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Hao Wu[1,2]
René van den Brink[2]
Arantza-Estévez-Fernández[2]

1 Hunan University China

2 Vrije Universiteit Amsterdam

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Hao Wu *[1,2], René van den Brink[2], and Arantza Estévez-Fernández[3]

[1]College of Finance and Statistics, Hunan University, Changsha 410006, China
[2]Department of Economics, Vrije Universiteit Amsterdam, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands
[3]Department of Operations Analytics, Vrije Universiteit Amsterdam, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands

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Abstract

Cooperative games with a permission structure are useful tools for analyzing the impact of hierarchical structures on allocation problems in Economics and Operations Research. In this paper, we propose a generalization of the local disjunctive and the local conjunctive permission approaches called the *k-local permission approach*. In this approach, every player needs permission from a certain number of its predecessors to cooperate in a coalition. The special case where every player needs permission from at least one of, respectively all, its predecessors coincides with the local disjunctive, respectively local conjunctive, approach in the literature. We define and characterize a corresponding *k-local permission value*. After that, we apply this value to define a new class of power measures for directed graphs. We axiomatize these power measures, and apply some of them to two classical networks in the literature.
Keywords: TU-game; Hierarchical structure; Shapley value; Axiomatization; Digraph; Power measure.

1 Introduction

In this paper, we consider games with a permission structure where players in a cooperative transferable utility (TU) game belong to a hierarchical structure that restricts the possibilities of coalition formation. There are many applications of games with a permission structure. Examples are the polluted river sharing problems of Ni and Wang (2007), see also van den Brink et al. (2018), the hierarchy revenue sharing problems of Hougaard et al. (2017), and the joint liability sharing problems of Dehez and Ferey (2013), see also Oishi et al. (2022). There are two major approaches to permission restriction: conjunctive and disjunctive. In the conjunctive approach, as developed in Gilles et al. (1992) and van den Brink and Gilles (1996), each player

*Corresponding author: wuhao9305@gmail.com

needs permission from all of its predecessors before it is allowed to cooperate or give permission to others. Essentially, the conjunctive approach is a tool for portraying the power to veto. In the disjunctive approach as developed in Gilles and Owen (1994) and van den Brink (1997), each player needs permission from at least one of its predecessors (if it has any) before it is allowed to cooperate or give permission to its own successors. Whereas in both approaches mentioned above a player needs permission (either from all or at least one) of its predecessors to cooperate and to give permission to its successors, van den Brink and Dietz (2014) and Wu et al. (2022) weaken these two approaches by assuming that a player needs permission from its predecessors to cooperate, but does not need approval to give permission to its own successors to cooperate. These weakened approaches are called the local conjunctive and local disjunctive approaches.

In the underlying work, we generalize these two local approaches by assuming that every player needs permission from at least a certain number of its predecessors to cooperate in a coalition if the player has more predecessors than this given number. The special case where this number is one, respectively the total number of players minus one, coincides with the local disjunctive, respectively local conjunctive, approach. Associated to each approval level, we define a value for games with a permission structure in the usual way. Given a game with a permission structure, we define a restricted game that takes the cooperation restrictions into account and consider its Shapley value. This methodology gives a new class of values for games with a permission structure which we name the *k-local permission values*. Besides, we provide an axiomatization for every k-local permission value.

As an application, we consider the k-local permission value on additive games with a digraph as permission structure. In this way, we obtain a new class of power measures for directed networks, called *k-local permission measures*. Two special cases are obtained by taking k equal to at least the total number of players minus one, yielding the reflexive β-measure (van den Brink and Borm (2002)), and taking k equal to one, yielding the local disjunctive permission measure. We provide axiomatizations of the k-local permission measures, showing that these new power measures have the equal share feature of the reflexive β-measure and the equal loss feature of the local disjunctive permission measure. Finally, we extend these measures to weighted digraphs and apply the generalized measures to two classical data sets of directed networks, illustrating how they can be used to identify the key nodes in those networks.

This paper is organized as follows. Section 2 contains preliminaries. Section 3 introduces games with a k-local permission structure and provides an axiomatization of the k-local permission value for these games. Section 4 applies games with a k-local permission structure and the associated k-local permission value to additive games, obtaining power measures for digraphs. Section 5 applies the power measures to two classical examples of directed networks. Finally, section 6 provides concluding remarks. All proofs of propositions, theorems, and corollaries are postponed to the appendix, including logical independence of the axioms in each axiomatization.

2 Preliminaries

2.1 Cooperative TU-games and permission structures

A situation in which a finite set of players $N \subseteq \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, simply a *TU-game*. A TU-game is defined as a pair (N, ν) where $\nu : 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $\nu(\emptyset) = 0$. For every coalition $E \subseteq N$, $\nu(E) \in \mathbb{R}$ is the worth of coalition E. Since we take the player set N to be fixed, we often write a TU game (N, ν) simply by its characteristic function ν. We denote the collection of all TU-games on N by $\mathcal{G}^N$.

A *payoff vector* for $\nu \in \mathcal{G}^N$ is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A *solution* for TU-games is a function $f\colon \mathcal{G}^N \to \mathbb{R}^N$, which maps each TU-game into a payoff vector. One of the most famous solutions for TU-games is the *Shapley value*, Shapley (1953), given by $\mathcal{S}h_i(\nu) = \sum_{\substack{E \subseteq N \\ E \ni i}} p(E)(\nu(E) - \nu(E \setminus \{i\}))$ where $p(E) = \frac{(|N|-|E|)!(|E|-1)!}{|N|!}$.

For every $T \subseteq N$, $T \neq \emptyset$, the *unanimity game* on coalition T is given by $u_T(E) = 1$ if $T \subseteq E$, and $u_T(E) = 0$ otherwise. It is well known that unanimity games form a basis for $\mathcal{G}^N$, specifically, for every $\nu \in \mathcal{G}^N$ it holds that $\nu = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_\nu(T) u_T$, where $\Delta_\nu(T) = \sum_{E \subseteq T} (-1)^{|T|-|E|} \nu(E)$ is the *Harsanyi dividend* of coalition $T \subseteq N$, $T \neq \emptyset$, see Harsanyi (1959).

A game with a permission structure describes a situation where some players are granted the right to permit other players to join cooperation. This interactive relationship can be characterized by a *permission structure*, which can be described by a *digraph*. A digraph is a pair (N, D) where $N \subseteq \mathbb{N}$ is a finite set of nodes (which corresponds to the set of players in this paper) and $D \subseteq N \times N$ is a binary relation on N. Since we take the player set to be fixed, a digraph (N, D) is simplified to D. The set $S_D(i) = \{j \in N \mid (i, j) \in D\}$ is called the set of *successors* of i. The set $P_D(i) = \{j \in N \mid (j, i) \in D\}$ is called the set of *predecessors* of i. The set $H_D^N = \{i \in N \mid P_D(i) = \emptyset\}$ is called the set of *top players*. For a coalition $E \subseteq N$, we define $S_D(E) = \bigcup_{i \in E} S_D(i)$ and $P_D(E) = \bigcup_{i \in E} P_D(i)$.

In this paper, we only consider *irreflexive digraphs*, meaning that $(i, i) \notin D$ for all $i \in N$. We denote the collection of irreflexive digraphs on N by $\mathcal{D}^N$. A triple (N, ν, D) with $N \subseteq \mathbb{N}$ a finite set of players, $\nu \in \mathcal{G}^N$ a TU-game, and $D \in \mathcal{D}^N$ an irreflexive digraph, is called a *game with a permission structure*. Because the player set is fixed in this paper, we often write a game with a permission structure simply as a pair (ν, D). Given a player set N, a *solution* for games with a permission structure, $f : \mathcal{G}^N \times \mathcal{D}^N \to \mathbb{R}^N$, assigns a payoff vector in $\mathbb{R}^N$ to each $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$.

2.2 The local disjunctive and conjunctive permission values

There is a large number of solutions for games with a permission structure in the literature, as mentioned in the introduction. The new solutions discussed in this paper are general forms of the *local conjunctive permission value* and the *local disjunctive permission value*. For completeness, we first briefly review these two solutions.

The local conjunctive permission value is based on the local conjunctive approach (see van den Brink and Dietz, 2014), which assumes that each player needs permission from all of its predecessors before cooperation. Thus, the permission approach limits the possibilities of coalition formation. The largest *local conjunctive feasible subset* of a coalition $E \subseteq N$ is $\sigma_D^{lc}(E) = \{i \in E \mid P_D(i) \subseteq E\}$ and consists of those players in E whose predecessors all belong to E. The set of *local conjunctive feasible coalitions* is given by $\Psi_D^{lc} = \{\sigma_D^{lc}(E) \mid E \subseteq N\}$.[1] The *local conjunctive permission value* is the solution ψ^{lc} given by $\psi^{lc}(\nu, D) = \mathcal{S}h(r_{\nu,D}^{lc})$ where the game $r_{\nu,D}^{lc}$ is given by $r_{\nu,D}^{lc}(E) = \nu\big(\sigma_D^{lc}(E)\big)$ for all $E \subseteq N$, and thus assigns to every coalition the worth of its largest local conjunctive feasible subset.[2]

Before recalling an axiomatization of ψ^{lc}, we remind the following definitions. Player $i \in N$ is a *null player* in game ν if $\nu(E) = \nu(E \backslash \{i\})$ for all $E \subseteq N$. Player $i \in N$ is a *necessary player* in ν if $\nu(E) = 0$ for every $E \subseteq N \backslash \{i\}$. Game $\nu \in \mathcal{G}^N$ is *monotone* if $\nu(E) \leq \nu(F)$ for all $E \subseteq F \subseteq N$. We denote the collection of all monotone TU-games on N by $\mathcal{G}_M^N$.

Efficiency: For every $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, $\sum_{i \in N} f_i(\nu, D) = \nu(N)$.

Additivity: For every $\nu, \omega \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, $f(\nu + \omega, D) = f(\nu, D) + f(\omega, D)$.

Necessary player property: For every $\nu \in \mathcal{G}_M^N$ and $D \in \mathcal{D}^N$, if $i \in N$ is a necessary player in ν, then $f_i(\nu, D) \geq f_j(\nu, D)$ for all $j \in N$.

Local inessential player property: For every $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, if $i \in N$ is such that every player $j \in S_D(i) \cup \{i\}$ is a null player in ν, then $f_i(\nu, D) = 0$.

Local structural monotonicity: For every $\nu \in \mathcal{G}_M^N$, $D \in \mathcal{D}^N$, and $i \in N$ such that at least one player in $S_D(i)$ is a necessary player in ν, we have $f_i(\nu, D) \geq f_j(\nu, D)$ for all $j \in S_D(i)$.

Notice that the necessary player property and local structural monotonicity consider monotone games.

