TI 2023-005/V
Tinbergen Institute Discussion Paper

Borrowing constraints, housing tenure choice and buy-to-let investors: An assignment model

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Borrowing constraints, housing tenure choice and buy-to-let investors: An assignment model*

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This version: January 21, 2023 First version: October 8, 2021

Abstract

We study the effect of borrowing constraints in an assignment model of the housing market. When constraints apply symmetrically to all households, these lead to lower prices but unchanged housing consumption. When households can invest their own wealth and may differ in tastes, borrowing constraints will in general result in lower house prices and higher housing consumption for unconstrained households, while housing consumption of constrained households may fall. Binding borrowing constraints result in profitable arbitrage possibilities for buy-to-let investors. They can buy houses that are preferred by constrained households unable to finance them, and make them available as rental housing. In an equilibrium with free entry of such investors, house prices and the allocation of houses to households is the same as without borrowing constraints.

Keywords: borrowing constraints, housing tenure, arbitrage, buy-to-let investment, assignment models.

JEL classifications: R31, R21, G51.

*Comments by participants of the 15^{th} North American Meeting of the Urban Economics Association are gratefully acknowledged.

1 Introduction

To the annoyance of many would-be homeowners with medium incomes and modest wealth, binding borrowing constraints are present in many booming housing markets all over the world. Although the presence of such constraints is motivated by the desire of banks (and authorities) to mitigate mortgage default risks, which may be in the interest of households as foreclosure can be a traumatic experience, many of the constrained households would like to bid more for their dream-house than banks allow them to do and feel constrained rather than protected.

Being unable to buy a house means that one has to rent, often at a cost that is comparable to that of the mortgage payments that would otherwise have been due. Moreover, the combination of a growing supply of rental housing and rapidly increasing prices of owner-occupied housing of comparable quality could suggest that buy-to-let investors drive up house prices, thereby worsening the situation for would-be owner-occupiers. Moreover, the rents to be paid may be comparable to or even higher than the mortgage payments that would have been due in case the household would have been allowed to buy it, which may call into question the motivation of the borrowing constraints.

Concerns like these appear to be present in many booming urban housing markets and it is not easy to evaluate them. It is clear that mitigating the risk of default and foreclosure is socially desirable, but so is the presence of affordable (owner-occupied) housing. There is the impression that these two concerns are in conflict with each other when rents are comparable or even higher than user costs for similar housing. On the one hand, one may argue that buy-to-let investors come to the rescue of households facing tight borrowing constraints, by allowing them to occupy their desired houses. On the other hand there is the concern that these investors are removing the opportunities of these households to become owner-occupiers by overbidding them on the housing market and thereby driving up the prices[1].

This paper sheds light on these issues by developing an assignment model for an urban housing market in which households that differ in income compete for a given number of owner-occupied houses differing in quality[2] and price. A given stock of housing is a realistic feature of housing markets in many core urban areas and that has implications for the way the housing market functions. The model we use was originally formulated in Braid (1981) where a rental market is analyzed in which the number of houses of any given quality level is fixed. Here the focus is on owner-occupied housing, implying that the user cost takes the place of the rent.

We contrast the assignment model with an alternative model proposed by Muth (1969) that considers housing as a standard commodity that is available at a constant price per unit in arbitrary quantities for all households. By choosing a particular num-

[1]Note that owner-occupiers have the opportunity to realize wealth gains associated with increasing house prices and often gradually attain full ownership by using self-amortizing mortgage loans.

[2]A scalar measure of quality is assumed.

ber of units, households determine the quality of their house individually and instead of a continuum of interrelated markets – one for each quality level – there is essentially only one market for housing services underlying the submarkets for qualities. Muth's model is therefore easier to handle, but this is mainly due to the unrealistic assumption that housing is malleable. That is, the model assumes implicitly that the housing stock is rebuilt from scratch in every period.

Associated with this are differences in the predictions of the two models in a number of relevant cases. Perhaps the most important one is the reaction to an income shock. With malleable housing supply, all households increase their housing consumption in response to a higher income. With fixed housing supply this is impossible, but prices of existing houses increase nevertheless, implying that households have to pay more for the same housing after the shock. Clearly, the latter possibility has relevance in many urban housing markets, where the built environment is determined by history. Changes are possible, but often take much time and effort, not just because renovation or demolition followed by replacement are in themselves complicated processes, but also because of hold-up problems, nimby-behavior and preservation measures. Although the assumption of a fixed housing stock may be regarded as the opposite extreme of that of perfectly malleable housing, it is much closer to reality in urban housing markets, at least in the short or medium run.

Existing assignment models for the housing market assume that all households have identical tastes, and differ only in income levels, which is clearly unrealistic. We therefore show how the model can be generalized to situations with an arbitrary number of household groups, each having different preferences and income distributions. Members of a given group have the same preferences. We then introduce borrowing constraints into the model. The starting point of the analysis is a simple case in which there is only one group of households, all households face the same constraint and possess no wealth. We show that in this case the borrowing constraint has no impact on the allocation of households to housing – that is, each household lives in the same house as in the market without the borrowing constraint – and that user costs will be lower if the constraint is binding for some households with a lower income. In particular if the households with the lowest incomes that are participating in the market experience a binding constraint, all households in the market will realize lower user cost of housing. This implies that the borrowing constraint will benefit some or even all households and harms none of them. This conclusion contrasts with the alternative (Muth) model with malleable housing in which borrowing constraints may harm some households and benefit no one. However, a paradoxical feature of the model is that constrained households are no longer in equilibrium but have a marginal willingness to pay for housing that exceeds its market price. They are therefore willing to pay for relaxation of the constraint, even though this would imply that in the alternative equilibrium their welfare is lower.

These conclusions change if households differ in tastes, or if borrowing constraints differ among households, for instance because some have wealth that may be invested in owner-occupied housing to circumvent or soften the borrowing constraint. The reason is that in this more general situation the assignment rule may change due to the presence of borrowing constraints. That is, two households that are otherwise similar will be assigned to houses of different qualities if they face different borrowing constraints. As a consequence constrained households may now be assigned to houses of lower quality compared to the equilibrium without constraints, while unconstrained households may be assigned to housing of higher quality. The lower housing consumption of constrained households has a negative impact on their welfare. Still, it can be shown that in the more general situation with heterogeneous constraints or tastes, the borrowing constraints decrease the average housing price as long as they are binding for some households and do not increase the price for any level of housing quality. However, the welfare effects for constrained households are now ambiguous.

Finally, we show that binding borrowing constraints imply the possibility for arbitrage by buy-to-let investors. The reason is that the constrained households' willingness to pay for housing exceeds the marginal price it faces. Hence they are willing to pay more for higher quality housing than they are currently allowed to do. Buy-to-let investors can solve their problem by giving them access to the higher quality housing, albeit only as renters. These agents can buy the higher quality houses at a price that exceeds the market level and let them to (until then) constrained households at their marginal willingness to pay. With free entry of such investors and negligible intermediation costs, the market will move towards an equilibrium in which housing consumption and expenditure are the same as in the absence of binding borrowing constraints, but where all households experiencing binding constraints on the owner-occupied market have become tenants. The restrictions on housing consumption have effectively been removed, but so have the lower housing prices implied by the borrowing constraints.

Assignment models have been applied for a long time to allocation of workers over jobs, see Sattinger (1993) for a review. The first application of such a model to the housing market, as far as we know, is Braid (1981), on which the model of the present paper is based. The assignment model studied in Landvoigt et al. (2015) differs in some important details from the one used here, perhaps the most important one being that they use a single wealth constraint, where wealth includes the borrowing capacity. Other recent applications of assignment models to housing are Määttänen and Terviö (2014) and Määttänen and Terviö (2021).

2 The return of the private rental sector in the Netherlands

To illustrate the empirical relevance of the analysis, this section contains a brief discussion of some recent developments in the Dutch housing market which triggered the analysis of the present paper. However, note that buy-to-let investment occurs in many other housing markets – such as the United Kingdom – and is possibly also related there with the impact of borrowing constraints.

In the second half of the 20^{th} century the Dutch housing market was characterized by a large social housing sector and a growing share of owner-occupied housing. Since social housing was subsidized and full mortgage deductibility lowered the user costs of owner-occupiers, there was little room left for the private rental sector. As a consequence, this part of the housing market, which was the most important one until the 1950s, was continuously shrinking over time[3]. A reasonable prediction in the 1990s would have been that the provision of short term housing for expats would be its only market niche in the long run. However, things have changed since then.

In the Netherlands house prices have been almost continuously increasing since the mid-1980s and, after a dip associated with the Global Financial Crisis and the euro crisis, reached unprecedented values. Since 2000, mortgage lenders have – under pressure of the government and consumers authorities, introduced a Code of Conduct that was tightened in 2011[4]. Although the principle of the Code is 'comply or explain' its rules only allow for exceptions in specific cases since the tightening took place in 2011 and in more recent years some of its rules have been transformed into official government regulations that are strictly binding for these lenders. The most important aspect of the code of conduct is that home buyers should not be allowed to get a loan implying a mortgage payment to income ratio that threatens their possibility of realizing other necessary consumer expenses[5]. With high demand pressure and strongly increasing prices, the implied mortgage qualification constraint becomes binding for a growing share of households. As a result there emerged a concern for the position of (what are referred to as) medium income households, that is households that are not eligible for social housing, but whose borrowing capacity is not sufficient to allow them access to (a reasonable part of) the owner-occupied markets in large cities, especially Amsterdam. For such households, rental housing in the private sector is an obvious alternative.

