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Abstract

This paper analyzes the allocation of the total toll collected in a highway among its segments. Based on different toll charging rules, we propose the Segments Equal Sharing method, the Exits Equal Sharing method, and the Entrances Equal Sharing method. We provide axioms and characterize these methods used to distribute the toll. Besides, we show how these methods can be obtained by applying the Shapley value to associated coalitional transferable utility games.

Keywords: Highway toll allocation problem; Axiomatic characterization; Shapley value.

1 Introduction

Highways are important channels to maintain the flow of people and goods between different regions. The distribution of highway tolls is a key factor to maintain the continuous operation of highways. There are many studies that focus on allocation problems of tolled transport systems. Examples are Rosenthal (2017) that investigates how to allocate costs of a regional transit system to its users, Algaba et al. (2019) that considers how to share revenues among transport companies in a multimodal public transport system, and Estañ et al. (2021) that explores how to distribute the fixed cost of a tram line among municipalities along that line. In this paper, based on Dong et al. (2012a), we analyze the allocation of the toll collected in a line highway among its segments.[1] Users can enter and leave from each segment, but must pay the corresponding toll. Whereas the *highway problem* of Dong et al. (2012a) studies how to set highway tolls that distribute the highway's building and maintenance costs among the users, we analyze how the toll that is collected from the users of the highway should be allocated over its different segments. We refer to our allocation model as a *highway toll allocation problem*.[2]

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[1]Here, a segment is a part of the highway which is between an entrance and the first available exit.

[2]Highway toll allocation problems are revenue sharing problems, while highway problems of Dong et al. (2012a) are cost sharing problems.

Toll roads have gradually got high interest. In many large cities, such as London, Copenhagen and Stockholm, toll roads have become a useful method to alleviate congestion. In some East Asian countries, such as China, toll highways connect almost all cities, and the operating model based on tolls has allowed for a rapid development of the construction of highways. Whether tolling highways helps to alleviate traffic congestion (Ren et al., 2020), to maintain operating repairs (Jaarsma and van Dijk, 2002), or even to control pollution (Chen et al., 2018), there is the underlying problem of how to allocate the toll to municipal governments or highway maintenance companies along the line.

One of the original studies on the problem of toll highways based on game theory is Dong et al. (2012a), in which a toll highway pricing model is designed to recover the highway construction costs. The axioms of Routing-proofness and Cost Recovery are key properties to define a highway toll pricing method. Kuipers et al. (2013) generalizes the model of Dong et al. (2012a) and proposes the *highway game*, where there is an ordered set of several indivisible segments, each segment has an associated fixed cost, and each agent uses some consecutive segments. Kuipers et al. (2013) designs algorithms to calculate the Shapley value and prenucleolus for highway games. Further, Sudholter and Zarzuelo (2017) characterizes the core, prenucleolus, and Shapley value of highway games, mainly based on the reduced highway problem property. The backgrounds of highway games and highway toll allocation games are similar, but the players and characteristic functions are different. The players in highway games are the users of the highway, while the players in highway toll allocation games are the highway segments. Whereas a highway game is a cost game which characteristic function depends on highway construction and maintenance costs, a highway toll allocation game is a profit game which characteristic function is determined by the toll that is collected from all users of the highway.

Although the highway toll allocation problem is inspired by the highway problem, it can be seen as an extension of the polluted river problem that is introduced in Ni and Wang (2007) and is widely studied in, e.g., Dong et al. (2012b), van den Brink et al. (2018), and Hou et al. (2020). A polluted river problem consists of a set of river segments located along a polluted river and a nonnegative cost associated with each river segment. All segments jointly bear the cost of cleaning up the river from pollution. Questions that are addressed are to what extent a segment should pay in the cleaning cost of its own segment, and how much it should contribute in the cleaning cost of upstream and downstream segments. Similarly, a highway toll allocation problem has an ordered line structure on the segments, and nonnegative tolls associated with every collection of consecutive segments instead of only individual segments. Therefore, a highway toll allocation problem is more general than a polluted river problem: a polluted river problem can be seen as a highway toll allocation problem where each user makes use of only one segment. Of course, one must also bear in mind that highway toll allocation problems deal with revenues instead of costs. Based on this point, we adopt principles behind classical (cost allocation) methods[3] for polluted river problems to define (revenue allocation) methods for highway toll allocation problems. The allocations determined by these extended methods coincide with the application of the Shapley value to associated games induced by different toll charging policies. These games can be related to two well-known classes of cooperative games with restricted cooperation. One is a nonnegative communication line-graph game as in Myerson (1977) where the line-graph is given by the consecutive segments of the highway. The other one is a game with a permission structure, see Gilles et al. (1992) and Gilles and Owen (1994), where the permission structure is given by the highway oriented in (respectively against) the driving direction.

The rest of this paper is organized as follows. Section 2 contains preliminaries. Section 3 describes the highway toll allocation model and proposes toll allocation methods inspired by underlying ideas of well-known methods for polluted river problems. Sections 4 and 5 provide characterizations of the methods introduced in Section 3. In fact, we present two different types of characterizations: axiomatic and methodological. The methodological characterizations are based on games that are defined under different charging policies and whose Shapley values coincide with the allocations induced by the proposed methods. Section 6 concludes. All proofs of propositions, theorems, and corollaries are postponed to the appendix, including the logical independence of axioms in each axiomatization.

2 Preliminaries

A situation in which a finite set of players can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, simply a *TU-game*. A TU-game is defined as a pair (N,ν) where $N=\{1,2,\ldots,n\}\subseteq\mathbb{N}$ is a *set of players*, and $\nu: 2^N\to\mathbb{R}$ is a *characteristic function* on N satisfying $\nu(\emptyset)=0$. For every coalition $E\subseteq N$, $\nu(E)\in\mathbb{R}$ is the worth of coalition E. Since we take the player set N to be fixed, we often write a TU-game (N,ν) simply by its characteristic function ν. We denote the collection of all TU-games on N by $\mathcal{G}^N$.

A *payoff vector* for $\nu\in\mathcal{G}^N$ is an $|N|$-dimensional vector $x\in\mathbb{R}^N$ assigning a payoff $x_i\in\mathbb{R}$ to each player $i\in N$. A *solution* for TU-games with player set $N\subseteq\mathbb{N}$ is a function $f\colon\mathcal{G}^N\to\mathbb{R}^N$, which maps each TU-game into a payoff vector. One of the most famous solutions for TU-games is the *Shapley value* (Shapley, 1953) given by $\mathcal{S}h_i(\nu)=\sum\limits_{E\subseteq N\setminus\{i\}} p(E)(\nu(E\cup\{i\})-\nu(E))$ for all $i\in N$ and $\nu\in\mathcal{G}^N$, where $p(E)=\frac{|E|!(|N|-|E|-1)!}{|N|!}$.

For every $E\subseteq N$, $E\neq\emptyset$, the *unanimity game* on coalition E is given by $u_E(F)=1$ if $E\subseteq F$, and $u_E(F)=0$ otherwise. It is well known that unanimity games form a basis

[3]These are the Local Responsibility Sharing (LRS) method, Upstream Equal Sharing (UES) method, and Downstream Equal Sharing (DES) method, see Ni and Wang (2007), Dong et al. (2012b), and Alcalde-Unzu et al. (2015).

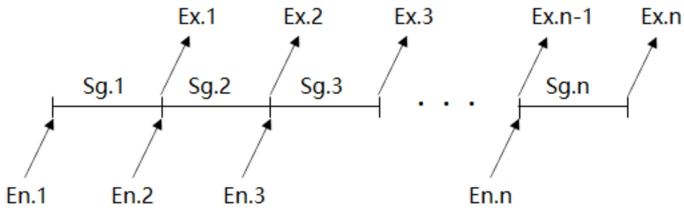

Figure 1: A linear highway

for $\mathcal{G}^N$, specifically, for every $\nu \in \mathcal{G}^N$ it holds that $\nu = \sum_{E \subseteq N, E \neq \emptyset} \Delta_\nu(E) u_E$, where $\Delta_\nu(E) = \sum_{F \subseteq E} (-1)^{|E|-|F|} \nu(F)$ is the *Harsanyi dividend* of coalition $E \subseteq N$, $E \neq \emptyset$, see Harsanyi (1959). An alternative expression for the Shapley value is $\mathcal{S}h_i(\nu) = \sum_{E \subseteq N, i \in E} \frac{\Delta_\nu(E)}{|E|}$ for all $i \in N$ and $\nu \in \mathcal{G}^N$.

