) = 0$ for all $i, j \in N$.

Expected utility We briefly recall the utility theory on mixture sets of (16) (see also (33; 34), and (21) for related works on linear utility representation theorems). Consider a set M. The (simple) *lottery* between two elements $a, b \in M$ where element a occurs with probability $p \in [0,1]$ (and element b occurs with probability $(1-p) \in [0,1]$) is denoted by $[pa; (1-p)b]$. A set M is a *mixture set* if for any $a, b \in M$ and any $p \in [0,1]$, the lottery $[pa; (1-p)b]$ also belongs to M. Notice that this implies that also all compound lotteries, i.e. lotteries over lotteries, etc., belong to M. It is assumed that for all $a, b \in M$ and $p, q \in [0,1]$, the following three standard equalities hold:

$$[1a; 0b] = a, \quad [pa; (1-p)b] = [(1-p)b; pa], \quad [q[pa; (1-p)b]; (1-q)b] = [pqa; (1-pq)b]. \tag{2}$$

The left hand side of the third equality is a so-called compound lottery where lottery $[pa; (1-p)b]$ occurs with probability p and element b occurs with probability $(1-p)$. The equality says that this is identical to the lottery where a occurs with probability pq and b occurs with probability $(1-pq)$.
A *preference relation* on M is a binary relation $\succeq$ with the interpretation that $a \succeq b$ means that "a is at least as good as b".

A function $u\colon M \to \mathbb{R}$ is an *expected utility function* representing the preference relation $\succeq$ if for all $a, b \in M$ and $p \in [0,1]$, it holds that

$$\begin{aligned} &(i)\ \ u(a) \geq u(b) \text{ if and only if } a \succeq b, \text{ and} \\ &(ii)\ u([pa; (1-p)b]) = pu(a) + (1-p)u(b). \end{aligned} \tag{3}$$

If u is an expected utility function representing the preference relation $\succeq$, then also every positive affine transformation $\bar{u}$ given by $\bar{u}(a) = \alpha u(a) + \beta$, for some $\alpha > 0,\ \beta \in \mathbb{R}$, represents this preference relation. We write $[a \succ b]$ if $[a \succeq b$ and $b \not\succeq a]$ for the strict preference relation, and $[a \sim b]$ if $[a \succeq b$ and $b \succeq a]$ for the indifference relation. The following axioms guarantee that an expected utility function representing $\succeq$ exists.

Axiom 1 (Completeness) *For any $a, b \in M$, either $a \succeq b$ or $b \succeq a$.*

Axiom 2 (Transitivity) *For any $a, b, c \in M$ such that $a \succeq b$ and $b \succeq c$, it holds that $a \succeq c$.*

Axiom 3 (Continuity) *For any $a, b, c \in M$, the sets $\{p \mid [pa; (1-p)b] \succeq c\}$ and $\{p \mid c \succeq [pa; (1-p)b]\}$ are closed.*

Axiom 4 (Substitutability) *If $a, a' \in M$ and $a \sim a'$, then for every $b \in M$, $[\frac{1}{2}a; \frac{1}{2}b] \sim [\frac{1}{2}a'; \frac{1}{2}b]$.*

We refer to preferences satisfying these axioms as *regular* preferences and assume throughout the paper that preferences are regular.

Preferences over network positions Following (7), we refer to a pair $(i, \omega) \in N \times \mathcal{WG}^N$ as a *(undirected) network position*. We assume that a preference relation $\succeq$ is defined on the set $N \times \mathcal{WG}^N$ of undirected network positions, and interpret $(i, \omega) \succeq (j, \omega')$ as "it is at least as good to be in the position of node i in network ω as it is to be in the position of node j in network ω'". Let M be the mixture set generated by all undirected network positions $(i, \omega) \in N \times \mathcal{WG}^N$ containing all simple and compound lotteries over network positions. This means that M contains all network positions (i, ω), all lotteries over network positions, all lotteries over those lotteries, etc.

For (i, ω), $(j, \omega') \in N \times \mathcal{WG}^N$ and $p \in [0, 1]$, the lottery $[p(i, \omega); (1-p)(j, \omega')]$ means that with probability p the agent takes the position of node i in network ω, and with probability $(1-p)$ it takes the position of node j in network ω'. Extending the preference relation over network positions to the mixture set M, an expected utility function for network positions is a function $\phi \colon M \to \mathbb{R}$ assigning a utility value to every mixture of network positions satisfying conditions (3).

Besides the standard axioms stated before, (7) consider the following adaptations of axioms from (27).

First, anonymity requires that relabelling the nodes in a network yields a corresponding reordering in the preference relation.

Axiom 5 (Anonymity) *For all $\omega \in \mathcal{WG}^N$, $i \in N$ and $\pi \in \Pi(N)$, it holds that $(i, \omega) \sim (\pi(i), \pi\omega)$.*

The second axiom compares different network positions, expressing preference with respect to connectedness. More specifically, an agent weakly prefers any position in any network above being isolated. Notice that this implies that an agent is indifferent between any two networks where it is isolated, irrespective of how other positions are connected among each other. Specifically, an agent is indifferent between being in the empty network ω^0 and being isolated in any other network.

Axiom 6 (Isolated is the worst) *For all $i \in N$, $\omega \in \mathcal{WG}^N$ and $\omega' \in \mathcal{WG}_i^N$, i.e., i is isolated in ω', it holds that $(i, \omega) \succeq (i, \omega')$.*

Whereas the previous axiom stated that being isolated as a worst element in the preference relation, the next axiom specifies at least one network position that is strictly better than being isolated. Specifically, it says that being the center of the star is strictly preferred to being in the empty network. By ω^i we denote the simple star network with i as center given by $\omega^i(\{i, j\}) = 1$ for all $j \in N \setminus \{i\}$ and $\omega^i(\{h, j\}) = 0$ if $i \notin \{h, j\}$.

Axiom 7 (Center of the star is strictly better than being isolated) *For all $i \in N$, it holds that $(i, \omega^i) \succ (i, \omega^0)$.*

Finally, neutrality to ordinary risk requires that an agent is indifferent between taking a position in a convex combination of two networks, and playing a lottery over the two networks with the corresponding probabilities.

Axiom 8 (Neutrality to ordinary risk) *For all $\omega, \omega' \in \mathcal{WG}^N$ and $i \in N$, it holds that $(i, p\omega + (1-p)\omega') \sim [p(i,\omega); (1-p)(i,\omega')]$.*

In (7) it is shown that the only utility functions that satisfy the above axioms are multiples of the degree measure.