[3]The slow disappearance of the sector despite its lack of profitability was due in large part to extensive protection of the rights of existing tenants. They could not be forced to depart, while rent control ensured the prolonged attractiveness of continuation of the contract.

[4]Rouwendal and Petrat (2022) show that the tightening of the Code of Conduct had a negative impact on Dutch house prices at the time of the euro crisis.

[5]In the Netherlands guidelines are provided by NIBUD, an institute specialized in the study of consumer expenses. These guidelines are part of the Code of Conduct for mortgage underwriting used by the banks.

It is perhaps not a coincidence that since the early 2000s the room for the private rental sector has increased. It was determined that the determination of rents that exceeded a particular threshold could be left to market forces. That is, rent control no longer referred to such housing. Moreover, tenant protection was reduced in this part of the market. Temporary contracts became common. Activities of private investors in rental housing, also from those of foreign origin, were welcomed by the national government. However, not only professional investors entered the market, many private persons possessing some wealth decided to invest in rental housing, which guarantees a stable flow of revenues and on top of that the possibility to realize a handsome indirect return via continued price increases[6].

Indeed the private rental sector in the Netherlands has shown a remarkable revival in recent years. Van der Harst and de Vries (2019) show that the turning point occurred in 2012, one year after the tightening of the Code of Conduct. In the municipality of Amsterdam, where social housing dominated the market until the 1990s and owner-occupied housing reached a share of 30% only in the 2010, the growth of the private rental sector started to reduce the share of owner-occupied housing again in the period 2017-2019.[7] To illustrate, in 2019 47% of the recent movers in Amsterdam entered a house in the private rental sector.[8] The growth continued in more recent years. In 2021 more than half of the housing moves in Amsterdam ended in a private rental house.[9]

The rapid increase in private rental housing caused concern among especially medium income would-be homebuyers that they were outbid from market segments that until then were accessible to them. Houses that switched from owner-occupied to private rental are often apartments. Moreover, these apartments are often located in the Center and South of Amsterdam, the most popular and expensive parts of the city.

In response to this concern the municipality looked for possibilities to limit the activity of buy-to-let investors. Initially, it was hard to find appropriate instruments. An obligation for the owner (first buyer) of newly constructed houses to live in them themselves was a modest start.[10] In 2021 the transfer tax on houses was changed. Since the beginning of that year, first time buyers were exempted, while other buyers who would live in the house have to pay 2% and those who do not have to pay 8%. Moreover, the Dutch parliament is considering measures to better protect the position of tenants in the private sector.

The change in attitude towards investors in private rental housing is largely driven

[6]Lankhuizen and Rouwendal (2020) report gross direct returns of 6% in the Amsterdam metropolitan area and somewhat lower values in the municipality of Amsterdam.

[7]From 32.4% in 2017 to 30.8%. See Berkers and Dignum (2020).

[8]Berkers and Dignum (2020)

[9]54.6% while private rental housing now covers 30.5% of the housing stock, a larger share than the owner-occupied sector.

[10]See https://www.amsterdam.nl/wonen-leefomgeving/vastgoedprofessionals/woningbouw-transformatie/verhuurverbod-zelfbewoningsplicht/

by the concern that they overbid medium income would-be homeowners and drive them out of the market for owner-occupied housing. A secondary concern is that some of these investors are simply private persons with some wealth in search of attractive investments in times of very low interest rates that are not professional landlords.

The public discussion about the measures that are introduced or considered is dominated by sentiments about the bad circumstances of medium income households. Initially the main sentiment was that rental housing should be made available for them, but when the private rental sector exploded there emerged a feeling that the position of would-be buyers became increasingly problematic. However, a systematic analysis of the pros and cons of the developments in the market and the policy proposals is lacking.

3 Two models of housing market allocation

In this section we develop an analysis of tenure choice on a housing market with borrowing constraints. In subsection 3.1 the general setup of te housing market will be introduced. Subsection 3.2 discusses Muth (1960)'s model of the housing market which considers housing essentially as a conventional (in microeconomic textbooks) homogeneous commodity. This model is standard in the economic literature. In subsection 3.3 we take into account that housing is durable and that its stock is given, at least in the short run. The model discussed there is essentially that of Braid (1981), although that study considers a rental market, while here the model is applied to owner-occupied housing.

3.1 The setup

We consider a market with a population of consumers who all have identical tastes over housing q and other consumption c that can be described by a utility function u:

$$u = u(q, c) \tag{1}$$

The utility function is assumed to be increasing in its two arguments, to have convex indifference curves and to be twice differentiable. Consumers maximize utility subject to a budget constraint. The budget will be referred to as income, but it should really be interpreted as the amount of money the consumer is willing to spend on consumption (of housing and other goods) in the period we consider.[11] Housing and other consumption are both normal goods. The consumers differ in incomes. The

[11] This budget can be thought of as being determined in an intertemporal utility maximizing framework, jointly with savings. See, for instance, Deaton and Muellbauer (1980). In this framework uncertainty about future house prices and the expected wealth effects of housing transactions can also be taken into account.

distribution of income is $F(y)$, which has positive support on an interval $[y^{min}, y^{max}]$. We treat income as a continuous variable, assume that F is differentiable and denote the density function as $f(y)$. The total number of households equals B, where $B = F(y^{max})$.

All consumers maximize utility subject to the budget constraint:

$$c + p(q) = y \tag{2}$$

Here $p(q)$ denotes the user cost of housing and y is the available budget. The budget will be referred to as income, but it should really be interpreted as the amojnt of mony the consumer is willing to spend on consumption (housing and other goods) in the period we consider. Think of the user cost $p(q)$ as a function of the sales price of the house. More specifically it is the product of the market value and the opportunity cost of the capital invested in the house, plus costs of maintenance and taxes minus the expected increase in value of the house. A common specification in the literature is: $p(q) = \gamma P(q) - E(\Delta P/(1+r))$, where $P(q)$ denotes the sales (transaction) price, γ reflects the various cost items (maintenance, insurance, taxes), ΔP is the (expected) change in the price of the house and r the rate of discount. In what follows, we focus on the user costs that equilibrate the market in the current period, without paying attention to its composition.[12] Note that it is not assumed that the user cost $p(q)$ (or the transaction price $P(q)$) is linear in the amount of housing. That is, the marginal price of a housing unit $\pi(q) = \partial p/\partial q$ may depend on the quantity of housing services consumed.[13]

Consumers have an outside option, for instance renting social housing or living in temporary housing. We include this in the model by assuming there exists a combination of housing consumption q^* and user cost p^* that is available for every household. Hence the reservation utility is $u^*(y) = u(q^*, y - p^*)$ and consumers will only participate in the (primary) housing market studied here if this offers them a higher utility than the outside option.

3.2 Malleable housing; Muth's model

In this subsection we disregard the durability of housing and assume that houses are supplied on a market with perfect competition according to a cost function $C(q)$. We

[12]Note that the first part of the user cost, $\gamma P(q)$, are mainly to out-of-pocket expenses like taxes, mortgage interest payments and maintenance, whereas expected price changes are not. The implication is that monetary outlays on housing can exceed the user cost when house prices are expected to increase. Borrowing restrictions, such as the mortgage qualification constraint discussed later in this paper refer to monetary expenses. Note also that there is in general not a one-to-one correspondence between users cost $p(q)$ and transaction price $P(q)$.

[13]In later subsections we will encounter situations in which the function $p(q)$ is not differentiable at some points.

focus on the case in which it is linear in the number of housing units:[14]

$$C(q) = bq \tag{3}$$

The price of a house with quality q is then equal to:

$$p(q) = bq \tag{4}$$

which implies that the budget constraint is linear in housing consumption q. Maximization of utility subject to this constraint leads to a housing demand function:

$$q = h(q, y) \tag{5}$$

We assume throughout that housing is a normal good: $\partial h/\partial y > 0$. In equilibrium all consumers are able to realize their demand for housing, which means that the distribution of housing reflects the distribution of income. More precisely: for any pair of households the one with the highest income will always consume more housing services.

Households will participate in the market if indirect utility $v(b, y) = u(h(b, y), y - bh(b, y))$ exceeds reservation utility u^*. The critical income y^c that is needed to participate in the housing market is defined implicitly by the equality: $v(b, y^c) = u^*(y^c)$. If this income level is smaller than y^{min} we take y^{min} as the critical income level. For later reference we note some other aspects of the equilibrium. All households with an income at least equal to y^c occupy a house. The household with the critical income inhabits a house of quality $q^{min} = h(b, y^c)$. Denoting the distribution of housing as $G(q)$, we must then have:

$$G(h(b, y)) = F(y) - F(y^c) \tag{6}$$

That is, housing consumption and income are aligned in the sense that the ordering of households on the basis of their consumption of regular housing is identical to their ordering in the basis of their incomes. If both distributions are differentiable and we denote their derivatives as $f(y) = \partial F/\partial y$ and $g(q) = \partial G/\partial q$, we must in equilibrium have $g(q)dq = f(y)dy$, and since all consumers are on their demand curves, this implies:

$$\partial q/\partial y = f(y)/g(q) \tag{7}$$

The left-hand side of this equation is the slope of the Engel curve for housing.