3 Highway toll allocation model and methods

Consider a one-way linear tolled highway which is divided into n segments indexed in a given order $i = 1, 2, \ldots, n$. We denote the set of segments by $N = \{1, 2, \ldots, n\}$. Assume that the entrance of segment $i \in N \setminus \{1\}$ is located at the same place as the exit of segment $i-1$, as in Figure 1. For $h < k$, we denote a sequence of consecutive segments $(i_h, i_{h+1}, \ldots, i_{k-1}, i_k)$, where $i_{j+1} = i_j + 1$ for all $j = h, \ldots, k-1$, as $[i_h, i_k]$ and refer to it as the *path* from i_h to i_k. Further, we denote the set of segments located on the path $[i_h, i_k]$ by $\mathcal{P}([i_h, i_k])$, i.e. $\mathcal{P}([i_h, i_k]) = \{i_h, \ldots, i_k\}$. Besides, an isolated segment is also seen as a path in our discussion, i.e. $\mathcal{P}([i, i]) = \{i\}$. We denote the toll collected from all users entering at entrance i and leaving at exit j by $t_{ij} \geq 0$. An $|N| \times |N|$-dimensional nonnegative matrix T is called a *one-way toll matrix* (or *toll matrix* for short) if $t_{ij} = 0$ for every $i > j$.[4] We denote the collection of all $|N| \times |N|$-dimensional toll matrices by $\mathcal{T}^N$.

A *highway toll allocation problem* is a pair (N, T) with $N = \{1, \ldots, n\}$ and $T \in \mathcal{T}^N$. A *toll allocation* for a problem (N, T) is a nonnegative vector $x \in \mathbb{R}^N_+$ assigning a share $x_i \in \mathbb{R}_+$ in the total toll to each segment $i \in N$. A *toll allocation method* is a mapping f that assigns a toll allocation $f(N, T)$ to each highway toll allocation problem (N, T). Since we take the set of segments N to be fixed, we often write a highway toll allocation problem (N, T) simply by its toll matrix T.

There are many different tolling systems for tolled highways. Tolling systems charge users[5] for entering or leaving tolled highways based on certain tolling policies. In this paper, we only

[4]This paper only focuses on studying one-way linear highways. A two-way highway can be seen as a combination of two one-way highways with opposite directions.

[5]In the following discussion, a user is one drive from some entrance to some exit of the highway. Thus, if a driver uses the highway twice, then these are considered two users.

consider toll allocation methods under the distance-based toll system and the fixed toll system, see Xiao et al. (2012) and Otaki et al. (2017). In the distance-based toll system, a user's toll depends on the number of used segments, while in the fixed toll system a user's toll is related to its entrance (or exit).

For the distance-based toll system, we propose the *Segments Equal Sharing method* that allocates the toll obtained from any user equally over the segments used by that user.

Definition 3.1. The Segments Equal Sharing method, shortly SES method, is given by

$$f_i^{Se}(N,T) = \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{t_{hk}}{k-h+1} \text{ for every } i \in N \text{ and } T \in \mathcal{T}^N.$$

Based on the idea of the fixed toll system, we introduce the following two methods, where also segments that are upstream, respectively downstream, of the used segments share in the revenues.

Definition 3.2. The Exits Equal Sharing method, shortly ExES method, is given by

$$f_i^{Ex}(N,T) = \sum_{k=i}^{n} \frac{\sum\limits_{h=1}^{k} t_{hk}}{k} \text{ for every } i \in N \text{ and } T \in \mathcal{T}^N.$$

Definition 3.3. The Entrances Equal Sharing method, shortly EnES method, is given by

$$f_i^{En}(N,T) = \sum_{h=1}^{i} \frac{\sum\limits_{k=h}^{n} t_{hk}}{n-h+1} \text{ for every } i \in N \text{ and } T \in \mathcal{T}^N.$$

Clearly, the ExES method allocates all the toll collected at an exit equally over the corresponding segment and all its upstream segments, while the EnES method allocates all the toll collected from users that enter at a segment equally over that segment and all its downstream segments. These two methods are suitable in a fixed toll system, specifically in situations where (i) the fixed toll is collected when exiting the highway, respectively (ii) the fixed toll is collected when entering the highway.

It is worth mentioning that the SES, ExES, and EnES methods can be seen as extended versions of the LRS, UES and DES methods for polluted river problems of Ni and Wang (2007) and Dong et al. (2012b) in the sense that they give the outcomes of these methods if $t_{ij} = 0$ whenever $i \neq j$.[6]

[6] A polluted river problem is a pair (N, C), where $N = \{1, \ldots, n\}$ is a finite set of agents located along a linear polluted river and $C = (c_i)_{i \in N}$ is an $|N|$-dimensional cost vector, with c_i the cost of cleaning the polluted river at segment $i \in N$. The LRS, UES, and DES methods are given by $p_i^{LRS}(N,C) = c_i$, $p_i^{UES}(N,C) = \sum\limits_{j \geq i} \frac{c_j}{j}$, and $p_i^{DES}(N,C) = \sum\limits_{j \leq i} \frac{c_j}{n-j+1}$ for all $i \in N$, respectively. Whereas Ni and Wang (2007) introduce the LRS and UES methods for linear (single spring, single sink) rivers, Dong et al. (2012b) considers more general rivers with either multiple springs or multiple sinks and, besides the LRS and UES methods, considers the DES method.

Example 1. *Consider a toll allocation problem* (N,T) *with* $N = \{1,2,3,4,5,6,7\}$, $t_{25} = 1$, *and* $t_{ij} = 0$ *for every* $(i,j) \neq (2,5)$. *We have* $f^{Se}(N,T) = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0)$, $f^{Ex}(N,T) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, 0)$, *and* $f^{En}(N,T) = (0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. ◇

4 Characterizations of the Segments Equal Sharing method

In this section, we first provide axiomatizations of the SES method. Second, we define a coalitional game for highway toll allocation problems whose Shapley value coincides with the allocation according to the SES method.

4.1 Axioms

The first axiom is additivity, which implies that the toll revenue allocation does not depend on the frequency (daily, weekly, monthly,...) with which the revenues are allocated.

Additivity. For every $T, T' \in \mathcal{T}^N$, $f(N, T+T') = f(N,T) + f(N,T')$.

The second axiom is efficiency, which says that the total toll will be fully allocated to all segments.

Efficiency. For every $T \in \mathcal{T}^N$, $\sum_{i\in N} f_i(N,T) = \sum_{h\in N}\sum_{k\in N} t_{hk}$.

Third, the inessential segment property states that, if a highway segment is not used by any user, the corresponding segment will not share in the toll.

Inessential segment property. For every $T \in \mathcal{T}^N$ and $i \in N$, if $t_{hk} = 0$ for all $h \leq i \leq k$, then $f_i(N,T) = 0$.

Fourth, segment symmetry requires that the segments used by all users should get an equal share in the toll.

Segment symmetry. For every $T \in \mathcal{T}^N$ and $i,j \in N$ such that $i,j \in \mathcal{P}([h,k])$ for all $h,k \in N$ with $t_{hk} > 0$, it holds that $f_i(N,T) = f_j(N,T)$.

The above four axioms characterize the SES method.

Theorem 4.1. *The SES method is the only method satisfying additivity, efficiency, the inessential segment property, and segment symmetry.*

Remark 1. As mentioned before, a highway toll allocation problem with $t_{ij} = 0$, whenever $i \neq j$, is equivalent to a polluted river problem. From Ni and Wang (2007), the LRS method for polluted river problems is determined by additivity, efficiency, and the inessential segment property.[7] However, these three axioms are not sufficient to characterize the SES method for highway toll allocation problems, because they do not put any restriction on how to allocate the toll t_{ij} over the segments on the path from i to j, $i < j$, if there is no toll collected at other segments (see the proof in the Appendix). This issue is solved using segment symmetry. $\diamond$

A stronger version of segment symmetry is the necessary segment property. It requires that a segment used by all users earns at least as much from the tolls as any other segment.[8]

Necessary segment property. For every $T \in \mathcal{T}^N$ and $i \in N$, if $t_{hk} = 0$ for all $h, k \in N$ with $i \notin \mathcal{P}([h,k])$, then $f_i(N,T) \geq f_j(N,T)$ for every $j \in N$.

Since the necessary segment property implies segment symmetry, and the SES method obviously satisfies the necessary segment property, we obtain the following characterization as a corollary.

Corollary 4.1.1. *The SES method is the only method satisfying additivity, efficiency, the inessential segment property, and the necessary segment property.*

4.2 Segments allocation game

In order to provide a game-theoretic analysis of the SES method, for every highway toll allocation problem, we define an associated cooperative game called *segments allocation game*, which assigns to every coalition $E \subseteq N$ the toll collected from using only segments of E, i.e.

$$\nu^{Se}(E) = \sum_{\substack{i,j \in N \\ \mathcal{P}([i,j]) \subseteq E}} t_{ij}. \tag{1}$$

Based on Harsanyi dividends, we argue that every segments allocation game is a nonnegative communication line-graph game (see Myerson, 1977): the Harsanyi dividends of coalitions of consecutive segments correspond with the collected toll of users that use exactly those segments, and the Harsanyi dividends of coalitions of nonconsecutive segments are zero, see also Owen (1986) and van den Brink et al. (2007).[9] Moreover, every nonnegative line-graph game is a segments allocation game.