Theorem 1 (van den Brink and Rusinowska (2022)) *A preference relation $\succeq$ over network positions $N \times \mathcal{WG}^N$ is regular and satisfies anonymity, isolated is the worst, center of the star is strictly better than being isolated, and neutrality to ordinary risk if and only if it can be represented by utility function $\phi(i,\omega) = d_i(\omega)$ for all $(i,\omega) \in N \times \mathcal{WG}^N$, where $d_i(\omega)$ is the degree of node i in network ω, see (1).*

This theorem gives the degree measure, which is a well-known centrality measure in social network theory, an interpretation as a von Neumann-Morgenstern expected utility function.

3 Degree centrality and externalities: the average degree externality measure

Before investigating what centrality measures we obtain as representing regular preference relations satisfying only anonymity and neutrality to ordinary risk, we introduce a class of centrality measures that are based on the degree measure but are modified for an externality. The average degree externality measure assigns to every node in a network its degree minus the average degree over all nodes.

Definition 1 *The average degree externality (ADE) measure $e \colon \mathcal{WG}^N \to \mathbb{R}^N$ is given by*

$$e_i(\omega) = d_i(\omega) - \frac{1}{n}\sum_{j \in N} d_j(\omega) \text{ for all } i \in N \text{ and } \omega \in \mathcal{WG}^N,$$

where $d_i(\omega)$ is the degree of node i in weighted network ω, see (1).

Next, we characterize the class of measures that satisfy the standard anonymity, scale invariance and additivity axioms.
Anonymity of a centrality measure requires that the centrality of nodes does not depend on their label.

Axiom 9 (Anonymity for centrality measures) *For every $\omega \in \mathcal{WG}^N$ and permutation $\pi \in \Pi(N)$, it holds that $f_i(\omega) = f_{\pi(i)}(\pi(\omega))$.*

Since the context makes clear whether we speak about anonymity of a preference relation or a centrality measure, we often simply refer to this as anonymity. The close relation between the two is expressed in Lemma 1.(i) in the next section.

Scale invariance requires that if the weights of all links in a network are multiplied by a common factor, then the centralities of the nodes in that network are multiplied by the same factor.

Axiom 10 (Scale invariance) *Let $\omega \in \mathcal{WG}^N$ and $\alpha \in \mathbb{R}_+$. Then $f(\alpha\omega) = \alpha f(\omega)$, where $\alpha\omega \in \mathcal{WG}^N$ is given by $\alpha\omega(i,j) = \alpha \cdot \omega(i,j)$ for all $i,j \in N,\ i \neq j$.*

Additivity requires that the centrality in the network that is obtained by adding two networks is equal to the sum of the centralities in these two networks.

Axiom 11 (Additivity) *For $\omega, \omega' \in \mathcal{WG}^N$ it holds that $f(\omega+\omega') = f(\omega)+f(\omega')$, where $(\omega+\omega')(\{i,j\}) = \omega(\{i,j\}) + \omega'(\{i,j\})$ for all $\{i,j\} \in L^c$.*

Theorem 2 *A measure f satisfies anonymity, scale invariance and additivity if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that*

$$f_i(\omega) = (\alpha - \beta)d_i(\omega) + \frac{\beta}{2}\sum_{j \in N} d_j(\omega) \text{ for all } i \in N \text{ and } \omega \in \mathcal{WG}^N. \tag{4}$$

Proof

It is straightforward to verify that centrality/power measures as given by (4) satisfy anonymity, scale invariance and additivity. Next, suppose that measure f satisfies anonymity, scale invariance and additivity, and consider $\omega \in \mathcal{WG}^N$. We show that it must be of the form as given by (4).

First, consider the empty network ω^0 and any network $\omega \in \mathcal{WG}^N$. Additivity implies that $f_i(\omega + \omega^0) = f_i(\omega) + f_i(\omega^0)$. Since $\omega + \omega^0 = \omega$, this implies that $f_i(\omega) = f_i(\omega) + f_i(\omega^0)$, and thus $f_i(\omega^0) = 0$ for all $i \in N$.

Next, take a pair $i,j \in N$, $i \neq j$, and consider the network $\omega \in \mathcal{WG}^N$ given by $\omega(i,j) = 1$ and $\omega(h,g) = 0$ for all $(h,g) \neq (i,j)$. By anonymity, there exist $\alpha, \beta \in \mathbb{R}$ such that $f_i(\omega) = f_j(\omega) = \alpha$ and $f_h(\omega) = \beta$ for all $h \in N \setminus \{i,j\}$. Next, take $\omega \in \mathcal{WG}_{ij}^N$ where $\mathcal{WG}_{ij}^N = \{\omega \in \mathcal{WG}^N \mid \omega(i,j) \neq 0 \text{ and } \omega(h,g) = 0 \text{ for all } (h,g) \neq (i,j)\}$ is the class of networks where only arc (i,j) has a nonzero weight. By scale invariance, $f_i(\omega) = f_j(\omega) = \alpha \cdot \omega(i,j)$ and $f_h(\omega) = \beta \cdot \omega(i,j)$ for all $h \in N \setminus \{i,j\}$.

Now take any $(h,g) \in N \times N, h \neq g,\ (h,g) \neq (i,j)$, and $\omega' \in \mathcal{WG}_{hg}^N$. By anonymity and the case $\omega \in \mathcal{WG}_{ij}^N$ above, we have $f_h(\omega') = f_g(\omega') = \alpha \cdot \omega'(h,g)$, and $f_k(\omega') = \beta \cdot \omega'(h,g)$ for all $k \in N \setminus \{h,g\}$.

