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The Demand for Programmable Payments

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Abstract

This paper studies the desirability of programmable payments where transfers are automatically executed conditional upon preset objective criteria. We do so by studying optimal payment arrangements in a framework that captures a wide range of economic relationships between two parties. Our framework stacks the cards in favor of programmable payments by considering an environment without legal recourse. The results show that the optimal payment arrangements for long-term economic relationships consist predominantly of simple direct payments. Direct payments increase the surplus by avoiding the liquidity cost of locking-up funds from the moment where the payer commits the funds in a programmable payment until the moment where the conditions are satisfied to release those funds to the payee. Programmable payments will be desirable, and may in fact be the only viable payment arrangement, in situations where economic relationships are of a short duration. Our results identifies a limit to the growth in the demand for payments as their cost decreases: While the number of feasible trading relationships will increase, existing trading relationships will optimally rely on fewer payments.

Keywords: Bill Payments, Blockchain, CBDC, Smart Contracts, Payment Economics
JEL Codes: E40, E42, E58

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1 Introduction

"Programmability" has become a popular catchword in payments. Programmable payments are transfers of funds that are automatically executed conditional upon preset objective criteria.[1] For some, programmability is one of the features driving the enthusiasm for private digital currencies. Programmability has also been raised as a potentially desirable feature for central bank digital currencies.[2] Programmable payments can provide assurance to both the payee and the payer by requiring the payer to commit funds while delaying the release of funds to the payee until services or goods are delivered. They essentially automate the process of committing funds for a given period of time and then releasing those funds to the payee at once or in small steps conditional upon the delivery of services or goods. The recent launch of the payment solution called "Yuan Steward" for China's digital yuan seeks to add such functionality to consumer wallets by employing smart contracts (Zhou, 2022). Often-raised use cases of programmable payments are enabling micropayments for pay-per-use concepts, "atomic" settlement and clearing for securities, currencies and derivatives, and automated escrow services. Although many have documented the wide range of technical possibilities, much less is known on the potential demand for programmable payments. Could programmable payments become the new default mode of making payments?

The purpose of this paper is to study, in a formal economic framework, the desirability of programmable payments for a set of situations requiring a payment arrangement between two counterparties. A seller provides a service for a period of time to a buyer in a continuous-time framework. The framework stacks the cards in favor of programmable payments by considering an environment where counterparties do not have any legal recourse in the event where either the

[1] There is no single accepted definition of programmable payments. Our description captures the main elements of the definitions of Bullock (2018, p. 4), Deutsche Bundesbank (2020a, p. 4), Arner et al. (2020, footnote 5) and Bechtel et al. (2020). Programmable payments may be enabled by programmable money, but they do not necessarily require programmable money (Deutsche Bundesbank, 2020a; Bechtel et al., 2020). Programmable money may also be used for other purposes such as limiting the purposes on which funds can be spent (e.g., food stamps) or implementing an expiry date on money (Kahn et al., 2021). The present paper focuses on demand for programmable payments, and the alternative purposes of programmable money are outside the scope.

[2] Reports that have looked at programmability include those of the Deutsche Bundesbank (2020b); Bank of England (2020); European Central Bank (2020); Wong and Maniff (2020); Usher et al. (2021).

buyer or the seller does not deliver upon their promises. This environment favours the potential value-added of the assurance that programmable payments can provide to both the payer and payee. Moreover, we study the extreme case where the degree of automation has evolved to the point where the cost of a programmable payment has fallen to a level comparable to that of a simple direct payment.[3] We use this framework to study the optimal payment arrangement and assess the conditions under which the optimal payment arrangement requires programmability.

We start by analyzing an environment where a buyer can make a single payment with the purpose of purchasing a service from a seller. Without legal enforcement, programmable payments will be the only viable payment arrangement. A direct payment cannot result in a viable trading relationship: Once the seller has received the funds, she has no incentives to continue providing the service. Similarly, the buyer has no incentives to pay using a direct payment knowing that the seller will stop providing the service as soon as she receives the funds. A programmable payment allows for a viable payment arrangement by temporarily locking-up the funds between the moment that funds are committed by the buyer and the moment they are received by the seller, while conditioning the release upon delivery of the service (akin to a traditional escrow account). Such an arrangement is feasible as long as the transaction cost is sufficiently low (Theorem 1).

Although a programmable payment enables a viable relationship, the maximum length of the arrangement between a payer and payee that can be supported by a single programmable payment is limited (Theorem 2). One might conjecture that one could always increase the amount of funds that will be released to the seller at the end of the arrangement in exchange for receiving services for a longer period of time. Even though this may be viable technically, economically it is not. The issue is that, at some point, the additional liquidity cost of locking up both *more* funds and for a *longer* period starts to exceed the additional surplus generated from extending the length of the arrangement. At this point, it is no longer economically beneficial to extend the length of the

[3] A simple direct payment is always implementable as a programmable payment where the funds committed by the payer are immediately and unconditionally released to the payee. Hence, it is reasonable to assume that a simple direct payment can always be implemented at the same cost as a programmable payment or less.

relationship. A single programmable payment can be an optimal arrangement when a buyer needs to purchase a service for a limited period of time.

Given the limited duration of relationships that can be supported by a single programmable payment, we continue by analyzing the optimal payment arrangement between a buyer and a seller when allowing for a chain of payments. The analysis of optimal chain-of-payments arrangements shows that, even though our framework treats programmable payments relatively favourably, there are many economic situations where the optimal payment arrangement consists predominantly of simple direct payments. The value of continuing a long-term economic relationship can be sufficient to establish incentives for both counterparties to deliver on their promises even without assurance of the payment and delivery by means of a programmable payment. The inefficiency of using programmable payments in such a context is the liquidity cost of locking up the funds, reducing the total surplus from the economic relationship between the two counterparties. With long-term relationships, it is therefore better to use direct payments.

The economics of finite chain-of-payments arrangements are well-illustrated by considering a two-payment arrangement. The last payment within a two-payment arrangement must be a programmable payment as before: Locking-up the funds of the last payment prevents the arrangement from unraveling as a finitely-repeated sequential-move game in a similar way as a single-payment arrangement with a direct payment would unravel. Things are different for the first of the two payments. The risk of losing the surplus of continuing the relationship after the release of the first payment helps to better align incentives. There are two possible cases for the optimal two-payment arrangement. In the first case, the first payment is also a programmable payment, but the surplus generated after the first payment allows delaying the moment the buyer is required to submit the first payment. This reduces the time the funds are locked-up and, hence, reduces the liquidity cost of the first payment (Theorem 4). The second case is that where the amount of surplus generated after the release of the first payment is large enough that the buyer can delay making the first payment up till the point where the money will be released to the seller. In this situation, the first payment would be released immediately so the it would simply be a direct payment (Theorem 5).

The tendency towards direct payments for earlier payments generalizes to all finite chain-of-payments arrangements. Sufficiently long optimal chain-of-payments arrangements always start with direct payments because of the lower liquidity costs (Theorem 7). Only towards the end of a relationship do the parties revert to the use of programmable payments. Moreover, the optimum for infinitely long payment arrangements is achieved by a chain of payments that consists of direct payments only (Theorem 8). These results suggest that programmable payments are unlikely to become the new "standard" for all payment arrangements due to the liquidity-savings offered by direct payments.

Our model also provides some important insights in the demand for payments more generally. Some have expressed the expectation that a strong reduction in transactions costs enabled by technological developments in the payment space could lead to an explosion in the number of payments, e.g., through so-called micro-payments in decentralized finance that could be used to approximate "streaming money".[4] The study of optimal payment arrangements suggests that the relationship between the transaction cost and the number of payments is more complex than that. In particular, we find different relationships for the *extensive* and the *intensive* margins. Lower transaction costs increase the number of payments for the extensive margin in the sense that the set of potential buyer-seller pairs where transaction costs are no longer prohibitively expensive increases as transaction costs drop (Corollary 1). For the intensive margin, that is, within buyer-seller pairs, we find the opposite effect: lower transaction costs are associated with fewer payments. Lower transaction costs increase in the surplus in any existing trading relationship between buyers and sellers. The higher surplus provides the buyer stronger incentives to pay when the time comes. These stronger incentives allow the seller to require the buyer to settle the bill by paying larger amounts at a lower frequency (Corollary 2). Whether lower transaction costs will lead to an increase

[4]Platforms that aim to provide the experience of nearly continuous payment streams in decentralized finance include Sablier (https://sablier.finance/), Superfluid (https://www.superfluid.finance/) and LlamaPay (https://llamapay.io/).

or decrease in the number of payments depends on the balance of the effects for new and existing relationships.

2 Related Literature

Our paper relates to several strands of literature. First, our paper contributes to the new literature on the economics of smart contracts. An important feature of smart contracts analyzed is the ability to commit funds that may be released based on preset conditions. Several papers study the importance of the ability to commit funds in a context of a non-repeat transaction.[5] Gans (2019) suggests a smart contract can implement a truth-revealing mechanism where the buyer correctly confirms the quality of the goods before the funds are released to a seller. Bakos and Hałaburda (2019) show how this feature allows contracting parties to commit not to hold-up. They demonstrate how algorithmic execution of programmable payments in combination with digital inputs from "Internet of Things" (IoT) sensors can enable an efficient trading arrangement that neither of the two technologies can achieve individually. Cong and He (2019) study how non-repeat transactions can be facilitated by smart contracts that execute payments based on consensus by third parties on whether goods are delivered. Lee et al. (2021) study the welfare impact of assuring settlement with a smart contract in the setting of a non-repeat transaction in securities trading. Different from those studies, we explore the benefits of programmable payments in a continuous-time environment that generalizes to lasting relationships with repeated and endogenously timed payments.

Second, our work relates to the growing literature on the economics of central bank digital currency (CBDC) (Andolfatto, 2021; Brunnermeier and Niepelt, 2019; Fernández-Villaverde et al., 2021; Schilling et al., 2020; Williamson, forthcoming). The survey by Kiff et al. (2020) highlights

[5] Chiu and Koeppl (2019) study non-repeat transactions in an environment where smart contracts achieve delivery-versus-payments in assets markets where the ownership of both the asset and the means of payment are recorded on the ledger.

important considerations and design choices. Many of these aspects have been studied using the lens of formal economic models, including policies around interest rates (Barrdear and Kumhof, 2022; Chiu et al., forthcoming; Jiang and Zhu, 2021; Garratt and Zhu, 2021; Keister and Sanches, forthcoming; Whited et al., 2022); privacy (Garratt and Van Oordt, 2021; Lee and Garratt, 2021; Tinn and Dubach, 2021; Ahnert et al., 2022); security and loss prevention (Kahn et al., 2020, 2021); holding limits (Assenmacher et al., 2021); and whether CBDC would take a form that would be more cash-like or more deposit-like (Agur et al., 2022; Li, 2021). Programmability is also raised as an important design choice in the literature review by Kiff et al. (2020). However, to the best of our knowledge, no formal theoretical analysis exists that incorporates the—according to our results crucial—aspect of repeat interactions on this design aspect. This paper aims to fill that gap.

Third, our paper relates to the growing literature on the economics of cryptocurrencies. The discussion around programmable payments has been ignited by the technology introduced in the realm of cryptocurrencies (Athey et al., 2016; Biais et al., forthcoming; Bolt and Van Oordt, 2020; Chiu and Koeppl, forthcoming; Garratt and Wallace, 2018; Hałaburda et al., 2022; Schilling and Uhlig, 2019). Popular cryptocurrencies have been subject to transaction fees and confirmation times that render them impractical to stand on their own as a retail payment system (Divakaruni and Zimmerman, 2020; Huberman et al., 2021; Hinzen et al., 2022). A fascinating development the payment protocol that operates on top of the Bitcoin blockchain known as the "lightning network" (Poon and Dryja, 2016). This protocol effectively allows participants to use a single operation on the Bitcoin blockchain to reserve funds for multiple future payments in a so-called bilateral payment channel. A payment channel allows for low cost and almost instant payments by simply changing the allocation of funds in the channel among two counterparties. Either counterparty may claim their share in the funds by initiating a second operation on the blockchain to close the channel. Guasoni et al. (2021) study how to optimally allocate funds to payment channels for exogenous payment patterns to save on transaction costs. Our analysis has implications for the relationship between the transaction cost and the number and size of payments. This suggests that a potentially interesting future avenue of research would be the study of optimal payment channels when payments are endogenous.

3 Model

Time is continuous. There are two risk-neutral agents, a "buyer" (he) and a "seller" (she) both with discount rate $\rho > 0$. The seller can provide a continuous flow of non-storable service to the buyer at a flow cost $c > 0$ per unit time. The buyer's flow utility from receiving the service is $b(t) \geq 0$. There is no asymmetric information and there are no legal enforcement powers. We assume that a counterparty will stop dealing with an agent if the agent does not stick to an arrangement. Buyer and seller have available to themselves a sophisticated payment system which imposes a fixed cost $K > 0$ on the buyer each time he submits funds to the seller. This fixed cost captures both transaction fees as well as potential administrative costs.