An important advantage of Muth's model is that it can easily deal with heterogeneity of tastes, borrowing constraints and buy-to-let investors as will now be discussed. The reason is that the housing stock adjusts easily to the implied changes in demand. If households differ in preferences, they can simply order the amount of housing that is suitable for them at the prevailing price. Those with more intense preferences for

[14]This implies no loss of generality. If the cost is increasing in the number of housing units produced, we can define the units so that their number is proportional to the production cost.

housing simply order a larger house. To make things concrete, if there are G groups of households $g = 1 \dots G$ and households in each group have identical tastes, but may differ in income, we can describe the market for each group in the same way as was done above. We get G market segments that have the same properties as the single one analyzed above.

Borrowing constraints impose a ceiling on housing demand. To analyze their impact we start by considering a simple situation in which housing expenditure is limited to a particular share μ of household income and there is no rental housing available. In this case, households experiencing a binding constraint must reduce their housing demand to the level that is compatible with the constraint. That is, borrowing constraints reduce housing consumption but have no impact on the price of housing. This remains true if we consider a more general model in which households may have some wealth available to invest in housing. They can use this to relax the borrowing constraint and if their wealth is large enough, avoid it completely. Again, the constraints do not have any impact on housing prices, but the average amount of housing consumption is reduced as long as the constraint is binding for some households. No household will ever increase its housing consumption because other households are experiencing binding restrictions.

However, the presence of binding borrowing constraints provides arbitrage opportunities for investors in rental housing. By offering rental housing of the qualities that are preferred by restricted households, but out of reach for them due to the borrowing constraints, they can make a profit. The reason is that the constrained households have a marginal willingness to pay for housing that exceeds the marginal production cost b. If enough investment in rental housing is realized, and the additional cost of making the housing available as rental housing is negligible, the market will return to the equilibrium without borrowing constraints as far as housing consumption and expenditure are concerned, but all households experiencing binding borrowing constraints in the owner-occupied market have shifted to the private rental sector.

3.3 Durable housing

In the previous subsection we have assumed that all houses are created in the period considered. In reality most of the housing dates back from the past and only a small percentage is constructed per period. To capture this, we will now consider the situation in which housing supply is completely fixed. That is, the housing stock – the distribution of houses, each with a given number of housing units – is given. As in the previous section, houses are available in a continuum of varieties and the distribution function of the quality of housing is still denoted $G(q)$. However, in contrast with the previous section, G is now given. It is assumed to have support on an interval $[q^{min}, q^{max}]$ and to be differentiable. The number of houses is $S, S = G(q^{max})$. We assume that the number of households is at least equal to the number of houses:

$B \geq S$. This setup is similar to Braid (1981) who considered a rental market. Here we assume the market refers to owner-occupied housing with user cost taking the place of rent.

Consumers are still assumed to be utility maximizers, but now have to compete for the available housing with other households instead of ordering housing construction. In this setting there is no reason to suppose that housing expenditure will be proportional to the number of housing units consumed, q, in equilibrium. The main purpose of this section is to find the equilibrium user cost function $p(q)$ on this market.

It is easy to see that $p(q)$ must be increasing and continuous in q.[15] That is, the user cost of housing is in the present framework a possibly nonlinear, but always increasing, function of the number of housing units offered by a house. Since housing is normal, a household with a higher income will in equilibrium consume more housing than a household with a lower income.[16] These two observations imply a simple assignment rule: in equilibrium the ranking of households on the basis of income corresponds to the ranking of households on the basis of housing consumption. That is, an equation like (6) is still valid in the market with durable housing. The assignment rule also implies that the households with incomes below the critical value y^c will choose the outside option. The critical income y^c is now determined by the condition that only S households can own a house:

$$B - F(y^c) = S \tag{8}$$

Using this, we can formalize the assignment rule as:

$$y(q) = y^c + F^{-1}(G(q)) \tag{9}$$

The equilibrium price function can be derived as follows. The household with the critical income has to be indifferent between housing of the lowest quality and the outside option. Hence:

$$u(q^{min}, y^c - p(q^{min}) = u^*(y^c) \tag{10}$$

This pins down the value $p(q^{min})$ of, the user cost of housing of the lowest quality. The value of the price at higher qualities is determined by the requirement that the slope of the price function, the marginal price of housing $\pi(q) = \partial p/\partial q$, must be equal to the marginal rate of substitution $M(q, c) = (\partial u/\partial q)/(\partial u/\partial c)$:

$$\pi(q) = M(q, y(q) - p(q)) \tag{11}$$

[15]Suppose a house with a better quality is less expensive than that of a lower quality. Then there will be no household choosing the lower quality house. Suppose there is a discontinuity in the house price function. Then the marginal price of housing is infinitely high at the point of the discontinuity, which means that there will be no demand for housing with quality just above that at the point of discontinuity.

[16]Consider a pair of households with different incomes and assume that the one with the highest income consumes less housing than the one with the lowest income. Then the high income household is able to compensate the lower income household for moving to the lower quality house and still reach a higher utility. Hence they can engage in a transaction that is beneficial to both, which shows that that initial situation is incompatible with equilibrium. See the Appendix for details.

The user cost at q^{min} and y^c is thus determined by (10) and this provides a starting point for deriving the housing price function. Equilibrium condition (11) pins down the slope of the housing price function and the assignment rule ((9)) implies:

$$f(y(q))dy = g(q)dq \tag{12}$$

This gives us the change in income dy/dq. We can trace out the housing price function by making use of the equilibrium condition and the assignment rule. That is, starting from the critical income y^c, the lowest housing quality q^{min}, and its price $p(q^{min})$, (11) determines the income associated with each housing quality and then (11) determines the equilibrium price for each housing quality. Since the marginal rate of substitution and the densities f and g are continuous in their arguments, the equilibrium price function is differentiable.

Let us now consider the special case in which the equilibrium price function is linear, as in Muth's model. This happens if the marginal price of housing is constant: $\partial\pi(q)/\partial q = 0$. It requires the marginal willingness to pay for housing to be constant as well. It can be shown (see Appendix A.2) that this implies that we must have:

$$\frac{\partial M/\partial q}{\partial M/\partial(y-p)} = \pi - \frac{g(q)}{f(y)} \tag{13}$$

for all possible combinations of q and $y-p(q)$. This requires that the distributions of income and housing are aligned to the marginal willingness to pay in a very specific way. In the model with malleable housing $g(q)$ adjusts so that (13) is always satisfied. However, with $g(q)$ fixed, we lose this flexibility.[17]

One way of interpreting this is that with a given stock of housing there needs to be equality of supply and demand for every housing quality. That is, instead of a single market for housing services that can be used to construct any desired quality of housing, we now have a continuum of markets for housing with given quality.[18] Equilibrium then requires a separate price for each submarket. This is the reason that we cannot expect the price function $p(q)$ to be linear when the housing stock is given.[19]

3.4 Effects of an income increase compared

To appreciate the difference between Muth's model and the assignment model, it is useful to briefly discuss some comparative statics. What happens if all incomes increase by the same percentage so that the ranking of the households on the basis of

[17]For instance, with Cobb-Douglas utility $u(q,c) = q^\alpha c^\beta$ the left-hand side of (13) is equal to $(y-p(q))/q$ and one may choose $p(q)$ for every q so that (13) is satisfied, but the first derivative of this price function will only by coincidence be equal to the constant π, implying a contradiction.

[18]Compare the plural in the title of Landvoigt et al. (2015).

[19]Our assumptions imply that the second order condition for utility maximization is satisfied with a linear budget constraint. With a nonlinear housing price function the budget constraint is also nonlinear. Appendix A3 shows that the second-order condition remains satisfied.

their income remains unchanged? In Muth's housing market all households increase their housing consumption, while the price per unit of housing remains unchanged. Moreover, some households formerly choosing the outside option will now order an owner-occupied house, thereby increasing the size of the housing stock. Now consider the assignment model. Here the size of the housing stock remains unchanged. Since the critical income increases, the price of housing of the lowest quality increases.[20] All households occupying a house in the given stock in the initial situation want to increase their housing consumption and bid up the price of housing. However, since the stock of housing is given and their position in the income distribution did not change, they end up in the same house, while paying a higher user cost. The reason is that housing of minimum quality has become more expensive, while on top of that the marginal willingness to pay for housing has increased for every value of housing consumption. Because of the latter effect, the price (user cost) increase will itself be increasing in q. This contrasts sharply in the market with malleable housing, but it seems to describe rather well what happens in many urban housing markets when household incomes increase.

A population with identical tastes is clearly restrictive. In the next section we discuss the extension of the model to situations with households differing in tastes.

4 Taste differences

The assumption that all households have the same tastes appears unrealistic and we will therefore in this section consider what changes in the model if we allow heterogeneity in tastes. More specifically, we assume that there are $n \geq 2$ groups of households. Within each group all households have the same tastes, but their incomes differ. We use a superfix $i = 1...n$ to refer to groups. Incomes belong to group-specific intervals $[y^{(i,min)}, y^{(i,max)}]$ and the income distributions $F^i(y^i)$ are differentiable and increasing on that interval. The total number of households in all groups exceeds the number of houses. Moreover, the housing distribution $G(q)$ is differentiable and increasing on that same interval.