[7] Additivity and efficiency as defined above are direct extensions of the corresponding axioms in Ni and Wang (2007). The inessential segment property implies the No blind cost axiom of Ni and Wang (2007) which says that an agent with zero cost contributes nothing.

[8] The necessary segment property extends the necessary agent property for polluted river problems in van den Brink et al. (2018), which is an application of the necessary player property of van den Brink and Gilles (1996) when considering a highway toll allocation problem as a game with a permission structure, see Section 5.2.

[9] For line-graph games, Owen (1986) and van den Brink et al. (2007) show that the Harsanyi dividend of a coalition of consecutive segments $\{i, i+1, \ldots, j-1, j\}$ is given by $\Delta_\nu(\{i, i+1, \ldots, j-1, j\}) = \nu(\{i, i+1, \ldots, j-1, j\}) + \nu(\{i+1, \ldots, j-1\}) - \nu(\{i+1, \ldots, j-1, j\}) - \nu(\{i, i+1, \ldots, j-1\})$, and the Harsanyi dividend of a coalition of nonconsecutive segments is zero.

Proposition 4.1. *(i) For every $T \in \mathcal{T}^N$ and $E \subseteq N$ such that $E = \mathcal{P}([i,j])$ for some $i, j \in N$, it holds that $\Delta_{\nu^{Se}}(E) = t_{ij}$;*

(ii) For every $T \in \mathcal{T}^N$ and $E \subseteq N$ such that there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$, it holds that $\Delta_{\nu^{Se}}(E) = 0$;

(iii) For every game (N, ν) such that $\Delta_\nu(E) \geq 0$ for every $E \subseteq N$, and $\Delta_\nu(E) = 0$ if there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$, it holds that (N, ν) is the segments allocation game of a highway toll allocation problem.

Using this proposition, it can be shown that the SES method provides the allocation that coincides with applying the Shapley value to the associated segments allocation game.

Proposition 4.2. *Let $T \in \mathcal{T}^N$. Then $f^{Se}(N,T) = \mathcal{S}h(\nu^{Se})$.*

Remark 2. By Proposition 4.2, the SES method "coincides" with the Shapley value of the associated game ν^{Se}. Moreover, by Proposition 4.1 (iii), the set of segments allocation games is equivalent to the class of nonnegative communication line-graph games. Based on these results, we can obtain new axiomatizations of the SES method by considering axiomatizations of the Myerson value for line-graph games. For example, the original axiomatization of Myerson (1977) using fairness and component efficiency can be adopted to characterize the SES method.[10]

In Appendix B, we provide a formal statement and the proof of the characterization of the SES method by means of toll fairness and sub-highway efficiency. Here, we give an informal description. To translate Myerson's fairness to our setting, "breaking a link" $\{i, i+1\}, i = 1, \dots, n-1$, in a highway toll allocation can be interpreted as a blocked road resulting in the loss of all toll collected from users entering at or before entrance i and leaving at or after exit $i+1$. This implies a reduction in tolls from all users passing through these two segments of the highway. Myerson's fairness can be interpreted as *toll fairness* requiring that this reduction has the same effect on the payoffs of these two disconnected neighboring segments.

To translate component efficiency to our setting, we first introduce the concept of "sub-highway". A coalition of consecutive segment $E \subseteq N$ is a *sub-highway* if there is no toll collected by users entering/exiting outside E and passing through segments in E. Myerson's component efficiency can be interpreted as *sub-highway efficiency* requiring that the toll of every sub-highway is exactly allocated over the segments in that sub-highway. ◇

[10]Notice that the characterization in Myerson (1977) is still valid for the subclass of line-graph games, see van den Brink et al. (2007).

5 Characterizations of the Exits Equal Sharing method and the Entrances Equal Sharing method

As in the previous section, we first provide axiomatizations of the two toll allocation methods. Second, for each toll allocation method, we define a coalitional game model for highway toll allocation problems whose Shapley value yields the corresponding method.

5.1 Axioms

The first axiom introduced in this section is independence of upstream exits, which states that the share of the toll revenue assigned to a segment depends only on tolls collected from users exiting at or after this segment's exit. This property becomes of interest in highway toll allocation problems with fixed toll systems where the toll is collected when exiting the highway.

Independence of upstream exits. For every $T, T' \in \mathcal{T}^N$ and $i \in N$, if $t_{hk} = t'_{hk}$ for all $k \geq i$ and $h \leq k$, then $f_j(N, T) = f_j(N, T')$ for all $j \geq i$.

For its counterpart, independence of downstream entrances states that the share of the toll revenue assigned to a segment only depends on tolls collected from users entering at or before this segment's entrance. This property becomes of interest in highway toll allocation problems with fixed toll systems where the toll is collected when entering the highway.

Independence of downstream entrances. For every $T, T' \in \mathcal{T}^N$ and $i \in N$, if $t_{hk} = t'_{hk}$ for all $h \leq i$ and $k \geq h$, then $f_j(N, T) = f_j(N, T')$ for all $j \leq i$.

Next, we present two symmetry properties, which require that two segments equally share the toll revenue if no user leaves, respectively enters, between the two segments. Again, these properties become relevant under the fixed toll system where tolls are collected at exits, respectively entrances.

Symmetry of exits. For every $T \in \mathcal{T}^N$ and $i, j \in N$ with $i < j$, if $t_{hk} = 0$ for all $h \leq k$ with $i \leq k < j$, then $f_i(N, T) = f_j(N, T)$.

Symmetry of entrances. For every $T \in \mathcal{T}^N$ and $i, j \in N$ with $i < j$, if $t_{hk} = 0$ for all $h \leq k$ with $i < h \leq j$, then $f_i(N, T) = f_j(N, T)$.

The next two results characterize the ExES, respectively EnES, methods.

Theorem 5.1. *The ExES method is the only method satisfying additivity, efficiency, independence of upstream exits, and symmetry of exits.*

Theorem 5.2. *The EnES method is the only method satisfying additivity, efficiency, independence of downstream entrances, and symmetry of entrances.*

Remark 3. Applied to the special case of polluted river problems, the axioms of Theorems 5.1 and 5.2 give the axioms of additivity, efficiency, independence of upstream costs, independence of downstream costs, upstream symmetry and downstream symmetry, that characterize the UES and DES methods for polluted river problems in Ni and Wang (2007) and Dong et al. (2012b).◇

Next, we translate two axioms on fairness defined for communication graph games and games with a permission structure which lead to alternative characterizations of the ExES and EnES methods.

From Example 1, it follows that the ExES and EnES methods do not satisfy the inessential segment property. However, both satisfy one of the following weaker versions of this property. The first requires that a segment gets a zero payoff if no user leaves at or after this segment's exit.

Downstream Inessential Segment Property. For every $T \in \mathcal{T}^N$ and $i \in N$ such that $t_{hk} = 0$ for all $k \geq i$ and $h \leq k$, it holds that $f_i(N,T) = 0$.

The counterpart of the above property requires that a segment gets a zero payoff if no user enters at or before this segment's entrance.

Upstream Inessential Segment Property. For every $T \in \mathcal{T}^N$ and $i \in N$ such that $t_{hk} = 0$ for all $h \leq i$ and $k \geq h$, it holds that $f_i(N,T) = 0$.

Since the downstream inessential segment property and the upstream inessential segment property together imply the inessential segment property, by Corollary 4.1.1, we have that the SES method is uniquely determined by the combination of additivity, efficiency, the necessary segment property, the downstream inessential segment property, and the upstream inessential segment property. The ExES method (respectively the EnES method) satisfies the downstream inessential segment property (respectively upstream inessential segment property) but does not satisfy the upstream inessential segment property (respectively the downstream inessential segment property). In order to uniquely determine the ExES and EnES methods using these weaker versions of the inessential segment property, we need an additional axiom.

As mentioned in the previous section, the SES method can be seen as the Myerson value of the associated (communication) line-graph game. Therefore, it satisfies toll fairness stating that "losing" the toll revenue of all users that travel from an entrance at or before segment i and leave at or after segment $i+1$, has the same effect on the payoff of i and $i+1$, see Remark 2. This is not satisfied by the ExES and EnES methods. Instead, they satisfy that the effect of a change in toll revenue between segments h and $k \geq h$ on the payoff of intermediate segment i with $h \leq i \leq k$, is equal to the sum of the effects on the payoffs of a segment upstream of h and a segment downstream of k. If there is no segment upstream, respectively downstream, of the path $[h,k]$, the fairness property is slightly different, equalizing the effect on the payoffs of intermediary agent i and a segment downstream of k, respectively upstream of h.