Theorem 2.1. *(van den Brink and Dietz, 2014) A solution is equal to the local conjunctive permission value ψ^{lc} if and only if it satisfies efficiency, additivity, the necessary player property, the local inessential player property, and local structural monotonicity.*

In the *local disjunctive approach* to games with a permission structure, each non-top player needs permission from at least one of its predecessors before it is allowed to cooperate. Therefore,

[1] The set of local conjunctive feasible coalitions is also defined as $\Psi_D^{lc} = \{E \subseteq N \mid P_D(i) \subseteq E \text{ for every } i \in E\}$.

[2] The fact that every coalition has a unique largest feasible subset follows from the set Ψ_D^{lc} being union closed, i.e. the union of every pair of feasible coalitions is also feasible.

for every coalition $E \subseteq N$, the largest *local disjunctive feasible subset* in $D \in \mathcal{D}^N$ is given by $\sigma^{ld}_D(E) = \{i \in E \mid P_D(i) \cap E \neq \emptyset\} \cup (E \cap H^N_D)$ and consists of all top players in E and all players in E with at least one predecessor in E. The set of *local disjunctive feasible coalitions* in D is given by $\Psi^{ld}_D = \{\sigma^{ld}_D(E) \mid E \subseteq N\}$.[3] The *local disjunctive permission value* is the solution ψ^{ld} given by $\psi^{ld}(\nu, D) = \mathcal{S}h(r^{ld}_{\nu,D})$ where the game $r^{ld}_{\nu,D}$ is given by $r^{ld}_{\nu,D}(E) = \nu\big(\sigma^{ld}_D(E)\big)$ for all $E \subseteq N$ and thus assigns to every coalition the worth of its largest local disjunctive feasible subset. An axiomatization of ψ^{ld} is provided by Wu et al. (2022) using the following weaker version of local structural monotonicity and local fairness to replace local structural monotonicity.

Weak local structural monotonicity: For every $\nu \in \mathcal{G}^N_M$ and $D \in \mathcal{D}^N$, if $j \in N$ is a necessary player in ν with $P_D(j) = \{i\}$, then $f_i(\nu, D) \geq f_j(\nu, D)$.

Local fairness[4]: For every $\nu \in \mathcal{G}^N$, $D \in \mathcal{D}^N$, $i \in N$ and $j \in S_D(i)$ with $|P_D(j)| \geq 2$, $f_i(\nu, D) - f_i(\nu, D \backslash \{(i,j)\}) = f_j(\nu, D) - f_j(\nu, D \backslash \{(i,j)\})$.

Theorem 2.2. *(Wu et al., 2022) A solution is equal to the local disjunctive permission value ψ^{ld} if and only if it satisfies efficiency, additivity, the necessary player property, the local inessential player property, weak local structural monotonicity, and local fairness.*

3 The k-local permission value

This section is divided into two parts. First, we introduce the k-local permission approach to cooperative games with a permission structure and explore relationships among their feasible coalitions. Second, we introduce and characterize the k-local permission value.

3.1 The k-local permission approach

In this subsection, we introduce the k-local permission approach, which is an extension of the local disjunctive and the local conjunctive approaches described in Section 2. In the k-local permission approach, given a number $k \in \mathbb{N}$, a player can cooperate only in the following two cases: (i) The player has permission from at least k of its predecessors (if the player has more than k predecessors), or (ii) the player has permission from all of its predecessors (if the player has at most k predecessors). Therefore, for every coalition $E \subseteq N$, the largest *k-local permission feasible subset* in $D \in \mathcal{D}^N$ is given by

$$\sigma^{lk}_D(E) = \big\{i \in E \mid |P_D(i) \cap E| \geq \min\{k, |P_D(i)|\}\big\},$$

[3]The set of local disjunctive feasible coalitions is also defined as $\Psi^{ld}_D = \{E \subseteq N \mid P_D(i) \cap E \neq \emptyset$ for every $i \in E \setminus H^N_D\}$.

[4]Notice that this axiom is similar to fairness for (undirected) communication graph games in Myerson (1977).

and the set of k-local permission feasible coalitions in D is given by[5]

$$\Psi_D^{lk} = \{\sigma_D^{lk}(E) \mid E \subseteq N\}.$$

The *k-local permission restricted game* induced by the game with a permission structure (ν, D) is the game $r_{\nu,D}^{lk}$ that assigns to every coalition $E \subseteq N$ the worth of its largest k-local permission feasible subset, i.e.

$$r_{\nu,D}^{lk}(E) = \nu\big(\sigma_D^{lk}(E)\big) \text{ for all } E \subseteq N.$$

Then, we define the *k-local permission value* ψ^{lk} as the solution that assigns to every game with a permission structure (ν, D) the Shapley value of the k-local permission restricted game, i.e.

$$\psi^{lk}(\nu, D) = \mathcal{S}h(r_{\nu,D}^{lk}) \text{ for all } \nu \in \mathcal{G}^N \text{ and } D \in \mathcal{D}^N.$$

Obviously, the local disjunctive and the local conjunctive permission approaches are special cases of the k-local permission approach by taking $k = 1$ and $k \geq |N| - 1$, respectively.

Example 3.1. *Let $k = 2$ and $N = \{1, 2, 3, 4\}$. Consider the game with a permission structure (ν, D) with $\nu = u_{\{4\}}$ and $D = \{(1, 4), (2, 4), (3, 4)\}$, see Figure 1.*

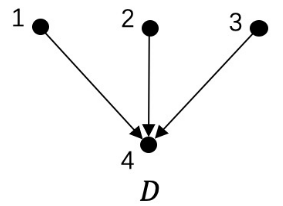

Figure 1: The digraph of Example 3.1.

We have $\Psi_D^{lk} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\big\}$ *and*

$$r_{\nu,D}^{lk}(E) = \begin{cases} 1 \text{ if } E \in \big\{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\big\}, \\ 0 \text{ otherwise.} \end{cases}$$

Thus, $\psi^{lk}(\nu, D) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2})$. ◇

For every $D \in \mathcal{D}^N$ and $E \subseteq N$, it is obvious that $\sigma_D^{lc}(E) \subseteq \sigma_D^{ld}(E)$. The following result provides statements relating different k-local permission feasible coalitions.

Proposition 3.1. *Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E \subseteq N$.*
(i) For every $F \subseteq E$, $\sigma_D^{lk}(F) \subseteq \sigma_D^{lk}(E)$.
(ii) For every $t \in \mathbb{N}$ with $t \leq k$, $\sigma_D^{lk}(E) \subseteq \sigma_D^{lt}(E)$.

[5]The set of k-local permission feasible coalitions is also defined as $\Psi_D^{lk} = \{E \subseteq N \mid |P_D(i) \cap E| \geq \min\{k, |P_D(i)|\}$ for every $i \in E\}$.

Based on Proposition 3.1, we obtain similar results for the set of k-local permission feasible coalitions to those for the locally conjunctive or disjunctive feasible coalitions in van den Brink and Dietz (2014), respectively Wu et al. (2022).

Proposition 3.2. *Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E \subseteq N$.*
(i) For every $F \subseteq E$ with $\sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big) \subseteq F$, $\sigma_D^{lk}(F) = \sigma_D^{lk}(E)$.
(ii) For every $F \subseteq E \backslash \sigma_D^{lk}(E)$, $\sigma_D^{lk}(F) = \emptyset$.

Proposition 3.3. *For every $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E, F \subseteq N$,*
(i) $\sigma_D^{lk}(E) \cup \sigma_D^{lk}(F) \subseteq \sigma_D^{lk}(E \cup F)$.
(ii) $\sigma_D^{lk}(E \cap F) \subseteq \sigma_D^{lk}(E) \cap \sigma_D^{lk}(F)$.

We conclude this subsection with an illustrative example.

Example 3.2. *Let $k = 2$, $N = \{1,2,3,4,5,6,7\}$, and $D = \{(1,2),(2,4),(3,4),(4,7),(5,7),(6,7)\}$, see Figure 2. Consider the two coalitions $E = \{2,3,5,6,7\}$ (the coalition of all triangle points) and $F = \{4,5,7\}$ (the coalition of all square points). We have $\sigma_D^{lk}(E) = \{3,5,6,7\}$, $\sigma_D^{lk}(F) = \{5,7\}$, $\sigma_D^{lk}(E \cup F) = \sigma^{lk}(\{2,3,4,5,6,7\}) = \{3,4,5,6,7\}$, and $\sigma_D^{lk}(E \cap F) = \sigma^{lk}(\{5,7\}) = \{5\}$. Thus, $\sigma_D^{lk}(E) \cup \sigma_D^{lk}(F) = \{3,5,6,7\} \subsetneq \{3,4,5,6,7\} = \sigma_D^{lk}(E \cup F)$ and $\sigma_D^{lk}(E \cap F) = \{5\} \subsetneq \{5,7\} = \sigma_D^{lk}(E) \cap \sigma_D^{lk}(F)$.*[6] $\diamond$

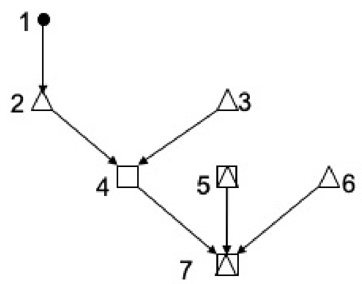

Figure 2: The digraph of Example 3.2.

Remark 3.1. Although we defined the k-local permission value only for $k \in \mathbb{N}$, notice that taking $k = 0$, $\sigma_D^{lk}(E) = E$ for all $E \subseteq N$ and $D \in \mathcal{D}^N$. Thus, when $k = 0$, the k-local permission value boils down to the Shapley value. Since not all results in the remainder of the paper hold for $k = 0$, we restrict to $k \in \mathbb{N}$.

3.2 An axiomatization of the k-local permission value

In this subsection, we characterize the k-local permission value inspired by Theorems 2.1 and 2.2. Since $\psi^{lk}(\nu, D)$ is obtained by applying the Shapley value to a modified game, the k-local

[6] From van den Brink and Dietz (2014), we have $\sigma_D^{lc}(E) \cap \sigma_D^{lc}(F) = \sigma_D^{lc}(E \cap F)$. However, from Example 3.2, the equality does not hold for arbitrary k.

permission value satisfies efficiency and additivity since $\sigma_D^{lk}(N) = N$ and $r_{\nu+\omega,D}^{lk} = r_{\nu,D}^{lk} + r_{\omega,D}^{lk}$ for every $\nu, \omega \in \mathcal{G}^N$. Besides, the k-local permission value satisfies the local inessential player property, see the proof of Theorem 3.1. However, the k-local permission value does not satisfy local fairness, as shown in the following example.