To provide intuition for the results that follow, it is helpful to recall the familiar Rosen (1974) analysis in which households that differ in tastes or incomes choose an equilibrium while taking a hedonic price function as given. Different households in general choose different positions. Equilibrium requires that all varieties present in the market will be chosen by some consumers and that for each variety there is equilibrium between supply and demand. The situation considered here is a simplified one in which there is only one continuous characteristic of the heterogeneous good while the supply of each variety is given. It seems plausible, and is confirmed by the analysis that follows, that for each group of households the equilibrium will be

[20] The reason is that the normality of housing implies that the willingness to pay for an increase in housing consumption from q^* to q^{min} increases.

similar to that studied in the previous section: housing consumption and expenditure are increasing in income and the marginal willingness to pay for housing is equal to its marginal price. The main issue that arises is that the curvature of the one-dimensional hedonic price function may be such that the implied budget constraint is more convex that the indifference curve for one or more groups of households. If this happens, there are 'holes' in the housing consumption of such groups in the sense that one or more housing quality intervals are skipped by the members of such a group. That is, housing consumption of the households in this group is not everywhere continuous in income, but may have 'jumps' at a limited number of incomes.

To provide a formal analysis, start by noting that in market equilibrium the housing price function $p(q)$ must still be increasing in q and that in all groups households with higher incomes consume more housing. The reasons are identical to the case with homogeneous households. This implies that for each group i households with incomes below a critical income level $y^{(i,c)}$ will choose the outside option, while housing consumption is strictly increasing in income for those realizing at least this critical income level. The critical income level may be identical to the minimum income in the group.

An equilibrium requires the existence of a housing price function $p(q)$ defined on the interval $[q^{min}, q^{max}]$ that is taken as given by all households. Moreover, all households maximize their utility subject to the budget constraint (2) that incorporates this price function. To have all houses filled, every q in the relevant interval must be the optimal choice for households in at least one group. That is, there must be at least one combination (i, y) with $y \in [y^{(i,min)}, y^{(i,max)}]$ for which q is the optimal choice. To have all households allocated, there must be intervals of housing qualities $[q^{(i,j,min)}, q^{(i,j,max)}]$ with $q^{(i,j,min)} \geq q^{min}$ and $q^{max} \leq q^{max}, j = 1...J(i)$, $J(i) \geq 1$, and $q^{(i,j,max)} < q^{(i,j+1,min)}, j = 1...J(i) - 1$ if $J(i) \geq 2$, such that households of type i choose houses of every quality q inside these intervals. Every $q \in [q^{min}, q^{max}]$ must belong to at least one such interval, possibly at its border.

The price function must be differentiable on every interval $[q^{(i,j,min)}, q^{(i,j,max)}]$ by the argument used in the analysis of the market with homogeneous households. On such intervals the marginal price must always be equal to the marginal willingness to pay of the households of the group to which this interval refers. The assumed properties of the utility function then ensure that the price function will be differentiable on such intervals. The equilibrium price function will hence be differentiable almost everywhere, that is except on a set of measure 0.

It is important to observe that not only the necessary first-order condition has to be satisfied, but also the sufficient second-order condition. The second-order condition requires that the budget line is less convex than the indifference curve at the point where both are tangent. Formally, the condition, which is discussed in Appendix A.3, requires:[21]

$$\frac{d\pi}{dq} \geq [\frac{\partial M^i}{\partial q} - \pi \frac{\partial M^i}{\partial (y-p)}] \tag{14}$$

If the first-order condition is satisfied, but the second-order condition fails, households express no demand for housing of quality q but they may demand housing of lower or higher quality. This is the reason why households of a particular group may demand no housing in specific quality intervals.

The assignment of households to housing is described by a generalization of (12):

$$g(q)dq = \sum_i \delta^i(q) f^i(y) dy^i \tag{15}$$

In this equation $\delta^i(q)$ is a 0-1 variable indicating that households of type i chose houses with quality q. Hence $\delta^i(q) = 1$ if the first-order condition is satisfied for an income $y \varepsilon [y^{i,min}, y^{i,max}]$ for which the second-order condtion is also satisfied.

Since there may be households of more than one group demanding housing of quality q, we cannot use this equation to determine the relationship between changes in housing consumption and income of specific groups. However, we can find an alternative way of doing so by considering the change in the first-order condition (11) that occurs if we move to a slightly higher housing quality. If households of group i continue to express demand at this higher quality, $d\pi/dq = dM^i/dq$ if we move along the housing price function. Elaboration of this condition gives:

$$\frac{d\pi}{dq} = [\frac{\partial M^i}{\partial q} - \pi \frac{\partial M^i}{\partial (y-p)}] + \frac{\partial M^i}{\partial (y-p)} \frac{dy^i}{dq} \tag{16}$$

This equation implies that $dy^i/dq > 0$ if the second-order condition is satisfied. (16) gives a relationship between the change in the marginal price of housing and the change in the incomes of households of group i assigned to the housing concerned. We can solve this equation for dy^i/dq and substitute the result into (15). This gives us an expression for the change in the marginal price of housing:

$$\frac{d\pi}{dq} = \frac{\sum_i \delta^i(q) f^i(y^i) (\frac{\partial M^i}{\partial (y^i - p)})^{-1} [\frac{\partial M^i}{\partial q} - \pi \frac{\partial M^i}{\partial (y^i - p)}] + g(q)}{\sum_i \delta^i(q) f^i(y^i) (\frac{\partial M^i}{\partial (y^i - p)})^{-1}} \tag{17}$$

Equations (16) and (17) give us the possibility to trace out the housing price function and the allocation of households from a given starting point. To find a suitable starting point, consider an arbitrary price $p(q^{min})$ for housing of minimum quality and check if there are incomes of households in the various groups for which indifference between the outside option and the housing of minimum quality obtains at the chosen price, that is if (11) is satisfied. This is not necessarily the case. The price may be too high for a specific group in the sense that even for households with

[21] If the q is at the border of the interval for which group i expresses housing demand, $d\pi/dq$ should be interpreted as the limit of $d\pi/dq$ when housing consumption approaches the border from inside the interval.

the maximum income the outside option is preferred. The price may also be too low in the sense that even the household with the minimum income prefers the housing of minimal quality to the outside alternative. If there is such a critical income at which the indifference occurs, then all households in de group with a higher income will also disregard the outside option in equilibrium. This means that we can determine a minimum number of households that will participate in the market for any chosen price $p(q^{min})$: the sum of all households with at least the critical income over all groups. This minimum demand is a continuous function of the price $p(q^{min})$. If we assume that all households prefer housing of minimum quality to the outside option if the price is sufficiently low, minimum demand can take on every value between 0 and the total number of households.

Now suppose that we choose the price $p^*(q^{min})$ so, that the minimum number of participants in the market equals the number of available houses. The marginal willingness to pay for housing at q^{min} will in general differ between the groups. Allocate the houses of minimum quality to the group with the lowest marginal willingness to pay and start tracing out the housing price function as in the previous section.[22] At each value of q, check if there are households belonging to a different group for whom the first- and second-order conditions for a utility maximum are satisfied. If so switch to the allocation process associated with (14)-(16) outlined above.

This algorithm will lead to an equilibrium if the household groups that start participating in the market at higher quality do so at the income levels at which they are indifferent between the outside option and housing of minimum quality at the price $p(q^{min})$. However, this will not necessarily be the case. We should expect the first households of such groups to start participating at a lower income. The reason is that the marginal willingness to pay for housing of households of this second group who are indifferent between housing of minimum quality and the outside option is higher than the marginal price of housing. This means that it is possible that the actual demand for housing at the chosen price exceeds the number of available houses. If so, we must increase the price $p^*(q^{min})$. We repeat this process until convergence occurs.

It may be noted that two different aspects of heterogeneity are important in this analysis. One is that households realize the same marginal utility of housing at different income levels, the other is that the convexity of the indifference curve (the first derivative of the marginal willingness to pay for housing) may also be different. It is possible that the first aspect is relevant, while the second is not. This situation occurs if the indifference curves of households belonging to different groups are parallel to each other, that is if $u^i(c, q) = u(\theta^i + c, q)$. Two households belonging to groups g and g' will then choose exactly the same location on the hedonic price function if the difference between their incomes is equal to the difference between their θ s.

[22] Allocation of these households to any other group will result in bunching of the group with the lowest marginal willingness to pay at the minimum quality, which is incompatible with equilibrium.

When a particular value of housing consumption q is chosen by households of group i with income y, that same value of housing consumption will also be chosen by households of group j with income $y - \theta^i + \theta^j$. Note that the share of households of the two groups may still vary over housing qualities, depending on their income distributions. For instance, with a linear demand function for housing,

$$q = \alpha^i + \beta\pi + \gamma y^v, \tag{18}$$

differences in the constant term α^g have a similar impact on the demand for housing as differences in income. Note that the share of households of the two groups living in housing of a particular quality may still vary over housing qualities, depending on their income distributions.

5 Borrowing constraints

The next step is the introduction of borrowing constraints, We start in subsection 5.1 by considering a uniform restriction on the share of user cost in household income and generalize this in subsection 5.2 to an arbitrary distribution of maximum purchase prices or user costs that may depend on household income. Throughout this section we assume all households have the same preferences.

5.1 A mortgage qualification constraint

The analysis thus far has assumed that households are not restricted in their choice behavior, except by the budget constraint. Many households need a mortgage loan to finance the purchase of their house and mortgage payments are an important element of their user cost. Banks usually impose restrictions on the size of these loans. As discussed in the introduction, in the Netherlands the ratio of mortgage payment to income is the most important indicator used by the lenders, and we will now consider the implications of such a constraint. In this subsection we do so in a simple way: we impose that user cost can at most be equal to a fraction μ of income for all households:[23]

$$p(q) \leq \mu y. \tag{19}$$

We refer to this restriction as the mortgage qualification constraint. If it is binding, we have:

$$p(q) = \mu y \tag{20}$$

for the housing consumption q of this households. In this situation (10) will be violated except in the special case in which the mortgage qualification constraint is just binding for the household's optimal choice without such a constraint imposed.