We are interested in finding optimal payment arrangements. Payments in the system work as follows. The buyer submits payment i by sending funds D_i at time T_i. The initiation of the payment is observable to the seller, although the seller may not have immediate access to the funds. Instead, the system allows for the possibility that the release of funds to the seller be programmed to occur at $S_i \geq T_i$, and can be conditional upon whether the seller has provided the service over the period $[T_i, S_i)$. This arrangement covers a broad range of real-world mechanisms, including smart contracts, where the release of funds is preprogrammed to occur based on information recorded on a distributed ledger by IoT sensors (Bakos and Hałaburda, 2019), a truth-revealing mechanism (Gans, 2019) or a consensus mechanism (Cong and He, 2019). The arrangement also covers more traditional escrow arrangements, where the release of funds occurs after a third-party verified whether the conditions have been satisfied. If the payment is structured such that $S_i = T_i$, then we speak of a *direct payment.* We stack the cards in favor of programmable payments in that we assume that the transaction cost of a programmable payment is not higher than that of a direct payment.

4 Optimal Single-Payment Arrangement

We start by considering a *single-payment* arrangement in a non-stationary environment, where $b(t)$ takes the following form

$$b(t) = \begin{cases} b, & \text{if } t < T_M \\ 0, & \text{otherwise} \end{cases}$$

for $b > c$ and some known $T_M > 0$. That is, the buyer obtains a constant benefit from the service up to some horizon date T_M.

An arrangement with a simple direct payment is ineffective in this environment. Once the payment is made, there is no incentive for the seller to honor a promise to produce, and if payment is delayed until after production, there is no incentive for the buyer to honor a promise to pay. Backward induction implies that neither the buyer nor the seller can commit to pay or to produce after the other player has completed their end of the deal.

Now consider the role of a programmable payment in this situation. The buyer and seller agree that the seller will start providing the service from $t = 0$. In return, the buyer will commit an amount D_1 at time T_1 using a programmable payment that releases the funds to the seller at time S_1 if she has provided the service continuously in the interim, where $T_1 \leq S_1 \leq T_M$. The utility of the buyer from such an arrangement would be

$$U(D_1, T_1, S_1) = b \int_0^{S_1} e^{-\rho t} \, dt - (D_1 + K)e^{-\rho T_1}. \tag{1}$$

The arrangement is subject to the following constraints:

$$b\int_{T_1}^{S_1} e^{-\rho t}\,dt \geq (D_1+K)e^{-\rho T_1}, \tag{2}$$

$$D_1 e^{-\rho S_1} \geq c\int_0^{S_1} e^{-\rho t}\,dt, \tag{3}$$

$$0 \leq T_1 \leq S_1 \leq T_M, \tag{4}$$

$$D_1 \geq 0. \tag{5}$$

The first constraint requires that, at the time he needs to pay, the prospective benefits to the buyer exceed the monetary cost of the arrangement. The second constraint requires that the discounted monetary benefit of the arrangement to the seller exceed her cost of providing the service. The third constraint says that the funds cannot be released to the seller before they have been committed by the buyer. The last constraint requires that the payment to be non-negative.

We call a single-payment arrangement (D_1, T_1, S_1) *self-enforcing* if conditions (2)-(5) hold true. Trading is *feasible* if there exists a self-enforcing single-payment arrangement. From these constraints, the following results arises.

Theorem 1. *Assume T_M is large.*[6] *Trading is feasible if and only if*

$$\sqrt{\rho K} + \sqrt{c} \leq \sqrt{b}. \tag{6}$$

Proof: Combining equations (2) and (3) to eliminate D_1 shows that we can find values (D_1, T_1, S_1) satisfying (2) and (3) if and only if there exists (T_1, S_1) such that $ce^{\rho S_1} + be^{-\rho S_1} < be^{-\rho T_1} + c - \rho K$. This condition is weakest if the right-hand-side of the inequality is maximized, that is, if $T_1 = 0$, and if the left-hand-side is minimized. The left-hand-side of the inequality is convex in S_1, while the partial derivative with respect to S_1 is negative at $S_1 = 0$ since $c < b$ and positive for large

[6]Lemma A1 in the appendix provides necessary and sufficient conditions for general T_M.

values of S_1. Its minimum is $S_1^* = \log(b/c)/(2\rho)$. Plugging $(T_1, S_1) = (0, S_1^*)$ into the inequality gives $2\sqrt{bc} < b + c - \rho K$ or equivalently $\sqrt{\rho K} + \sqrt{c} \leq \sqrt{b}$. ■

Intuitively, a self-enforcing single-payment arrangement does not exist unless the benefit of the service to the buyer exceeds the cost to the seller of providing the service, that is, unless $b > c$. Theorem 1 shows that this is not sufficient, however. The difference between the benefit and the cost should be sufficiently large to exceed the cost of payment. The cost of payment has both a transaction cost component, K, and a liquidity cost component, ρ. We will refer to Eq. (6) as the "feasibility-condition"; this condition will recur in different circumstances. Note that

Corollary 1. *Lowering the transaction cost K relaxes the feasibility-condition.*

Even if there is no technical limitation to the amount committed in a programmable payment, the maximum economically-feasible length of a single-payment arrangement is limited. Define

$$x =: \frac{1}{2}(b + c - \rho K).$$

The following theorem provides the maximum length of a self-enforcing single-payment arrangement.

Theorem 2. *Assume T_M is large. The length of a self-enforcing single-payment arrangement is limited to*

$$S_1^{\max} = \frac{1}{\rho} \log \left(\frac{x}{c} + \sqrt{\left(\frac{x}{c}\right)^2 - \frac{b}{c}} \right). \tag{7}$$

Proof: The value of $S_1^{\max}$ is the maximum of the two solutions to the pair of conditions (2) and (3) holding with equality. This solution is a function of T_1 that is maximized within the range of possible values specified in condition (4) when setting $T_1 = 0$. ■

The maximum length of a self-enforcing single-payment arrangement is limited because the longer the agreement the greater the dissipation of the value of the payment due to the liquidity cost of holding the funds locked up in a programmable payment. Extending the length of the arrangement increases the liquidity cost both in the length of time the funds will be locked up as well as the amount that will be locked up. If funds are held for the maximum feasible time $S_1^{\max}$, then the present value of the future release of the amount transferred with the programmable payment must just equal the present value of the accumulated costs borne by the seller over the interval $[0, S_1^{\max}]$ while the immediate value of the funds committed equals the present value of benefits received by the buyer, less transaction costs. At $S_1^{\max}$, the surplus from the match is eaten up entirely by the opportunity cost of the sequestering of the funds for the length of time.

Choosing the longest possible arrangement is in general not optimal, and for two reasons. First, on the margin, reducing the length of the agreement slightly increases the benefits by freeing the funds locked-up in the programmable payment more quickly. Second, the surplus generated by the match incentivizes the buyer to pay, which enables the buyer to credibly delay committing the funds. This reduces the liquidity cost as well as the present value of the transaction cost of making the payment.[7]

We next consider the *optimal single-payment arrangement* which maximizes the buyers utility $U(D_1, T_1, S_1)$ in (1) subject to the constraints for a self-enforcing single-payment arrangement in equations (2)-(5).

Theorem 3. *Suppose the length of time horizon satisfies*

$$T_M < \frac{1}{\rho} \log \left(\frac{x}{c} \right).$$

Then the optimal single-payment arrangement is constrained by the time horizon and equals

$$(D_1, T_1, S_1) = \left(\frac{ce^{\rho T_M} - c}{\rho}, T_M - \frac{1}{\rho} \log \left(\frac{b}{b - \rho(D_1 + K)} \right), T_M \right). \tag{8}$$

[7]This second manner in which costs can be reduced depends on the ability that the seller can identify the buyer. Our framework can be adjusted to consider environments where identification is impossible by, for example, forcing the buyer to commit funds into the programmable payments at the start of the relationship.

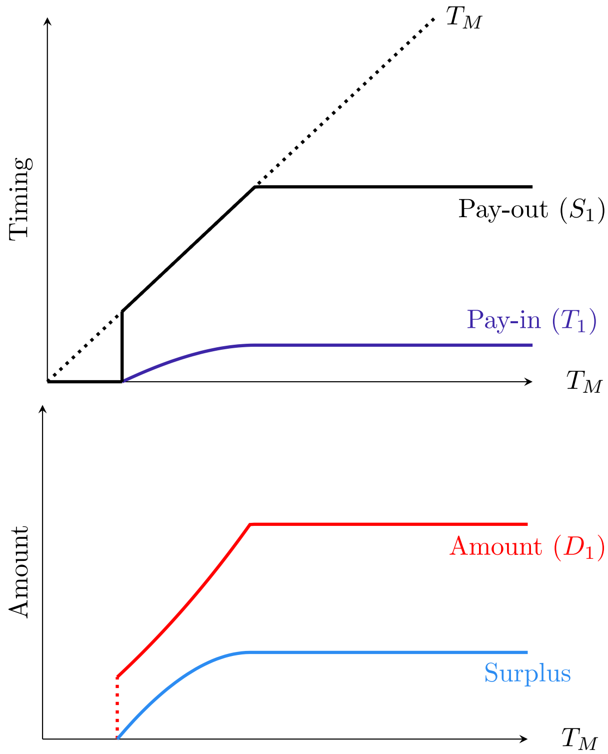
Otherwise, the optimal single-payment arrangement is

$$(D_1, T_1, S_1) = \left(\frac{x-c}{\rho}, S_1 - \frac{1}{\rho} \log\left(\frac{b}{x}\right), \frac{1}{\rho} \log\left(\frac{x}{c}\right) \right). \tag{9}$$

Proof: See Appendix A.1. ■

The qualitative characteristics of the optimal single-payment arrangement are graphically illustrated in Figure 1. The figure considers comparative statics of the solution as a function of the duration of the interval $[0, T_M]$ over which the buyer values obtaining the service. For short horizons, the net benefit of setting up a relationship is inadequate to cover the transaction cost. At a critical level, the surplus generated in the interval less the costs of the payment is just enough to justify transferring the amount D_1, leaving no surplus for the buyer. As the duration of the relationship increases beyond this minimum length, the present value of the benefit exceeds the amount that must be deposited, and so the arrangement generates a surplus for the buyer; however, the fact that the relationship does generate a surplus enables the buyer to delay committing the funds to the point T_1 where the remaining benefit of the match just equals the necessary payment. The longer the relationship is anticipated to continue, the longer the payment can be delayed. This occurs until the critical point where the optimal arrangement switches from the "horizon-constrained solution" in Eq. (8) to the "horizon-unconstrained solution" in Eq. (9). At this point, the surplus no longer increases, because any further delay in releasing the funds locked up in the programmable payment does not justify the additional benefit for the buyer. No further extension of the relationship is desirable and the arrangement and its duration remains fixed as T_M increases.

Figure 1: Optimal Single-Payment Arrangement



Note: The figure shows the optimal single-payment arrangement (D_1, T_1, S_1) as a function of the point in time until which the buyer derives utility from the service, T_M. The upper panel reports the time the buyer commits the funds, T_1, and the time of the release of the funds to the seller, S_1. The lower panel reports the amount paid, D_1, and the total surplus from the arrangement.

5 Optimal Two-Payment Arrangement

We have seen that there is a maximum length of time $S_1^{\max}$ that service can be supported by a single programmable payment. Beyond that limit, the immediate transactions cost plus the liquidity cost of tying up the funds exceeds the surplus generated by the match. Once we have found the apparent maximum possible duration for the use of the sequestered funds, we might expect that the relationship could be extended by repeated programmable payments at intervals of $S_1^{\max}$; thus the relationship could be supported over an interval of length $2S_1^{\max}$ by two programmable payments, etc. Indeed this is feasible, but it is suboptimal. In general, a better solution would be to set up repeated arrangements with each programmable payment following the solution in Theorem 3. However, there are further improvements that can be made, reducing the wastage of the liquidity costs of locking up funds in programmable payments. To explore this, we consider the general problem of supporting the relationship by an arrangement existing of a chain of payments.

The optimal arrangement with a chain of two payments provides important insights that generalize to chains with multiple payments. With a two-payment arrangement, the last of the two payments needs to be a programmable payment that delays the release of the funds until the relationship ends. Otherwise, the seller would have incentives to stop providing the service before the agreed-upon point in time. However, in some cases, the earlier of the two payments may be a simple payment—that is, it can be the case that $S_1 = T_1$ in the optimal arrangement.