Externality fairness I. For every $T, T' \in \mathcal{T}^N$ such that there exist $h, k \in N$, $1 < h \leq k$, with $t_{ij} = t'_{ij}$ for all $(i,j) \neq (h,k)$, $i, j \in N$,

(i) when $k < n$, we have

$$f_s(N,T) - f_s(N,T') = f_p(N,T) - f_p(N,T') + f_q(N,T) - f_q(N,T')$$

for all $s \in \{h, \ldots, k\}$, $p \in \{1, \ldots, h-1\}$, and $q \in \{k+1, \ldots, n\}$;

(ii) when $k = n$, we have

$$f_s(N,T) - f_s(N,T') = f_p(N,T) - f_p(N,T')$$

for all $s \in \{h, \ldots, k\}$ and $p \in \{1, \ldots, h-1\}$.

Externality fairness II. For every $T, T' \in \mathcal{T}^N$ such that there exist $h, k \in N$, $h \leq k < n$, with $t_{ij} = t'_{ij}$ for all $(i,j) \neq (h,k)$, $i, j \in N$,

(i) when $h > 1$, we have

$$f_s(N,T) - f_s(N,T') = f_p(N,T) - f_p(N,T') + f_q(N,T) - f_q(N,T')$$

for all $s \in \{h, \ldots, k\}$, $p \in \{1, \ldots, h-1\}$, and $q \in \{k+1, \ldots, n\}$;

(ii) when $h = 1$, we have

$$f_s(N,T) - f_s(N,T') = f_q(N,T) - f_q(N,T')$$

for all $s \in \{h, \ldots, k\}$ and $q \in \{k+1, \ldots, n\}$.

The ExES method satisfies externality fairness I. A characterization for the ExES method is obtained by replacing the independence and symmetry axioms in Theorem 5.1 by the necessary segment property, the downstream inessential segment property, and externality fairness I.

Theorem 5.3. *The ExES method is the only method satisfying additivity, efficiency, the necessary segment property, the downstream inessential segment property, and externality fairness I.*

The counterpart characterization of the EnES method uses the upstream inessential segment property and externality fairness II instead of the downstream inessential segment property and externality fairness I.

Theorem 5.4. *The EnES method is the only method satisfying additivity, efficiency, the necessary segment property, the upstream inessential segment property, and externality fairness II.*

Theorems 5.3 and 5.4 differ both in the type of the inessential segment as well as the externality fairness axiom that is used. To have comparable axiomatizations of the two methods, differing in only one axiom, we weaken the two externality axioms such that it is satisfied by both methods, by only considering changes that do not involve the two endpoint segments, i.e. the part (i) in both axioms.

Externality fairness. For every $T, T' \in \mathcal{T}^N$ such that there exist $h, k \in N$, $1 < h \leq k < n$, with $t_{ij} = t'_{ij}$ for all $(i,j) \neq (h,k)$, $i, j \in N$, it holds that

$$f_s(N,T) - f_s(N,T') = f_p(N,T) - f_p(N,T') + f_q(N,T) - f_q(N,T'),$$

for all $s \in \{h, \ldots, k\}$, $p \in \{1, \ldots, h-1\}$, and $q \in \{k+1, \ldots, n\}$.

Remark 4. Notice that the ExES method and the EnES method both satisfy externality fairness, but the SES method does not satisfy externality fairness. ◇

By weakening externality fairness, we add a monotonicity axiom with respect to extending the highway that is satisfied by both methods. The monotonicity axiom requires that, adding a segment at the beginning or the end of the highway, such that there are no toll revenues earned that involve these new segments and no change in the toll revenue of the other segments, this should not benefit any of the original segments.

Before formalizing the monotonicity axiom, we first define extended highway toll allocation problems. For notational convenience, we allow to index 0 in segment sets of extended highway toll allocation problems. For every $N = \{1, \ldots, n\} \subseteq \mathbb{N}$, $T \in \mathcal{T}^N$, and $m \in \{0, n+1\}$, a extended highway toll allocation problem is a pair (N^{+m}, T^{+m}), where $N^{+m} = N \cup \{m\}$ and $T^{+m} \in \mathcal{T}^{N^{+m}}$ such that $t^{+m}_{ij} = t_{ij}$ for all $i, j \in N$, and $t^{+m}_{ij} = 0$ if $m \in \{i, j\}$.

Segment monotonicity. For every $T \in \mathcal{T}^N$ and $m \in \{0, n+1\}$, $f_i(N,T) \geq f_i(N^{+m}, T^{+m})$ for all $i \in N$.

Segment monotonicity is both satisfied by the ExES and EnES methods. In the axiomatizations of the ExES and EnES method below, the only difference is with regard to the inessential segment property.[11]

Theorem 5.5. *The ExES method is the only method satisfying additivity, efficiency, the necessary segment property, the downstream inessential segment property, externality fairness, and segment monotonicity.*

Theorem 5.6. *The EnES method is the only method satisfying additivity, efficiency, the necessary segment property, the upstream inessential segment property, externality fairness, and segment monotonicity.*

[11] Notice that, by segment monotonicity, these theorems axiomatize the ExES and EnES methods on highway toll allocation problems where we allow to extend the highway with a segment on one of its endpoints.

5.2 Exits allocation game and entrances allocation game

In highway toll allocation problems with fixed toll systems, where tolls are collected at exits, one can argue that downstream segments of j have no right to share in the toll t_{ij}. Based on this idea, we introduce the *exits allocation game* which assigns to every coalition, the total toll collected at exits that are in the coalition or downstream a member of the coalition. Formally,

$$\nu^{Ex}(E) = \sum_{j=minE}^{n} \sum_{i=1}^{j} t_{ij} \text{ for all } E \subseteq N, \tag{2}$$

where $minE = \min_{i \in E} i$ is the most upstream segment in coalition E.

It turns out that the ExES method can be obtained by applying, to every highway toll allocation problem, the Shapley value to the corresponding exits allocation game.

Proposition 5.1. *Let $T \in \mathcal{T}^N$. Then $f^{Ex}(N,T) = \mathcal{S}h(\nu^{Ex})$.*

Remark 5. The game ν^{Ex} is the dual game[12] of the conjunctive (or disjunctive) restricted game of (N, ν^{Se}, D) where ν^{Se} is the segments allocation game given by (1), and $D = \{(i, i+1) \mid i = 1, \ldots, n-1\}$ is the digraph where the edges on the highway are oriented from upstream to downstream.[13] Consequently, the ExES method equals the conjunctive (respectively disjunctive) permission value of this dual game, see van den Brink and Gilles (1996) (respectively van den Brink, 1997). For completeness, we give a self-contained proof in Appendix C.

Using this observation, we could directly apply one of the axiomatizations of the (conjunctive) permission value in van den Brink and Gilles (1996), stating that the ExES method is the only method satisfying additivity, efficiency, the downstream inessential segment property, the necessary segment property and *upstream monotonicity*. The last axiom states that upstream segments always earn at least as much as downstream segments. ◇

Similarly, a cooperative game underlying the EnES method, called *entrances allocation game*, can be given by

$$\nu^{En}(E) = \sum_{i=1}^{maxE} \sum_{j=i}^{n} t_{ij} \text{ for all } E \subseteq N,$$

where $maxE = \max_{i \in E} i$ is the most downstream segment in the coalition E. This game theoretical model is of interest in highway toll allocation problems with fixed toll systems where tolls are collected at the entrances.

Proposition 5.2. *Let $T \in \mathcal{T}^N$. Then $f^{En}(N,T) = \mathcal{S}h(\nu^{En})$.*

[12] The dual game (N, ν^*) of game (N, ν) is given by $\nu^*(E) = \nu(N) - \nu(N \setminus E)$ for all $E \subseteq N$.

[13] A game with a permission structure is a triple (N, ν, D) where N is a finite set of players, ν is the characteristic function of a cooperative game on N, and $D \subseteq N \times N$ is a digraph on N. Specifically, if $D = \{(i, i+1) \mid i = 1, \ldots, n-1\}$ is a linear order, the conjunctive (or disjunctive) restricted game of ν on D is given by $r_{\nu,D}(E) = \nu(\{i \in E \mid \{1, \ldots, i-1\} \subseteq E\})$ for all $E \subseteq N$, see Gilles et al. (1992) and Gilles and Owen (1994).