Finally, consider any $\omega \in \mathcal{WG}^N$. For every $i, j \in N$, $i \neq j$, define $\omega^{ij}(i,j) = \omega(i,j)$ and $\omega^{ij}(h,g) = 0$ for all $(h,g) \neq (i,j)$. Then, additivity implies that for all $i \in N$,

$$\begin{aligned}
f_i(\omega) = \sum_{\substack{h,g \in N \\ h \neq g}} f_i(\omega^{hg}) &= \sum_{j \in N \setminus \{i\}} f_i(\omega^{ij}) + \sum_{h,g \in N \setminus \{i\}} f_i(\omega^{hg}) \\
&= \sum_{j \in N \setminus \{i\}} \alpha \cdot \omega(i,j) + \sum_{h,g \in N \setminus \{i\}} \beta \cdot \omega(h,g) \\
&= \alpha d_i(\omega) + \beta \left(\frac{\sum_{j \in N \setminus \{i\}} d_j(\omega) - d_i(\omega)}{2} \right) \\
&= \alpha d_i(\omega) + \beta \sum_{j \in N} \left(\frac{d_j(\omega)}{2} - d_i(\omega) \right) \\
&= (\alpha - \beta) d_i(\omega) + \frac{\beta}{2} \sum_{j \in N} d_j(\omega).
\end{aligned}$$

□

This theorem characterizes a class of measures that are based on the degree measure but are modified for an externality by the term $\frac{\beta}{2}\sum_{j \in N} d_j(\omega)$. The parameter $\frac{\beta}{2}$ can be considered as an *externality parameter*. Notice that this parameter can be negative as well as positive, and negative (respectively, positive) values of the externality parameter express negative (respectively, positive) externalities. If $\beta = 0$ then we simply have the degree measure expressing no externalities. Taking $\alpha = \frac{n-2}{n}$, $\beta = -\frac{2}{n}$ gives the average degree externality (ADE) measure as defined in Definition 1. In the next section we will give a foundation of these measures as utility functions over network positions, and we give extra conditions that imply the parameters to be within a certain range, specifically when they are negative or positive.

In the remainder of this section, we want to introduce an axiom that explicitly brings in negative externalities of other nodes being linked with each other into the measure. We do this by requiring that the sum of the values assigned to all nodes is always a constant, specifically is zero. So, if one node gets a higher value, then there must be at least one node that gets a lower value. This is satisfied by, for example, the average degree externality (ADE) measure.

Axiom 12 (Null normalization) *For every $\omega \in \mathcal{WG}^N$, it holds that*

$$\sum_{i \in N} f_i(\omega) = 0.$$

Notice that the degree measure satisfies the normalization that the sum of the values assigned to all nodes always equals twice the number of links. So, the 'sum of powers' depends positively on the number of links, allowing all nodes to (weakly) increase their power.

Next, we characterize the class of measures in Theorem 2 that satisfy additionally the null normalization.

Proposition 1 *Let $n > 2$. A measure f satisfies null normalization, anonymity, scale invariance and additivity if and only if there exists an $\alpha \in \mathbb{R}$ such that*

$$f_i(\omega) = \frac{n\alpha}{n-2} d_i(\omega) - \frac{\alpha}{n-2} \sum_{j \in N} d_j(\omega) \text{ for all } i \in N \text{ and } \omega \in \mathcal{WG}^N.$$

Proof

By Theorem 2, it follows that the measures of Proposition 1 satisfy anonymity, scale invariance and additivity. Null normalization follows since

$$\begin{aligned}\sum_{i \in N} \left(\frac{n\alpha}{n-2} d_i(\omega) - \frac{\alpha}{n-2} \sum_{j \in N} d_j(\omega) \right) &= \frac{n\alpha}{n-2} \sum_{i \in N} d_i(\omega) - \frac{\alpha}{n-2} \sum_{i \in N} \sum_{j \in N} d_j(\omega) \\ &= \frac{n\alpha}{n-2} \sum_{i \in N} d_i(\omega) - \frac{\alpha}{n-2} n \sum_{j \in N} d_j(\omega) = 0.\end{aligned}$$

Next, suppose that measure f satisfies anonymity, scale invariance, additivity and null normalization, and consider $\omega \in \mathcal{WG}^N$. By Theorem 2, there exist $\alpha, \beta \in \mathbb{R}$ such that measure f can be written as $f_i(\omega) = (\alpha - \beta) d_i(\omega) + \frac{\beta}{2} \sum_{j \in N} d_j(\omega)$. Null normalization then implies that

$$\begin{aligned}\sum_{i \in N} \left((\alpha - \beta) d_i(\omega) + \frac{\beta}{2} \sum_{j \in N} d_j(\omega) \right) &= (\alpha - \beta) \sum_{i \in N} d_i(\omega) + \frac{\beta}{2} \sum_{i \in N} \sum_{j \in N} d_j(\omega) \\ &= (\alpha - \beta) \sum_{i \in N} d_i(\omega) + \frac{n\beta}{2} \sum_{j \in N} d_j(\omega) \\ &= (\alpha - \beta + \frac{n\beta}{2}) \sum_{i \in N} d_i(\omega) = 0\end{aligned}$$

$$\Leftrightarrow$$

$$(\alpha - \beta + \frac{n\beta}{2}) = 0 \Leftrightarrow \frac{2\alpha - 2\beta + n\beta}{2} = 0 \Leftrightarrow 2\alpha + (n-2)\beta = 0 \Leftrightarrow \beta = -\frac{2\alpha}{n-2},$$

where the first equivalence follows from the fact that the equation before must hold for every $\omega \in \mathcal{WG}^N$, specifically when taking any network other than the empty network. Thus,

$$\begin{aligned}f_i(\omega) &= (\alpha - \beta) d_i(\omega) + \frac{\beta}{2} \sum_{j \in N} d_j(\omega) \\ &= \left(\alpha + \frac{2\alpha}{n-2} \right) d_i(\omega) - \frac{\alpha}{n-2} \sum_{j \in N} d_j(\omega) \\ &= \left(\frac{n\alpha - 2\alpha + 2\alpha}{n-2} \right) d_i(\omega) - \frac{\alpha}{n-2} \sum_{j \in N} d_j(\omega) \\ &= \frac{n\alpha}{n-2} d_i(\omega) - \frac{\alpha}{n-2} \sum_{j \in N} d_j(\omega).\end{aligned}$$

□

Taking $\alpha = \frac{n-2}{n}$ in Proposition 1 gives the average degree externality (ADE) measure. Observe that the degree measure does not belong to this class since it does not satisfy null normalization. In the next section we will discuss other ways to deal with externalities, also allowing positive externalities.

4 Degree utility, ordinary risk neutrality and externalities

Similarly to (7) and following (27), in this section we will interpret centrality measures as von Neumann-Morgenstern expected utility functions. However, whereas the above mentioned references add several axioms regarding what is the worst network position, here we want to require in some sense the least that is needed to speak about von Neumann-Morgenstern preferences. That is, besides the axioms that define a regular preference relation, we will only consider anonymity and neutrality to ordinary risk.

It turns out that a preference relation that satisfies these axioms must be representable by a centrality measure as considered in Theorem 2. This gives our main theorem of this paper.