Example 3.3. *Let $N = \{1, 2, 3, 4\}$. Consider the game with permission structure (ν, D) with $\nu = u_{\{4\}}$ and $D = \{(1, 4), (2, 4), (3, 4)\}$, see Figure 3.*

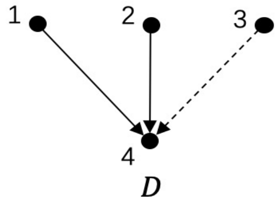

Figure 3: The digraph of Example 3.3.

We have

$$\psi^{lk}(\nu, D) = \begin{cases} (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{3}{4}) & \text{if } k = 1, \\ (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}) & \text{if } k = 2, \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & \text{if } k = 3, \end{cases} \quad \text{and} \quad \psi^{lk}\big(\nu, D \backslash \{(3, 4)\}\big) = \begin{cases} (\frac{1}{6}, \frac{1}{6}, 0, \frac{2}{3}) & \text{if } k = 1, \\ (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}) & \text{if } k = 2, \\ (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}) & \text{if } k = 3. \end{cases}$$

Notice that

$$\psi_3^{lk}(\nu, D) - \psi_3^{lk}\big(\nu, D \setminus \{(3, 4)\}\big) = \psi_4^{lk}(\nu, D) - \psi_4^{lk}\big(\nu, D \setminus \{(3, 4)\}\big) \text{ for every } k < 3 = |P_D(4)|,$$

but

$$\psi_3^{lk}(\nu, D) - \psi_3^{lk}\big(\nu, D \setminus \{(3, 4)\}\big) \neq \psi_4^{lk}(\nu, D) - \psi_4^{lk}\big(\nu, D \setminus \{(3, 4)\}\big) \text{ if } k = 3 = |P_D(4)|.$$

$\diamond$

As hinted in Example 3.3, the fairness equalities hold when deleting an arc where the successor on the arc has more than k predecessors. Let $k \in \mathbb{N}$. k-fairness is defined as follows.

k-**fairness**: For every $\nu \in \mathcal{G}^N$, $D \in \mathcal{D}^N$, $i \in N$, and $j \in S_D(i)$ with $|P_D(j)| > k$, $f_i(\nu, D) - f_i\big(\nu, D \setminus \{(i, j)\}\big) = f_j(\nu, D) - f_j\big(\nu, D \setminus \{(i, j)\}\big)$.

This weaker version of local fairness expresses the "equal treatment" with respect to the benefits (or loss) that are obtained from creating (or deleting) an arc only in case the number of predecessors of the successor on the arc exceeds a certain threshold.

In order to characterize the k-local permission value, we also need a "k-version" of structural monotonicity. Let $k \in \mathbb{N}$. k-structural monotonicity is defined as follows.

k**-structural monotonicity**: For every $\nu \in \mathcal{G}_M^N$, $D \in \mathcal{D}^N$, and $j \in N$ a necessary player in ν with $0 < |P_D(j)| \leq k$, $f_i(\nu, D) \geq f_j(\nu, D)$ for every $i \in P_D(j)$.

The k-local permission value is characterized by replacing local fairness and weak local structural monotonicity in Theorem 2.2 by the corresponding k-fairness and k-structural monotonicity axioms.

Theorem 3.1. *Let $k \in \mathbb{N}$. A solution is equal to the k-local permission value ψ^{lk} if and only if it satisfies efficiency, additivity, the necessary player property, the local inessential player property, k-fairness, and k-structural monotonicity.*

Remark 3.2. When $k = 1$, k-fairness boils down to local fairness, and k-structural monotonicity becomes weak local structural monotonicity. The corresponding solution is the local disjunctive permission value: $\psi^{lk}(\nu, D) = \psi^{ld}(\nu, D)$. When $k \geq |N| - 1$, k-fairness has no meaning, and k-structural monotonicity turns into local structural monotonicity. The corresponding solution is the local conjunctive permission value: $\psi^{lk}(\nu, D) = \psi^{lc}(\nu, D)$.

Remark 3.3. Notice that for $D = \emptyset$, the k-local permission value boils down to the Shapley value ($\psi^{lk}(\nu, D) = Sh(\nu)$) since $r^{lk}_{\nu,D} = \nu$. In that case, the axioms of Theorem 3.1 boil down to axioms characterizing the Shapley value for TU-games. Efficiency and additivity are already the classic axioms, the necessary player property implies symmetry for unanimity games, and the local inessential player property is reduced to the classical null player property. Further, k-fairness and k-structural monotonicity have no meaning.

4 Additive games with a permission structure: locally partial permission power measures for directed networks

Digraphs are often used to describe networks with asymmetric relationships, such as buying and selling relationships in exchange networks, asymmetric relationships in social networks, and winning or losing matches in sports competitions. There is an extensive literature that studies these digraphs and defines related measures, such as Kondratev and Mazalov (2002), Kinne (2012), and Riquelme et al. (2018). In this section, we aim to apply the k-local permission value to define a new class of power or dominance measures for digraphs.

4.1 The k-local permission measure

A *power measure* for digraphs is a function $m : \mathcal{D}^N \to \mathbb{R}^N$ that assigns a real number to every node in a digraph. These numbers can be seen as measures of 'power', 'importance', or 'dominance' of the nodes in the digraph. Examples of power measures can be found in, e.g. Bonacich (1987), van den Brink and Gilles (2000), and Boldi and Vigna (2014). Based on the k-local permission value, we introduce a new class of power measures that have some similarity with the *reflexive β-measure* (van den Brink and Borm (2002)). Therefore, we first recall this measure.

The *reflexive β-measure*, $\beta^{refl} : \mathcal{D}^N \to \mathbb{R}^N$, shares the power over a node equally among itself and its predecessors, and is given by

$$\beta_i^{refl}(D) = \sum_{j \in S_D(i) \cup \{i\}} \frac{1}{|P_D(j)| + 1} \text{ for all } i \in N.$$

It is worth mentioning that the reflexive β-measure can also be seen as the local conjunctive permission value of the game with a permission structure (ν^1, D), where $D \in \mathcal{D}^N$ and ν^1 is the additive game given by

$$\nu^1(E) = |E| \text{ for all } E \subseteq N,$$

i.e. $\beta_i^{refl}(D) = \psi^{lc}(\nu^1, D)$.

We can define a new measure, called the *local disjunctive permission measure* $l : \mathcal{D}^N \to \mathbb{R}^N$, in a similar way by applying the local disjunctive permission value to the game with a permission structure (ν^1, D), and thus is given by

$$l_i(D) = \psi_i^{ld}(\nu^1, D) = \begin{cases} 1 + \sum_{j \in S_D(i)} \frac{1}{|P_D(j)|(|P_D(j)|+1)} & \text{if } P_D(i) = \emptyset, \\ \frac{|P_D(i)|}{|P_D(i)|+1} + \sum_{j \in S_D(i)} \frac{1}{|P_D(j)|(|P_D(j)|+1)} & \text{otherwise.} \end{cases}$$

Analogously, we define a class of power measures containing the above two measures by applying the k-local permission value to the game with a permission structure (ν^1, D).

Definition 4.1. The *k-local permission measure* on N is the function $p^k : \mathcal{D}^N \to \mathbb{R}^N$ given by $p^k(D) = \psi^{lk}(\nu^1, D)$.

The k-local permission measure can be expressed as in the next result.

Proposition 4.1. *For every $D \in \mathcal{D}^N$,*

$$p_i^k(D) = \frac{|P_D(i)| + 1 - \min\{k, |P_D(i)|\}}{|P_D(i)| + 1} + \sum_{j \in S_D(i)} \frac{\min\{k, |P_D(j)|\}}{|P_D(j)|\big(|P_D(j)| + 1\big)} \text{ for all } i \in N.$$

Following the relationships between the k-local permission value and the local conjunctive or disjunctive permission values, we can easily verify the following proposition.

Proposition 4.2. *For every $D \in \mathcal{D}^N$, $p^1(D) = l(D)$, and $p^t(D) = \beta^{refl}(D)$ for $t \geq |N| - 1$.*

Next, we want to characterize the k-local permission value. The first three axioms introduced are efficiency norm, local determinateness and symmetry, which are also satisfied by the reflexive β-measure.

Efficiency norm: If $D \in \mathcal{D}^N$, then $\sum_{i \in N} m_i(D) = |N|$.

Local determinateness: If $D, D' \in \mathcal{D}^N$ and $i \in N$ are such that $S_D(i) = S_{D'}(i)$ and $P_D(j) = P_{D'}(j)$ for every $j \in \{i\} \cup S_D(i)$, then $m_i(D) = m_i(D')$.

Symmetry: If $D \in \mathcal{D}^N$ and $i, j \in N$ are such that $S_D(i) = S_D(j)$ and $P_D(i) = P_D(j)$, then $m_i(D) = m_j(D)$.

The above axioms commonly appear in the literature. Next, we state the following two axioms. The first one is the k-equal loss property which considers the power change when an arc is broken and the related successor has more than k predecessors.

k-equal loss property: If $D \in \mathcal{D}^N$, $i \in N$, and $j \in S_D(i)$ with $|P_D(j)| > k$, then $m_i(D) - m_i\big(D\backslash\{(i,j)\}\big) = m_j(D) - m_j\big(D\backslash\{(i,j)\}\big)$.

The k-equal loss property is obtained by applying k-fairness to the (additive) game ν^1 with a permission structure associated to the digraph.

The next axiom is called k-equal sharing property, and requires that deleting an arc has the same effect on the power of the predecessor and successor on the arc in case the successor on the arc has at most k predecessors.

k-equal sharing property: If $D \in \mathcal{D}^N$ and $i \in N$ with $0 < |P_D(i)| \leq k$, then $m_i\big(D \setminus \{(i,j) \mid j \in S_D(i)\}\big) = m_h(D) - m_h\big(D \setminus \{(h,i)\}\big)$ for every $h \in P_D(i)$.

Next, we can state an axiomatization of the k-local permission measure.

Theorem 4.1. *Let $k \in \mathbb{N}$. A power measure is equal to the k-local permission measure p^k if and only if it satisfies efficiency norm, local determinateness, symmetry, the k-equal loss property, and the k-equal sharing property.*

4.2 Extending k-local permission measures to node-weighted digraphs

Since nodes are not equally important in many applications, in this section, we consider extending k-local permission measures to node-weighted digraphs (simply weighted digraphs). Denote the set of weight vectors $\mathcal{W}^N = \{\omega \in \mathbb{R}^N \mid \sum_{i \in N} \omega_i = 1, \text{ and } 0 \leq \omega_i \leq 1\}$. For every $\omega \in \mathcal{W}^N$, denote the game $\nu^\omega = \sum_{i \in N} \omega_i u_{\{i\}}$. Notice that these are additive games. A *generalized power measure* for weighted digraphs is a function $gm : \mathcal{D}^N \times \mathcal{W}^N \to \mathbb{R}^N$ that assigns a real number to every node in a weighted digraph. In a straightforward manner, we extend the k-local permission measure to weighted digraphs by applying the k-local permission value to the game ν^ω.