To set the stage and get a first impression of the impact of constraint (19) on house prices, we consider the housing price as a function of income. The assignment

[23] We will later consider cases in which households can face different constraints.

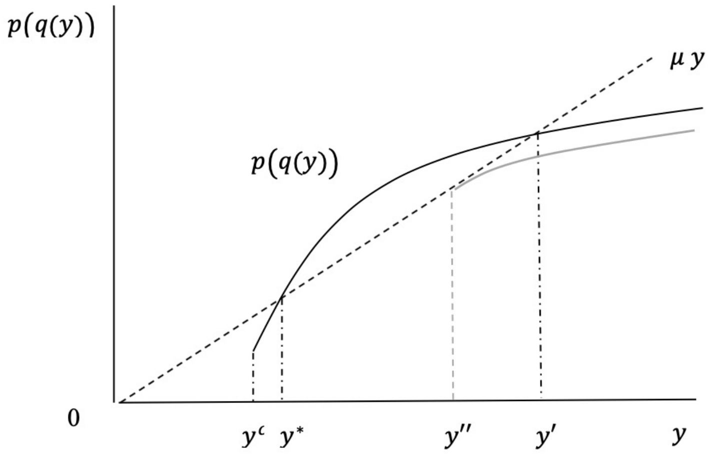

Figure 1: The mortgage qualification constraint and the price of housing

rule (9) is an increasing relationship between housing quality and income and we write its inverse as $q(y)$. Using this, we derive the user cost of housing as a function of income, $p(q(y))$. Since the equilibrium housing price is increasing in quality and, according to the assignment rule, quality is increasing in income, the user cost $p(q)$ must also be increasing in income. Its slope is equal to:

$$\frac{dp}{dy} = \pi(q)\frac{dq}{dy} = \pi(q)\frac{f(y)}{g(q(y))} \tag{21}$$

Figure 1 illustrates the relationship between user cost and income and the mortgage qualification constraint (19). The equilibrium price function in the initial situation (without borrowing constraints) is $p(q(y))$. In the situation pictured, the constraint is binding for incomes between y^* and y^{**} for such incomes the user cost implied by $p(q(y))$ is higher than μy. This suggests that after the introduction the equilibrium price function will coincide with the constraint on the interval $[y^*, y^{**})]$ while for incomes higher than y^{**} the initial price function is still valid. As will be shown now, this conjecture is not completely valid.

We will now derive the equilibrium price function in the presence of a mortgage qualification constraint. That is, we assume that initially the market is in equilibrium and there is no borrowing constraint, and consider what changes if such a constraint is imposed. We start by assuming that the assignment rule does not change because of the mortgage qualification constraint and verify the validity of this assumption later on. We will refer from now on to the price function derived in the previous section for the situation without borrowing constraints as $p^m(q)$ and to the one that is relevant with the mortgage qualification constraint present as $p^{bc}(q)$. We start by discussing the situation pictured in Figure 1, and will discuss other cases (for instance, the one in which the borrowing constraint is binding for the critical income) later on.

Since the assignment tule did not change, the minimum income of owner-occupiers is the same as in the equilibrium without a borrowing constraint. That is: $p^{bc}(q^{min}) = p^m(q^{min})$. We can now follow the same logic as in the previous subsection to show that until income y^*, where the borrowing constraint becomes binding, and the associated housing quality $q(y^*)$, the functions $p^{bc}(q)$ and $p^m(q)$ will coincide. At income y^* the borrowing constraint is satisfied as an equality. Using (21) we see that $\pi(q(y^*))\frac{f(y^*)}{g(q(y^*))} > \mu$. A household with income slightly higher than y^* would thus like to spend a larger income share than μ on housing, but is restricted. Since it can only spend the maximum share μ, $p^{bc}(q)$ and $p^m(q)$ are now diverging. The slope of $p^{bc}(q)$ is determined by the borrowing constraint and equal to:

$$\frac{\partial p^{bc}}{\partial q} = \mu\frac{g(q(y))}{f(y)} < M(q(y), y - p^{bc}(q(y))) \tag{22}$$

The equality uses the fact that the user cost as a function of income equals the borrowing constraint, which implies that its slope must be equal to μ, and (21). Because of the binding borrowing constraint, the slope of $p^{bc}(q)$ must be smaller that that of $p^m(q)$, which equals the marginal willingness to pay for housing. Hence for incomes slightly above y^* the price function is equal to the maximum user cost that is consistent with the borrowing constraint. This will be the case as long as the marginal willingness to pay for housing exceeds the slope of $p^{bc}(q)$. Households that find themselves in this situation want to consume more housing, but are unable to realize this desire. Note that for all households experiencing a binding borrowing constraint, $p^{bc}(q) < p^m(q)$. That is: due to the binding borrowing constraint they pay less for the same house they would have occupied in the unconstrained market equilibrium. Their utility will therefore be higher, although they are no longer in equilibrium since their marginal willingness to pay for housing exceeds the marginal price.

Note, however, that at income y^{**} in Figure 1, where the borrowing constraint is again just binding, the marginal willingness to pay for housing is smaller than $\mu\frac{g(q(y))}{f(y)}$. To verify this, note that the slope of $p^m(q(y'))$ equals $M(q(y), y - p^{bc}(q(y)))\frac{f(y)}{g(q(y))}$ which is smaller than μ. This tells us that the borrowing constraint will not be binding at income y^{**}. Hence there must be a lower income, say y'' for which this constraint stops being restrictive. The marginal willingness to pay for housing is a continuous function of income, so the function $p^{bc}(q(y))$ will not have a kink at y''. For incomes higher than y'' the slope of the price function $p^{bc}(q)$ is equal to the marginal willingness to pay for housing. It follows then that $p^{bc}(q) < p^m(q)$ for $q > q(y'')$ as long as the borrowing constraint does not become binding again. This is illustrated by the grey bended line that departs from the borrowing constraint at y'' in Figure 1. The households for which this part of $p^{bc}(q(y))$ is relevant pay a lower price for the same housing they would have occupied in the unconstrained market equilibrium and they are satisfied with their situation, as the marginal willingness to pay for housing equals the marginal price they face.

Summarizing, we have shown that the function $p^{bc}(q(y))$ coincides with $p^m(q(y))$ until this function hits the borrowing constraint. Then $p^{bc}(q(y))$ follows the borrowing constraint until the marginal willingness to pay for housing is so low that households prefer to spend less on housing than is allowed by this constraint. This happens at an income y'' that is lower than the income y^{**} at which $p^m(q(y))$ crosses the borrowing constraint. There is a kink in $p^{bc}(q(y))$ at income y^* but not at y''.

Let us now verify the validity of the assignment rule that is relevant in the unconstrained market equilibrium. To be valid, it must be the case that no household can reach a higher utility by deviating from this rule. For households who do not face a binding budget constraint, this is the case. Households experiencing a binding constraint would like to consume more housing, but are unable to do so and can therefore not improve their position. Hence the assignment rule of the market equilibrium is still relevant when a binding borrowing constraint is present. This completes our discussion of the case pictured in Figure 1.

Which other possibilities are there? First, it may be the case that the borrowing constraint is binding for households with income y^{min}. Then the price of housing of the minimum quality q^{min} must equal μy^{min}. Depending on value of the marginal willingness to pay and the densities of income and housing, the price function may follow the borrowing constraint in the same way as discussed above, or continue as an unconstrained function at a lower level. Second, it may be the case that the borrowing constraint remains binding until income y^{max} is reached. This doesn't bring up any new issues. Third, as hinted at above, it may be the case that the borrowing constraint is binding for a particular income interval, is non-binding for an ensuing interval, and then becomes binding again. The analysis for the second binding interval is entirely similar to that of the first one.

What happens if the mortgage qualification constraint is relaxed? Consider again the situation pictured in Figure 1. Households for whom the constraint is no longer binding will attempt to increase their housing consumption until the constraint binds again, or until they are on their housing demand function. However, if housing supply does not adjust, all households will stay in the same house, which will have become more expensive. Households for whom the constraint is no longer binding will attempt to increase their housing consumption until the constraint binds again, or until they are on their housing demand function. However, if housing supply does not adjust, all households will stay in the same house, which will have become more expensive. If in the new situation the constraint is no longer binding for any household if the initial equilibrium price function $p^m(q(y))$ would be valid, the market (re)turns to that equilibrium. If some households are still constrained, then there will in the new equilibrium be a smaller interval for which the constraint is binding. Prices will increase for all households with an income higher than y^*. The welfare of all these household will decrease, since their housing consumption and income do not change. However, the number of constrained households (for whom the borrowing constraints is binding) will be smaller than with the tighter constraint.

Note that the results of this section are sensitive to the assumption that all households have to borrow all the money needed for purchasing their houses. If some households with a given income experience a binding credit constraint while others own some wealth and are willing to invest it in their houses, the latter group may not experience a binding credit constraint while the former group does. In such a case the allocation of households over the housing stock will be affected by the borrowing constraint, as will be discussed below.