The buyer needs to solve the following program

$$\max_{D_1,D_2,T_1,T_2,S_1,S_2} b\int_0^{S_2} e^{-\rho t}\,dt - (D_1+K)e^{-\rho T_1} - (D_2+K)e^{-\rho T_2},$$

subject to

$$\begin{aligned} c\int_0^{S_2} e^{-\rho t}\,dt &\leq D_1\,e^{-\rho S_1} + D_2\,e^{-\rho S_2}, \\ c\int_{S_1}^{S_2} e^{-\rho t}\,dt &\leq D_2\,e^{-\rho S_2}, \\ D_1 + K + (D_2+K)e^{-\rho(T_2-T_1)} &\leq b\int_{T_1}^{S_2} e^{-\rho(t-T_1)}\,dt, \\ D_2 + K &\leq b\int_{T_2}^{S_2} e^{-\rho(t-T_2)}\,dt, \\ D_i \geq 0, \quad T_i \leq S_i &\leq T_M, \quad i=1,2, \\ 0 \leq T_1 &\leq T_2. \end{aligned}$$

Note the similarity of the constraints to those in the single-payment arrangement. The first and second constraints ensure that at every point in time, the present value of cost to the seller of continuing to provide the services is less than or equal to the present value of the future releases of funds. The third and fourth constraints ensure it is rational for the buyer to pay the funds at the specified times under the arrangement. The remaining constraints ensure, among other things, that the release of funds from a programmable payment does not occur before the funds are committed.

The following theorems characterize the optimal two-payment arrangement. We focus on the case with *unconstrained horizon*, that is, where the time horizon T_M is distant enough for the agents to take full benefit of the relationship. This is the case which is of interest for generalization to multiple payments.[8] The following condition is sufficient for an unconstrained horizon in the two-payment problem (see Corollary A2 in the appendix):

$$T_M \geq \frac{2}{\rho} \log \frac{\sqrt{2}x}{c}. \tag{10}$$

There are two possibilities, depending on the value of the parameter Ψ, where

$$\Psi = \frac{x^3}{b^2 c}.$$

The optimum involves either two programmable payments or a direct transfer and a subsequent programmable payment. In characterizing the optimum we make use of the parameter $\overline{T}$ where

$$\overline{T} = \frac{2}{\rho} \log \frac{x}{c}.$$

Theorem 4. *(Case I: Two programmable payments) Suppose the horizon is unconstrained. If* $\Psi < 1$*, then the optimal two-payment arrangement is the following: The seller provides the service to the buyer from time* 0 *until time* $\overline{T}$*. She receives compensation in two equal amounts, one halfway*

[8]See Appendix A.2 for the solution of the horizon-constrained case.

through ($S_1 = \overline{T}/2$) and one at the end of the arrangement ($S_2 = \overline{T}$). The amounts are

$$D_1 = D_2 = \frac{x - c}{\rho}.$$

The buyer commits the funds using programmable payments at time $T_1 = \overline{T} - 2L$ and $T_2 = \overline{T} - L$, where

$$L = \frac{1}{\rho} \log \frac{b}{x}.$$

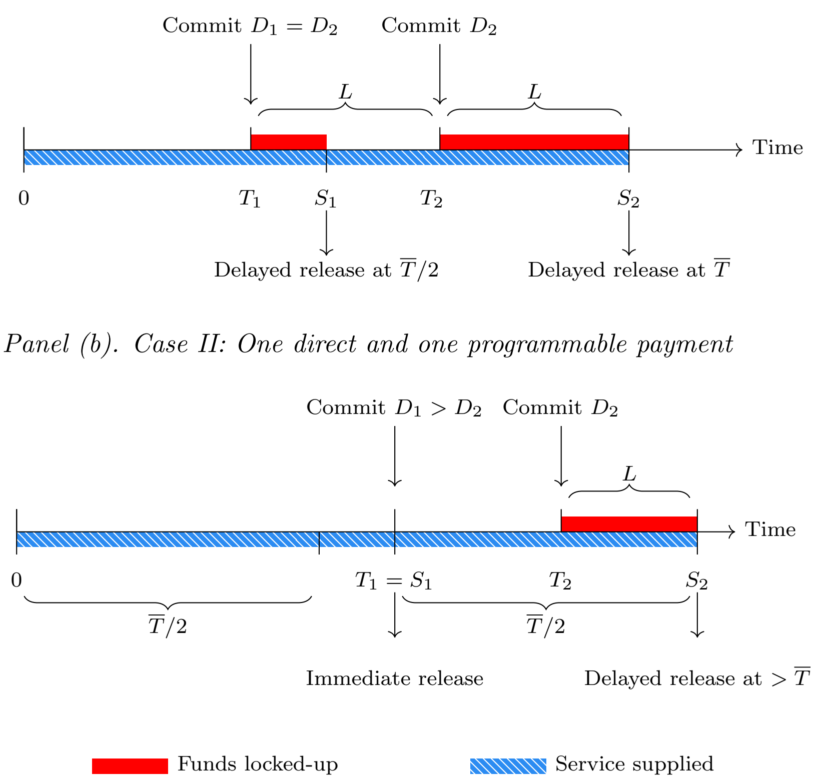
Proof: See Appendix A.2. ■

The amount of funds transferred in each payment is equal to the amount transferred in the optimal horizon-unconstrained single-payment arrangement. Figure 2, panel (a) shows the timing when both payments in the optimal two-payment arrangement are programmable payments. There is a subtle difference in the timing of the payments compared to that in the horizon-unconstrained single-payment solution. Like in the single-payment arrangement, the two-payment arrangement starts with the seller supplying the service for a limited period before the buyer commits the funds for the first programmable payment. However, the buyer is able to delay committing the funds for the first payment for a longer period of time than in the single-payment arrangement. Hence, the funds for the first payment are locked-up for a shorter period of time. The reason is that the buyer is better incentivized to pay the first payment since he risks foregoing a longer beneficial relationship if he fails to commit funds for the first payment. There are no such additional incentives for the second payment, because the arrangement stops anyway after the second payment. The timing of the second payment is therefore comparable to that of the optimal payment in the horizon-unconstrained single-payment arrangement: The delay in committing the funds for the second payment in the optimal two-payment arrangements is equal that in the horizon-unconstrained single-payment arrangement, and the funds also remain locked-up for the same amount of time.

Figure 2: Timeline of the Optimal Two-Payment Arrangement

Panel (a). Case I: Two programmable payments

Panel (b). Case II: One direct and one programmable payment



Note: The figure illustrates the timeline for the optimal two-payment arrangement. Panel (a) describes the optimal timeline for parameter values where both payments are programmable payments (Theorem 4). Panel (b) describes the optimal timeline for parameter values where the surplus generated by the last programmable payment is sufficiently large so that the first payment can be a direct payment (Theorem 5).

The interval L between payins is a decreasing function of b because, the higher the benefits enjoyed by the consumer, the stronger the incentives to pay, and the shorter the second payment needs to remain locked-up. Note that, as panel (a) of Figure 2 illustrates, in the optimal arrangement with two programmable payments, L is also the interval between the second payin and the end of the agreement.

When considering smaller and smaller values of L, one could imagine that, eventually, the moment to commit the first payment would coincide with the moment of its release, i.e., b becomes large enough that $2L = \overline{T}/2$ or, equivalently, $\Psi = 1$. At this point, the trust of the seller in the willingness of the buyer to make the first payment will be strong enough that there is no need to lock up the first payment, and the first payment in the optimal arrangement would be a direct payment. The optimal solution when the first payment is a direct payment is summarized in Theorem 5.

Theorem 5. *(Case II: One direct payment, one programmable payment) Suppose the horizon is unconstrained. If $\Psi > 1$, then the optimal arrangement is the following: The second transfer D_2 has the same formula as in the previous theorem, the formula for the length of time the funds are locked up in the programmable payment is also unchanged. The first transfer is a direct payment and is greater than the second payment. The seller waits longer for the direct payment than for the subsequent payout of the programmable payment.*

Proof: See Appendix A.2.

The timeline of the arrangement when the first payment is a direct payment is illustrated in panel (b) of Figure 2. The benefit of the first payment being a direct payment is that there is no liquidity cost of locking up funds. The absence of the liquidity cost from locking up funds allows the first payment to cover the bill for obtaining the service for a longer period of time by paying a larger sum (i.e., $T_1 \geq \overline{T}/2$ and $D_1 \geq D_2$).

6 Multiple Payments

The core insight of the example of the optimal two-payment arrangement is that, if the continuation value of the relationship to the buyer is sufficiently high, then the seller can trust that the buyer will make the first payment without requiring him to temporarily locking up the funds in a programmable payment. This is beneficial due to the reduction in liquidity costs. The insight—

that earlier payments can be direct payments when the continuation value of the relationship to the buyer is sufficiently high—generalizes to any feasible chain of non-overlapping payments that covers a sufficiently long horizon.

One can think about the value to the buyer of continuing the relationship as generating trust. Let W denote the continuation value of the relationship to the buyer the moment after a payment is released to the seller. The level of W acts as a substitute for the requirement to guarantee the payment by locking-up funds in a programmable payment. The higher the level of W, the shorter the period funds need to be locked up in the next programmable payment.[9] Whenever W is sufficiently high, then the trust-effect is so strong that programmable payments—with their liquidity costs from locking up funds—become inferior to direct payments.

Theorem 6. *Assume the feasibility-condition holds. In an optimal chain of payments, suppose* $W \geq (b - x)/\rho$ *immediately after some payment is released to the seller. Then that payment and any earlier payments are direct payments. Any payment that comes after the last payment for which this holds true is a programmable payment.*

Proof: See Appendix A.4.

The previous theorem documents the minimum continuation value of the relationship to the buyer that is necessary in order for all earlier payments in the optimal payment chain to be direct payments. The theorem does not report under which parameter values it would be possible to achieve this value. Clearly the feasibility-condition for the single-payment arrangement needs to hold true: For a multiple payments arrangement not to unravel, there must be a net benefit for the buyer and seller to continue the relationship once they arrive at the point where there is only one payment left. It turns out that the feasibility-condition for the single-payment arrangement is also a sufficient condition provided that the trading relationship covers a sufficiently long period.

[9]The environment we consider stacks the cards in favour of programmable payments. The value of W in our model comes exclusively from the continuation value of the relationship with the same counterparty. One could think of legal enforcement and reputation formation as aspects that could further contribute to the value of W. Such aspects would make direct payments more attractive compared to programmable payments.

As the next theorem states, given the feasibility condition for the single-payment arrangement, it is always possible to achieve the necessary level of surplus for direct payments as long as enough time is available.

Theorem 7. *Assume the feasibility-condition holds true. If the horizon T_M is sufficiently remote, every optimal chain of payments will start with direct payments.*

Proof: See Appendix A.4.

Theorem 7 shows that sustainable trading arrangements—that is, trading arrangements that last sufficiently long in the future—create sufficient trust in order to avoid programmable payments. However, such relationships require the use of programmable payments when they get closer to the terminal date.

In a stationary environment, where $b(t) = b$ for all t, programmable payments are not required at all for the optimal payment arrangement. The following theorem establishes how the optimum can be reached in a stationary environment with a payment arrangement that consists of direct payments only.

Theorem 8. *If the feasibility-condition holds true and $b(t) = b$ for all t, then the optimum is reached by a payment arrangement consisting of direct payments only. The amount of each payment equals*

$$D = \frac{x-c}{\rho} + \sqrt{\left(\frac{x-c}{\rho}\right)^2 - \frac{cK}{\rho}}$$

and payments occur at a regular interval

$$\Delta T = \frac{1}{\rho} \log \frac{\rho D + c}{c}.$$

Proof: See Appendix A.4.

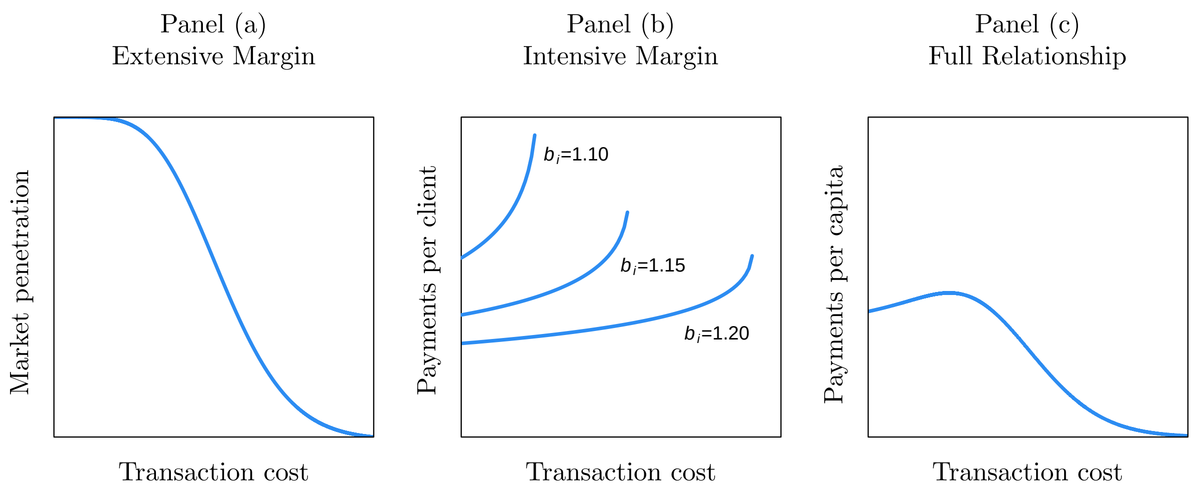
Corollary 2. *Within a stationary environment, the size of each payment increases and the frequency of payments decreases as the transaction cost decreases.*

Proof: Recall that $x = (b + c - \rho K)/2$. We have $\partial D/\partial K < 0$, and, hence, $\partial \Delta T/\partial K > 0$. ■

Some have expressed the expectation that a strong reduction in transactions costs enabled by technological developments in the payment space and the use of information technology to settle payments could lead to an explosion in the number of payments. In an extreme case, new payment technologies could lead to the use of so-called micro-payments in decentralized finance that essentially approximates "streaming money" through a constant flow of small payments. This expectation is not affirmed by Theorem 8, which reveals a more complex relationship between level of the transaction cost and the number of payments in an economy.