Definition 4.2. The generalized k-local permission measure on N is the function $gp^k : \mathcal{D}^N \times \mathcal{W}^N \to \mathbb{R}^N$ given by $gp^k(D, \omega) = \psi^{lk}(\nu^\omega, D)$.

Based on Proposition 4.1, the expression of the generalized measures can be obtained as follows by multiplying the terms in this expression by the relevant node weights.

Proposition 4.3. *For every $D \in \mathcal{D}^N$ and $w \in \mathcal{W}^N$,*

$$gp_i^k(D,\omega) = \frac{|P_D(i)| + 1 - \min\{k, |P_D(i)|\}}{|P_D(i)| + 1}\omega_i + \sum_{j \in S_D(i)} \frac{\min\{k, |P_D(j)|\}}{|P_D(j)|(|P_D(j)| + 1)}\omega_j \text{ for all } i \in N.$$

Also, we can generalize the axioms of Theorem 4.1 to weighted digraphs as expected.

Generalized efficiency norm: If $D \in \mathcal{D}^N$ and $\omega \in \mathcal{W}^N$, then $\sum_{i \in N} gm_i(D,\omega) = \sum_{i \in N} w_i$.

Generalized local determinateness: If $D, D' \in \mathcal{D}^N$, $\omega \in \mathcal{W}^N$ and $i \in N$ are such that $S_D(i) = S_{D'}(i)$ and $P_D(j) = P_{D'}(j)$ for every $j \in \{i\} \cup S_D(i)$, then $gm_i(D,\omega) = gm_i(D',\omega)$.

Generalized symmetry: If $D \in \mathcal{D}^N$, $\omega \in \mathcal{W}^N$, and $i, j \in N$ are such that $S_D(i) = S_D(j) = \emptyset$ and $P_D(i) = P_D(j) = \emptyset$, then $\omega_j \cdot gm_i(D,\omega) = \omega_i \cdot gm_j(D,\omega)$.

Generalized k-equal loss property: If $D \in \mathcal{D}^N$, $\omega \in \mathcal{W}^N$, $i \in N$, and $j \in S_D(i)$ with $|P_D(j)| > k$, then $gm_i(D,\omega) - gm_i(D \backslash \{(i,j)\},\omega) = gm_j(D,\omega) - gm_j(D \backslash \{(i,j)\},\omega)$.

Generalized k-equal sharing property: If $D \in \mathcal{D}^N$, $\omega \in \mathcal{W}^N$ and $i \in N$ with $0 < |P_D(i)| \leq k$, then $gm_i(D \setminus \{(i,j) \mid j \in S_D(i)\},\omega) = gm_h(D,\omega) - gm_h(D \setminus \{(h,i)\},\omega)$ for every $h \in P_D(i)$.

Theorem 4.2. *Let $k \in \mathbb{N}$. A power measure for weighted digraphs is equal to the generalized k-local permission measure gp^k if and only if it satisfies generalized efficiency norm, generalized local determinateness, generalized symmetry, the generalized k-equal loss property, and the generalized k-equal sharing property.*

5 Applications

In this section, we apply the (generalized) k-local permission measures to find key nodes in two classical weighted digraphs. To illustrate these measures, we compare them with two classical local measures, outdegree[7] and Copeland score[8].

First, we apply these locally partial measures on the Countries Trade Network collected by Wasserman and Faust (1994) to estimate trade powers, where the directed network is constructed based on the relations of imports of minerals, fuels, and other petroleum products, and the weights of nodes (countries) are the proportions of each country's energy consumption in 1980[9]. The network is shown in Figure 4, and the values of the trade power of these 24 countries based on some of our local measures for weighted digraphs are given in Table 1.

[7]For every $D \in \mathcal{D}^N$ and $w \in \mathcal{W}^N$, the outdegree of node i is defined by $out_i(D,\omega) = \sum_{j \in S_D(i)} w_j$.

[8]For every $D \in \mathcal{D}^N$ and $w \in \mathcal{W}^N$, the Copeland score of node i is defined by $cop_i(D,\omega) = \sum_{j \in S_D(i)} w_j - \sum_{j \in P_D(i)} w_j$.

[9]The sample consists of 24 countries, the details can be seen in Wasserman and Faust (1994).

Table 1: Scores of 24 countries in the trade network based on various measures

Nation	ω	gp^1	gp^2	gp^5	gp^{10}	gp^{n-1}	out	cop
N01	0.0132	0.0197	0.0263	0.0459	0.0591	0.0626	**0.5641**	0.1100
N02	0.0351	0.0299	0.0247	0.0161	0.0235	0.0270	0.2543	-0.0754
N03	0.0179	0.0200	0.0222	0.0264	0.0332	0.0363	0.3417	-0.0661
N04	0.0100	0.0195	0.0290	0.0572	0.0669	0.0704	0.5357	0.1863
N05	0.1111	**0.0867**	0.0623	0.0446	0.0462	0.0462	0.1237	-0.0854
N06	0.0112	0.0100	0.0088	0.0074	0.0126	0.0157	0.1886	-0.1787
N07	0.0097	0.0131	0.0165	0.0269	0.0379	0.0414	0.4362	-0.0380
N08	0.0004	0.0003	0.0001	0.0001	0.0001	0.0001	0.0000	-0.2740
N09	0.1031	0.0861	0.0692	0.0391	0.0500	0.0534	0.3674	0.0356
N10	0.0047	0.0032	0.0016	0.0016	0.0016	0.0016	0.0000	-0.2641
N11	0.0043	0.0103	0.0163	0.0343	0.0471	0.0506	0.4814	0.0371
N12	0.0456	0.0382	0.0307	0.0083	0.0087	0.0087	0.0097	-0.4911
N13	0.0754	0.0747	0.0739	0.0652	0.0666	0.0701	0.5157	**0.2117**
N14	0.0081	0.0075	0.0068	0.0047	0.0068	0.0068	0.0486	-0.3758
N15	0.0012	0.0010	0.0007	0.0002	0.0002	0.0002	0.0000	-0.2524
N16	0.0781	0.0669	0.0557	0.0220	0.0238	0.0238	0.0754	-0.2884
N17	0.0036	0.0032	0.0027	0.0013	0.0015	0.0015	0.0060	-0.4012
N18	0.0478	0.0545	0.0612	0.0743	0.0658	0.0649	0.5026	-0.0768
N19	0.0847	0.0831	**0.0815**	0.0582	0.0637	0.0637	0.2936	-0.1437
N20	0.0156	0.0142	0.0127	0.0083	0.0140	0.0143	0.1331	-0.2408
N21	0.0060	0.0062	0.0065	0.0071	0.0115	0.0146	0.1898	-0.1776
N22	0.0854	**0.1045**	**0.1235**	**0.1547**	**0.1258**	**0.1292**	**0.7982**	**0.2379**
N23	0.1886	**0.1969**	**0.2051**	**0.2181**	**0.1611**	**0.1211**	**0.7003**	**0.2423**
N24	0.0390	0.0505	0.0620	**0.0780**	**0.0723**	**0.0758**	0.5067	-0.0419

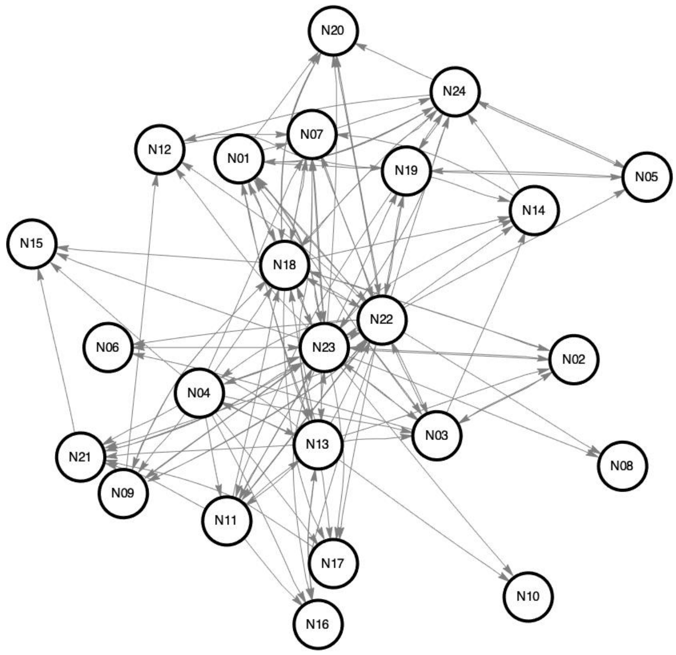

Figure 4: Trade of minerals, fuels, and other petroleum products between countries.

In each column of Table 1, the three highest scores are marked in bold. As we see from Table 1, N23 (the United States) and N22 (the United Kingdom) get the two highest scores in all these measures, which suggests that the (generalized) k-local permission measure can identify the key nodes, similar as the classical measures. Since each measure has a different attitude towards relatives (predecessors and successors), the node with the third-highest score is different for different measures[10]. Coincidentally, most of these countries with bold high scores are classified into the core block of the interaction network in Snyder and Kick (1979).

Second, we apply the two measures to Freeman's EIES network collected by Freeman and Freeman (1979). It is a classical network widely used in the literature to illustrate the usefulness of new measures. Here, we focus on the network representing a message-sending relation among 32 researchers who participated in an early study on the Electronic Information Exchange System. The weight of a node (researcher) is defined as the proportion of messages he/she sent. The network is shown in Figure 5. The scores of these 32 researchers based on our local measures are given in Table 2.

In each column of Table 2, the two highest scores are marked in bold. As we see from Table 2, R01 (Lin Freeman) and R29 (Barry Wellman) get the two highest scores, except in the outdegree and Copeland score measures. One fact that has been confirmed by the literature, such as Opsahl et al. (2010), Qi et al. (2012), Wei et al. (2013), is that these two researchers are the most active in the network, which is also successfully identified by the local permission

[10]These nodes include N01 (Algeria), N05 (Czechoslovakia), N13 (Japan), N19 (Switzerland), and N24 (Yugoslavia).