5.2 General borrowing constraints

In this subsection we consider a more general situation in which households can experience borrowing constraints of a general nature. The borrowing constraint is a household-specific maximum imposed on the purchase price of a house. Since there is a one-to-one relationship between purchase price and user cost,[24] we include this in our model as a maximum user cost ρ. We assume that for each household $\rho \in [\rho^{min}, \rho^{max}]$ for some $\rho^{min} \geq 0$ and $\rho^{max} \leq \infty$. For instance, ρ^{min} may be determined by the mortgage qualification constraint discussed in the previous section for a household with minimum income y^{min} and ρ^{max} similarly for a household with the maximum income. Households are characterized by their income and maximum user cost, an ordered pair (y, ρ). We denote the simultaneous distribution of income and maximum user cost as $F(y, \rho)$, that is: $F(y, \rho)$ is the number of households with income at most equal to y who can bid at most ρ for a house. The corresponding density is $f(y, \rho)$. In the previous subsection we discussed a case in which ρ is a function of income, $\rho = \rho(y)$. This is a special case of the situation considered here, in which the density $f(y, \rho)$ is only positive for $\rho = \rho(y)$. The discussion that follows considers a different and much more general case in which $f(y, \rho)$ is a continuous function of its two arguments. Such a situation is compatible with the mortgage constraint discussed in the previous section if households may have wealth that can be used, possibly in addition to a mortgage loan to finance a house. If the simultaneous distribution of wealth and income has a density that is continuous than this will also be the case for $f(y, \rho)$.

The ρ that is relevant for a particular household should now be interpreted as the user cost that the household can afford. One relevant situation is that in which the mortgage qualification constraint of the previous section is valid. Denoting the user cost permitted by the mortgage loan now as $\rho^m (= \rho^m(y)) = \mu y$, for households without any wealth the relevant constraint is still $\rho < \mu y$. If the household has wealth that can be used to help finance the house, there is a second part of the user cost, to be denoted ρ^w which is a function of the household's wealth w. The constraint is now that the actual user cost p is at most equal to the sum $\rho = \rho^m + \rho^w$.

For simplicity, one may assume that user cost is proportional to the purchase price of a house, with the constant of proportionality equal to the mortgage interest rate, which is equal for all households. If it is further assumed that the opportunity

[24] The household-specific maximum is conditional on the mortgage underwriting rules and the mortgage interest rate.

coast of wealth is equal to this interest rate there is a conveniently simple relationship between the purchase price of a house and its user cost. The discussion below refers to this simplified case. However, we note that the analysis is also relevant for other cases. For instance, if there is a down-payment constraint instead of a mortgage qualification constraint, the household must have enough wealth to pay a share σ that equals one minus the maximum loan-to-value ratio. Using the assumptions of the previous paragraph, the maximum user cost may be determined as follows. The down-payment constraint is: $\sigma P < w$, where P denotes the purchase price. Multiplication of both sides of the inequality by the mortgage interest rate and dividing by σ gives: $rP < rw/\sigma$. The left-hand side of this inequality is the user cost. The right hand side gives the maximum ρ of the user cost in this situation, which is now independent of income. This shows that the model discussed below is as relevant for situations with a down-payment constraint as with the mortgage qualification constraint of the previous section.

The supply side of the market is unchanged. The number of households assigned to a house must therefore be equal to the number of houses that is available. These households must have an income that is at least is as high as the critical level at which housing of the lowest quality is consumed and a maximum user cost that is larger than that of the lowest quality housing. For the household with the critical income level y^c, (10) must again be valid. Moreover, we must have:

$$\rho \geq p(q^{min}) \tag{23}$$

and:

$$F(y^{max}, \rho^{max}) - F(y^c, p(q^{min})) = S. \tag{24}$$

The value of $p(q^{min})$ is determined in the same way as before, viz. by the condition that a household with the critical income must be indifferent between the housing of minimum quality and the outside option. This means that $p(q^{min})$ is an increasing function of the critical income y^c. It follows then that the left-hand side of (24) is an increasing function of income and that y^c is uniquely determined.

There are potentially two groups of households demanding housing of minimum quality: (i) those with a maximum user cost $\rho = p(q^{min})$ and (ii) those with income $y = y^c$. The first type will always demand this housing, but the second type will only do so if (23) also holds true. To trace out the user cost function we consider what happens at a combination of income y, housing units q and an associated user cost $p(q)$. The idea is that all households with an income lower than y or a maximum user cost lower than $p(q)$ either have been assigned a house, or will not participate in the housing market. The supply of housing of quality q is $g(q)dq$ and this must be equal to the demand. Demand originates both from households experiencing borrowing constraints and from those who do not. We write:

$$g(q)dq = f^{bc}(y, p(q))dp + f^{uc}(y, p(q))dy \tag{25}$$

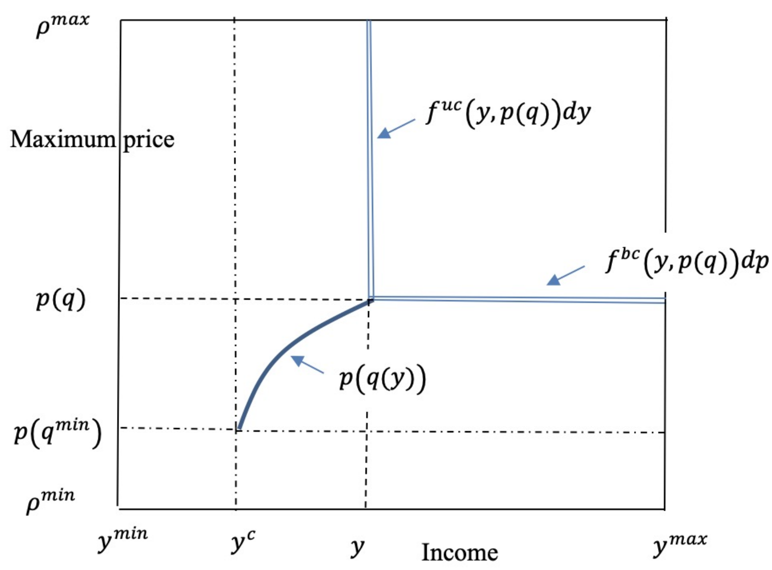

Figure 2: Constrained and unconstrained housing demand

In this equation, $f^{bc}(y, p(q))$ is the density of households not having been assigned a house who are constrained at user cost $p(q)$:

$$f^{bc}(y, p(q)) = \int_{y}^{y^{max}} f(y, p(q))dy \tag{26}$$

while $f^{uc}(y, p(q))$ is the density of unconstrained households choosing a house with q units:

$$f^{uc}(y, p(q)) = \int_{p(q)}^{\rho^{max}} f(y, \rho)d\rho. \tag{27}$$

Figure 2 illustrates. The box indicates the combinations of income $y \in [y^{min}, y^{max}]$ and maximum prices $\rho \in [\rho^{min}, \rho^{max}]$ for which the distribution $F(y, \rho)$ has positive support. The housing price is given as a function of income. It starts at the critical income and is shown until some higher value y corresponding to housing demand q, that commands price $p(q)$. The two narrow blue boxes indicate the demand for housing at this point. The vertical one refers to unconstrained households, that is households with income y who are able to bid at least $p(q)$. The horizontal box refers to constrained households, who can just afford to bid $p(q)$ but cannot afford more expensive housing because of a borrowing constraint. Total demand for housing of quality q is equal to the number of households whose combinations of income and maximum loan belong to these two boxes.

To analyze the situation further, observe that unconstrained households will only choose the combination $(q, p(q))$ of housing units and user cost if the first-order condition (11) holds. Using the definition of the marginal price $\pi(q) = dp/dq$, this implies:

$$dp = M(q, y - p(q))dq. \tag{28}$$

If this equality does not hold, either no unconstrained household will choose housing of quality q (if the equality is replaced with $a < sign$) or there will be bunching (if the equality is replaced by the $a > sign$). The latter possibility is incompatible with equilibrium. The former is immediately seen to imply $dy = 0$. Suppose, (28) holds, then we can rewrite (25) as:

$$[g(q) - f^{bc}(y, p(q))M(q, y - p(q))]dq = f^{uc}(y, p(q))dy \tag{29}$$

Since the right-hand side is non-negative, the expression in square brackets on the left-hand side must also be nonnegative. If this condition is satisfied, there are enough houses offering q units available for all the constrained and unconstrained households interested in it. This corresponds to a 'mixed' equilibrium in which a given type of housing is inhabited by both types of households, a situation that did not occur with the uniform mortgage qualification constraint studied in the previous subsection.

If the expression in square brackets on the left-hand side of (29) is negative, a mixed equilibrium is not feasible. In that situation there are so many households with a binding borrowing constraint at the prevailing housing price that nothing is left for unconstrained households. The constrained households will consume all housing with q units. This means that the second term on the right-hand side of (25) disappears ($f^{uc} = 0$) and instead we have:

$$g(q)dq = f^{bc}(y, p(q))dp \tag{30}$$

The evolution of the price will now be determined by the densities of housing and of borrowing-constrained households in such a way that all constrained households are exactly on their constraint:

$$\pi(q)(= \frac{dp}{dq}) = \frac{g(q)}{f^{bc}(y, p(q))}. \tag{31}$$

Note that (30) allows the density of borrowing-constrained households to be larger than the density of housing. If many borrowing-constrained households are clustered in a particular price-quality range, they may occupy all housing for a range of qualities, as happened in the previous subsection. Unconstrained households will be set 'on hold' until all the constrained households have been served and $f^{bc}(y, p(q))$ returns to a level below $g(q)$ at which the first-order condition (28) can be satisfied.