The impact of the transaction cost on the demand for payments in an economy differs for the extensive and the intensive margins. The impact along the *extensive* margin refers to the question as to whether a reduction in the transaction costs increases the number of buyer-seller relationships. This effect along the extensive margin is dominated by the feasibility-condition as expressed in Corollary 1 which can be used to assess whether the transaction cost is prohibitively expensive. The lower the transaction cost K, the larger the set of feasible buyer-seller relationships within an economy. Low-cost payments enables economic relationships with smaller margins, which contributes positively to the total number of payments in the economy. The impact along the *intensive* margin refers to how a reduction in the transaction cost impacts the total number of payments within an existing trading relationship. Corollary 2 shows that the payment frequency decreases as the transaction cost decreases. In other words, the optimal payment pattern within an existing trading relationship does not approximate "streaming money" as the transaction cost decreases. Instead, payments become less frequent (i.e., ΔT increases) with each payment increasing in size (i.e., D increases). The reason is that the reduction in the transaction cost increases the surplus from the buyer-seller relationship. A higher surplus increases the trust of the seller that the

Figure 3: Illustration of the Demand for Payments



Note: The figure provides for illustrative purposes the optimal number of payments in an economy based on Theorem 8. The figure is based on the following parameterization: $c = 1$; $\rho = 0.2$; heterogeneous consumers derive a flow benefit b_i that is drawn from a normal distribution with parameters $\mu = 1.15$ and $\sigma = 0.025$. The transaction cost K on the horizontal axis ranges from $(0, 0.05)$. Panel (a) reports the fraction of consumers for which trading is feasible. Panel (b) reports the optimal number of payments for clients with various levels of b_i in the range of values of the transaction cost where trading is feasible. Panel (c) reports the total number of payments per capita.

buyer pays when the time comes. As a consequence, the buyer pays the seller less frequently when the transaction costs are lower, which contributes to a reduction in the total number of payments in the economy.

The total impact of a decrease in the transaction cost on the demand for payments depends on the balance between the increase along the extensive margin and the decrease along the intensive margin. Figure 3 provides an illustration of the impact of the transaction cost on the number of payments along both the extensive and the intensive margins. The figure considers the number of payments in an economy with a continuum of heterogeneous buyers. The buyers derive a benefit b_i from consuming a service that a seller provide at a cost $c = 1$, where b_i is drawn from a normal distribution with $\mu = 1.15$ and $\sigma = 0.025$. The discount rate is set at $\rho = 0.2$. Panel (a) shows the relationship between the transaction cost K and market penetration, defined as the fraction of all consumers for which trading with the seller is feasible. Trading is feasible for fewer consumers when the transaction cost is higher (the extensive margin). Panel (b) shows the optimal number of payments per period within a trading relationship for clients with various levels of b_i. Clients must pay more often as the transaction cost increases (the intensive margin), at least, that is, until trading becomes infeasible. Panel (c) shows the full relationship by reporting the number of payments per capita as a function of the transaction cost. For this particular parameterization, the relationship is non-monotone. The effect along the intensive margin dominates for low transaction costs resulting in a positive relationship with the total number of payments. The effect along the extensive margin dominates for high transaction costs resulting in a negative relationship with the total number of payments. This parameterization is just one illustration of the potentially complex relationship between transaction costs and the total number of payments. Other paramaterizations can result in relationships that are exclusively positive or exclusively negative.[10]

7 Other Applications

When new payment methods are developed, it becomes important to understand the circumstances under which they are attractive. Our results can be thought of as illustrating when programmable payments will be desirable and when payments arrangements without programmable features will be successful. While our focus has been on programmable payments, our analysis has implications for mainstream payments arrangements as well.

[10] For a simple example where the effect along the intensive margin dominates, set $b_i = 1.2$ for all consumers while keeping all other parameters unchanged. This parameterization switches off the effect along the extensive margin and, hence, the number of payments per capita in such an environment will be a increasing function of the transaction cost (that is, until the transaction cost increases so much that trading becomes infeasible, after which the number of payments per capita drops to zero). For an example where the effect along the extensive margin dominates, let $b_i \sim U[1.0, 1.2]$ while keeping all other parameters unchanged. This parameterization results in an exclusively negative relationship.

7.1 Bill Payments

Bill payment refers to the process by which a purchaser extinguishes a debt established with a vendor. Bill payments are distinct from the more-commonly studied "DVP transactions" (delivery vs payment transactions)—transactions in which a spot exchange is made of a good or service for a monetary asset. While DVP transactions have been at the center of much of micro-founded monetary theory,[11] examinations of bill payments are much rarer (for some exceptions, see the references in Kahn and Roberds, 2009). Nonetheless, bill payments are a significant portion of the total value of consumer payments.[12] Relationships between suppliers and producers are also dominated by bill payments.

The fundamental economic distinction between the two types of payments is the existence or non-existence of a credit relationship. In a DVP transaction, no credit relationship need exist; indeed the individuals can be anonymous to one another with no prospect of any subsequent relationship. Bill payment on the other hand requires that the creditor have some prospect of a future meeting with the debtor, at least to present the bill, and some prospect that the presented bill will be honored. Moreover, there are important institutional differences between the two types of transactions. Different payments media are better suited to one or the other type of transaction: for example, physical cash is relatively inconvenient for bill payments, while online or electronic fund transfers tend to be relatively inconvenient for point-of-sale transactions.

Our model can be thought of as a microfoundation for the act of paying bills. On the face of it, the question "why do people pay their bills?" has an obvious answer: if buyer and seller know each other's identity then there is a legal threat when goods are not paid for. However, the legal remedy

[11] A natural and important framework for studying such transactions is the "day-night" models of Lagos and Wright (2005) and subsequent papers.

[12] Based on diaries of consumer payment choice in the United States, Greene and Stavins (2020) find that, when distinguishing between purchases (typically DVP) and bill payments, bill payments account for 20 percent of all payments by number and 63 percent of all payments by value. Other empirical research into consumer payment behavior distinguishes between point-of-sale transactions (typically DVP) and remote transactions (typically bill payments).

when bills remain unpaid may be expensive and therefore ineffective.[13] Important cases where this may arise are small-value transactions on one extreme and international trade on the other. As in our model, when repeat interactions are anticipated, the real threat enforcing bill payments is likely to be the loss of future value: Bills are paid in order to continue enjoying the benefits of the relationship. Theorem 6 describes the circumstances where the future benefits are sufficient to support direct payments of bills.

7.2 Middlemen

When the sticks of legal sanction and the carrots of continued cooperation prove inadequate to enforce payment, the parties can turn to a variety of third-party options, which we can summarize by the term "escrow." Essentially, when simultaneous payment and delivery is infeasible, the payment is made before the good or service is provided and a trusted custodian holds on to the payment until the buyer receives delivery. Traditional examples include real estate transactions and letters of credit where a creditworthy third-party assures payment for goods upon the seller providing proof of shipment.[14] From this point of view, programmable payments can be considered as an automated version of escrow. Theorem 1 describes the circumstances where reductions in transactions costs make trading through an escrow arrangement feasible.

Our model emphasizes the importance of repeated interaction in enforcing payment. An application of our results is in the context of arrangements that redirect payment flows. Credit cards provide an interesting example: a consumer repeatedly interacts with the card issuer even though the consumer does not interact repeatedly with a particular retailer. The consolidation of interactions means that it will be possible to sustain a direct payment arrangement (Theorem 7) instead

[13]Standard advice to creditors whose accounts have been sent to a collection agency is to negotiate for a partial payment to settle the debt rather then to repay the debt in full (Equifax, 2020). A report by the Federal Trade Commission found that going-concern debt buyers not specialized in bankruptcy debt purchased in the period 2006-09 paid on average only 4 cents for each dollar of debt (Leibowitz et al., 2013).

[14]Besides the traditional examples, a host of real-world institutions can be understood in this light, from, at the wholesale payment extreme, the CLS bank (Kahn and Roberds, 2001), down to guarantees offered by some payment cards when customers make purchases using the card.

of the more costly escrow or programmable payment arrangements that would be required in the absence of repeated interactions (Theorem 3). Moreover, the consolidation of payments makes trading feasible in situations where the transaction costs for individual payments would be prohibitively high. When interacting with $n > 1$ identical counterparties but making payments to a single card issuer, the feasibility-condition generalizes to the weaker condition

$$\sqrt{\rho K} + \sqrt{nc} \leq \sqrt{nb}.$$

This gives an argument for the network benefits of a payment system, not just from the possibility to receive payments from more individuals, but from the trust that it generates. There is an interesting analogy of this network benefit with the results of Koeppl et al. (2012) in the context of clearing houses.

8 Conclusion

The objective of this paper was to study the demand for programmable payments where transfers are automatically executed conditional upon preset objective criteria. We did so in a framework that allows for a wide range of economic situations. Our results show that programmable payments may be the only viable payment arrangement in situations where economic relationships are of a short duration. However, payment arrangements with direct payments dominate in long-term relationships. These results call into question whether sufficiently cheap programmable payments would replace most direct payments.

Our results also call into question the prediction that payments will be made more frequently as the transaction cost drops. The model shows that there is an important distinction between the impact of the transaction cost along the extensive and the intensive margins. Although the number of payments increases when the transaction cost drops for situations where the transaction cost would otherwise have stopped the buyer from making a purchase (the extensive margin), the optimal frequency to make payments in existing long-term relationships decreases when the transaction cost decreases (the intensive margin). This effect comes from the fact that cheaper payments increase the value of the relationship to the buyer, so that the seller can be more confident that the buyer will continue to make payments. The total effect of cheaper payments on the demand for payments will therefore depend on the difference between the effects along the extensive and the intensive margins.

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A Appendix: Proofs

In the appendix, we will use the following shorthand in order to simplify presentation of calculations:

$$B = \frac{b}{\rho}, \qquad C = \frac{c}{\rho}$$

$$\tau_i = e^{-\rho T_i}, \qquad \nu_i = e^{-\rho S_i}$$

Recall we assume B, C and K to be positive. We assume $B \geq C + K$, otherwise autarky would be optimal. We also define

$$X = \frac{1}{2}(B + C - K)$$

so that

$$X - C \geq 0$$

We let W denote the value received by the buyer upon a satisfactory end to the relationship. We assume $B > W \geq 0$. (Otherwise the buyer places more value on the post relationship phase than on the relationship itself.) The earlier sections of the paper consider $W = 0$. In the latter sections W will represent the continuation value from a new arrangement anticipated at the end of the first arrangement.

Terminology A *simple payment* is one where $\tau_i = \nu_i$, that is, the payin and payout are made simultaneously. A *programmable payment* is one which requires an escrow facility: $\tau_i > \nu_i$. In all arrangements $\nu_i \geq \tau_M$, that is, payouts must be made before the terminal date. When this constraint binds for the final payout, we will refer to the contract as *horizon-constrained.* An arrangement is *individually rational* if the payoff to the buyer is greater than 0. It meets the seller's *participation constraint at* ν_i, where $\nu_i = e^{-\rho S_i}$, if the arrangement going forward from S_i (and ignoring payout received at S_i) has non negative NPV.

A.1 Solution for Nonstationary Environment with One Payment

The **single-payment problem** is

$$\max_{D_1, \tau_1, \nu_1} B(1 - \nu_1) + W\nu_1 - (D_1 + K)\tau_1$$

subject to

$$C(1 - \nu_1) \leq D_1\nu_1, \tag{11}$$

$$(D_1 + K)\tau_1 \leq B(\tau_1 - \nu_1) + W\nu_1, \tag{12}$$

$$\tau_M \leq \nu_1 \leq \tau_1 \leq 1, \tag{13}$$

$$0 \leq D_1. \tag{14}$$

When $W = 0$ the program reduces to the maximization of $U(D_1, T_1, S_1)$ in (1) and conditions (2)-(5) in the main text.