Theorem 3 *A preference relation $\succeq$ over network positions $N \times \mathcal{WG}^N$ satisfies anonymity and is neutral to ordinary risk if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\succeq$ can be represented by utility function $\phi(i, \omega) = f_i(\omega)$ for all $(i, \omega) \in N \times \mathcal{WG}^N$, where $f_i(\omega)$ is as given in (4).*

To prove Theorem 3, we use the following lemma.

Lemma 1 *Consider an expected utility function $\phi\colon M \to \mathbb{R}$ for positions in a network that is determined by a centrality measure f as follows: $\phi(i, \omega) = f_i(\omega)$.*

(i) *If expected utility function ϕ represents a preference relation $\succeq$ satisfying anonymity (Axiom 5), then centrality measure f satisfies anonymity (Axiom 9).*
(ii) *If expected utility function ϕ represents a preference relation $\succeq$ satisfying anonymity and neutrality to ordinary risk (Axioms 5 and 8), then f is a centrality measure that satisfies scale invariance and additivity (Axioms 10 and 11).*

Proof

(i) This is already stated in (7) and follows immediately from Axiom 5.
(ii) The proof of this part follows the same lines as the proof of Lemma 2 in (7), following (27), except that they also used the axiom of 'isolated is worst' which we do not assume here. Consequently, (i) although they also considered Cases 1 and 2 the proof below is slightly different because we cannot use 'isolated is worst', and (ii) we need to separately consider Case 3 below which cannot occur under 'isolated is worst'. (For completeness, we also add the proofs of Cases 1 and 2 although they are only slightly different from (7).)

Consider $\omega \in \mathcal{WG}^N$ and $c > 1$. Suppose that $\succeq$ satisfies anonymity and neutrality to ordinary risk. Taking $p = \frac{1}{c}$, $\omega' = \omega^0$ and considering the network $c\omega$, neutrality to ordinary risk implies that

$$\left(i, \left(\frac{1}{c}c\omega + (1 - \frac{1}{c})\omega^0\right)\right) \sim \left[\frac{1}{c}(i, c\omega); (1 - \frac{1}{c})(i, \omega^0)\right]$$

which is equivalent to

$$(i, \omega) \sim \left[\frac{1}{c}(i, c\omega); (1 - \frac{1}{c})(i, \omega^0)\right]. \tag{5}$$

Now, let u be a utility function representing a preference relation $\succeq$ satisfying Axioms 1-4 from the preliminaries. From (16), it follows that there exist $r_0, r_1 \in M$ with $r_1 \succ r_0$ such that an expected utility function ϕ over the positions in a digraph ω can be written as

$$\phi(i, \omega) = \frac{p_{ab}(i, \omega) - p_{ab}(r_0)}{p_{ab}(r_1) - p_{ab}(r_0)} \tag{6}$$

for some $a, b \in M$ with $a \succeq (i, \omega) \succeq b$ and $a \succeq r_1 \succ r_0 \succeq b$ with probabilities $p_{ab}(i, \omega)$ defined such that $(i, \omega) \sim [p_{ab}(i, \omega)a; (1 - p_{ab}(i, \omega))b]$.

In the following, we take $r_0 = (i, \omega^0)$ and take any $r_1 \succ r_0$. To show that the preference relation can be represented by a scale invariant power measure, we distinguish the following three cases with respect to $(i, \omega) \in \mathcal{WG}^N$ and $c > 1$.

Case 1: Suppose that $(i, c\omega) \succeq r_1$.
Take $a = (i, c\omega)$ and $b = r_0 = (i, \omega^0)$. Then $p_{ab}(i, c\omega) = p_{ab}(a) = 1$ and $p_{ab}(r_0) = p_{ab}(b) = 0$, and thus by (6), we have $\phi(i, c\omega) = \frac{p_{ab}(i,c\omega)}{p_{ab}(r_1)} = \frac{p_{ab}(a)}{p_{ab}(r_1)} = \frac{1}{p_{ab}(r_1)}$.
By (5), we have $(i, \omega) \sim [\frac{1}{c}(i, c\omega); (1 - \frac{1}{c})(i, \omega^0)]$, so $p_{ab}(i, \omega) = \frac{1}{c}$. But then $f_i(\omega) = \phi(i, \omega) = \frac{p_{ab}(i,\omega)}{p_{ab}(r_1)} = \frac{1}{c} \cdot \frac{1}{p_{ab}(r_1)} = \frac{1}{c}\phi(i, c\omega) = \frac{1}{c}f_i(c\omega)$. So, scale invariance is satisfied in this case.

Case 2: Suppose that $r_1 \succeq (i, c\omega) \succeq r_0$.
Take $a = r_1$ and again $b = r_0 = (i, \omega^0)$. Then $p_{ab}(r_1) = p_{ab}(a) = 1$ and $p_{ab}(r_0) = p_{ab}(b) = 0$, and so $\phi(i, c\omega) = p_{ab}(i, c\omega)$. By (5), we have $(i, \omega) \sim [\frac{1}{c}(i, c\omega); (1 - \frac{1}{c})(i, \omega^0)] \sim [\frac{1}{c}[p_{ab}(i, c\omega)a; (1 - p_{ab}(i, c\omega))b]; (1 - \frac{1}{c})(i, \omega^0)] = [\frac{1}{c}p_{ab}(i, c\omega)a; (1 - \frac{1}{c}p_{ab}(i, c\omega))b]$, where the second equivalence follows from the definition of p_{ab} and the equality follows from the third equality in (2) and the fact that we took $b = (i, \omega^0)$. So, $p_{ab}(i, \omega) = \frac{1}{c}p_{ab}(i, c\omega)$. Then, by (6), we have $\phi(i, \omega) = p_{ab}(i, \omega) = \frac{1}{c}p_{ab}(i, c\omega)$, and thus $f_i(\omega) = \phi(i, \omega) = \frac{1}{c}p_{ab}(i, c\omega) = \frac{1}{c}\phi(i, c\omega) = \frac{1}{c}f_i(c\omega)$. So, scale invariance is also satisfied in this case.