Table 2: Scores of 32 researchers of Freeman's EIES network based on various measures

No.	ω	gp^1	gp^2	gp^5	gp^{10}	gp^{n-1}	out	cop
R01	0.2059	**0.2022**	**0.1984**	**0.1870**	**0.1669**	**0.0549**	**1.0000**	0.0060
R02	0.0726	0.0724	0.0722	0.0715	0.0692	0.0469	0.9355	-0.0307
R03	0.0009	0.0014	0.0019	0.0033	0.0057	0.0142	0.3821	-0.2988
R04	0.0216	0.0226	0.0236	0.0266	0.0317	0.0427	0.9341	-0.0083
R05	0.0100	0.0115	0.0130	0.0176	0.0267	0.0475	0.9834	**0.2480**
R06	0.0176	0.0183	0.0190	0.0211	0.0244	0.0381	0.8412	0.0960
R07	0.0013	0.0014	0.0016	0.0021	0.0033	0.0093	0.2732	-0.2844
R08	0.1035	0.1012	0.0989	0.0921	0.0801	0.0492	0.9632	0.0144
R09	0.0131	0.0137	0.0142	0.0157	0.0182	0.0342	0.8009	-0.0364
R10	0.0235	0.0244	0.0253	0.0281	0.0325	0.0424	0.9255	-0.0073
R11	0.0559	0.0553	0.0547	0.0530	0.0494	0.0446	0.9252	0.0034
R12	0.0055	0.0060	0.0066	0.0082	0.0109	0.0207	0.4787	-0.3303
R13	0.0010	0.0013	0.0017	0.0028	0.0047	0.0120	0.3291	-0.3005
R14	0.0054	0.0056	0.0058	0.0064	0.0074	0.0166	0.4404	-0.1468
R15	0.0197	0.0186	0.0176	0.0144	0.0109	0.0210	0.4783	-0.1195
R16	0.0137	0.0143	0.0149	0.0168	0.0198	0.0328	0.7710	-0.0276
R17	0.0150	0.0159	0.0167	0.0193	0.0236	0.0376	0.8586	-0.0518
R18	0.0124	0.0128	0.0132	0.0144	0.0163	0.0312	0.7444	-0.0279
R19	0.0045	0.0050	0.0055	0.0069	0.0092	0.0192	0.4846	-0.1514
R20	0.0005	0.0007	0.0008	0.0013	0.0022	0.0055	0.1441	-0.5289
R21	0.0014	0.0019	0.0024	0.0039	0.0064	0.0157	0.4183	-0.2078
R22	0.0045	0.0049	0.0054	0.0067	0.0088	0.0193	0.4735	-0.2846
R23	0.0037	0.0041	0.0045	0.0055	0.0077	0.0192	0.5175	-0.1550
R24	0.0423	0.0426	0.0428	0.0434	0.0433	0.0509	**1.0000**	**0.1815**
R25	0.0077	0.0076	0.0076	0.0074	0.0096	0.0185	0.4241	-0.0376
R26	0.0044	0.0046	0.0048	0.0054	0.0074	0.0189	0.5121	-0.2018
R27	0.0197	0.0206	0.0215	0.0242	0.0286	0.0401	0.8809	-0.0433
R28	0.0017	0.0022	0.0027	0.0043	0.0069	0.0153	0.3804	-0.1509
R29	0.1427	**0.1401**	**0.1375**	**0.1297**	**0.1160**	**0.0523**	0.9893	0.0064
R30	0.0184	0.0190	0.0196	0.0214	0.0241	0.0364	0.8182	-0.0249
R31	0.0673	0.0676	0.0678	0.0686	0.0688	0.0483	0.9327	0.0091
R32	0.0825	0.0802	0.0779	0.0710	0.0593	0.0445	0.9059	0.0000

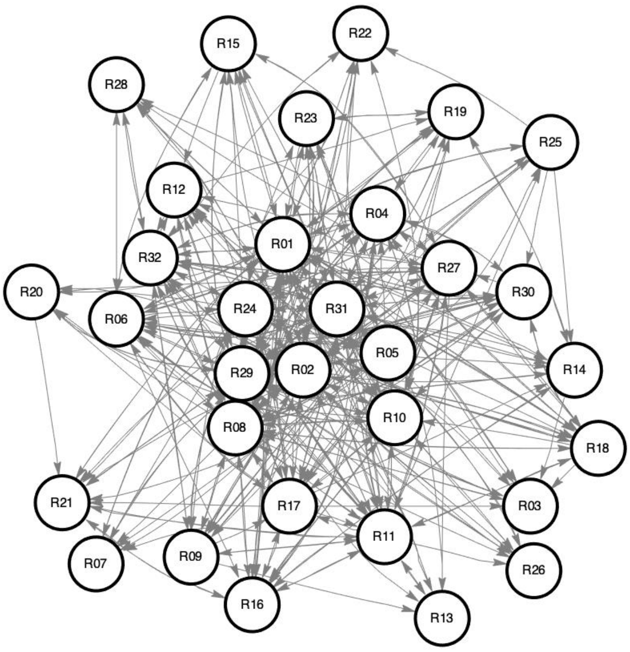

Figure 5: Freeman's EIES network.

measures.

6 Concluding remarks

To end this paper, we first summarize the axioms used in the characterizations of the local conjunctive, local disjunctive, and k-local permission values. We take k with $1 < k < |N| - 1$ since for $k = 1$ and $k = |N| - 1$, the k-local permission value coincides with the local disjunctive, respectively local conjunctive, value. In Table 3, we provide the summary, where a "+" means that the permission value satisfies the axiom, a '$-$' has the converse meaning, and a '$\oplus$' indicates the axiom is used in the characterization of a permission value in this paper.

Table 3: Characterizing properties of permission values

Properties	ψ^{ld}	ψ^{lk}	ψ^{lc}
Efficiency	$\oplus$	$\oplus$	$\oplus$
Additvity	$\oplus$	$\oplus$	$\oplus$
Necessary player property	$\oplus$	$\oplus$	$\oplus$
Local inessential player property	$\oplus$	$\oplus$	$\oplus$
k-structural monotonicity	$-$	$\oplus$	$+$
Local structural monotonicity	$-$	$-$	$\oplus$
Weak local structural monotonicity	$\oplus$	$+$	$+$
k-fairness	$+$	$\oplus$	$-$
Local fairness	$\oplus$	$-$	$-$

An advantageous feature of the k-local permission approach is that results hold for arbitrary permission structures, also for permission structures with cycles. This is not the case for the "global" disjunctive approach, where most results in the literature are stated only for acyclic permission structures.

As future research, we propose a variant of the k-local permission approach, which requires that every player needs to get permission from at least a fraction α of its predecessors before cooperation. We call this approach the α-permission approach. The α-permission approach seems appropriate in situations where the necessary number of approving predecessors of a player depends on the total number of his predecessors. For instance, for $k = 2$ and $D = \{(i, 1) \mid i = 3, 4\} \cup \{(j, 2) \mid j = 5, \ldots, 9\}$, in the k-local permission approach, player 1 needs to get permission from all its predecessors. However, player 2 can cooperate even if more than half of its predecessors do not agree. This may seem unreasonable in some situations.

In the α-permission approach for $\alpha \in [0, 1]$, the largest α-permission feasible subset in $D \in \mathcal{D}^N$ is given by

$$\sigma_D^\alpha(E) = \{i \in E \mid |P_D(i) \cap E| \geq \alpha |P_D(i)|\} \text{ for every } E \subseteq N,$$

The *α-permission restricted game* of (ν, D) is the game $r^\alpha_{\nu,D}$ that assigns to every coalition $E \subseteq N$ the worth of its largest α-permission feasible subset, i.e. $r^\alpha_{\nu,D}(E) = \nu\big(\sigma^\alpha_D(E)\big)$ for all $E \subseteq N$. Applying the Shapley value to the α-permission restricted game, we obtain the *α-permission value* ψ^α given by $\psi^\alpha(\nu, D) = \mathcal{S}h(r^\alpha_{\nu,D})$ for all $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$. Future studies can address the axiomatization and applications of the α-permission values.

Appendix: Proofs and logical independence

Proposition 3.1. *Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E \subseteq N$.*
(i) For every $F \subseteq E$, $\sigma^{lk}_D(F) \subseteq \sigma^{lk}_D(E)$.
(ii) For every $t \in \mathbb{N}$ with $t \leq k$, $\sigma^{lk}_D(E) \subseteq \sigma^{lt}_D(E)$.

Proof. Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E \subseteq N$.
(i) For every $i \in \sigma^{lk}_D(F)$, we have $|P_D(i) \cap F| \geq \min\{k, |P_D(i)|\}$. Since $F \subseteq E$, we have $P_D(i) \cap F \subseteq P_D(i) \cap E$ and, thus, $|P_D(i) \cap E| \geq |P_D(i) \cap F| \geq \min\{k, |P_D(i)|\}$. Hence, $i \in \sigma^{lk}_D(E)$. Thus, $\sigma^{lk}_D(F) \subseteq \sigma^{lk}_D(E)$.
(ii) For every $i \in \sigma^{lk}_D(E)$ and $k \geq t$, we have $|P_D(i) \cap E| \geq \min\{k, |P_D(i)|\} \geq \min\{t, |P_D(i)|\}$. Hence, $i \in \sigma^{lt}_D(E)$. Thus, $\sigma^{lk}_D(E) \subseteq \sigma^{lt}_D(E)$. ∎

Proposition 3.2. *Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E \subseteq N$.*
(i) For every $F \subseteq E$ with $\sigma^{lk}_D(E) \cup P_D\big(\sigma^{lk}_D(E)\big) \subseteq F$, $\sigma^{lk}_D(F) = \sigma^{lk}_D(E)$.
(ii) For every $F \subseteq E \backslash \sigma^{lk}_D(E)$, $\sigma^{lk}_D(F) = \emptyset$.

Proof. Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E \subseteq N$.
(i) By Proposition 3.1 (i), we know that

$$\sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big) \subseteq F \subseteq E \text{ implies } \sigma_D^{lk}\Big(\sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big)\Big) \subseteq \sigma_D^{lk}(F) \subseteq \sigma_D^{lk}(E).$$

By definition of the largest k-local permission feasible subset, we have

$$\begin{aligned}
& \sigma_D^{lk}\Big(\sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big)\Big) \\
= & \Big\{i \in \sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big) \mid |P_D(i) \cap \Big(\sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big)\Big)| \geq \min\{k, |P_D(i)|\}\Big\} \\
\supseteq & \Big\{i \in \sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big) \mid |P_D(i) \cap P_D\big(\sigma_D^{lk}(E)\big)| \geq \min\{k, |P_D(i)|\}\Big\} \\
\supseteq & \Big\{i \in \sigma_D^{lk}(E) \mid |P_D(i) \cap P_D\big(\sigma_D^{lk}(E)\big)| \geq \min\{k, |P_D(i)|\}\Big\} \\
= & \Big\{i \in \sigma_D^{lk}(E) \mid |P_D(i)| \geq \min\{k, |P_D(i)|\}\Big\} \\
= & \sigma_D^{lk}(E),
\end{aligned}$$

where the second equality follows since $i \in \sigma_D^{lk}(E)$ implies $P_D(i) \subseteq P_D(\sigma_D^{lk}(E))$.

Thus, we have $\sigma_D^{lk}(E) \subseteq \sigma_D^{lk}\Big(\sigma_D^{lk}(E) \cup P_D\big(\sigma_D^{lk}(E)\big)\Big) \subseteq \sigma_D^{lk}(F) \subseteq \sigma_D^{lk}(E)$ and thus all inclusions are equalities.