Lemma A1. *The following are necessary and sufficient conditions for the existence of a solution to the single-payment problem:*

$$X^2 \geq (B - W)C \tag{15}$$

and

$$\tau_M \leq \frac{1}{B - W}(X + \sqrt{X^2 - (B - W)C}). \tag{16}$$

Proof. By condition (12), a triple (D_1, τ_1, ν_1) is infeasible unless $D_1 < B - K$ (since $B > W$). From (12) and $\tau_1 \leq 1$, a feasible triple (D_1, τ_1, ν_1) exists if and only if the triple $(D_1, 1, \nu_1)$ is feasible, that is, if and only if

$$1 \geq \nu_1 \geq \tau_M$$

and

$$\frac{B - K - D_1}{B - W} \geq \nu_1 \geq \frac{C}{C + D_1}.$$

See Figure A.1. The feasible set of pairs (D_1, ν_1) is non-empty if and only if 1) the curve $C/(C+D_1)$ intersects the line $(B - K - D)/(B - W)$, 2) the lower intersection lies below the line $\nu_1 = 1$ (equivalently, that the lower intersection lies to the right of the vertical axis) and 3) the upper intersection lies above the line $\nu_1 = \tau_M$. These requirements are equivalent to the two conditions of the lemma. (To see this, solve

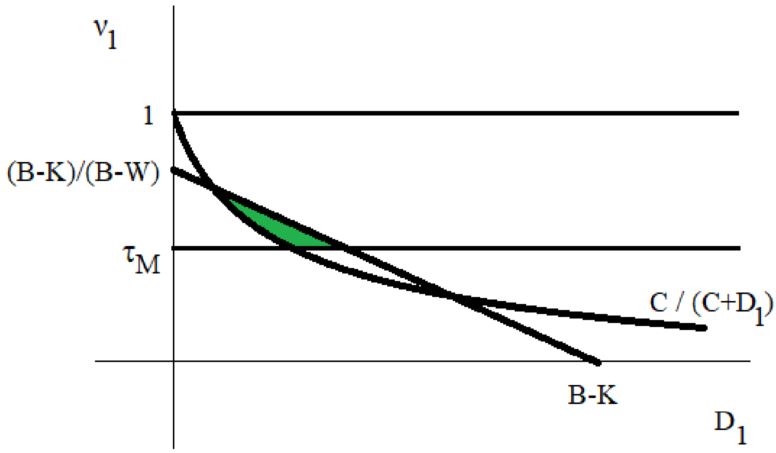
$$\frac{B - K - D_1}{B - W} = \frac{C}{C + D_1}$$

for D_1:

$$D_1 = X - C \pm \sqrt{X^2 - C(B - W)}.$$

The intersection occurs if and only if the discriminant is non-negative; this is condition (15). Since the larger root is positive, the second requirement is automatically satisfied. The third requirement

Figure A.1: Feasible Set of (D_1, ν_1) for the Single-Payment Arrangement



Note: The green area indicates the feasible set of values of (D_1, ν_1) for the single-payment arrangement.

is precisely condition (16).) With the feasible set non-empty, existence follows by continuity and compactness. ■

In particular, note that if $\tau_M \leq X/(B-W)$, (15) implies (16) so that (15) by itself is necessary and sufficient. When $W = 0$, condition (15) can be restated as

$$\sqrt{B} \geq \sqrt{C} + \sqrt{K}.$$

Thus this lemma generalizes Theorem 1.

The objective in the single-payment problem is decreasing in all arguments; reducing D_1 demonstrates that either (11) binds or $D_1 = 0$. In the latter case, (11) implies $\nu_1 = 1$, so (11) binds anyway. Hence, we must have

$$\nu_1 = \frac{C}{C + D_1}. \tag{17}$$

Reducing τ_1 demonstrates that either $\tau_1 = \nu_1$ (so that the payment is a simple direct payment) or (12) binds. Thus

Corollary A1. *If $W < K$ then a solution to the single-payment problem must be programmable.*

Proof. If $\tau_1 = \nu_1$ then (12) is violated when $W < K$. ■

On the other hand, when $W - K \geq 0$ a feasible solution *always* exists (cf. Figure A.1); in particular the direct payment $(D_1, \tau_1, \nu_1) = (0, 1, 1)$ is feasible and provides non-negative payoff to the buyer.

Next we characterize the solution when it is programmable, and provide a necessary condition for a solution to be programmable.

Lemma A2. *If the optimal single-payment arrangement is programmable, then*

$$D_1 = \min\left\{C(\tau_M^{-1} - 1), X - C\right\}, \tag{18}$$

$$\nu_1 = \max\left\{\tau_M, \frac{C}{X}\right\}, \tag{19}$$

$$\tau_1 = \frac{(B - W)\nu_1}{B - D_1 - K}. \tag{20}$$

Furthermore,

$$D_1 > W - K. \tag{21}$$

Proof. If $\tau_1 > \nu_1$ then (12) binds. Together these imply conditions (20)-(21). Using (17) and (20) to define ν_1 and τ_1, the problem can be rewritten as follows:

$$\max_{D_1} B\left(1 - \frac{(B - W)C}{(C + D_1)(B - D_1 - K)}\right)$$

subject to

$$D_1 \leq C\tau_M^{-1} - C \tag{22}$$

$$W - K \leq D_1 \tag{23}$$

$$(B - W)C \leq (B - D_1 - K)(D_1 + C) \tag{24}$$

$$0 \leq D_1. \tag{25}$$

The unrestricted maximum occurs at

$$D_1 = \frac{1}{2}(B - K - C) = X - C.$$

Restriction (25) does not bind, nor, assuming (21), does (23). Restriction (24) is satisfied iff the objective is non-negative; therefore it affects whether or not the problem is feasible, but it does not affect the choice of D_1 given that the problem is feasible.

The remaining restriction (22) may bind; since the objective is quasiconcave, (18) says that the optimal D_1 is either the unrestricted optimizer or the binding value from (22). Finally, (19) is obtained from plugging the solution for D_1 into (17). ■

An optimum must be a programmable payment if a simple direct payment is infeasible—for example, if $W = 0$. This fact and the above characterization prove Theorem 3.

Now we characterize the optimal simple direct payment contract, by substituting (11) and $\tau_1 = \nu_1$ into the single-payment problem, which becomes:

$$\max_{D_1} B - \frac{C(B - W + D_1 + K)}{D_1 + C}$$

subject to

$$D_1 \leq W - K, \tag{26}$$

$$\tau_M \leq \frac{C}{D_1 + C}, \tag{27}$$

$$0 \leq D_1. \tag{28}$$

From this we deduce the following

Lemma A3. *Suppose* $W - K \geq 0$.

1) If $W - K < B - C$ *then the optimum among direct payment contracts is*

$$D_1 = \min\{W - K, C(\tau_M^{-1} - 1)\}$$

$$\tau_1 = \nu_1 = \max\left\{\frac{C}{C + W - K}, \tau_M\right\}.$$

The payoff is

$$(1 - \tau_M)(B - C) + \tau_M(W - K)$$

if $\tau_1 = \tau_M$ *and*

$$\frac{B(W - K)}{C + W - K} \tag{29}$$

otherwise.

2) If $W - K > B - C$, *then the optimum among direct payment contracts is* $(D_1, \tau_1, \nu_1) = (0, 1, 1)$ *and the payoff is* $W - K$.

In the borderline case, the buyer is indifferent between the two direct payment contracts described in the lemma (as well as among all intermediate contracts).

When $W - K < 0$ a simple direct payment contract is infeasible, and the optimal contract, if it exists, must be programmable. If $W - K \geq 0$ a simple direct payment contract is always feasible. In this case, it remains to determine whether there exists a programmable contract which dominates simple payment contracts. For our purposes it suffices to focus on the situation where the problem is not horizon-constrained, that is $\tau_M \leq \nu_1$ does not bind. A sufficient condition for this is

$$\tau_M \leq \min\left\{\frac{C}{X}, \frac{C}{C + W - K}\right\}.$$

Lemma A4. *Suppose $W - K \geq 0$ and the horizon-constraint does not bind. Then the optimal contract is a programmable payment if and only if*

$$W - K < X - C \tag{30}$$

Otherwise the optimal contract is a direct payment.

Proof. Lemma A2 implies that in any optimal programmable payment arrangement

$$X - C \geq D_1 > W - K.$$

Conversely, given (30), a direct calculation shows that the contract in Lemma A2 when the solution is not horizon-constrained is feasible and that the payoff from the optimal programmable payment is

$$B\left(1 - \frac{(B - W)C}{X^2}\right). \tag{31}$$

Condition (30) implies $W - K < B - C$ so that the payoff in the optimal simple payment contract is (29). But then the difference between the two payoffs is

$$B\left(1 - \frac{(B - W)C}{X^2}\right) - \frac{B(W - K)}{C + W - K}$$

$$= B\left(\frac{C}{C+W-K} - \frac{(B-W)C}{X^2}\right)$$

$$= \frac{BC}{X^2(C+W-K)}(X^2 - (B-W)(C+W-K))$$

$$= \frac{BC}{X^2(C+W-K)}(X^2 - (B-W)(2X-B+W))$$

$$= \frac{BC}{X^2(C+W-K)}(B-X-W)^2$$

$$= \frac{BC}{X^2(C+W-K)}((X-C)-(W-K))^2 > 0$$

In other words under (30) a programmable payment dominates. ■

In short, the following is a complete characterization of the single-payment problem when the situation is not horizon-constrained: If $X^2 < (B-W)C$ the problem is infeasible. If $X^2 \geq (B-W)C$ and $W-K < X-C$ the solution is a programmable payment. If $B-C > W-K \geq X-C$ then the solution is a simple direct payment of a positive amount. If $W-K \geq B-C$ then the solution is a simple direct payment of 0 (immediate move to the post-contract relationship). When a solution with non-zero payment exists, then

$$(D_1, \nu_1, \tau_1) = \left(\max\{X-C, W-K\},\, \frac{C}{C+D_1},\, \frac{(B-W)\nu_1}{B-D_1-K}\right).$$

A.2 Solution for Non-Stationary Environment with Two Payments

Assuming

$$\nu_1 \geq \nu_2 \tag{32}$$

The **two-payment problem** is

$$\max_{D_1,D_2,\tau_1,\tau_2,\nu_1,\nu_2} B(1-\nu_2) + W\nu_2 - (D_1+K)\tau_1 - (D_2+K)\tau_2$$

subject to

$$C(1-\nu_2) \le D_1\nu_1 + D_2\nu_2 \tag{33}$$

$$C(\nu_1-\nu_2) \le D_2\nu_2 \tag{34}$$

$$(D_1+K)\tau_1 + (D_2+K)\tau_2 \le B(\tau_1-\nu_2) + W\nu_2 \tag{35}$$

$$(D_2+K)\tau_2 \le B(\tau_2-\nu_2) + W\nu_2 \tag{36}$$

$$D_i \ge 0, \quad i=1,2 \tag{37}$$

$$\nu_i \le \tau_i, \quad i=1,2 \tag{38}$$

$$\tau_M \le \nu_2 \tag{39}$$

$$\tau_2 \le \tau_1 \le 1 \tag{40}$$

(We will add (32) to the set of requirements. After solving we can verify (32) does not bind.)

The following lemma relates the feasibility of the single-payment problem and the feasibility of the two-payment problem:

Lemma A5. *If there exists a (individually-rational) solution to the two-payment problem either with $\tau_2 \le \nu_1$ or with conditions (34) and (36) binding, then there exists a (individually-rational) solution to the single-payment problem.*

Proof. Suppose $(D_1^*, D_2^*, \tau_1^*, \tau_2^*, \nu_1^*, \nu_2^*)$ satisfies the two-payment problem restrictions (32-40). If $\tau_2^* \le \nu_1^*$ then the triple

$$D_1 = D_2^*, \qquad \tau_1 = \frac{\tau_2^*}{\nu_1^*}, \qquad \nu_1 = \frac{\nu_2^*}{\nu_1^*}$$

satisfies the single-payment problem restrictions (11-14). (Verification: (11) follows from (34), (12) follows from (36),

$$\nu_1 = \frac{\nu_2^*}{\nu_1^*} \ge \nu_2^* \ge \tau_M$$

and the rest of the conditions are immediate.) Furthermore, non-negativity of the single-payment objective follows from (36), which implies (12), and $\tau_2^* \leq \nu_1^*$, which implies $\tau_1 \leq 1$.

Otherwise, if (34) and (36) bind, then $(D_1^*, \tau_1^*, \nu_1^*)$ directly satisfies the single-payment problem restrictions. (Subtracting (36) from (35), we have

$$(D_1 + K)\tau_1 \leq B(\tau_1 - \tau_2)$$

which in turn is less than the right hand side of (12). Similarly, (11) follows from (34) and (33). The other conditions are immediate.) Furthermore, when (36) binds, the objective in the two-payment problem can be rewritten as

$$B(1 - \tau_2) - (D_1 + K)\tau_1.$$

As long as $\tau_2 > \nu_1$, then the payoff in the single-payment problem is non-negative whenever the payoff in the two-payment problem is non-negative. ■

In this subsection we will find the optimum for the two-payment problem for the case where $W = 0$. In the following subsection we will solve the problem for $W > 0$ and τ_M sufficiently small.

Restriction (36) demonstrates that feasibility requires $\tau_2 > \nu_2$. Considering the effect of reducing τ_2 demonstrates that either $\tau_2 = \nu_2$ or (36) binds. We conclude that (36) must bind. Restriction (36) (or individual rationality) also guarantees that $D_2 < B - K$.

Considering the effect of reducing D_1 demonstrates that either $D_1 = 0$ or (33) binds. We will allow a single-payment arrangement as an alternative, in which case a two-payment arrangement with either $D_1 = 0$ or $D_2 = 0$ is dominated by a single-payment arrangement. Thus we can drop restrictions (64), and conclude that (33) binds in a two-payment optimal arrangement.