This is similar to what happens with the uniform mortgage qualification constraint discussed in the previous subsection at point B or C (depending on the tightness of the constraint) when unrestricted households take over. Note that at such a point the user cost function need not be differentiable.

As will be clear by now, the equilibrium with credit constraints in the model of the present subsection differs substantially from that in the previous section with a uniform mortgage qualification constraint. In particular, the assignment rule in the constrained equilibrium now differs from that in the unconstrained one. Households experiencing a binding borrowing constraint will in general consume less housing than unconstrained households with the same income level. Some of them may even be pushed out of the housing market, while other households with lower incomes but less tight borrowing constraints will be able to enter. However, with the generalized borrowing constraints of the present subsection it is still true that - relative to the corresponding equilibrium without constraints, house prices will be lower.

The key observation here is that the income of unconstrained households demanding housing of any quality q will never be higher than in the unconstrained equilibrium and will be lower if some borrowing constraints are binding at prices below $p(q)$. To see this, consider first the critical income. This will never be higher than in the unconstrained equilibrium and therefore $p(q^{min})$will never be higher than in the unconstrained equilibrium. If the critical income is lower in the constrained equilibrium, the price for housing of minimum quality will also be lower. Moreover, the marginal price at $p(q^{min})$ will also be lower. In the present version of the model constrained households will consume less housing than unconstrained households with the same income level. This implies that unconstrained households will at least consume the same housing quality than in the unconstrained equilibrium and therefore that their marginal willingness to pay for housing is lower than in the unconstrained equilibrium. Houses of all qualities will be inhabited by households have at most the same income as in the unconstrained equilibrium. Hence the marginal price of housing will be at most equal to that in the unconstrained equilibrium. Since we have already drawn a similar conclusion for the price level of housing of minimum quality, it follows that for all levels of housing quality the price will at most be equal to that in the unconstrained equilibrium. If some households are driven out of the market by the borrowing constraints, the equilibrium housing price will be lower for all quality levels. If this is not the case, but some households are forced to accept a lower housing quality by the borrowing constraints, then the prices for all higher quality levels are lower than they would be in the unconstrained equilibrium.

The mechanism through which this happens is that the unconstrained households are enabled to consume housing of higher quality relative to the unconstrained equilibrium by the fact that constrained households are forced to accept lower housing consumption. If there is a quality range where all houses are occupied by borrowing-constrained households, the marginal price is also lower than the marginal willingness to pay for unconstrained households interested in this housing.[25]

On the other hand, it is easy to verify that the price of housing of quality q or lower will not be affected by the presence of binding borrowing constraints that refer exclusively to households assigned to houses of higher quality. However, this appears to be an exceptional situation because binding borrowing constraints push households toward lower quality housing and should therefore be expected to con-

[25] To see this, note that in this range the expression in square brackets on the left-hand side of (29) is negative implying that $\frac{g(q)}{f^{bc}(y,p(q))} < M(q, y - p(q))$ and use (31).

centrate constrained demand at the low quality range. We may conclude that the presence of borrowing constraints that are binding for some households will never result in higher house prices for unconstrained households and will result in strictly lower (total and marginal) house prices, and therefore increased housing consumption for unconstrained households with actual user costs higher than the maximum user cost of some constrained other households.[26] The first conclusion is in line with the analysis of the previous subsection, but the second is a substantial deviation. The benefits of the unconstrained households are related to the lower housing consumption of the constrained households, which suggests that they may no longer be better off, due to the constraints.

Now consider a borrowing-constrained household. The house they would have preferred in the situation without borrowing constraints is no longer available to them, notwithstanding the lower price. Instead, they had to accept a lower level of housing units at a total price that is low enough for their marginal willingness to pay for housing to exceed the actual marginal price. Their utility will certainly be lower than they could have reached at the currently prevailing user cost function (should their borrowing constraint be relaxed while those of all others remained in place), but it is unclear if their utility is also lower than in the situation without borrowing constraints for any household. If their constraint is mild, in the sense that housing consumption is modestly reduced, the net effect may be positive, as is always the case with a uniform mortgage qualification constraint (see the previous subsection), but if the constraint forces them to reduce their housing consumption substantially, it will be negative. Hence the welfare effect of the borrowing constraints on the constrained households is ambiguous in the situation studied in the present subsection.

6 Buy-to-let investors

Next, let us consider what happens if buy-to-let investors enter the market. We assume they do not experience borrowing constraints and have enough capital available to buy any house they want. If they buy houses at the prevailing market price $P(q)$, they will be able to let them against a rent $p(q)$, which offers the investor a return γ. We assume that this return is not attractive enough to trigger large investments on the housing market. This implies that a market like the one described in subsection 3.3, on which no borrowing constraints are present buy-to-let investors will not enter in large numbers. However, the situation is different when some households experience a binding borrowing constraint. As we have seen above, households restricted by a borrowing constraint have a marginal willingness to pay for housing that exceeds the marginal price. Then there are households willing to pay more than the user cost

[26] Note that these constrained households may either consume lower quality housing or may have been pushed out of the market because they are not even allowed to occupy housing of minimum quality.

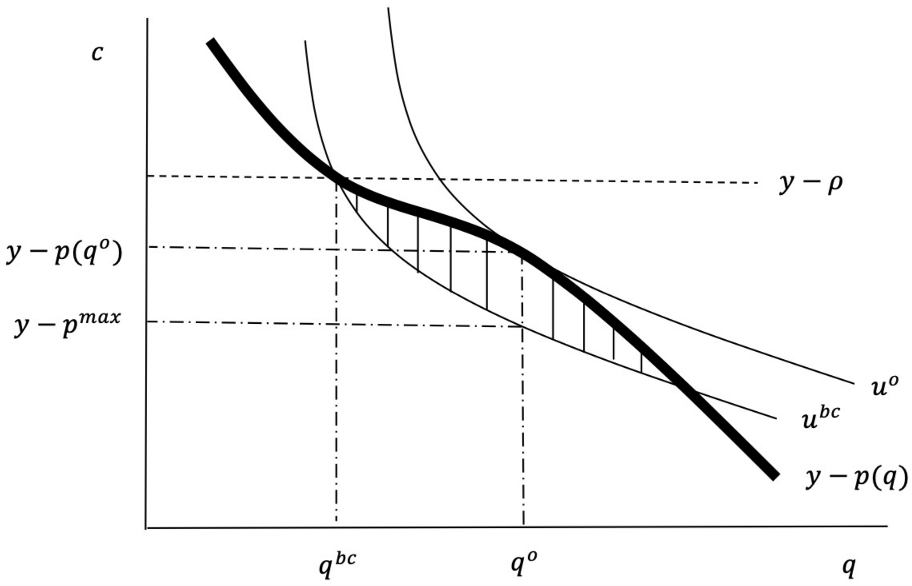

Figure 3: Borrowing constraint and profitable buy-to-let

$p(q)$ as rent if this offers them the possibility to consume more housing than they are able to do in owner-occupied housing with the borrowing constraint present. Hence the return of the buy-to-let investor will be higher than γ.

The model thus predicts that buy-to-let investors will enter the market when borrowing constraints are binding for some households. This amounts to an arbitrage process that ends when the user cost is again equal to $p(q)$ and for all households marginal willingness to pay equals marginal user cost.

The situation is illustrated in Figure 3. The bold line indicates the budget line of a household, that is the difference between income and the user cost of housing, $y - p(q)$. In the absence of a binding borrowing constraint the household would choose housing consumption q^o. However, with a binding borrowing constraint maximum user cost is ρ and the household can only consume q^{bc} units of housing. A buy-to-let investor can purchase a house of quality q^0 at the prevailing market price, finance it with user cost $p(q^o)$ and offer it to the borrowing-constrained household at a rent between $p(q^o)$ and p^{max}. This will offer him an additional return on the house and it will give the household the possibility to increase its utility. In fact, any offer to rent a house implying that the household will reach a combination of housing and other consumption somewhere in the shaded area means a possibility to improve utility relative to the present state of constrained owner-occupied housing consumption for the household.

The activity of buy-to-let investors will, in the longer run, affect the housing price. The prices of houses that were low-priced because demand for them was depressed by the borrowing constraints will increase until the possibility for profit making buy-to-let activities will have disappeared. In this situation all households who were initially borrowing constrained will have avoided the implied restriction by moving to rental housing. The allocation of housing over households will be identical to that in a pure owner-occupied market without borrowing constraints as a result of the arbitrage of buy-to-rent investors. Borrowing constraints are nevertheless useful as they protect households (and banks) against the risks associated with mortgage default.

7 Conclusion

This paper provides an analysis of the interaction between binding borrowing constraints and buy-to-let investment behavior in the context of urban housing markets where the housing stock can be considered as given, at least in the short run. In such a market positive income shocks make housing more expensive, even if population size remains unchanged. With high house prices borrowing constraints may become binding for many low income and/or low wealth households. Although these households dislike such restrictions, they imply lower house prices and it is shown that in a simple benchmark case their impact on utility is positive or zero, implying a Pareto-improvement. In the general setting borrowing constraints still decrease house prices, and improve welfare for all unrestricted households, while they may decrease welfare of the most restricted households. Binding borrowing restrictions open up possibilities for profitable arbitrage by buy-to-let investors. By offering the houses preferred by the restricted households as rental housing, they allow them to reach the same level of housing consumption as in the case without borrowing constraints, albeit at a higher price than is relevant in the situation when borrowing constraints are binding. In the equilibrium with free entry of buy-to-let investors the allocation of housing over households is the same as in the situation without borrowing constraints.