Similar to Proposition 5.1, this result is a direct consequence of ν^{En} being the dual game of the conjunctive (or disjunctive) restricted game of (N, ν^{Se}, D) where $D = \{(i, i-1) \mid i = 2, \ldots, n\}$ is the digraph where the edges on the highway are oriented from downstream to upstream. Consequently, the EnES method can be characterized by similar axioms as mentioned at the end of Remark 5. [14]

6 Concluding remarks

Whereas Dong et al. (2012a) consider the allocation of building and maintenance costs of highways over their users, in this paper, we analyze the allocation of the collected tolls over the different segments of a highway. The model and associated games in this paper generalize those for polluted river problems in Ni and Wang (2007) and Dong et al. (2012b).

Based on different toll systems, we propose and axiomatize toll allocation methods, which are based on well-known pollution cost allocation methods for polluted river problems. To explore the relationship between these methods from a (cooperative) game viewpoint, we consider tailor-made games whose Shapley values yield the corresponding toll allocation methods.

The games used in this paper to model highway toll allocation problems have a strong connection with the communication graph games of Myerson (1977) and the games with a permission structure of Gilles et al. (1992). Specifically, the class of segments allocation games, which supports the SES method, coincides with the class of nonnegative communication line-graph games. Considering the framework of games with a permission structure, we obtain two new games: the exits allocation game and the entrances allocation game. The exits allocation game is the restricted game corresponding to the segments allocation game with the linear permission structure where highway segments are oriented from upstream to downstream. Similar, the entrances allocation game is the restricted game corresponding to the segments allocation game with the linear permission structure where highway segments are oriented from downstream to upstream. We exploit these relationships in several of our characterizations. Using this relationship between highway toll allocation problems and games with a communication or permission structure may bring important new insights on highway toll allocation problems. The following topics are of relevance for further development of this field.

(i) Application of other solutions for communication graph games or games with a permission structure to highway toll allocation problems gives alternatives to the methods considered in this paper. Examples are the hierarchical outcomes of Demange (2004), and combinations of hierarchical outcomes such as their average (see Herings et al., 2008), and weighted average (see

[14] Specifically, the EnES method is the only method satisfying additivity, efficiency, the upstream inessential segment property, the necessary segment property and *downstream monotonicity*. The last axiom states that downstream segments always earn at least as much as upstream segments.

Béal et al., 2010). Specifically, it is interesting to compare the methods in this paper with three solutions: the upper-equivalent solution, the lower-equivalent solution, and the average of these two solutions (van den Brink et al., 2007). The upper-equivalent solution assigns the marginal vector corresponding to the order where the players enter from upstream to downstream. The lower-equivalent solution assigns the marginal vector corresponding to the order where the players enter from downstream to upstream. As an illustration, in Example 1, the upper-equivalent solution assigns the allocation $(0, 0, 0, 0, 1, 0, 0)$, the lower-equivalent solution assigns the allocation $(0, 1, 0, 0, 0, 0, 0)$, and thus their average assigns $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0)$. Compared to the SES method, the average solution does not reward the middle segments of the highway, but only the segments where a user enters and leaves (segments 2 and 5 in this example).

(ii) In the other direction, translation of the methods considered in this paper to other applications of communication line-graphs, such as the river games of Ambec and Sprumont (2002) or the one-machine sequencing games of Curiel et al. (1993) and Curiel et al. (1994) may lead to valuable new insights for these applications. Solutions mentioned under (i) are known in the literature under different names, depending on the application. For example, the upper-equivalent solution is known as the downstream incremental solution for river sharing problems, and the average of the upper- and lower-equivalent solutions is known as the equal gains split rule in sequencing problems. It is interesting to compare our new methods, introduced here for highway toll allocation problems, with these known methods for other applications.

(iii) Inspired by the ExES and EnES methods subsequent extension of these solutions to communication graph games with a more general structure, such as cycle-free graph games, opens a new line of research with promising results.

To conclude, we summarize the toll allocation methods and axioms discussed in this paper in Table 1. "+" means that the method satisfies the axiom, while "−" has the converse meaning. "⊕" indicates the axiom is used in the characterization of a method in this paper.

Table 1: Characterizing properties of toll allocation methods.

Properties	f^{Se}	f^{Ex}	f^{En}
Additivity	$\oplus$	$\oplus$	$\oplus$
Necessary player property	$+$	$\oplus$	$\oplus$
Segment monotonicity	$+$	$\oplus$	$\oplus$
Efficiency	$\oplus$	$\oplus$	$\oplus$
Sub-highway efficiency	$\oplus$	$-$	$-$
Segment symmetry	$\oplus$	$+$	$+$
Symmetry of Exits	$-$	$\oplus$	$-$
Symmetry of Entrances	$-$	$-$	$\oplus$
Inessential segment property	$\oplus$	$-$	$-$
Downstream Inessential segment property	$+$	$\oplus$	$-$
Upstream Inessential segment property	$+$	$-$	$\oplus$
Independence of Upstream Exits	$+$	$\oplus$	$-$
Independence of Downstream Entrances	$+$	$-$	$\oplus$
Toll fairness	$\oplus$	$-$	$-$
Externality fairness I	$-$	$\oplus$	$-$
Externality fairness II	$-$	$-$	$\oplus$
Externality fairness	$-$	$\oplus$	$\oplus$

Appendix A: Proofs

Theorem 4.1. *The SES method is the only method satisfying additivity, efficiency, the inessential segment property, and segment symmetry.*

Proof. First, we prove that f^{Se} satisfies the four axioms.

Additivity directly follows from f_i^{Se} being a linear function of t_{hk}, $h \leq k$, with $i \in \mathcal{P}([h,k])$.

To show that f^{Se} satisfies efficiency, let $T \in \mathcal{T}^N$. Then,

$$\begin{aligned}\sum_{i=1}^{n} f_i^{Se}(N,T) &= \sum_{i=1}^{n}\sum_{h=1}^{i}\sum_{k=i}^{n} \frac{t_{hk}}{k-h+1} \\ &= \sum_{h=1}^{n}\sum_{k=h}^{n}(k-h+1)\frac{t_{hk}}{k-h+1} \\ &= \sum_{h=1}^{n}\sum_{k=h}^{n} t_{hk} = \sum_{h\in N}\sum_{k\in N} t_{hk},\end{aligned}$$

where the last equality follows by $t_{hk} = 0$ if $h > k$. Thus, f^{Se} satisfies efficiency.[15]

The inessential segment property directly follows since $f_i^{Se}(N,T) = \sum\limits_{h=1}^{i}\sum\limits_{k=i}^{n} \frac{t_{hk}}{k-h+1} = 0$ if $t_{hk} = 0$ for all $h \leq i \leq k$.

[15] Using Proposition 4.2, this also follows from the fact that $\nu^{Se}(N) = \sum\limits_{h=1}^{n}\sum\limits_{k=h}^{n} t_{hk}$ and efficiency of the Shapley value.

To show that f^{Se} satisfies segment symmetry, let $T \in \mathcal{T}^N$ and $i,j \in N$ such that $i,j \in \mathcal{P}([h,k])$ for all $h,k \in N$ with $t_{hk} > 0$. Let $i < j$. We have

$$\begin{aligned} f_i^{Se}(N,T) &= \sum_{h=1}^{i}\sum_{k=i}^{n}\frac{t_{hk}}{k-h+1} \\ &= \sum_{h=1}^{i}\sum_{k=i}^{j-1}\frac{t_{hk}}{k-h+1} + \sum_{h=1}^{i}\sum_{k=j}^{n}\frac{t_{hk}}{k-h+1} \\ &= \sum_{h=1}^{i}\sum_{k=j}^{n}\frac{t_{hk}}{k-h+1} \\ &= \sum_{h=1}^{i}\sum_{k=j}^{n}\frac{t_{hk}}{k-h+1} + \sum_{h=i+1}^{j}\sum_{k=j}^{n}\frac{t_{hk}}{k-h+1} \\ &= \sum_{h=1}^{j}\sum_{k=j}^{n}\frac{t_{hk}}{k-h+1} = f_j^{Se}(N,T), \end{aligned}$$

where the third equality is a direct consequence of $t_{hk} = 0$ if $j \notin \mathcal{P}([h,k])$, and the fourth equality is a direct consequence of $t_{hk} = 0$ if $i \notin \mathcal{P}([h,k])$. This shows that f^{Se} satisfies segment symmetry.

Therefore, we conclude that f^{Se} satisfies the four axioms.