Considering the effect of reducing τ_1 demonstrates that either (35) binds or $\tau_1 = \nu_1$ or $\tau_1 = \tau_2$. However substituting (36) as equality into (35) yields the following condition:

$$(D_1 + K)\tau_1 \leq B(\tau_1 - \tau_2)$$

from which we conclude $\tau_1 > \tau_2$ so that only the two remaining alternatives are possible: (35) binds or $\tau_1 = \nu_1$ (we also conclude $D_1 < B - K$). Furthermore $1 \geq \tau_1$ will not bind (although the condition does have to be checked for feasibility in any proposed solution).

For future reference we also record the following lemma:

Lemma A6. *In an optimum with $\tau_1 = \nu_1$ and D_2 strictly positive,* (34) *binds.*

Constraint (34) is the seller's participation constraint at ν_1.

Proof. Suppose (34) does not bind. Decrease D_2 by a small amount Δ while increasing D_1 by $\Delta\nu_2/\nu_1$. Constraint (33) is unaffected. Constraint (36) is relaxed. The quantity $D_1\tau_1 + D_2\tau_2$ changes by the amount

$$\Delta\frac{\nu_2}{\nu_1}\tau_1 - \Delta\tau_2 < 0$$

relaxing (35) and improving the objective. Contradiction. ∎

Of the various conditions in (32), (38)-(40), the following remain as potentially binding:

$$\tau_1 \geq \nu_1 \geq \nu_2 \geq \tau_M.$$

Using the binding constraints (33) and (36) to define ν_1 and τ_2 we simplify the problem:

$$\max_{D_1, D_2, \tau_1, \nu_2} B - (D_1 + K)\tau_1 - \frac{B^2\nu_2}{B - D_2 - K}$$

subject to

$$C^2 \leq (C + D_2)(C + D_1)\nu_2 \tag{41}$$

$$B^2\nu_2 \leq (B - D_1 - K)(B - D_2 - K)\tau_1 \tag{42}$$

$$C \leq D_1\,\tau_1 + (C + D_2)\nu_2 \tag{43}$$

$$(C + D_1 + D_2)\nu_2 \leq C \tag{44}$$

$$\tau_M \leq \nu_2 \tag{45}$$

The objective is decreasing in all four variables. Thus (44) does not bind and can be omitted until the check for feasibility at the end. Reducing τ_1 demonstrates (42) or (43) binds.

Suppose (43) binds; use it to substitute for τ_1 :

$$\max_{D_1, D_2, \nu_2} B - (D_1 + K)\frac{C - (C + D_2)\nu_2}{D_1} - \frac{B^2\nu_2}{B - D_2 - K}$$

subject to

$$C^2 \leq (C + D_2)(C + D_1)\nu_2 \tag{46}$$

$$B^2\nu_2 \leq (B - D_1 - K)(B - D_2 - K)\frac{C - (C + D_2)\nu_2}{D_1} \tag{47}$$

$$\tau_M \leq \nu_2 \tag{48}$$

Now the objective *increases* with D_1, demonstrating that (47) (equivalently, (42)) binds. Therefore return to the previous version of the problem and use (42) as an equality. Substituting it into the objective function we see that the objective equals $B(1 - \tau_1)$ so that maximizing the objective is the same as minimizing τ_1 (and the result is individually rational for the buyer iff $\tau_1 \leq 1$.) Using (42) to eliminate τ_1 from the problem we have

$$\min_{D_1, D_2, \nu_2} \frac{B^2 \nu_2}{(B - D_1 - K)(B - D_2 - K)} \tag{49}$$

subject to

$$C^2 \leq (C + D_2)(C + D_1)\nu_2 \tag{50}$$

$$C - (C + D_2)\nu_2 \leq D_1 \frac{B^2 \nu_2}{(B - D_1 - K)(B - D_2 - K)} \tag{51}$$

$$\tau_M \leq \nu_2 \tag{52}$$

where either (50) or (51) binds.

Constraint (50) is a disguised version of the supplier's participation constraint at ν_1. Constraint (51) says a payout cannot precede the payin. If it binds, the first payment is a direct payment. However, if the first payment is a direct payment, then by the preceding lemma, the supplier's participation constraints bind. We conclude

Lemma A7. *In an optimal two payment arrangement, the supplier's participation constraints bind.*

We use the binding constraint to eliminate ν_2. The problem becomes

$$\min_{D_1, D_2} \frac{B^2 C^2}{(B - D_1 - K)(B - D_2 - K)(C + D_1)(C + D_2)}$$

subject to

$$C - \frac{C^2}{C + D_1} \leq \frac{D_1 B^2 C^2}{(B - D_1 - K)(B - D_2 - K)(C + D_1)(C + D_2)} \tag{53}$$

$$\tau_M \leq \frac{C^2}{(C + D_1)(C + D_2)} \tag{54}$$

Constraint (54) is the horizon-constraint. The problem can be restated more simply as

$$\max_{D_1, D_2} (2X - C - D_1)(2X - C - D_2)(C + D_1)(C + D_2)$$

subject to

$$(2X - C - D_1)(2X - C - D_2)(C + D_2) \leq B^2 C \tag{55}$$

$$(C + D_1)(C + D_2) \leq C^2 \tau_M^{-1} \tag{56}$$

Depending on the values of the three key parameters X, $C^2\tau_M^{-1}$, and B^2C, there are four possibilities, illustrated in Figure A.2. If neither constraint binds, the optimum is achieved by $D_1 = D_2 = X - C$ (top-left). If only the horizon-constraint (56) binds (bottom-left), then $D_1 = D_2 = C\tau_M^{-1/2} - C < X - C$. If only the prepayment-constraint (55) binds (top-right), then $D_1 > D_2 = X - C$. If constraints (55) and (56) both bind, then $D_2 < X - C$ as shown in the bottom-right panel of Figure A.2. (If the optimal values of D_1 or D_2 are not positive, that indicates that a single-payment arrangement or autarky automatically dominates all two-payment arrangements).

More specifically, the results so far in effect have demonstrated that in the original problem, the constraints (33)-(36) are all binding.

Constraints (33)-(36) can be solved simultaneously to find formulas for $\tau_1, \tau_2, \nu_1, \nu_2$ as functions of D_1, D_2:

$$\begin{aligned}
\nu_1 &= \frac{C}{D_1 + C},\\
\nu_2 &= \frac{C^2}{(D_1 + C)(D_2 + C)},\\
\tau_2 &= \frac{BC^2}{(B - D_2 - K)(D_1 + C)(D_2 + C)},\\
\tau_1 &= \frac{B^2C^2}{(B - D_1 - K)(B - D_2 - K)(D_1 + C)(D_2 + C)}.
\end{aligned}$$

It remains to specify when each of the four possibilities in Figure A.2 arises, and when the two-payment solutions are feasible and dominate the single-payment solution.

1) The unconstrained optimum has

$$D_1 = D_2 = X - C.$$

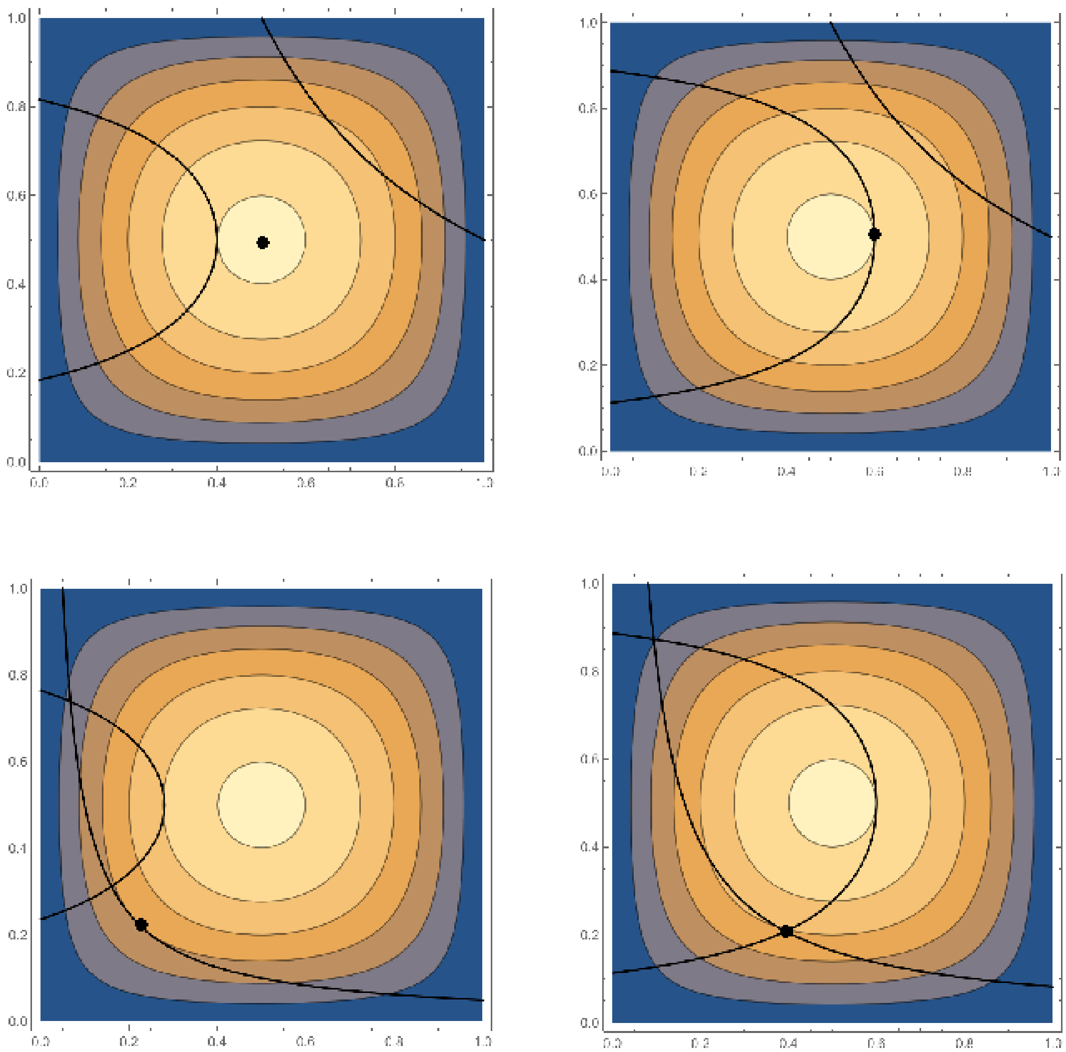
The top-left solution applies if both constraints (55) and (56) are satisfied when these values of D_1, D_2 are substituted. These conditions reduce to

$$\begin{aligned}
X^3 &\leq B^2 C,\\
X^2 &\leq C^2 \tau_M^{-1}.
\end{aligned}$$

2) Ignoring the prepayment-constraint, the optimum occurs when

$$D_1 = D_2 = \min\{C(\tau_M^{-1/2} - 1), X - C\}$$

Figure A.2: Possible Cases for the Two-Payment Solution



Note: The top-left panel illustrate the values of (D_1, D_2) where neither the prepayment-constraint in (55) nor the horizon-constraint in (56) bind (Theorem 4). The top-right panel illustrates the situation where only the prepayment-constraint binds so that the first payment is a direct payment (Theorem 5). The remaining panels illustrate the horizon-constrained solutions where the prepayment-constraint does not bind (bottom-left) or where it binds (bottom-right).

or equivalently, when $D_1 = D_2 = C(\nu_2^{-1/2} - 1)$ and

$$\nu_2 = \max\left\{\frac{C^2}{X^2}, \tau_M\right\}.$$

The prepayment-condition is satisfied when

$$B(1 - \nu_2^{1/4}) < C(\nu_2^{-1/2} - 1) + K$$

Therefore, necessary and sufficient conditions for the solution to be in the bottom-left corner are

$$\begin{aligned} &X^2 > C^2\tau_M^{-1}, \\ &2X \leq B\tau_M^{1/4} + C\tau_M^{-1/2}. \end{aligned}$$

These first two cases arise when both payments are programmable. The remaining two cases arise when the first payment is direct and the second is programmable.

3) Ignoring the horizon-constraint, the optimum occurs when $D_2 = X - C$ and

$$D_1 = \max\{X - C, 2X - C - \frac{B^2C}{X^2}\}. \tag{57}$$

Plugging this into the horizon-constraint, the top right solution applies if the top left conditions are violated and

$$\max\{X^2, 2X^2 - \frac{B^2C}{X}\} \leq C^2\tau_M^{-1}.$$

Therefore necessary and sufficient conditions for the solution to be in the top right corner are

$$\begin{aligned} X^3 &> B^2C \\ 2X^2 - \frac{B^2C}{X} &\leq C^2\tau_M^{-1}. \end{aligned}$$

Note therefore, that in this case, the payments are not equal. The second payment is the optimal payment in the single-payment problem, and the first payment is *greater*:

$$\begin{aligned} D_1 &> D_2 \\ \Leftrightarrow B - K - \frac{B^2C}{X^2} &> \frac{1}{2}(B - C - K), \\ \Leftrightarrow -\frac{B^2C}{X^2} &> -X \\ \Leftrightarrow B^2C &< X^3 \end{aligned}$$

which is the binding prepayment-constraint.