(ii) For $i \in E \setminus \sigma_D^{lk}(E)$, we have $|P_D(i) \cap E| < \min\{k, |P_D(i)|\}$. Since $F \subseteq E \setminus \sigma_D^{lk}(E) \subseteq E$, we have $|P_D(i) \cap F| \leq |P_D(i) \cap E| < \min\{k, |P_D(i)|\}$ for all $i \in F$. Therefore, $\sigma_D^{lk}(F) = \emptyset$. ■

Proposition 3.3. *For every $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E, F \subseteq N$,*
(i) $\sigma_D^{lk}(E) \cup \sigma_D^{lk}(F) \subseteq \sigma_D^{lk}(E \cup F)$.
(ii) $\sigma_D^{lk}(E \cap F) \subseteq \sigma_D^{lk}(E) \cap \sigma_D^{lk}(F)$.

Proof. Let $k \in \mathbb{N}$, $D \in \mathcal{D}^N$, and $E, F \subseteq N$.
(i) The first statement follows since

$$\begin{aligned}
& \sigma_D^{lk}(E \cup F) \\
= & \{i \in E \cup F \mid |P_D(i) \cap (E \cup F)| \geq \min\{k, |P_D(i)|\}\} \\
= & \{i \in E \cup F \mid |\big(P_D(i) \cap E\big) \cup \big(P_D(i) \cap F\big)| \geq \min\{k, |P_D(i)|\}\} \\
\supseteq & \{i \in E \cup F \mid |P_D(i) \cap E| \geq \min\{k, |P_D(i)|\}\} \cup \{i \in E \cup F \mid |P_D(i) \cap F| \geq \min\{k, |P_D(i)|\}\} \\
\supseteq & \{i \in E \mid |P_D(i) \cap E| \geq \min\{k, |P_D(i)|\}\} \cup \{i \in F \mid |P_D(i) \cap F| \geq \min\{k, |P_D(i)|\}\} \\
= & \sigma_D^{lk}(E) \cup \sigma_D^{lk}(F)
\end{aligned}$$

(ii) For every $i \in \sigma_D^{lk}(E \cap F)$, we have $|E \cap F \cap P_D(i)| \geq \min\{k, |P_D(i)|\}$. Since $E \cap F \cap P_D(i) \subseteq E \cap P_D(i)$, we have $|E \cap P_D(i)| \geq |E \cap F \cap P_D(i)| \geq \min\{k, |P_D(i)|\}$. Similarly, we have $|F \cap P_D(i)| \geq |E \cap F \cap P_D(i)| \geq \min\{k, |P_D(i)|\}$. Thus, $i \in \sigma_D^{lk}(E)$ and $i \in \sigma_D^{lk}(F)$. Therefore, $\sigma_D^{lk}(E \cap F) \subseteq \sigma_D^{lk}(E) \cap \sigma_D^{lk}(F)$. ■

Theorem 3.1. *Let $k \in \mathbb{N}$. A solution is equal to the k-local permission value ψ^{lk} if and only if it satisfies efficiency, additivity, the necessary player property, the local inessential player property, k-fairness, and k-structural monotonicity.*

Proof. Let $k \in \mathbb{N}$. First, we prove that ψ^{lk} satisfies the six axioms.

Efficiency of ψ^{lk} directly follows from efficiency of the Shapley value and the fact that $\sigma_D^{lk}(N) = N$ for all $D \in \mathcal{D}^N$.

Additivity of ψ^{lk} directly follows from additivity of the Shapley value and $r_{\nu,D}^{lk} + r_{\omega,D}^{lk} = r_{\nu+\omega,D}^{lk}$ for all $\nu, \omega \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$.

To show that ψ^{lk} satisfies the necessary player property, let $\nu \in \mathcal{G}_M^N$, and $D \in \mathcal{D}^N$. It follows:

(i) For every $E \subseteq N$ and $i \in N$, by irreflexivity of D, we have $i \notin \sigma_D^{lk}(E \setminus \{i\})$. This implies that $r_{\nu,D}^{lk}(E \setminus \{i\}) = \nu\big(\sigma_D^{lk}(E \setminus \{i\})\big) = 0$ if i is a necessary player in ν.

(ii) For every $E \subseteq N$, we have $r_{\nu,D}^{lk}(E) = \nu\big(\sigma_D^{lk}(E)\big) \geq \nu(\emptyset) = 0$ since $\nu \in \mathcal{G}_M^N$.

Let $i \in N$ be a necessary player in monotone game ν. Then, for every $j \in N$,

$$\begin{aligned}
\psi_i^{lk}(\nu, D) &= Sh_i(r_{\nu,D}^{lk}) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{i\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{i\})\right) + \sum_{\substack{E \subseteq N \\ E \ni i \\ E \not\ni j}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{i\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E) r_{\nu,D}^{lk}(E) + \sum_{\substack{E \subseteq N \\ E \ni i \\ E \not\ni j}} p(E) r_{\nu,D}^{lk}(E) \\
&\geq \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E) r_{\nu,D}^{lk}(E) \\
&\geq \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{j\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{j\})\right) + \sum_{\substack{E \subseteq N \\ E \not\ni i \\ E \ni j}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{j\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni j}} p(E) \left(r_{\nu,D}^{lk}(E) - r_{\nu,D}^{lk}(E \setminus \{j\})\right) \\
&= Sh_j(r_{\nu,D}^{lk}) = \psi_j^{lk}(\nu, D) \text{ for every } j \in N,
\end{aligned}$$

where the fourth equality follows from (i), the inequalities follow from (ii), and the fifth equality from (i). Thus, ψ^{lk} satisfies the necessary player property.

The local inessential player property of ψ^{lk} directly follows from the null player property of the Shapley value, and the fact that if a player $i \in N$ is such that every player $j \in S_D(i) \cup \{i\}$ is a null player in ν, then i is a null player in $r^{lk}_{\nu,D}$.

To show that ψ^{lk} satisfies k-fairness, let $\nu \in \mathcal{G}^N$, $D \in \mathcal{D}^N$, $i \in N$, and $j \in S_D(i)$ with $|P_D(j)| > k$. We have

$$\begin{aligned}
& \psi^{lk}_i(\nu, D) - \psi^{lk}_i\big(\nu, D \setminus \{(i,j)\}\big) \\
= \;& \mathcal{S}h_i(r^{lk}_{\nu,D}) - \mathcal{S}h_i(r^{lk}_{\nu,D\setminus\{(i,j)\}}) \\
= \;& \sum_{\substack{E\subseteq N\\ E\ni i}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_D(E\setminus\{i\})\big)] - \sum_{\substack{E\subseteq N\\ E\ni i}} p(E)[\nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E\setminus\{i\})\big)] \\
= \;& \sum_{\substack{E\subseteq N\\ E\ni i}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] - \sum_{\substack{E\subseteq N\\ E\ni i}} p(E)[\nu\big(\sigma^{lk}_D(E\setminus\{i\})\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E\setminus\{i\})\big)] \\
= \;& \sum_{\substack{E\subseteq N\\ E\ni i\\ E\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] - \sum_{\substack{E\subseteq N\\ E\ni i\\ E\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E\setminus\{i\})\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E\setminus\{i\})\big)] \\
& + \sum_{\substack{E\subseteq N\\ E\ni i\\ E\not\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] - \sum_{\substack{E\subseteq N\\ E\ni i\\ E\not\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E\setminus\{i\})\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E\setminus\{i\})\big)] \\
= \;& \sum_{\substack{E\subseteq N\\ E\ni i\\ E\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] - \sum_{\substack{E\subseteq N\\ E\not\ni i\\ E\ni j}} p(E\cup\{i\})[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] \\
& + \sum_{\substack{E\subseteq N\\ E\ni i\\ E\not\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] - \sum_{\substack{E\subseteq N\\ E\not\ni i\\ E\not\ni j}} p(E\cup\{i\})[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] \\
= \;& \sum_{\substack{E\subseteq N\\ E\ni i\\ E\ni j}} p(E)[\nu\big(\sigma^{lk}_D(E)\big) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)].
\end{aligned}$$

where the last equality follows since $\sigma^{lk}_D(E) = \sigma^{lk}_{D\setminus\{(i,j)\}}(E)$ for every $E \subseteq N$ with $\{i,j\} \not\subseteq E$. Analogously, we have

$$\begin{aligned}
\psi^{lk}_j(\nu, D) - \psi^{lk}_j\big(\nu, D \setminus \{(i,j)\}\big) &= \sum_{\substack{E\subseteq N\\ E\ni i\\ E\ni j}} p(E)[\nu(\sigma^{lk}_D(E)) - \nu\big(\sigma^{lk}_{D\setminus\{(i,j)\}}(E)\big)] \\
&= \psi^{lk}_j(\nu, D) - \psi^{lk}_j\big(\nu, D\setminus\{(i,j)\}\big).
\end{aligned}$$

Thus, ψ^{lk} satisfies k-fairness.

To show that ψ^{lk} satisfies k-structural monotonicity, let $\nu \in \mathcal{G}^N_M$, and $D \in \mathcal{D}^N$. It follows:

(i) For every $E \subseteq N$ and j a necessary player in ν, $r^{lk}_{\nu,D}(E) = 0$ if $E \not\ni j$.

(ii) For every $E \subseteq N$, and $i \in N$, if there is a necessary player $j \in S_D(i)$ in ν such that $0 < |P_D(j)| \leq k$, then $r^{lk}_{\nu,D}(E \setminus \{i\}) = 0$.

(iii) For every $E \subseteq N$, we have $r^{lk}_{\nu,D}(E) = \nu\big(\sigma^{lk}_D(E)\big) \geq \nu(\emptyset) = 0$.

Let $j \in N$ be a necessary player in ν with $0 < |P_D(j)| \leq k$, and let $i \in P_D(j)$. Then,

$$\begin{aligned}
\psi^{lk}_i(\nu, D) &= Sh_i(r^{lk}_{\nu,D}) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E)\left(r^{lk}_{\nu,D}(E) - r^{lk}_{\nu,D}(E \setminus \{i\})\right) + \sum_{\substack{E \subseteq N \\ E \ni i \\ E \not\ni j}} p(E)\left(r^{lk}_{\nu,D}(E) - r^{lk}_{\nu,D}(E \setminus \{i\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E)\left(r^{lk}_{\nu,D}(E) - r^{lk}_{\nu,D}(E \setminus \{i\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E) r^{lk}_{\nu,D}(E) \\
&\geq \sum_{\substack{E \subseteq N \\ E \ni i \\ E \ni j}} p(E)\left(r^{lk}_{\nu,D}(E) - r^{lk}_{\nu,D}(E \setminus \{j\})\right) \\
&= \sum_{\substack{E \subseteq N \\ E \ni j \\ E \ni i}} p(E)\left(r^{lk}_{\nu,D}(E) - r^{lk}_{\nu,D}(E \setminus \{j\})\right) + \sum_{\substack{E \subseteq N \\ E \ni j \\ E \not\ni i}} p(E)\left(r^{lk}_{\nu,D}(E) - r^{lk}_{\nu,D}(E \setminus \{j\})\right) \\
&= Sh_j(r^{lk}_{\nu,D}) = \psi^{lk}_j(\nu, D),
\end{aligned}$$

where the third equality follows from (i), the fourth equality follows from (ii), the inequality follows from (iii), and the fifth equality follows from (ii). Thus, ψ^{lk} satisfies k-structural monotonicity.