Case 3: Suppose that $r_0 \succeq (i, c\omega)$. (As mentioned, this case does not occur in (7) since they assume 'isolated is worst'.)
Take $a = r_1$ and $b = (i, c\omega)$. Then $p_{ab}(r_1) = p_{ab}(a) = 1$ and $p_{ab}(i, c\omega) = p_{ab}(b) = 0$, and so $\phi(i, c\omega) = \frac{0 - p_{ab}(r_0)}{1 - p_{ab}(r_0)} = -\frac{p_{ab}(r_0)}{1 - p_{ab}(r_0)}$ where, as in all cases, $r_0 = (i, \omega^0)$. Then, we have $(i, \omega) \sim [\frac{1}{c}(i, c\omega); (1 - \frac{1}{c})(i, \omega^0)] = [\frac{1}{c}b; (1 - \frac{1}{c})(i, \omega^0)] \sim [\frac{1}{c}b; (1 - \frac{1}{c})[p_{ab}(i, \omega^0)a; (1 - p_{ab}(i, \omega^0))b]] \sim [(1 - \frac{1}{c})p_{ab}(i, \omega^0)a; (\frac{1}{c} + (1 - \frac{1}{c})(1 - p_{ab}(i, \omega^0)))b]$, where the first equivalence follows from (5), the equality follows from the assumption that $b = (i, c\omega)$, the second equivalence

follows by definition of p_{ab}, and the third equivalence follows from the third equality in (2). Thus, $p_{ab}(i,\omega) = (1-\frac{1}{c})p_{ab}(i,\omega^0)$. By (5), we have $\phi(i,\omega) = \frac{p_{ab}(i,\omega)-p_{ab}(i,\omega^0)}{1-p_{ab}(i,\omega^0)} = \frac{(1-\frac{1}{c})p_{ab}(i,\omega^0)-p_{ab}(i,\omega^0)}{1-p_{ab}(i,\omega^0)} = -\frac{1}{c}\cdot\frac{p_{ab}(i,\omega^0)}{1-p_{ab}(i,\omega^0)} = -\frac{1}{c}\cdot\frac{p_{ab}(r_0)}{1-p_{ab}(r_0)} = \frac{1}{c}\phi(i,c\omega)$. So, scale invariance is also satisfied in this case.

The three cases together show that the utility function satisfies scale invariance.

To prove that the preference relation can be represented by an additive power measure, consider any $\omega,\omega' \in \mathcal{WG}^N$. Note that for every $i \in N$, neutrality to ordinary risk implies that $(i,\frac{1}{2}\omega+\frac{1}{2}\omega') \sim [\frac{1}{2}(i,\omega);\frac{1}{2}(i,\omega')]$ and thus $\phi(i,\frac{1}{2}\omega+\frac{1}{2}\omega') = \phi([\frac{1}{2}(i,\omega);\frac{1}{2}(i,\omega')]) = \frac{1}{2}\phi(i,\omega)+\frac{1}{2}\phi(i,\omega')$, where the first equality follows from neutrality to ordinary risk and the second from (3). But then $f_i(\omega+\omega') = f_i(2(\frac{1}{2}\omega+\frac{1}{2}\omega')) = 2f_i(\frac{1}{2}\omega+\frac{1}{2}\omega') = 2\phi(i,\frac{1}{2}\omega+\frac{1}{2}\omega') = 2(\frac{1}{2}\phi(i,\omega)+\frac{1}{2}\phi(i,\omega')) = \phi(i,\omega)+\phi(i,\omega') = f_i(\omega)+f_i(\omega')$, where the second equality follows from scale invariance of f. So, additivity is satisfied. $\square$

Proof of Theorem 3

The 'only if' part follows immediately from Lemma 1 and Theorem 2.
To prove the 'if' part, let $\succeq$ be the preference relation based on $\phi(i,\omega) = f_i(\omega)$ as given by (4), i.e. there exist $\alpha,\beta \in \mathbb{R}$ such that $(i,\omega) \succeq (j,\omega')$ if and only if $(\alpha-\beta)d_i(\omega) + \frac{\beta}{2}\sum_{h\in N} d_h(\omega) \geq (\alpha-\beta)d_j(\omega') + \frac{\beta}{2}\sum_{h\in N} d_h(\omega')$. It is straightforward to check that $\succeq$ satisfies anonymity (Axiom 5). To prove neutrality to ordinary risk, consider $\omega,\omega' \in \mathcal{WG}^N$ and $i \in N$. Then, for $p \in [0,1]$ we have $\phi(i,p\omega+(1-p)\omega') = (\alpha-\beta)d_i(p\omega+(1-p)\omega') + \frac{\beta}{2}\sum_{h\in N} d_h(p\omega+(1-p)\omega') = (\alpha-\beta)d_i(p\omega) + (\alpha-\beta)d_i((1-p)\omega') + \frac{\beta}{2}\sum_{h\in N} d_h(p\omega) + \frac{\beta}{2}\sum_{h\in N} d_h((1-p)\omega') = p(\alpha-\beta)d_i(\omega) + p\frac{\beta}{2}\sum_{h\in N} d_h(\omega) + (1-p)(\alpha-\beta)d_i(\omega') + (1-p)\frac{\beta}{2}\sum_{h\in N} d_h(\omega') = p\phi(i,\omega) + (1-p)\phi(i,\omega') = \phi([p(i,\omega);(1-p)(i,\omega')])$, where the last equality follows from (3). $\square$

Next, we add additional axioms on the preferences, to specify the range for the parameters α and β in these utility functions, specifically when they are negative or positive.

As discussed before, a major question is if there are externalities of connections between other positions. Other positions being better connected can have positive as well as negative externalities. Positive externalities can arise from the fact that if you are connected with a better connected network, some measures of centrality (such as closeness) will improve. On the other hand, the closeness of other positions also increase, and therefore your relative closeness might decrease, and this might give a negative externality on your utility. For betweenness, it is not clear beforehand if other positions being better connected increase or decrease your betweenness.

In any case, it seems that if you are isolated, and other positions get better connected by having more links, there is no positive externality for the isolated position, but negative externalities might be possible. Therefore, we impose the following axiom which requires that, if you are isolated, you weakly prefer the other positions to be less connected in the sense that you prefer their links to have less weight.

Axiom 13 (Negative externalities for isolated positions) *For every pair $\omega,\omega' \in \mathcal{WG}^N_i$, i.e. i is isolated in ω and ω', satisfying $\omega(h,j) \leq \omega'(h,j)$ for all (h,j) with $i \notin \{h,j\}$, it holds that $(i,\omega) \succeq (i,\omega')$.*

Additional to the other axioms, this requires β to be nonpositive, expressing the negative externalities.