4) Finally, if none of these possibilities hold, then the bottom right solution applies. Necessary and sufficient conditions are therefore

$$\begin{aligned} 2X &> B\tau_M^{1/4} + C\tau_M^{-1/2} \\ 2X^2 - \frac{B^2C}{X} &> C^2\tau_M^{-1} \end{aligned}$$

Table 1 provides the formulas for the optimal choice variables in each of these four cases and, for comparison, in the single-payment case. Note that the payoff is $B(1 - \tau_1)$.

Corollary A2. *The solution for the optimal two-payment arrangement is horizon unconstrained iff*

$$C^2\tau_M^{-1} \geq \max\{X^2, 2X^2 - \frac{B^2C}{X}\}.$$

Proof. The horizon-constraint does not bind iff the solution belongs to Case 1 or Case 3 above. ■

The condition (10) in the main text reduces to $C^2\tau_M^{-1} \geq 2X^2$ implying the condition in Corollary A2 holds true. The formulas in Table 1 in the "unconstrained" columns for Cases 1 ("none direct") and 3 ("once direct") prove Theorems 4 and 5, respectively.

Table 1: Solutions of the One-Payment and Two-Payment Problem for $W = 0$

Choice variable	One payment (constrained)	One payment (unconstrained)	Two payments (constrained) None direct	Two payments (unconstrained) None direct	Two payments (constrained) Once direct	Two payments (unconstrained) Once direct
ν_1	τ_M	$\frac{C}{X}$	$\tau_M^{1/2}$	$\frac{C}{X}$	$\frac{C}{D_1+C}$	$\frac{CX^2}{2X^3-B^2C}$
τ_1	$\frac{\tau_M B}{B-D_1-K}$	$\frac{BC}{X^2}$	$\tau_M\left(\frac{B}{B-D_2-K}\right)^2$	$\frac{B^2C^2}{X^4}$	$\frac{C}{D_1+C}$	$\frac{CX^2}{2X^3-B^2C}$
D_1	$C(\tau_M^{-1}-1)$	$X-C$	$C(\tau_M^{-1/2}-1)$	$X-C$	$\frac{C^2}{\tau_M(D_2+C)}-C$	$2X-C-\frac{B^2C}{X^2}$
ν_2			τ_M	$\frac{C^2}{X^2}$	τ_M	$\frac{XC^2}{2X^3-B^2C}$
τ_2			$\tau_M\frac{B}{B-D_2-K}$	$\frac{BC^2}{X^3}$	$\tau_M\frac{B}{B-D_2-K}$	$\frac{BC^2}{2X^3-B^2C}$
D_2			$C(\tau_M^{-1/2}-1)$	$X-C$	$X-C-M$	$X-C$

Note: "Constrained" refers to the horizon-constrained case. "None direct" means two programmable payments, "once direct" means that only the last payment is programmable. The M in the solution to the horizon-constrained two-payment solution with one direct payment is defined as $M =: \left(\sqrt{16X^4 + C^4\tau_M^{-2} - 8B^2CX - 8C^2X^2\tau_M^{-1}} - C^2\tau_M^{-1}\right)/(4X)$.

A.3 Two Payments with W Positive and τ_M Small

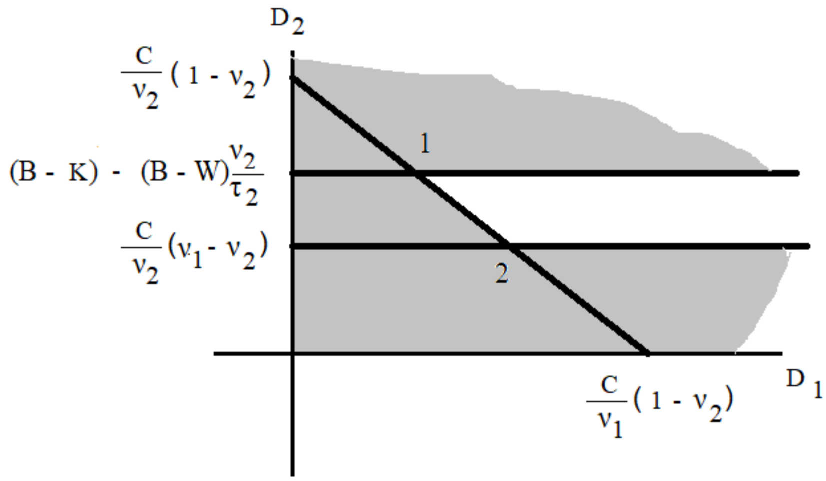
In this subsection, we suppose that an optimal solution exists and that it has two payments (so that D_1, D_2 are positive) and that $\nu_2 > \tau_M$.

Since the objective is decreasing in τ_2, either (36) binds, or $\tau_2 = \nu_2$. It follows that

Corollary A3. *The second payment is a programmable payment iff $D_2 > W - K$ (it is a simple direct payment iff $D_2 \leq W - K$).*

Proof. If $\tau_2 = \nu_2$ then from (36) $B - W \leq B - D_2 - K$. If $\tau_2 > \nu_2$ then (36) binds and $(B-W)\nu_2 = (B - D_2 - K)\tau_2$, so that $B - W > B - D_2 - K$. ∎

Figure A.3: Feasible Set of (D_1, D_2) for the Two-Payment Arrangement



Note: The unshaded area indicates the feasible set of values of (D_1, D_2) for the two-payment arrangement in a scenario where the continuation value can be positive ($W \geq 0$) and the horizon-constraint does not bind (τ_M small).

Lemma A8. *Constraint (33) binds. If there is no optimum in which constraint (34) binds then in any optimum, constraint (36) binds;*

Proof. The objective is decreasing in D_1 and D_2. Figure A.3 shows the restrictions that affect the choice of D_1 and D_2. The horizontal lines reflect, respectively, constraints (34) and (36) binding. The diagonal line reflects constraint (33) binding and has slope $-\nu_1/\nu_2$. (The remaining constraint involving D_1 and D_2 is (35). But as this constraint lies on an isoprofit line, it affects feasibility but not the choice of D_1, D_2.) The slope of the isoprofit lines when rewriting in terms of the value of D_2 as a function of D_1 is $-\tau_1/\tau_2$. If $\tau_1/\tau_2 \leq \nu_1/\nu_2$ then Point 2 is an optimum, and constraints (33) and (34) bind]. If $\tau_1/\tau_2 > \nu_1/\nu_2$ then Point 1 is the only optimum, and constraints (33) and (36) bind. ■

Lemma A9. $\tau_1 > \tau_2$.

Proof. Suppose instead $\tau_1 = \tau_2$. Then (36) cannot bind, otherwise (35) is violated. But if (36) does not bind, then $\tau_2 = \nu_2$ as mentioned before. Since $\tau_1 \geq \nu_1 \geq \nu_2$, they all must be equal. But a single payment is better than two simultaneous direct payments. ■

Since the objective is decreasing in τ_1, either (35) binds, or $\tau_1 = \nu_1$ (the previous lemma rules out $\tau_1 = \tau_2$). If (35) binds the objective can be rewritten as $B(1 - \tau_1)$ so that maximizing it is the same as minimizing τ_1.

Next we demonstrate that in fact

Lemma A10. *Constraint* (34) *always binds.*

Proof. If there is an optimum where (35) does not bind, then $\tau_1 = \nu_1$ and by an argument identical to that in the proof of Lemma A6, constraint (34) binds. The remainder will be proven by contradiction. Suppose there is an optimum where constraint (35) binds but constraint (34) does not; in this optimum constraints (33) and (36) would be binding (Lemma A8). We can use (33),(35),(36) as definitions of ν_1, τ_1, τ_2. We would have

$$\tau_1 = \frac{B}{B - D_1 - K}\tau_2$$
$$\tau_2 = \frac{B - W}{B - D_2 - K}\nu_2$$
$$\nu_1 = \frac{C - (C + D_2)\nu_2}{D_1}$$

and dropping the constraints assumed to be non-binding, the only ones remaining would be $\nu_2 \leq \tau_2$, $\nu_2 \leq \nu_1$, and $\tau_1 \leq 1$. Substituting the above definitions, the problem would become

$$\max_{D_1, D_2, \nu_2} B\left(1 - \frac{B(B - W)\nu_2}{(B - D_1 - K)(B - D_2 - K)}\right)$$

subject to

$$
\begin{aligned}
D_2 &\geq W - K \\
1 &\geq \frac{B(B-W)\nu_2}{(B-D_1-K)(B-D_2-K)} \\
\nu_2 &\leq \frac{C}{C+D_1+D_2}
\end{aligned}
$$

The objective would be declining in ν_2 with ν_2 unconstrained from below unless (34) binds. ■

In summary, we have established

$$\nu_1 = \frac{C}{C+D_1}, \tag{58}$$

$$\nu_2 = \nu_1 \frac{C}{C+D_2}, \tag{59}$$

$$\tau_2 = \max\left\{\nu_2, \frac{(B-W)\nu_2}{B-D_2-K}\right\}, \tag{60}$$

$$\tau_1 = \max\left\{\nu_1, \frac{(D_2+K)\tau_2 + (B-W)\nu_2}{B-D_1-K}\right\}. \tag{61}$$

Furthermore, the conditions of Lemma A5 are satisfied (if $\tau_2 = \nu_2$, (59) implies $\nu_1 \geq \tau_2$; otherwise (36) binds, as well as (34)). Thus a solution to the single-payment problem is feasible whenever a solution to the two-payment problem is feasible, as long as τ_M is small.

Corollary A4. *If $X^2 < (B-W)C$ a solution to the two-payment problem is infeasible.*

Proof. When τ_M is small, the result follows directly from the necessary condition in Lemma A1. But as τ_M increases, the feasible set shrinks, so the result holds, *a fortiori.* ■

A.3.1 Both Payments Programmable

If $\tau_1 > \nu_1$ and $\tau_2 > \nu_2$ we use (58-61) to define τ_i, ν_i and, recalling that when (35) binds the objective is equivalent to minimizing τ_1, the problem can be reformulated as

$$\min_{D_1, D_2} \frac{B(B-W)C^2}{(B-D_1-K)(B-D_2-K)(C+D_1)(C+D_2)}$$

The unconstrained minimum occurs at

$$D_1 = D_2 = X - C.$$

Payouts occur at

$$\nu_1 = \frac{C}{X}, \nu_2 = \frac{C^2}{X^2}.$$

Payins occur at

$$\tau_1 = \frac{B(B-W)C^2}{X^4}, \tau_2 = \frac{(B-W)C^2}{X^3}.$$

Assuming that τ_M is small and given the hypothesis $\tau_i > \nu_i$, $i = 1, 2$, these formulas satisfy all the conditions (32-40). As noted before, the second payment is a programmable payment (that is, $\tau_2 > \nu_2$) iff $D_2 > W - K$ which, in this case translates to $W < B - X$. Given this condition, the first payment is programmable iff

$$W < B - X\frac{X^2}{BC}.$$

The buyer's payoff from this solution is

$$B(1-\tau_1) = B - \frac{B^2C^2}{X^4}(B-W).$$

A.3.2 First Payment Direct, Second Programmable

Assume $\tau_1 = \nu_1$ while $\tau_2 > \nu_2$. In this situation, condition (36) will be binding and the objective can be rewritten as

$$\max_{D_1,D_2,\tau_1,\tau_2,\nu_1,\nu_2} B\left(1 - \frac{B-W}{B-D_2-K}\frac{C}{C+D_2}\frac{C}{C+D_1}\right) - \frac{C(D_1+K)}{C+D_1}$$

subject to (32-40). The optimal value for D_2 is

$$D_2 = X - C.$$

independent of D_1.