Therefore, we conclude that ψ^{lk} satisfies the six axioms for every $k \in \mathbb{N}$.

Second, we prove uniqueness. Let $k \in \mathbb{N}$. Assume that a solution f satisfies the six axioms. Consider the permission structure $D \in \mathcal{D}^N$ and game $w_T = c_T u_T$, where u_T is the unanimity game of $T \subseteq N$ and $c_T \geq 0$. We define

$$\alpha(T) = P_D(T) \setminus T,$$

and

$$\beta^k(T) = \{i \in \alpha(T) \mid \text{ there exists } j \in S_D(i) \cap T \text{ with } |P_D(j)| \leq k\}.$$

Notice that $\sum_{i \in N} |P_D(i)| + |H^N_D| \geq |N|$ for all $D \in \mathcal{D}^N$. We proceed by induction on $\sum_{i \in N} |P_D(i)| + |H^N_D|$.

First, if $\sum_{i \in N} |P_D(i)| + |H^N_D| = |N|$, then $|P_D(i)| = 1$ for all $i \in N \setminus H^N_D$. In this case, $\alpha(T) = \beta^k(T)$. Moreover, $\alpha(T) \cup T$ and $N \setminus \big(\alpha(T) \cup T\big)$ form a partition of N. Since all

$j \in S_D(i) \cup \{i\}$ with $i \in N \setminus (\alpha(T) \cup T)$ are null players in w_T, by the local inessential player property, we have

$$f_i(w_T, D) = 0 \text{ for every } i \in N \setminus \big(\alpha(T) \cup T\big). \tag{1}$$

Besides, for $i \in T$ and $j \in \alpha(T)$, (i) the necessary player property implies $f_i(w_T, D) \geq f_j(w_T, D)$, and (ii) k-structural monotonicity implies $f_i(w_T, D) \leq f_j(w_T, D)$ since $|P_D(i)| = 1 \leq k$ for all $i \in N \setminus H_D^N$. Thus, for a given $j \in \alpha(T) \cup T$, we have

$$f_i(w_T, D) = f_j(w_T, D) \text{ for every } i \in (\alpha(T) \cup T) \setminus \{j\}.$$

With efficiency, this gives $1 + (|\alpha(T) \cup T| - 1) = |\alpha(T) \cup T|$ independent linear equations in the $|\alpha(T) \cup T|$ unknown payoffs $f_i(w_T, D)$, $i \in \alpha(T) \cup T$. Thus, with (1) $f(w_T, D)$ is uniquely determined.

Proceeding by induction, assume that $f(w_T, D')$ is uniquely determined for all $D' \in \mathcal{D}^N$ with $\sum_{i \in N} |P_{D'}(i)| + |H_{D'}^N| < \sum_{i \in N} |P_D(i)| + |H_D^N|$.

If $\sum_{i \in N} |P_D(i)| + |H_D^N| > |N|$ with $\alpha(T) = \beta^k(T)$, then uniqueness follows in the same way as when $\sum_{i \in N} |P_D(i)| + |H_D^N| = |N|$. Therefore, we only consider $\sum_{i \in N} |P_D(i)| + |H_D^N| > |N|$ with $\alpha(T) \neq \beta^k(T)$. In this case, $N \setminus \big(\alpha(T) \cup T\big)$, $\beta^k(T) \cup T$, and $\alpha(T) \setminus \beta^k(T)$ form a partition of N. Let $i \in N$. We consider three cases depending on the set to which i belongs.

Case 1: If $i \in N \setminus \big(\alpha(T) \cup T\big)$, by the local inessential player property, $f_i(w_T, D) = 0$.

Case 2: If $i \in \beta^k(T) \cup T$, by the necessary player property and k-structural monotonicity, we have,

$$f_j(w_T, D) = f_i(w_T, D) \text{ for every } j \in (\beta^k(T) \cup T) \setminus \{i\},$$

similar as in the case $\sum_{i \in N} |P_D(i)| + |H_D^N| = |N|$ above.

Case 3: If $i \in \alpha(T) \setminus \beta^k(T)$, there exists a $j \in S_D(i)$ such that $j \in T$ and $|P_D(j)| > k$. In that case, k-fairness implies that

$$f_i(w_T, D) - f_i\big(w_T, D \setminus \{(i,j)\}\big) = f_j(w_T, D) - f_j\big(w_T, D \setminus \{(i,j)\}\big),$$

where $f_i\big(w_T, D \setminus \{(i,j)\}\big)$ and $f_j\big(w_T, D \setminus \{(i,j)\}\big)$ are determined by the induction hypothesis.

Based on Cases 2 and 3, and using efficiency, we get $(|\beta^k(T) \cup T| - 1) + (|\alpha(T)| - |\beta^k(T)|) + 1 = |\alpha(T) \cup T|$ independent linear equations in the $|\alpha(T) \cup T|$ unkown payoffs $f_i(w_T, D)$, $i \in \alpha(T) \cup T$, which are uniquely determined. Thus, with Case 1, $f(w_T, D)$ is uniquely determined.

Next, we consider $c_T < 0$ and game $w_T = c_T u_T$ where u_T is the unanimity game of $T \subseteq N$. Since $-c_T > 0$, $f(-w_T, D)$ is uniquely determined for every $D \in \mathcal{D}^N$ as above. Additivity implies

$f(-w_T, D) + f(w_T, D) = f(-w_T + w_T, D)$. Since all players are null players in $-w_T + w_T$, by the local inessential player property, $f(-w_T + w_T, D) = 0$. Thus, $f(w_T, D) = -f(-w_T, D)$ is uniquely determined.

Since every game $\nu \in \mathcal{G}^N$ can be expressed as a linear combination of unanimity games in a unique way, it follows by additivity that $f(\nu, D)$ is uniquely determined for every $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$. Since ψ^{lk} satisfies the axioms, it must hold that $f(\nu, D) = \psi^{lk}(\nu, D)$ for every $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$. ∎

We show logical independence of the axioms stated in Theorem 3.1 by presenting six alternative solutions.

1. Let $f^1 : \mathcal{G}^N \times \mathcal{D}^N \to \mathbb{R}^N$ be given by $f^1(\nu, D) = \mathcal{S}h(\nu)$ for all $\nu \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$. This solution satisfies all axioms in Theorem 3.1 except k-structural monotonicity.

2. For, $1 \leq k < |N| - 1$, the local $(k+1)$-permission value satisfies all axioms in Theorem 3.1 except k-fairness. Notice that for $k = |N| - 1$, k-fairness has no meaning.

3. Let $f^2 : \mathcal{G}^N \times \mathcal{D}^N \to \mathbb{R}^N$ be given by $f^2_i(\nu, D) = \frac{\nu(N)}{|N|}$ for all $i \in N$, $\nu \in \mathcal{G}^N$, and $D \in \mathcal{D}^N$. This solution satisfies all axioms in Theorem 3.1 except the local inessential player property.

4. Let $f^3 : \mathcal{G}^N \times \mathcal{D}^N \to \mathbb{R}^N$ be given by, for all $D \in \mathcal{D}^N$,

$$f^3(\nu, D) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) g(u_T, D),$$

where u_T is the unanimity game of $T \subseteq N$, $T \neq \emptyset$, and g is defined as follows. If $T \cap H_D^N \neq \emptyset$,

$$g_i(u_T, D) = \begin{cases} \frac{1}{|T \cap H_D^N|} & \text{if } i \in T \cap H_D^N, \\ 0 & \text{if } i \in N \setminus (T \cap H_D^N). \end{cases}$$

If $T \cap H_D^N = \emptyset$

$$g_i(u_T, D) = \psi_i^{kd}(u_T, D) \text{ for every } i \in N.$$

f^3 satisfies all axioms in Theorem 3.1 except the necessary player property.

5. For $T \subseteq N$, consider $z_T \in \mathcal{G}^N$ given by

$$z_T(E) = \begin{cases} 1 & \text{if } E \cap T \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f^4 : \mathcal{G}^N \times \mathcal{D}^N \to \mathbb{R}^N$ be given by

$$f^4(\nu, D) = \begin{cases} f^3(\nu, D) & \text{if } \nu = z_T \text{ with } |T| \geq 2, \\ \psi^{lk}(\nu, D) & \text{otherwise.} \end{cases}$$

f^4 satisfies all axioms in Theorem 3.1 except additivity.

6. Let $f^5 : \mathcal{G}^N \times \mathcal{D}^N \to \mathbb{R}^N$ be given by $f^5_i(\nu, D) = 0$ for all $i \in N$, $\nu \in \mathcal{G}^N$, and $D \in \mathcal{D}^N$. This solution satisfies all axioms in Theorem 3.1 except efficiency.

Proposition 4.1. *For every* $D \in \mathcal{D}^N$*,*

$$p^k_i(D) = \frac{|P_D(i)| + 1 - \min\{k, |P_D(i)|\}}{|P_D(i)| + 1} + \sum_{j \in S_D(i)} \frac{\min\{k, |P_D(j)|\}}{|P_D(j)|\big(|P_D(j)| + 1\big)} \text{ for all } i \in N.$$

Proof. According to Definition 4.1, we have $p^k(D) = \psi^{lk}(\nu^1, D)$. By additivity of ψ^{lk}, and $\nu^1 = \sum_{i \in N} u_{\{i\}}$, $\psi^{lk}_j(\nu^1, D) = \sum_{i \in N} \psi^{lk}_j(u_{\{i\}}, D)$. Fix $i \in N$.

(i) For every $j \in N \setminus \big(\{i\} \cup P_D(i)\big)$, we have that every $t \in \{j\} \cup S_D(j)$ is a null player in $u_{\{i\}}$. Then, $\psi^{lk}_j(u_{\{i\}}, D) = 0$, by the local inessential property.