Proposition 2 *A preference relation $\succeq$ over network positions $N \times \mathcal{WG}^N$ satisfies anonymity, neutrality to ordinary risk and negative externalities for isolated positions if and only if there exist $\alpha, \beta \in \mathbb{R}$ with $\beta \leq 0$ such that $\succeq$ can be represented by utility function $\phi(i,\omega) = f_i(\omega)$ for all $(i,\omega) \in N \times \mathcal{WG}^N$, where $f_i(\omega)$ is as given in (4).*

Proof

A preference relation that can be represented by a utility function as given in (4) with $\beta \leq 0$ satisfying anonymity and neutrality to ordinary risk follows from Theorem 3. It is straightforward to verify that a preference relation that can be represented by a utility function as given in (4) with $\beta \leq 0$ satisfies negative externalities for isolated positions. Now, suppose that preference relation $\succeq$ on $N \times \mathcal{WG}^N$ satisfies the axioms. From Theorem 3 we know that a preference relation that satisfies anonymity and neutrality to ordinary risk can be represented by a utility function as given by (4). Negative externalities for isolated positions implies that for $i \notin \{h,j\}$, it holds that $f_i(\omega^0) \geq f_i(\omega^{hj})$, where $\omega^{hj} \in \mathcal{WG}^N_{hj}$ is given by $\omega^{hj}(h,j) = 1$, and $\omega^{hj}(g,k) = 0$ for all $(g,k) \neq (h,j)$. But then $f_i(\omega^0) = 0 \geq f_i(\omega^{hj}) = \frac{\beta}{2} \cdot 2$, and thus $\beta \leq 0$. $\square$

Notice, that we can similarly use the weaker (negative) externality axiom which requires that you prefer to be in the empty network then to be in the simple network where you are isolated, but all other positions are fully connected to each other. Let $\omega^{-i} \in \mathcal{WG}^N$ be given by $\omega^{-i}(i,j) = 0$ for all $j \in N \setminus \{i\}$, and $\omega^{-i}(h,j) = 1$ for all $h,j \in N \setminus \{i\}$.

Axiom 14 (Negative externalities for isolated positions in complete subnetworks) *For every $i \in N$ and simple network ω, it holds that $(i,\omega) \succeq (i,\omega^{-i})$.*

Proposition 3 *A preference relation $\succeq$ over network positions $N \times \mathcal{WG}^N$ satisfies anonymity, neutrality to ordinary risk and negative externalities for isolated positions in complete subnetworks if and only if there exist $\alpha, \beta \in \mathbb{R}$ with $\beta \leq 0$ such that $\succeq$ can be represented by utility function $\phi(i,\omega) = f_i(\omega)$ for all $(i,\omega) \in N \times \mathcal{WG}^N$, where $f_i(\omega)$ is as given in (4).*

The proof is similar to that of Proposition 2, and is therefore omitted.

Analogously, we can have positive externalities for isolated positions (in complete subnetworks) to conclude that $\beta \geq 0$.

Instead of looking at externalities with respect to other position's relations, we can also look at the effect of a position's own relations. For example, we could consider the axiom that requires that getting more relatives is always preferred.

Axiom 15 (Monotonicity) *For every pair $\omega, \omega' \in \mathcal{WG}^N$ with $\omega(i,j) \geq \omega'(i,j)$, and $\omega(h,j) = \omega'(h,j)$ for all (h,j) with $i \notin \{h,j\}$, it holds that $(i,\omega) \succeq (i,\omega')$.*

Additional to the other axioms, this requires α to be nonnegative.

Proposition 4 *A preference relation $\succeq$ over network positions $N \times \mathcal{WG}^N$ satisfies anonymity, neutrality to ordinary risk and monotonicity if and only if there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$ such that $\succeq$ can be represented by utility function $\phi(i,\omega) = f_i(\omega)$ for all $(i,\omega) \in N \times \mathcal{WG}^N$, where $f_i(\omega)$ is as given in (4).*

Proof

A preference relation that can be represented by a utility function as given in (4) with $\alpha \geq 0$ satisfying anonymity and neutrality to ordinary risk follows from Theorem 3.
It is straightforward to verify that a preference relation that can be represented by a utility function as given in (4) with $\alpha \geq 0$ satisfies monotonicity.
Now, suppose that preference relation $\succeq$ on $N \times \mathcal{WG}^N$ satisfies the axioms. From Theorem 3 we know that a preference relation that satisfies anonymity and neutrality to ordinary risk can be represented by a utility function as given by (4). Let $\omega, \omega' \in \mathcal{WG}^N$ be such that, for some $i, j \in N, i \neq j$, $\omega(i,j) = \omega'(i,j) + 1$, and $\omega(h,g) = \omega'(h,g)$ for all $(h,g) \neq (i,j)$. Monotonicity implies that $f_i(\omega) = (\alpha - \beta) d_i(\omega) + \frac{\beta}{2} \sum_{j \in N} d_j(\omega) = (\alpha - \beta)(d_i(\omega') + 1) + \frac{\beta}{2}(\sum_{j \in N} d_j(\omega') + 2) \geq f_i(\omega') = (\alpha - \beta) d_i(\omega') + \frac{\beta}{2} \sum_{j \in N} d_j(\omega')$, and thus $(\alpha - \beta) + 2 \cdot \frac{\beta}{2} = \alpha \geq 0$. $\square$

5 Hypergraphs, cooperative games and the Shapley value

In Section 4 we described the class of centrality measures (with externalities) that can represent regular preferences that are neutral to ordinary risk. As mentioned before, (27) characterized the Shapley value as a von Neumann-Morgenstern expected utility function for cooperative transferable utility games under the additional neutrality to strategic risk, and a *null-veto* axiom that requires that it is worst to be a null player, and it is strictly better to be a dictator than to be a null player. Since cooperative transferable utility games are equivalent to weighted hypergraphs, this also gives a foundation of the Shapley value as centrality measure for weighted hypergraphs.

A *weighted hypergraph* or *cooperative transferable utility game* (*cooperative game* for short) is a pair (N, v) where $N \subset \mathbb{N}$ is a finite set of *nodes* or *players* and $v \colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For every *hyperlink* or *coalition* $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of S. Since the set of nodes/players is fixed, we represent a hypergraph/cooperative game (N, v) by its characteristic function v.

Depending on the context, the literature uses terminology 'nodes' and 'hyperlink' if the pair (N, v) is referred to as a weighted hypergraph, and the terminology 'players' and 'coalition' if the pair (N, v) is referred to as a cooperative game. In the remainder of this section, we use the game terminology, but we remark that everything could be stated in terms of hypergraphs.