Substituting $D_2 = X - C$ into the objective function yields an expression that is increasing in D_1 provided that $CB(B-W)/X^2 + K \geq C$ (if this condition does not hold true, it would be optimal to set $D_1 = 0$ and a two-payment solution where the first payment is direct and the second payment is programmable would be suboptimal). Plugging the solutions for the other choice variables into constraint (35) and increasing D_1 until (35) binds gives

$$D_1 = 2X - C - (B-W)\frac{BC}{X^2}.$$

Substituting into the constraints gives

$$\tau_1 = \nu_1 = \frac{CX^2}{2X^3 - (B-W)BC}$$

and

$$\nu_2 = \nu_1\frac{C}{X} \quad \text{and} \quad \tau_2 = \nu_1\frac{C(B-W)}{X^2}.$$

Substituting the solutions for ν_2 and τ_2 into (61) gives us the following condition for when the first payment must be a direct payment (i.e., $\tau_1 = \nu_1$): $BC(B-W) \leq X^3$, which is the converse of the condition obtained in the previous subsection, assuming the second payment is programmable. Altogether, with the conditions for a nonzero D_1 and $\tau_2 > \nu_2$, the condition on W that results in a direct first payment is

$$B - X\frac{X^2}{BC} \leq W < B - \max\left\{X, (C-K)\frac{X^2}{BC}\right\}.$$

We have $X > (C-K)X^2/(BC) \Leftrightarrow B(C+K) > (C-K)^2$, so the condition on W can be summarized as

$$B - X\frac{X^2}{BC} \leq W < B - X.$$

Combining this with the results from the previous subsection we have

Corollary A5. *The second payment is a programmable payment iff $W < B - X$. If the second payment is programmable, $D_2 = X - C$.*

A.3.3 Second Payment a Direct Payment

Now assume $\tau_2 = \nu_2$. Using conditions (58-61), the two-payment problem can be rewritten as

$$\max_{D_1, D_2, \tau_1} B - \frac{C^2(B - W + D_2 + K)}{(C + D_1)(C + D_2)} - (D_1 + K)\tau_1$$

subject to

$$\frac{C^2(B-W+D_2+K)}{(C+D_1)(C+D_2)} \le (B-D_1-K)\tau_1 \tag{62}$$

$$D_2 \le W-K \tag{63}$$

$$D_i \ge 0, \quad i=1,2 \tag{64}$$

$$\frac{C}{C+D_1} \le \tau_1, \tag{65}$$

$$\tau_1 \le 1, \tag{66}$$

where (62) and (65) are derived from conditions (60) and (61), respectively, and (63) from condition (36). Holding the other variables constant for the moment, notice that when we move D_2 so as to improve the objective, we also relax constraint (62). The objective is monotonic for D_2 in the permitted range $[0, W-K]$; if it is monotonically increasing then the optimal $D_2 = W-K$. (If it is monotonically decreasing, then $D_2 = 0$, in which case the second payment is suboptimal). We conclude that a necessary condition for a solution of the form where $D_2 = W-K$ is

$$B-C \ge W-K \ge 0 \tag{67}$$

in which case

$$\tau_1 = \max\left\{\frac{C}{C+D_1}, \frac{BC^2}{(W+C-K)(B-D_1-K)(C+D_1)}\right\}. \tag{68}$$

Suppose $\tau_1 > \nu_1$. Then condition (62) would bind, and the objective would be equivalent to minimizing τ_1, which would equal the second term in the above expression. The minimum occurs at $D_1 = X-C$, which would give

$$\tau_1 = \frac{BC^2}{X^2(W+C-K)}; \qquad \nu_1 = \frac{C}{X}.$$

However, it turns out that for the parameter values that are relevant for us, this outcome is not possible:

Lemma A11. *If $X^2 > (B-W)C$, then a two-payment arrangement where a programmable payment is followed by a direct payment is not optimal.*

Proof. Consider the case $X^2 \geq BC$. Suppose a programmable payment is followed by a direct payment is optimal.

$$\tau_1 = \nu_1 \frac{BC}{X(W+C-K)} \leq \nu_1 \frac{BC}{X(B-X+C-K)} = \nu_1 \frac{BC}{X^2} \leq \nu_1$$

contradicting $\tau_1 > \nu_1$ (the first inequality holds by Corollary A5).

Consider the case $(B-W)C < X^2 < BC$. In this case, the payoff of the two-payment arrangement where a programmable payment followed by a direct payment is strictly dominated by a direct single-payment arrangement. The arrangement with the smallest value for τ_1 has the highest payoff. So, a direct single-payment arrangement strictly dominates a programmable payments followed by a direct payment iff

$$\frac{C}{W+C-K} < \frac{BC^2}{X^2(W+C-K)} \Leftrightarrow X^2 < BC.$$

Combining the two cases proves that a two-payment arrangement where a programmable payments is followed by a direct payment is not optimal if $(B-W)C < X^2$. ■

On the other hand, if $\tau_1 = \nu_1$, the objective reduces to

$$\max_{D_1} B - \frac{BC^2}{(C+D_1)(C+W-K)} - \frac{C(D_1+K)}{C+D_1}$$

or

$$\max_{D_1} B - \frac{C}{C+D_1}\left(\frac{BC}{C+W-K} + K + D_1\right)$$

subject to

$$\frac{BC}{C+W-K} + K + D_1 \leq B$$
$$D_1 \geq 0.$$

The objective is increasing in D_1 provided

$$\frac{BC}{C+W-K} \geq C-K$$

which follows from (67) (since $BC/(C+W-K) \geq BC/(C+B-C) = C \geq C-K$). Thus the solution is

$$D_1 = B-K-\frac{BC}{C+W-K}$$

provided (67) holds. (The value of D_1 is positive provided that $X^2 \geq BC$:

$$\begin{aligned}\frac{BC}{C+W-K} &\leq \frac{BC}{C+B-X-K} = \frac{BC}{X} \\ &\leq X = \frac{1}{2}(B-K+C) \\ &\leq B-K\end{aligned}$$

where the first inequality follows from Corollary A5).

A.4 Multiple Payments

To analyze the general case of multiple payments, we use the following insight: Suppose that T_M is so large as to pose no restriction on the calculations–that is, the problem is not horizon-constrained. Look for the optimal arrangement that uses N payments, and suppose the first payment in the arrangement occurs at T and withdrawal from escrow occurs at S. Then the arrangement starting at S is an optimal arrangement with $N-1$ payments. Therefore, summarize the $N-1$ payment optimum by the payoff it yields to the buyer. We treat this payoff as the value the buyer would receive at the end of a single-payment arrangement, thereby deriving the optimal terms for the first payment in an N-payment arrangement.

For this intuition to apply, it must be the case that there is no "overlap" between individual payments (hence, a chain of payments): each payout occurs before the next payin. We begin by verifying there is no overlap in the two-payment case and that the second payment indeed mimics the optimal single-payment solution.

Lemma A12. *Suppose $X^2 \geq BC$. In an optimal two-payment arrangement without horizon-constraint, $\tau_2 \leq \nu_1$.*

Proof. From the preceding subsection, there are two cases to consider:

1. If the first payment is programmable then the second payment is as well and

$$\tau_2 = \frac{(B-W)C^2}{X^3} \leq \frac{C}{X} = \nu_1$$

2. If the first payment is direct then $\nu_1 = \tau_1 > \tau_2$.

∎

The reason we are restricting attention to the case $X^2 \geq BC$ is that this condition is necessary for a single-payment arrangement to be feasible in the final round as the continuation value W reaches zero in the final round.

Lemma A13. *Suppose $X^2 \geq BC$. Then in an optimal horizon-unconstrained two-payment arrangement, the second payment constitutes an optimal horizon-unconstrained single-payment arrangement starting from time S_1*

Proof. From the preceding subsection, we have that in an optimal two-payment arrangement $D_2 = X - C$ iff $W < B - X$ and $D_2 = W - K$ otherwise. (Since $B - X = X - C + K$, this can also be written as $D_2 = \max\{X - C, W - K\}$.) This is identical to the optimal value for D_1 in the horizon-unconstrained single-payment arrangement in all cases where an optimal two-payment arrangement exists (i.e., for $W - K < B - C$; when $W - K > B - C$ a two-payment arrangement is inferior to paying 0 immediately to move to the post-arrangement payoff). Since there is no overlap in all cases, dividing τ_2 and ν_2 by ν_1 in the solution for the optimal two-payment arrangement yields the optimal values for τ_1 and ν_1 for the horizon-unconstrained optimal single-payment arrangement. ■

Now we can put these single-payment arrangements together. A *chain of payments* is a non-overlapping sequence of triples (D_n, ν_n, τ_n), $n = 1, \ldots, N$, where the subscript n now represents the nth payment *from the end.* Let W_N be the maximum buyer payoff in an N-payment chain, with $W_0 = 0$. Define

$$\begin{aligned} \delta_n &= \nu_n / \nu_{n+1} && (< 1) \\ \epsilon_n &= \tau_n / \nu_n && (\geq 1) \end{aligned}$$

With $W_0 = 0$, the last payment must be programmable. The preceding lemma implies that $(D_n, \delta_n, \delta_n \epsilon_n)$ is the optimal single-payment arrangement with terminal payoff W_{n-1}. Thus, by Lemma A2, as long as we are in the programmable payment case (and given that the problem is assumed horizon-unconstrained), we have

$$\begin{aligned} D_n &= X - C \\ \delta_n &= \frac{C}{X} \\ \epsilon_n &= \frac{B - W_{n-1}}{X}. \end{aligned}$$

The formula for the buyer's payoff (31) for the optimal horizon-unconstrained single-payment arrangement implies

$$W_n = B\left(1 - \frac{(B - W_{n-1})\,C}{X^2}\right)$$

where the last equation can also usefully be written as

$$B - W_n = (B - W_{n-1})\frac{BC}{X^2} = (B - W_0)\left(\frac{BC}{X^2}\right)^n. \tag{69}$$

So long as we are in the programmable payment region, D_n is constant. Moreover, δ_n is constant as well, meaning that the interval $S_n - S_{n-1}$ between payouts to the seller is constant. In this region,

$$\frac{\epsilon_{n+1}}{\epsilon_n} = \frac{B - W_n}{B - W_{n-1}} = \frac{BC}{X^2},$$

which has the interpretation that payins by the buyer are also made at constant frequency (although a different frequency from the payouts). Iterating this condition yields

$$\epsilon_n = \frac{B}{X}\left(\frac{BC}{X^2}\right)^{n-1}.$$

The feasibility condition in the single-payment case is necessary for $\epsilon_1 \geq 1$; otherwise this sequence could not get started. But if payment $n = 1$ (the final payment) is feasible, then this arrangement can be repeated backwards in time until the prepayment-constraint is violated. The process for W_n in (69) is monotonic; therefore extending the relationship with additional rounds of payments is always welfare-improving. Since the difference equation for W_n starts at 0 for $n = 1$ and asymptotes to B as n increases, the necessary condition for programmable payments in Lemma A4 will be violated in finite time. Suppose the condition is violated for $n+1$ but not for smaller (i.e. later) n, so that $n+1$ is the last direct payment. Then

$$\epsilon_n = \frac{B}{X}\left(\frac{BC}{X^2}\right)^{n-1} > 1$$

but

$$\frac{B}{X}\left(\frac{BC}{X^2}\right)^n \leq 1.$$

Thus this critical value of n satisfies

$$\left(\frac{BC}{X^2}\right)^n \leq \frac{X}{B} < \left(\frac{BC}{X^2}\right)^{n-1}.$$

In the region with simple direct payments,

$$\begin{aligned} D_n &= W_{n-1} - K \\ \delta_n &= \frac{C}{C + W_{n-1} - K} \\ \epsilon_n &= 1. \end{aligned}$$

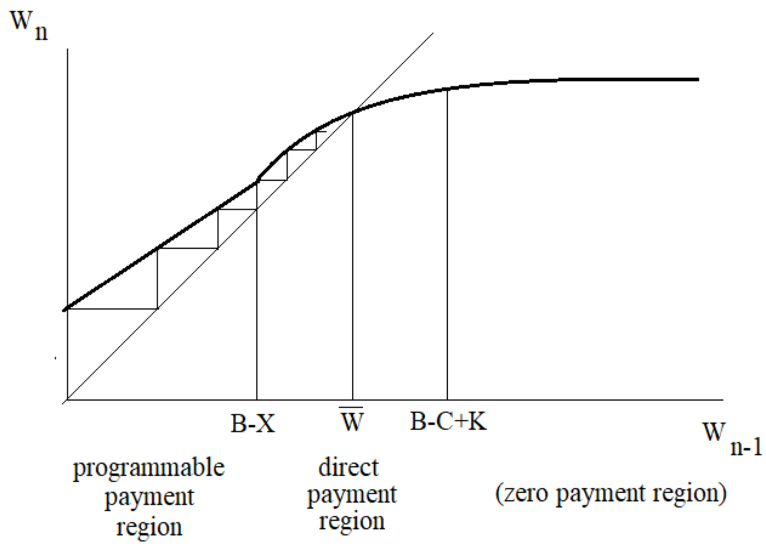
In this region, the dynamics are determined by the non-linear difference equation derived from (29):

$$W_n = B\left(\frac{W_{n-1} - K}{C + W_{n-1} - K}.\right)$$

The full difference equation is illustrated in Figure A.4. The feasibility constraint guarantees that the difference equation lies above the forty-five degree line at at the boundary between the two regions $B - X$. The series starts at $W_0 = 0$. Monotonicity of the function proves Theorem 6 where the expression for the threshold value of W for a direct payment comes from the condition in Lemma A4. The fact, already established, that the series crosses the boundary in finite time, proves Theorem 7. The series monotonically increases towards $\overline{W}$, the larger root of the steady state equation:

$$\overline{W} = \frac{1}{2}\left(B + K - C + \sqrt{(B + K - C)^2 - 4BK}\right) \tag{70}$$

Figure A.4: The Difference Equation for Chains of Multiple Payments



so that the payoff from the steady state arrangement is greater than the payoff of any finite payment arrangement. In the steady state

$$\overline{D} = \overline{W} - K$$
$$\bar{\delta} = \frac{C}{C + \overline{W} - K}$$

proving Theorem 8.