(ii) For every $j \in P_D(i)$, we have

$$r^{lk}_{u_{\{i\}},D}(E \cup \{j\}) - r^{lk}_{u_{\{i\}},D}(E) = \begin{cases} 1 & \text{if } E \ni i \text{ and } |P_D(i) \cap E| = \min\{k, |P_D(i)|\} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we determine the proportion of coalitions $E \subseteq N$ with $r^{lk}_{u_{\{i\}},D}(E \cup \{j\}) - r^{lk}_{u_{\{i\}},D}(E) = 1$. If $|P_D(i)| \leq k$, we have $|P_D(i)| = \min\{k, |P_D(i)|\}$, and thus this proportion is

$$\frac{|(P_D(i) \setminus \{j\}) \cup \{i\}|!}{|P_D(i) \cup \{i\}|!} = \frac{|P_D(i)|!}{(|P_D(i)| + 1)!} = \frac{1}{|P_D(i)| + 1} = \frac{\min\{k, |P_D(i)|\}}{|P_D(i)|\big(|P_D(i)| + 1\big)}.$$

If $|P_D(i)| \geq k$, we have $k = \min\{k, |P_D(i)|\}$, and thus this proportion is

$$\begin{aligned} & \binom{|P_D(i)| - 1}{k - 1} \cdot \frac{k!\big(|P_D(i)| - k\big)!}{\big(|P_D(i)| + 1\big)!} \\ = & \frac{\big(|P_D(i)| - 1\big)!}{(k-1)!\big(|P_D(i)| - k\big)!} \cdot \frac{k!\big(|P_D(i)| - k\big)!}{\big(|P_D(i)| + 1\big)!} \\ = & \frac{k}{|P_D(i)|\big(|P_D(i)| + 1\big)} \\ = & \frac{\min\{k, |P_D(i)|\}}{|P_D(i)|\big(|P_D(i)| + 1\big)}. \end{aligned}$$

Thus, $\psi^{lk}_j(u_{\{i\}}, D) = \frac{\min\{k, |P_D(i)|\}}{|P_D(i)|\big(|P_D(i)| + 1\big)}$ for every $j \in P_D(i)$.

(iii) Finally, by efficiency,

$$\begin{aligned} \psi^{lk}_i(u_{\{i\}}, D) &= 1 - \sum_{j \in N \setminus \{i\}} \psi^{lk}_j(u_{\{i\}}, D) \\ &= 1 - \sum_{j \in P_D(i)} \frac{\min\{k, |P_D(i)|\}}{|P_D(i)|\big(|P_D(i)| + 1\big)} \\ &= 1 - \frac{\min\{k, |P_D(i)|\}}{|P_D(i)| + 1} \end{aligned}$$

$$= \frac{|P_D(i)| + 1 - \min\{k, |P_D(i)|\}}{|P_D(i)| + 1}.$$

From (i)-(iii), we conclude that

$$\begin{aligned} p_i^k(D) &= \sum_{j \in N} \psi_i^{lk}(u_{\{j\}}, D) \\ &= \psi_i^{lk}(u_{\{i\}}, D) + \sum_{j \in S_D(i)} \psi_i^{lk}(u_{\{j\}}, D) \\ &= \frac{|P_D(i)| + 1 - \min\{k, |P_D(i)|\}}{|P_D(i)| + 1} + \sum_{j \in S_D(i)} \frac{\min\{k, |P_D(j)|\}}{|P_D(j)|\big(|P_D(j)| + 1\big)}. \end{aligned}$$

■

Theorem 4.1. *Let $k \in \mathbb{N}$. A power measure is equal to the k-local permission measure p^k if and only if it satisfies efficiency norm, local determinateness, symmetry, the k-equal loss property, and the k-equal sharing property.*

Proof. Let $k \in \mathbb{N}$. First, we prove that p^k satisfies the five axioms.

To show that p^k satisfies efficiency norm, let $D \in \mathcal{D}^N$. Then,

$$\sum_{i \in N} p_i^k(D) = \sum_{i \in N} \psi_i^{lk}(\nu^1, D) = \nu^1(N) = |N|,$$

where the second equality follows from efficiency of ψ^{lk}. Thus, p^k satisfies efficiency norm.

Local determinateness and symmetry directly follow since for every $i \in N$, $p_i^k(D)$ only depends on $|P_D(i)|$ and $|P_D(j)|$, $j \in S_D(i)$.

To show that p^k satisfies the k-equal loss property, let $D \in \mathcal{D}^N$, $i \in N$, and $j \in S_D(i)$ with $|P_D(j)| > k$. By definition of p^k,

$$\begin{aligned} p_i^k(D) - p_i^k\big(D \setminus \{(i,j)\}\big) &= \psi_i^{lk}(\nu^1, D) - \psi_i^{lk}\big(\nu^1, D \setminus \{(i,j)\}\big) \\ &= \psi_j^{lk}(\nu^1, D) - \psi_j^{lk}\big(\nu^1, D \setminus \{(i,j)\}\big) = p_j^k(D) - p_j^k\big(D \setminus \{(i,j)\}\big), \end{aligned}$$

where the second equality follows from k-fairness of ψ^{lk}. Thus, p^k satisfies the k-equal loss property.

To show that p^k satisfies the k-equal sharing property, let $D \in \mathcal{D}^N$, $i \in N$ with $0 < |P_D(i)| \leq k$, and $h \in P_D(i)$. By Proposition 4.1,

$$p_i^k\big(D \setminus \{(i,j) \mid j \in S_D(i)\}\big) = \frac{|P_D(i)| + 1 - \min\{k, |P_D(i)|\}}{|P_D(i)| + 1} = \frac{1}{|P_D(i)| + 1}$$

and

$$p_h^k(D) - p_h^k\big(D \setminus \{(h,i)\}\big) = \frac{\min\{k, P_D(i)\}}{|P_D(i)|\big(|P_D(i)| + 1\big)} = \frac{1}{|P_D(i)| + 1}.$$

Thus, p^k satisfies the k-equal sharing property.

Therefore, we conclude that p^k satisfies the five axioms.

Second, we prove uniqueness. Assume that a measure m satisfies the five axioms. Let $D \in \mathcal{D}^N$ and $k \in \mathbb{N}$. We proceed by induction on $|D|$.

If $|D| = 0$, then $D = \emptyset$. By the efficiency norm and symmetry, we have $m_i(\emptyset) = 1$ for all $i \in N$. Thus, $m(D)$ is uniquely determined.

Proceeding by induction, assume $m(D')$ is uniquely determined for all $D' \in \mathcal{D}^N$ with $|D'| < |D|$.

Since $|D| > 0$, there is a $j \in N$ with $|P_D(j)| \geq 1$. Denote $D^* = D \setminus \{(h, j)\}$ for some $h \in P_D(j)$.

First, let $i \in N \setminus (\{j\} \cup P_D(j))$. By local determinateness,

$$m_i(D) = m_i(D^*), \tag{2}$$

which is determined by the induction hypothesis.

To determine the values for $i \in \{j\} \cup P_D(j)$, we distinguish two possible cases: (i) $|P_D(j)| > k$, and (ii) $0 < |P_D(j)| \leq k$.

<u>Case 1</u>: Let $|P_D(j)| > k$, and consider $i \in P_D(j)$.

By the k-equal loss property,

$$m_i(D) - m_i\big(D \setminus \{(i, j)\}\big) = m_j(D) - m_j\big(D \setminus \{(i, j)\}\big) \tag{3}$$

where $m_i\big(D \setminus \{(i, j)\}\big)$ and $m_j\big(D \setminus \{(i, j)\}\big)$ are determined by the induction hypothesis. With the efficiency norm, we have

$$\sum_{i \in \{j\} \cup P_D(j)} m_i(D) = |N| - \sum_{i \in N \setminus (\{j\} \cup P_D(j))} m_i(D), \tag{4}$$

where the last term is determined by (2). Thus, (3) and (4) give $|P_D(j)| + 1$ independent linear equations with the same number of unknowns, $m_j(D)$ and $m_i(D)$, $i \in P_D(j)$, which thus are determined.

<u>Case 2</u>: Let $0 < |P_D(j)| \leq k$, and consider $i \in P_D(j)$.

By the k-equal sharing property,

$$m_j\big(D \setminus \{(j, t) \mid t \in S_D(j)\}\big) = m_i(D) - m_i\big(D \setminus \{(i, j)\}\big) \tag{5}$$

If $S_D(j) \neq \emptyset$, $m_j\big(D \setminus \{(j, t) \mid t \in S_D(j)\}\big)$ and $m_i\big(D \setminus \{(i, j)\}\big)$ are determined by the induction hypothesis. Thus, $m_i(D)$ is uniquely determined by equation (5). By the efficiency norm and (2), $m_j(D) = |N| - \sum_{i \in P_D(j)} m_i(D) - \sum_{i \in N \setminus (P_D(j) \cup \{j\})} m_i(D^*)$ is determined.

If $S_D(j)=\emptyset$, $m_j\big(D\setminus\{(j,t)\mid t\in S_D(j)\}\big)=m_j(D)$. With the efficiency norm, we have

$$\sum_{i\in\{j\}\cup P_D(j)} m_i(D)=|N|-\sum_{i\in N\setminus(\{j\}\cup P_D(j))} m_i(D)=|N|-\sum_{i\in N\setminus(\{j\}\cup P_D(j))} m_i(D^*), \tag{6}$$

where the last term is determined by (2). Thus, with (5) and (6), we also get $|P_D(j)|+1$ independent linear equations with the same number of unknowns, $m_j(D)$ and $m_i(D)$, $i\in P_D(j)$, which thus are determined.

Therefore, $m(D)$ is uniquely determined. Since p^k satisfies the five axioms, it must hold that $m(D)=p^k(D)$. ■

Proposition 4.3. *For every $D\in\mathcal{D}^N$ and $w\in\mathcal{W}^N$,*

$$gp_i^k(D,\omega)=\frac{|P_D(i)|+1-\min\{k,|P_D(i)|\}}{|P_D(i)|+1}\omega_i+\sum_{j\in S_D(i)}\frac{\min\{k,|P_D(j)|\}}{|P_D(j)|\big(|P_D(j)|+1\big)}\omega_j \text{ for all } i\in N.$$

The proof of this proposition follows the same lines as that of Proposition 4.1, and is therefore omitted. The only difference is that, the marginal contributions in the restricted game $r^{lk}_{\omega_i u_{\{i\}},D}$ are either ω_i or 0 (instead of 1 and 0).

Theorem 4.2. *Let $k\in\mathbb{N}$. A measure for weighted digraphs is equal to the generalized k-local permission measure gp^k if and only if it satisfies the generalized efficiency norm, generalized local determinateness, generalized symmetry, the generalized k-equal loss property, and the generalized k-equal sharing property.*

The proof of this theorem follows the same lines as that of Theorem 4.1, and is therefore omitted. The only difference is in replacing 1 by ω_i at several places. Notice that generalized symmetry is satisfied by the generalized k-permission measure since $gp_i^k(D,\omega)=\omega_i$ for every $i\in N$ with $P_D(i)=S_D(i)=\emptyset$. In the proof of uniqueness, similar independent equations are derived as in the proof of Theorem 4.1. At the beginning of the uniqueness proof, applying generalized symmetry to the empty graph gives $gm_i(\emptyset,\omega)=\omega_i$ for all $i\in N$.

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