We denote by $\mathcal{G}^N$ the class of all cooperative games on N. A *payoff vector* for cooperative game v on N is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for cooperative games is a function f that assigns a payoff vector $f(v) \in \mathbb{R}^N$ to every cooperative game v on N. One of the most famous solutions for cooperative games is the *Shapley value* (31) given by

$$Sh_i(v) = \sum_{S \subseteq N : i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \left(v(S) - v(S \setminus \{i\}) \right).$$

In terms of hypergraphs, the Shapley value can be seen as a centrality measure for weighted hypergraphs. Specifically, (10) argue that any undirected network can be represented by a so-called *2-game* v, where the worth of any two player coalition is the weight of the link, and the worth of any other coalition equals the sum of the worths of all two player subcoalitions. In other words, the cooperative game v^ω associated to undirected network $\omega \in \mathcal{WG}^N$ is given by (i) $v(\{i\}) = 0$ for all $i \in N$, (ii) $v(\{i,j\}) = \omega(\{i,j\})$ for all $S = \{i,j\},\ i \neq j$, and (iii) $v^\omega(S) = \sum_{T \subseteq S, |T|=2} v(T) = \sum_{i,j \in S, i \neq j} \omega(\{i,j\})$ if $|S| \geq 3$. As (10) show, the Shapley value of the associated cooperative game v^ω assigns to every player half of its degree in ω.

Next, we want to explore what solutions for cooperative games are obtained when, besides the regularity axioms, we only assume anonymity and both risk neutrality axioms of (27). In previous sections we saw that when we consider networks, i.e. 2-games, requiring only anonymity and neutrality to ordinary risk allowed to take account of externalities, but only in the way as described by (4).

For completeness, we also state anonymity and neutrality to ordinary risk for general cooperative games. (Notice that these axioms are the same as Axioms 5 and 8 by just replacing ω and ω' by v and v'.)

Axiom 16 (Anonymity for cooperative games) *For all $v \in \mathcal{G}^N$, $i \in N$ and $\pi \in \Pi(N)$, it holds that $(i,v) \sim (\pi(i), \pi v)$.*

Axiom 17 (Neutrality to ordinary risk for cooperative games) *For all $v, v' \in \mathcal{G}^N$ and $i \in N$, it holds that $(i, pv + (1-p)v') \sim [p(i,v); (1-p)(i,v')]$.*

Next, we see what we can do when we go back to the framework of (27), and characterize a class of solutions for arbitrary cooperative games that, for the special class of network games (i.e. 2-games) gives the measures as given by (4). We do this by adding neutrality to strategic risk to the axioms of Theorem 3. First, we formally recall neutrality to strategic risk from (27). It requires that a player is indifferent between being the dictator in its own unanimity game, and being one of the unanimity players in $|S|$ times the unanimity game of coalition S. The *unanimity game* associated to coalition $T \subseteq N,\ T \neq \emptyset$, is the game (N, u_T) with characteristic function u_T given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise.

Axiom 18 (Neutrality to strategic risk) *For $S \subseteq N$ and $i \in S$, it holds that $(i, u_{\{i\}}) \sim (i, |S| u_S)$.*

Notice that this axiom only deals with players in the unanimity coalition. Since we do not require the null-veto axiom, we extend neutrality to strategic risk also for null players. Specifically, we require that also a null player is indifferent between being in a dictator game of another player, or in $|S|$ times the unanimity game of a coalition S that does not contain him/her.

Axiom 19 (Strong neutrality to strategic risk) *For $S \subseteq N$ and $i,j \in N$, it holds that (i) $(i, u_{\{i\}}) \sim (i, |S| u_S)$ if $i \in S$, and (ii) $(i, u_{\{j\}}) \sim (i, |S| u_S)$ if $i \notin S$, $i \neq j$.*

Notice that (27)'s null-veto axiom explicitly requires that a player is indifferent between two game positions (i,v) and (i,w) if player i is a null player in both games v and w. This implies the additional part of strong neutrality to strategic risk. The following preference relation shows that the implication is not the other way around and thus strong neutrality to strategic risk is weaker than neutrality to strategic risk and the null-veto axiom together: the preference relation $(i, v) \succeq (j, w) \Leftrightarrow \mathit{Sh}_i(v) \leq \mathit{Sh}_j(w)$ satisfies strong neutrality to strategic risk but not the null-veto axiom.

The next theorem characterizes the solutions for cooperative games that satisfy all axioms mentioned before and strong neutrality to strategic risk for classes of cooperative games with at least three players. (We consider games with at least three players since two-player games are networks.)

Theorem 4 *Consider $N \subset \mathbb{N}$ with $|N| \geq 3$. A preference relation $\succeq$ over game positions $N \times \mathcal{G}^N$ satisfies anonymity, neutrality to ordinary risk and strong neutrality to strategic risk if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\succeq$ can be represented by utility function $\phi(i, v) = f_i(v)$ for all $(i, v) \in N \times \mathcal{G}^N$, where $f_i(v)$ is given by*

$$f_i(v) = 2(\alpha - \beta)\mathit{Sh}_i(v) + 2\beta \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \frac{\Delta_v(T)}{t} \text{ for all } i \in N \text{ and } \omega \in \mathcal{WG}^N, \tag{7}$$

where $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ is the so-called Harsanyi dividend ((15)) of coalition T in game v.

Proof

It is straightforward to verify that preferences that are represented by utility functions as in (7) satisfy the axioms.

Now, suppose that f is a solution that represents regular preferences that satisfy anonymity, neutrality to ordinary risk as well as strong neutrality to strategic risk.