These conclusions have relevance for housing markets in large cities like Amsterdam where buy-to-let investors have been very active since the recovery of the housing market from the Global Financial Crisis and the ensuing euro crisis. It has recently been argued that house prices in the Netherlands are stronger associated with borrowing constraints than with housing shortages, which may be interpreted as pointing to the relevance of the assignment model. Although buy-to-let investors were initially welcomed as offering renters more possibilities on the Amsterdam housing market, where social housing is only available for low-income households and private renting was until recently mostly used by expats, a much more critical attitude has emerged in recent years when the strong growth of the private rental resulted in a decline of the share of owner-occupied housing. Our analysis suggests that the suspicion that buy-to-let investors drive up prices and let houses to households that would have been owner-occupiers, had they not been restricted by borrowing constraints has merit. Indeed the equilibrium with borrowing constraint is better for all except the most tightly restricted would-be owners.

On the other hand, our analysis differs from the opinion of many housing market watchers in the relatively positive verdict on borrowing constraints. In addition to protecting the homeowners against taking too much risk, they also help to mitigate the level of house prices in a market with fixed supply. It is true that te unrestricted households benefit most from this effect, while for restricted households there are not only lower house prices but – except in the benchmark case with identical tastes and identical mortgage qualification constraints – also in many cases a lower housing consumption while the net impact on welfare is ambiguous. For those experiencing the tightest borrowing restrictions the net effect is likely negative. Given the negative sentiment about buy-to-let investors it is paradoxical that the impact of their behavior is unambiguously positive for exactly those households, while the less severely or unrestricted households loose welfare because of buy-to-let activity.

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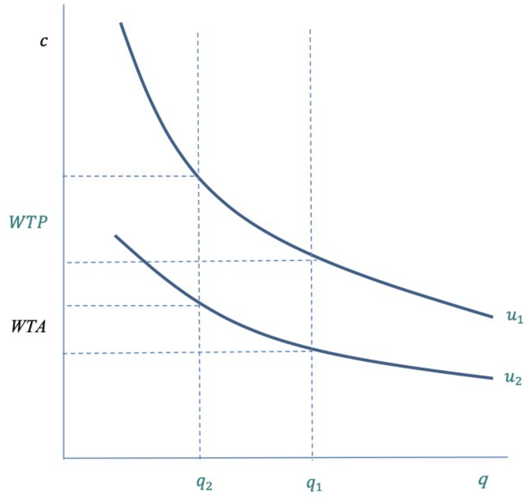

Figure A1: Illustration of switching houses

Appendix

A.1 Higher income households consume more housing

In the assignment model the marginal price of housing is not given. To show that housing consumption is still increasing in income, recall that normality of housing implies that the marginal willingness to pay for housing $M(q, c)$ is increasing in c for given q. Consider two households, 1 and 2, with incomes y_1 and y_2, $y_1 > y_2$ and housing consumption q and q_2, $q_1 < q_2$. It will be shown that both households can benefit from switching houses. Household 1 reaches a higher utility, u_1, than household 2, u_2. The willingness to pay of household 1 for the larger house can be written as:

$$WTP = \int_{q_1}^{q_2} M(q, c(q, u_1))dq \tag{A1.1}$$

where $c(q, u_1)$ denotes the value of other consumption that keeps the household on its initial indifference curve when housing consumption is q. Similarly, we can write the minimum required compensation (willingness to accept) of household 2 for the smaller house as:

$$WTA = \int_{q_1}^{q_2} M(q, c(q, u_2))dq \tag{A1.1}$$

Where the interpretation of $c(q, u_2)$ is analogous. Since $M(q, c(q, u_1)) > M(q, c(q, u_2))$ for all q, we must have $WTP > WTA$, which implies that both households can reach a higher utility level if they switch houses. Figure A1 illustrates.

A.2 Derivation of equation (13)

Note that the slope of the housing price function is always equal to the marginal willingness to pay of the households that have been assigned to the houses with the quality considered, see (10). Starting from the total derivative of the marginal rate of substitution, we can therefore derive:

$$\begin{aligned} d\pi &= \frac{\partial M}{\partial q}dq + \frac{\partial M}{\partial (y-p)}d(y-p) \\ &= \frac{\partial M}{\partial q}dq + \frac{\partial M}{\partial (y-p)}(dy - \pi dq) \\ &= [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}]dq + \frac{\partial M}{\partial (y-p)}dy \\ &= [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}]dq + \frac{\partial M}{\partial (y-p)}\frac{g(q)}{f(y)}dq. \end{aligned} \tag{A2.1}$$

The right-hand side of the first line is the total derivative of the marginal willingness to pay. The second line uses the definition of the marginal price of housing. The third line is a rearrangement of terms. The fourth line uses the assignment rule (12). Re-writing the last line gives:

$$\frac{d\pi}{dq} = [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}] + \frac{\partial M}{\partial (y-p)}\frac{g(q)}{f(y)} \tag{A2.2}$$

It is now easy to verify that (13) holds if $d\pi/dq = 0$.

A.3 The second-order condition

The first-order condition (11) requires that the slop of the budget line is equal to that of the indifference curve. The second-order condition requires that a move along the budget line starting from the point where the first-order condition is satisfied results in a lower utility. This is the case if the budget line is locally less convex than the indifference curve. We show here that this is always the case if the assignment rule is followed.

By the definition of the marginal willingness to pay, it must be true on an indifference curve that:

$$dc = -Mdq.$$

Now how M changes along an indifference curve:

$$\begin{aligned} dM &= \frac{\partial M}{\partial q}dq + \frac{\partial M}{\partial (y-p)}dc \\ &= \frac{\partial M}{\partial q}dq - \frac{\partial M}{\partial (y-p)}Mdq \\ &= [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial (y-p)}]dq. \end{aligned} \tag{A3.1}$$

The first line is the total derivative of M, now with the notation c for $y-p(q)$. The second line imposes a move along the indifference curve and the third line uses the equality between the marginal price and the marginal willingness to pay for housing. The expression in square brackets in (A3.1) is the second derivative of the indifference curve with reversed sign.

In Appendix A2 we considered the second derivative of the equilibrium housing price function, $d\pi/dq$, also with reversed sign, see (A2.1). The second-order condition is satisfied if the budget line is less convex than the indifference curve in the optimum, that is if:

$$-\frac{d\pi}{dq} \leq -\frac{dM}{dq_{u,constant}} \tag{A3.2}$$

Comparison of the two equations makes clear that this is equivalent $\frac{\partial M}{\partial(y-p)}\frac{g(q)}{f(y)} > 0$, which is true because housing is normal and demand is only expressed for existing housing by existing households, implying that both $g(q)$ and $f(y)$ are positive. Condition (A3.2) can alternatively be formulated as:

$$\frac{d\pi}{dq} > [\frac{\partial M}{\partial q} - \pi\frac{\partial M}{\partial(y-p)}]. \tag{A3.3}$$

Consider the housing of minimal quality q^{min}. For any given price $p(q^{min})$ we set, there will (within limits imposed by the income distribution and tastes) be critical incomes for both groups at which households are indifferent between the outside option and living in the housing of minimal quality. However, the marginal willingness to pay for housing will differ. Allocate the housing to the group with the highest marginal willingness to pay for housing. We can then use the method proposed in section 3.2 to trace out the housing price function for this group. At some quality it may be the case that there are households of the other group opting for such housing. The marginal willingness to pay for housing of both groups is identical and if the relevant indifference curves are locally similar so that (A3.3) is satisfied, there will be a segment of mixed housing occupation.

A.4 The simulation model

This appendix provides details about the indirect utility and demand functions used to illustrate various aspects of the assignment model. The demand function is linear:

$$q = \alpha + \beta\pi + \gamma y \tag{A4.1}$$

The associated indirect utility function is (see Hausman, 1981):

$$v = e^{-\gamma\pi}[y + \frac{1}{\gamma}(\beta\pi + \frac{\beta}{\gamma} + \alpha)] \tag{A4.2}$$

Demand functions are derived on the basis of a linear budget constraint, but they can also be used to describe te consumer's optimal choice under a nonlinear budget constraint, if one uses the marginal price of the commodity concerned (housing) and virtual income y^v, that is the income that corresponds with the linearized budget constraint that passes through the consumer's optimum:[27]

$$y^v = y - p(q) + \pi q \tag{A4.3}$$

[27] Note that virtual income equals actual income if the budget constraint is linear, that is if the marginal price of housing is constant.

Substitution of (A4.3) into (A4.1) gives:

$$q = \alpha + (\beta + \gamma q)\pi + \gamma(y - p(q)) \tag{A4.4}$$

The coefficient for the marginal price in the equation is the Slutsky term. Solving this equation for the marginal price gives:

$$\pi = \frac{q - \alpha - \gamma(y - p(q))}{\beta + \gamma q} \tag{A4.5}$$

Since the demand equation indicates consumption bindles at which the marginal price is equal to the marginal willingness to pay for housing, the right-hand side of (A4.5) must be equal to the latter variable.

To carry out the simulations for the baseline case with identical preferences, we need to be able to determine the user cost of housing of minimum quality. This requires a comparison of the utility of the outside option with that of living in housing with the minimum quality at the critical income level. To do this we have to substitute (A4.5) into (A4.2) and then use the known values of income, quantity and full price of the outside option and the minimum quality. This leaves us with only the full price $p(q^{min})$ as unknown variable for which we can solve numerically. We can then use the (A4.5) to compute the marginal price.