Second, we prove that f^{Se} is the only method satisfying these four axioms. Assume that method f satisfies the four axioms. Let $T \in \mathcal{T}^N$ and $h,k \in N$, $h \le k$. We first consider the toll matrix $T^{hk} \in \mathcal{T}^N$ given by $t_{hk}^{hk} = t_{hk}$, and $t_{ij}^{hk} = 0$ if $(i,j) \neq (h,k)$. By the inessential segment property,

$$f_i(N,T^{hk}) = 0 \text{ if } i < h \text{ or } i > k. \tag{3}$$

By segment symmetry, there is a constant $c \ge 0$ such that

$$f_i(N,T^{hk}) = c \text{ if } h \le i \le k. \tag{4}$$

Since efficiency implies $\sum\limits_{i=1}^{n} f_i(N,T^{hk}) = \sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n} t_{ij}^{hk} = t_{hk}^{hk} = t_{hk}$, with equations (3) and (4), we have

$$f_i(N,T^{hk}) = \begin{cases} \frac{t_{hk}}{k-h+1} & \text{if } h \le i \le k, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Since $T = \sum\limits_{h=1}^{n}\sum\limits_{k=h}^{n} T^{hk}$, we have

$$\begin{aligned} f_i(N,T) &= f_i(N,\sum_{h=1}^{n}\sum_{k=h}^{n} T^{hk}) \\ &= \sum_{h=1}^{n}\sum_{k=h}^{n} f_i(N,T^{hk}) \\ &= \sum_{h=1}^{i}\sum_{k=i}^{n}\frac{t_{hk}}{k-h+1} = f_i^{Se}(N,T) \text{ for all } i \in N, \end{aligned}$$

where the second equality follows by additivity, and the third equality from equation (5). ■

We show logical independence of the axioms stated in Theorem 4.1 by presenting four alternative methods.

1. Let f^1 be the combined method that is given by

$$f^1(N,T) = \begin{cases} f^{Ex}(N,T) & \text{if } T \in \mathcal{T}^N \text{ with } t_{ij} > 0 \text{ for all } 1 \leq i \leq j \leq n, \\ f^{Se}(N,T) & \text{otherwise.} \end{cases}$$

   This method satisfies all axioms in Theorem 4.1 except additivity.

2. Let f^2 be the method in which no toll is distributed to any segment,

$$f_i^2(N,T) = 0 \text{ for all } i \in N \text{ and } T \in \mathcal{T}^N. \tag{6}$$

   This method satisfies all axioms in Theorem 4.1 except efficiency.

3. Let f^3 be the method that equally distributes all tolls to each segment,

$$f_i^3(N,T) = \frac{\sum_{h=1}^{n} \sum_{k=h}^{n} t_{hk}}{|N|} \text{ for every } i \in N \text{ and } T \in \mathcal{T}^N. \tag{7}$$

   This method satisfies all axioms in Theorem 4.1 except the inessential segment property.

4. Let f^4 be the method in which each segment is allocated the tolls paid by users entering at its entrance,

$$f_i^4(N,T) = \sum_{j=i}^{n} t_{ij} \text{ for every } i \in N \text{ and } T \in \mathcal{T}^N. \tag{8}$$

   This method satisfies all axioms in Theorem 4.1 except segment symmetry.

Corollary 4.1.1. *The SES method is the only method satisfying additivity, efficiency, the inessential segment property, and the necessary segment property.*

Proof. From the proof of Theorem 4.1, we know that the SES method satisfies additivity, efficiency, and the inessential segment property. We only need to show that the SES method satisfies the necessary segment property. Let $T \in \mathcal{T}^N$ and $i \in N$ such that $t_{hk} = 0$ for all

$h, k \in N$ with $i \notin \mathcal{P}([h,k])$. Then, for every $j > i$, we have

$$\begin{aligned}
f_j^{Se}(N,T) &= \sum_{p=1}^{j}\sum_{q=j}^{n}\frac{t_{pq}}{q-p+1} \\
&= \sum_{p=1}^{i}\sum_{q=j}^{n}\frac{t_{pq}}{q-p+1} + \sum_{p=i+1}^{j}\sum_{q=j}^{n}\frac{t_{pq}}{q-p+1} \\
&= \sum_{p=1}^{i}\sum_{q=j}^{n}\frac{t_{pq}}{q-p+1} \\
&\leq \sum_{p=1}^{i}\sum_{q=j}^{n}\frac{t_{pq}}{q-p+1} + \sum_{p=1}^{i}\sum_{q=i}^{j-1}\frac{t_{pq}}{q-p+1} \\
&= \sum_{p=1}^{i}\sum_{q=i}^{n}\frac{t_{pq}}{q-p+1} = f_i^{Se}(N,T),
\end{aligned}$$

where the inequality follows from T being a nonnegative matrix. Similarly, it can be shown that $f_j^{Se}(N,T) \leq f_i^{Se}(N,T)$ for every $j < i$. Thus, the SES method satisfies the necessary segment property.

The proof of uniqueness follows since, as mentioned before the corollary in Section 4, the necessary segment property implies segment symmetry,[16] and the SES method satisfies the necessary segment property as shown above. ■

The alternative methods after Theorem 4.1 can also be used to show logical independence of the axioms stated in Corollary 4.1.1. Notice that f^1, f^2 and f^3 satisfy the necessary segment property, while f^4 does not.

Proposition 4.1. *(i) For every $T \in \mathcal{T}^N$ and $E \subseteq N$ such that $E = \mathcal{P}([i,j])$ for some $i, j \in N$, it holds that $\Delta_{\nu^{Se}}(E) = t_{ij}$;*

(ii) *For every $T \in \mathcal{T}^N$ and $E \subseteq N$ such that there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$, it holds that $\Delta_{\nu^{Se}}(E) = 0$;*

(iii) *For every game (N, ν) such that $\Delta_\nu(E) \geq 0$ for every $E \subseteq N$, and $\Delta_\nu(E) = 0$ if there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$, it holds that (N, ν) is the segments allocation game of a highway toll allocation problem.*

Proof. (i) Let $T \in \mathcal{T}^N$ and $i, j \in N$, $i \leq j$. We prove that $\Delta_{\nu^{Se}}(\mathcal{P}([i,j])) = t_{ij}$ by induction on $|\mathcal{P}([i,j])|$. If $|\mathcal{P}([i,j])| = 1$, i.e. $i = j$, we have $\Delta_{\nu^{Se}}(\mathcal{P}([i,j])) = \Delta_{\nu^{Se}}(\{i\}) = \nu^{Se}(\{i\}) = t_{ii}$. If

[16] Let $T \in \mathcal{T}^N$ and $i, j \in N$ such that $i, j \in \mathcal{P}([h,k])$ for all $h, k \in N$ with $t_{hk} > 0$. Considering i, the necessary segment property implies that $f_i(N,T) \geq f_r(N,T)$ for all $r \in N$, and thus, $f_i(N,T) \geq f_j(N,T)$. Conversely, considering j, the necessary segment property implies $f_j(N,T) \geq f_i(N,T)$. Thus, $f_i(N,T) = f_j(N,T)$ showing that the necessary segment property implies segment symmetry.

$|\mathcal{P}([i,j])| = 2$, i.e. $j = i+1$, we have $\Delta_{\nu^{Se}}(\mathcal{P}([i,j])) = \Delta_{\nu^{Se}}(\{i,j\}) = \nu^{Se}(\{i,j\}) - \nu^{Se}(\{i\}) - \nu^{Se}(\{j\}) = t_{ij} + t_{ii} + t_{jj} - t_{ii} - t_{jj} = t_{ij}$.

Proceeding by induction, assume that $\Delta_{\nu^{Se}}(\mathcal{P}([i',j'])) = t_{i'j'}$ for all i', j' with $|\mathcal{P}([i',j'])| < |\mathcal{P}([i,j])|$. Then, we have

$$\begin{aligned}
\Delta_{\nu^{Se}}(\mathcal{P}([i,j])) &= \nu^{Se}(\mathcal{P}([i,j])) - \sum_{\substack{F \subseteq \mathcal{P}([i,j]) \\ F \notin \{\emptyset, \mathcal{P}([i,j])\}}} \Delta_{\nu^{Se}}(F) \\
&= \sum_{h=i}^{j} \sum_{k=h}^{j} t_{hk} - \sum_{h=i}^{j} \sum_{\substack{k=h \\ (h,k) \neq (i,j)}}^{j} t_{hk} = t_{ij},
\end{aligned}$$

where the second equality follows from equation (1) and the induction hypothesis.