First, consider a unanimity game u_T, $\emptyset \neq T \subseteq N$, with $t = |T|$. If $t = 2$, then by Theorem 3 and by $\mathit{Sh}_i(u_T) = \frac{1}{t}$ if $i \in T$, and $\mathit{Sh}_i(u_T) = 0$ if $i \in N \setminus T$, we know that there exist $\alpha, \beta \in \mathbb{R}$ such that these preferences are represented by utility function/solution

$$\begin{aligned} f_i(u_T) &= (\alpha - \beta)2\mathit{Sh}_i(u_T) + \beta \sum_{j \in N} \mathit{Sh}_j(u_T) = 2(\alpha - \beta)\mathit{Sh}_i(u_T) + \beta u_T(N) \\ &= \begin{cases} (\alpha - \beta) + \beta = \alpha & \text{if } i \in T \\ \beta & \text{if } i \in N \setminus T. \end{cases} \end{aligned}$$

For the dictator game $u_{\{i\}}$, $i \in N$, for any $j \in N \setminus \{i\}$, we have

$$f_i(u_{\{i\}}) = f_i(2u_{\{i,j\}}) = 2f_i(u_{\{i,j\}}) = 2\alpha,$$

where the first equality follows from strong neutrality to strategic risk, and the second equality follows from neutrality to ordinary risk implying scale invariance of the utility function. Moreover, for some $h \in N \setminus \{i, j\}$ (note that such h exists since $|N| \geq 3$),

$$f_j(u_{\{i\}}) = f_j(2u_{\{i,h\}}) = 2f_j(u_{\{i,h\}}) = 2\beta.$$

For arbitrary coalition $\emptyset \neq T \subseteq N$, strong neutrality to strategic risk implies that

$$f_i(u_T) = f_i\left(\frac{1}{t}u_{\{i\}}\right) = \frac{1}{t}f_i\left(u_{\{i\}}\right) = \frac{2\alpha}{t} \text{ if } i \in T,$$

and

$$f_j(u_T) = f_j\left(\frac{1}{t}u_{\{i\}}\right) = \frac{1}{t}f_j\left(u_i\right) = \frac{2\beta}{t} \text{ if } j \in N \setminus T,\ i \in T.$$

Since neutrality to ordinary risk implies scale invariance and additivity, we have for arbitrary $v \in \mathcal{G}^N$ that

$$\begin{aligned}
f_i(v) &= \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} f_i(\Delta_v(T)u_T)\\
&= \sum_{\substack{T\subseteq N\\ i\in T}} \Delta_v(T)f_i(u_T) + \sum_{\substack{T\subseteq N\\ i\notin T}} \Delta_v(T)f_i(u_T)\\
&= \sum_{\substack{T\subseteq N\\ i\in T}} \frac{2\alpha\Delta_v(T)}{t} + \sum_{\substack{T\subseteq N\\ i\notin T}} \frac{2\beta\Delta_v(T)}{t}\\
&= \sum_{\substack{T\subseteq N\\ i\in T}} \frac{2\alpha\Delta_v(T)}{t} + \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \frac{2\beta\Delta_v(T)}{t} - \sum_{\substack{T\subseteq N\\ i\in T}} \frac{2\beta\Delta_v(T)}{t}\\
&= \sum_{\substack{T\subseteq N\\ i\in T}} \frac{(2\alpha-2\beta)\Delta_v(T)}{t} + \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \frac{2\beta\Delta_v(T)}{t}\\
&= 2(\alpha-\beta)\sum_{\substack{T\subseteq N\\ i\in T}} \frac{\Delta_v(T)}{t} + 2\beta\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \frac{\Delta_v(T)}{t}\\
&= 2(\alpha-\beta)Sh_i(v) + 2\beta\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \frac{\Delta_v(T)}{t}.
\end{aligned}$$

□

We remark that the Harsanyi dividends $\Delta_v(T)$ can alternatively be interpreted as the unique coefficients in the expression of a cooperative game as a linear combination of unanimity games in the sense that every cooperative game v can be expressed as $v = \sum_{T\subseteq N} \Delta_v(T)u_T$ (31), (15).

Similar as neutrality to ordinary risk determines a centrality measure based on the degree for networks, ordinary and strong neutrality to strategic risk determine a solution for cooperative games that is based on the Shapley value and a term $\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \frac{\Delta_v(T)}{t}$ (the total per capita dividend) that does not depend on the players. Notice that, taking $\alpha = 1$ and $\beta = 0$ gives the Shapley value. Taking $\alpha = \beta$ implies equal payoffs for all players.

We also remark that for $t = 2$, the utility function boils down to $f_i(v) = 2(\alpha - \beta)Sh_i(v) + \beta\sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \Delta_v(T) = 2(\alpha-\beta)Sh_i(v) + \beta v(N)$ which we saw before.

6 Concluding remarks

In this paper, we first gave a utility foundation for network centrality measures as von Neumann-Morgenstern expected utility functions using as few axioms as possible. Specifically, besides usual regularity axioms, we only assumed anonymity and neutrality to ordinary risk of the preference relation. We saw that the only utility functions that can represent such a preference relation are obtained as a combination of the degree of a node and the average degree in the network. This implies that neutrality to ordinary risk allows to take account of externalities regarding the relations of other nodes, but only through the average degree. In other words, the centrality of the nodes is fully determined by the degree sequence in the network. We obtained this main result (Theorem 3) by first characterizing this class of utility functions as centrality measures for networks (Proposition 1) and using a lemma that relates properties of preference relations to properties of utility functions representing these preferences (Lemma 1). In this way, this lemma can be seen as a bridge between the economic literature on preferences and utility functions and the social network literature on centrality measures.

Second, after characterizing the class of utility functions that satisfy anonymity and neutrality to ordinary risk, we discussed some axioms that imply how the parameters in the centrality measure depend on the externalities from the relations of other nodes (Propositions 2 and 3) and ones own relations (Proposition 4).

Finally, we extended our main result to the class of all weighted hypergraphs, also known as cooperative games, obtaining a utility foundation for a class of cooperative game solutions that are a combination of the famous Shapley value and a total per capita dividend that does not depend on individual players. This also generalizes the utility foundation of the Shapley value in (27).

Other solutions for cooperative games, such as the *nucleolus* (30), the τ*-value* (32) and the proportional allocation of nonseparable contributions (PANSC) value (6) also give the degree measure on the class of 2-games, see e.g. (25) and (8). Therefore, the results in this paper can also be used to give a utility foundation for these solutions for applications of 2-games, such as queueing games (18), telecommunication games (25), and broadcasting rights games (2). Although the equal allocation of nonseparable costs value (see (20), (35)) does not give the degree for 2-games, variations such as first assigning each individual half of its separable cost and splitting the remainder equally over the players (see (6)), do give the degree for 2-games. However, since these solutions do not coincide with the Shapley value for general cooperative games, from Theorem 4 it is obvious that these other solutions cannot be interpreted as von Neumann-Morgenstern expected utility functions in the way done in this paper for arbitrary cooperative games.

We finally remark that (7) also considered directed networks. Although these are not special cases of cooperative games, a similar analysis supported linear combinations of the indegree and outdegree measures as von Neumann Morgenstern expected utility functions. A study similar as in the underlying paper for directed networks is a plan for future research.

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