(ii) For every $T \in \mathcal{T}^N$ and $E \subseteq N$, by equation (1), we have

$$\nu^{Se}(E) = \sum_{\substack{i,j \in N \\ \mathcal{P}([i,j]) \subseteq E}} t_{ij} = \sum_{F \in \mathcal{C}_L(E)} \sum_{\substack{i,j \in N \\ \mathcal{P}([i,j]) \subseteq F}} t_{ij} = \sum_{F \in \mathcal{C}_L(E)} \nu^{Se}(F)$$

where $\mathcal{C}_L(E)$ is the set of consecutive sets $F \subseteq E$ such that there exist $i, j \in E$ with $F = \mathcal{P}([i,j])$ and $i-1, j+1 \notin E$. Thus, ν^{Se} coincides with a line-graph restricted game. From Owen (1986) and van den Brink et al. (2007), in a line-graph restricted game, the Harsanyi dividend of an unconnected coalition[17] is zero. Thus, $\Delta_{\nu^{Se}}(E) = 0$ if there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$.

(iii) For every game (N, ν) such that $\Delta_\nu(E) \geq 0$ for every $E \subseteq N$, and $\Delta_\nu(E) = 0$ if there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$, consider the highway toll allocation problem (N, T^*) where $t^*_{ij} = \Delta_\nu(\mathcal{P}([i,j]))$ if $i, j \in N$ with $i \leq j$, and $t^*_{ij} = 0$ otherwise. The game (N, ν) is given by $\nu(E) = \sum_{\substack{i,j \in N \\ \mathcal{P}([i,j]) \subseteq E}} t^*_{ij}$ for all $E \subseteq N$, and thus is the segments allocation game associated to (N, T^*). ■

Proposition 4.2. *Let $T \in \mathcal{T}^N$. Then $f^{Se}(N,T) = \mathcal{S}h(\nu^{Se})$.*

Proof. Based on Proposition 4.1, $\Delta_{\nu^{Se}}(E) = t_{hk}$ if there are $h, k \in N$ such that $E = \mathcal{P}([h,k])$,

[17] A coalition, $E \subseteq N$, is unconnected in a line-graph restricted game if there does not exist $i, j \in N$ with $\mathcal{P}([i,j]) = E$.

and $\Delta_{\nu^{Se}}(E) = 0$ otherwise. Therefore, the Shapley value of game (N, ν^{Se}) for segment i is

$$\begin{aligned} \mathcal{S}h_i(\nu^{Se}) &= \sum_{\substack{E \subseteq N \\ i \in E}} \frac{\Delta_{\nu^{Se}}(E)}{|E|} \\ &= \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{\Delta_{\nu^{Se}}(\mathcal{P}([h,k]))}{|\mathcal{P}([h,k])|} \\ &= \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{t_{hk}}{k-h+1} = f_i^{Se}(N,T) \text{ for all } i \in N. \end{aligned}$$

■

Theorem 5.1. *The ExES method is the only method satisfying additivity, efficiency, independence of upstream exits, and symmetry of exits.*

Proof. First, we prove that f^{Ex} satisfies the four axioms.

Additivity and independence of upstream exits both follow since f_i^{Ex} is a linear function of t_{hk} with $k \geq i$ and $h \leq k$.

Efficiency[18] follows since $\sum_{i=1}^{n} f_i^{Ex}(N,T) = \sum_{i=1}^{n} \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = \sum_{k=1}^{n} k \frac{\sum_{h=1}^{k} t_{hk}}{k} = \sum_{k=1}^{n} \sum_{h=1}^{k} t_{hk}$.

Symmetry of exits follows since $t_{hk} = 0$ for all $h \leq k$ with $i \leq k < j$, implies that $f_i^{Ex}(N,T) = \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = \sum_{k=i}^{j-1} \frac{\sum_{h=1}^{k} t_{hk}}{k} + \sum_{k=j}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = \sum_{k=j}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = f_j^{Ex}(N,T)$.

Second, we prove that f^{Ex} is the only method satisfying these axioms. Assume that method f satisfies the four axioms. Let $T \in \mathcal{T}^N$ and $h, k \in N$, $h \leq k$. We first consider the matrix $T^{hk} \in \mathcal{T}^N$ given by $t_{hk}^{hk} = t_{hk}$, and $t_{ij}^{hk} = 0$ if $(i,j) \neq (h,k)$. Besides, let $T^0 \in \mathcal{T}^N$ be the null matrix given by $t_{ij}^0 = 0$ for all $i, j \in N$.

By efficiency and $f(N,T) \in \mathbb{R}_+^N$, we have $f_i(N,T^0) = 0$ for every $i \in N$. By independence of upstream exits, $f_i(N,T^{hk}) = f_i(N,T^0)$ for every $i > k$. Thus,[19]

$$f_i(N,T^{hk}) = 0 \text{ for all } i > k. \tag{9}$$

Since efficiency implies $\sum_{i=1}^{n} f_i(N,T^{hk}) = \sum_{i=1}^{n} \sum_{j=i}^{n} t_{ij}^{hk} = t_{hk}$, we have with (9) that $\sum_{i=1}^{k} f_i(N,T^{hk}) = t_{hk}$. By symmetry of exits, we have[20]

$$f_i(N,T^{hk}) = \begin{cases} \frac{t_{hk}}{k} & \text{if } i \leq k, \\ 0 & \text{if } i > k. \end{cases} \tag{10}$$

[18] Using Proposition 5.1, this also follows from the fact that $\nu^{Ex}(N) = \sum_{h=1}^{n} \sum_{k=h}^{n} t_{hk}$ and efficiency of the Shapley value.

[19] If $k = n$, the result has no meaning.

[20] If $k = 1$, independence of upstream exits and efficiency are sufficient to get equation (10).

Since $T = \sum_{h=1}^{n} \sum_{k=h}^{n} T^{hk}$, by additivity and equation (10), we have $f_i(N,T) = \sum_{h=1}^{n} \sum_{k=h}^{n} f_i(N,T^{hk}) = \sum_{k=i}^{n} \sum_{h=1}^{k} \frac{t_{hk}}{k} = f_i^{Ex}(N,T)$ for all $i \in N$. ■

We show logical independence of the axioms stated in Theorem 5.1 by presenting four alternative methods.

1. Let f^5 be the combined method given by

$$f^5(N,T) = \begin{cases} f'(N,T) & \text{if } T \in \widetilde{\mathcal{T}}^N, \\ f^{Ex}(N,T) & \text{otherwise.} \end{cases}$$

where $\widetilde{\mathcal{T}}^N = \{T \in \mathcal{T}^N \mid t_{ij} > 0 \text{ if } i \leq j \leq 2, \text{ and } t_{ij} = 0 \text{ otherwise}\}$, and f' is given by

$$f'_i(N,T) = \begin{cases} t_{11} + \frac{2(t_{12}+t_{22})}{3} & \text{if } i = 1, \\ \frac{t_{12}+t_{22}}{3} & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$$

f^5 satisfies all axioms in Theorem 5.1 except additivity.

2. f^2 given by (6) satisfies all axioms in Theorem 5.1 except efficiency.

3. f^3 given by (7) satisfies all axioms in Theorem 5.1 except independence of upstream exits.

4. f^{Se} satisfies all axioms in Theorem 5.1 except symmetry of exits.

Theorem 5.2. *The EnES method is the only method satisfying additivity, efficiency, independence of downstream entrances, and symmetry of entrances.*

Proof. The proof follows similar lines as those of Theorem 5.1 and is therefore omitted. Also, logical independence can be shown with similar alternative methods as those mentioned after the proof of Theorem 5.1.[21] ■

Theorem 5.3. *The ExES method is the only method satisfying additivity, efficiency, the necessary segment property, the downstream inessential segment property, and externality fairness I.*

Proof. From the proof of Theorem 5.1, we know that the ExES method satisfies additivity and efficiency.

[21] The proof of Theorem 5.2 and logical independence of the axioms can be obtained from the authors on request.

To show that f^{Ex} satisfies the necessary segment property, let $T \in \mathcal{T}^N$, $i \in N$, and $t_{hk} = 0$ for all $h, k \in N$ with $i \notin \mathcal{P}([h,k])$. For every $j < i$, we have[22]

$$\begin{aligned} f_j^{Ex}(N,T) &= \sum_{k=j}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} \\ &= \sum_{k=j}^{i-1} \frac{\sum_{h=1}^{k} t_{hk}}{k} + \sum_{k=i}^{n} \frac{\sum_{h=1}^{i} t_{hk} + \sum_{h=i+1}^{k} t_{hk}}{k} \\ &= \sum_{k=i}^{n} \frac{\sum_{h=1}^{i} t_{hk}}{k} \leq \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = f_i^{Ex}(N,T) \end{aligned}$$

where the third equality follows from the fact that $t_{hk} = 0$ for all $h, k \in N$ with $i \notin \mathcal{P}([h,k])$. Similar, for every $j > i$, we have $f_i^{Ex}(N,T) = \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = \sum_{k=i}^{j-1} \frac{\sum_{h=1}^{k} t_{hk}}{k} + \sum_{k=j}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} \geq \sum_{k=j}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = f_j^{Ex}(N,T)$. Thus, f^{Ex} satisfies the necessary segment property.

The downstream inessential segment property directly follows since $t_{hk} = 0$ for every $k \geq i$ and $h \leq k$, implies that $f_i^{Ex}(N,T) = \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = 0$.

To show that f^{Ex} satisfies externality fairness I, let $T, T' \in T^N$ be such that there exist $h, k \in N$, $1 < h \leq k$, with $t_{ij} = t'_{ij}$ for all $(i,j) \neq (h,k)$, $i, j \in N$. For every $s \in \{h, \ldots, k\}$ and $p \in \{1, \ldots, h-1\}$,

$$f_s^{Ex}(N,T) - f_s^{Ex}(N,T') = \sum_{l=s}^{n} \frac{\sum_{g=1}^{l} t_{gl}}{l} - \sum_{l=s}^{n} \frac{\sum_{g=1}^{l} t'_{gl}}{l} = \frac{t_{hk} - t'_{hk}}{k}, \text{ and}$$

$$f_p^{Ex}(N,T) - f_p^{Ex}(N,T') = \sum_{l=p}^{n} \frac{\sum_{g=1}^{l} t_{gl}}{l} - \sum_{l=p}^{n} \frac{\sum_{g=1}^{l} t'_{gl}}{l} = \frac{t_{hk} - t'_{hk}}{k}.$$

If $k = n$, then this directly implies that $f_s^{Ex}(N,T) - f_s^{Ex}(N,T') = f_p^{Ex}(N,T) - f_p^{Ex}(N,T')$, showing part (ii). Otherwise $k < n$, and for every $q \in \{k+1, \ldots, n\}$, we have $f_q^{Ex}(N,T) - f_q^{Ex}(N,T') = 0$, showing part (i). Thus, f^{Ex} satisfies externality fairness I.

Second, we prove that the ExES method is the only method satisfying these five axioms. Assume that method f satisfies the five axioms. Let $T^0 \in \mathcal{T}^N$ be the null matrix where $t^0_{ij} = 0$ for all $i, j \in N$. By the downstream inessential segment property, we have

$$f_i(N, T^0) = 0 \text{ for all } i \in N. \tag{11}$$

[22]We take $\sum_{h=i+1}^{k} t_{hk}$ to be equal to zero if $k = i$.

Let $T \in \mathcal{T}^N$ and $h, k \in N$, $h \leq k$. We first consider the matrix $T^{hk} \in \mathcal{T}^N$ given by $t_{hk}^{hk} = t_{hk}$, and $t_{ij}^{hk} = 0$ if $(i,j) \neq (h,k)$.

We distinguish the following four cases with respect to h and k.

Case 1: Let $h = 1$ and $k < n$. By the downstream inessential segment property, we have $f_i(N, T^{hk}) = 0$ for all $i > k$.

By the necessary segment property, there is a $c' \geq 0$ such that $f_i(N, T^{hk}) = c'$ for all $i \leq k$. Efficiency then implies that $c' = \frac{t_{hk}}{k}$, and thus

$$f_i(N, T^{hk}) = \begin{cases} \frac{t_{hk}}{k} & \text{if } i \leq k, \\ 0 & \text{if } i > k. \end{cases} \tag{12}$$

Case 2: Let $h > 1$ and $k < n$. By the downstream inessential segment property, we have $f_i(N, T^{hk}) = 0$ for all $i > k$. By equation (11), externality fairness I (part (i)) then implies

$$f_i(N, T^{hk}) = f_j(N, T^{hk}) \text{ for all } j < h \leq i \leq k.$$

Notice that externality fairness I also implies that $f_i(N, T^{hk}) = f_j(N, T^{hk})$ if $j < i < h$ or $h \leq j < i \leq k$. By efficiency, we have $\sum\limits_{i=1}^{k} f_i(N, T^{hk}) = t_{hk}$. Thus, also in this case $f(N, T^{hk})$ is defined as in (12).

Case 3: Let $h = 1$ and $k = n$. By the necessary segment property, $f_i(N, T^{1,n}) = f_j(N, T^{1,n})$ for all $i, j \in N$. By efficiency, we have $f_i(N, T^{1,n}) = \frac{t_{1,n}}{n}$ for all $i \in N$.

Case 4: Let $h > 1$ and $k = n$. By externality fairness I (part (ii)) and equation (11), we have

$$f_i(N, T^{hk}) = f_j(N, T^{hk}) \text{ for all } j < h \leq i \leq k = n.$$

Similar as in Case 2, externality fairness I also implies that $f_i(N, T^{hk}) = f_j(N, T^{hk})$ if $j < i < h$ or $h \leq j < i \leq k$. By efficiency, we have $\sum\limits_{i=1}^{n} f_i(N, T^{hk}) = t_{hk}$. Thus, $f_i(N, T^{h,n}) = \frac{t_{h,n}}{n}$ for every $i \in N$.

From Cases 1-4, $f(N, T^{hk})$ are defined as in (12) for every $h, k \in N$ with $h \leq k$. Since $T = \sum\limits_{h=1}^{n} \sum\limits_{k=h}^{n} T^{hk}$, by additivity and equation (12), we have $f_i(N, T) = \sum\limits_{h=1}^{n} \sum\limits_{k=h}^{n} f_i(N, T^{hk}) = \sum\limits_{k=i}^{n} \frac{\sum\limits_{h=1}^{k} t_{hk}}{k} = f_i^{Ex}(N, T)$, for all $i \in N$. ■

We show logical independence of the axioms stated in Theorem 5.3 by presenting four alternative methods.

1. Let f^6 be the modified version of f^{Ex}, in which the total toll is divided into two parts: the first part being the sum of tolls collected by the users entering at the entrance of segment 1, and the second part being the sum of all other tolls. Let $N^1 = \{j \in N \mid t_{1j} > 0\}$ be the set of exits $j \in N$ with $t_{1j} > 0$. The first part is shared equally among segments

$\{1, 2, \ldots, maxN^1\}$, and the second part is distributed based on method f^{Ex}. Thus, the method f^6 is given by

$$f_i^6(N,T) = \begin{cases} f_i^{Ex}(N,T) & \text{if } N^1 = \emptyset, \\ \frac{\sum\limits_{k=1}^{n} t_{1k}}{maxN^1} + \sum\limits_{k=2}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } i = 1, \\ \frac{\sum\limits_{k=1}^{n} t_{1k}}{maxN^1} + \sum\limits_{k=i}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } 1 < i \leq maxN^1, \\ \sum\limits_{k=i}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } i > maxN^1. \end{cases} \tag{13}$$

This method satisfies all axioms in Theorem 5.3 except additivity.

2. f^2 given by (6) satisfies all axioms in Theorem 5.3 except efficiency.

3. Let f^7 be another modified version of method f^{Ex}, in which the total toll is divided into two parts similar as in method f^6. But now, the first part (involving segment 1) is fully allocated to the segment 1, and the second part is distributed based on method f^{Ex}. Thus, the method f^7 is given by

$$f_i^7(N,T) = \begin{cases} f_i^{Ex}(N,T) & \text{if } N^1 = \emptyset, \\ \sum\limits_{k=1}^{n} t_{1k} + \sum\limits_{k=2}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } i = 1, \\ \sum\limits_{k=i}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } i > 1. \end{cases} \tag{14}$$

This method satisfies all axioms in Theorem 5.3 except the necessary segment property.

4. Let f^8 be the modified version of method f^{Ex}, in which the total toll is again divided into the two parts as in method f^6. However, now the first part (involving segment 1) is equally distributed over segments $\{1, \ldots, n\}$, and the second part is still distributed based on method f^{Ex}. Thus, the method f^8 is given by

$$f_i^8(N,T) = \begin{cases} f_i^{Ex}(N,T) & \text{if } N^1 = \emptyset, \\ \frac{\sum\limits_{k=1}^{n} t_{1k}}{n} + \sum\limits_{k=2}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } i = 1, \\ \frac{\sum\limits_{k=1}^{n} t_{1k}}{n} + \sum\limits_{k=i}^{n} \frac{\sum\limits_{h=2}^{k} t_{hk}}{k} & \text{if } N^1 \neq \emptyset \text{ and } i > 1. \end{cases}$$

This method satisfies all axioms in Theorem 5.3 except the downstream inessential segment property.

5. f^{Se} satisfies all axioms in Theorem 5.3 except externality fairness I.

Theorem 5.4. *The EnES method is the only method satisfying additivity, efficiency, the necessary segment property, the upstream inessential segment property, and externality fairness II.*

Proof. The proof follows similar lines as those of Theorem 5.3 and is therefore omitted. Also, logical independence can be shown with similar alternative methods as mentioned after the proof of Theorem 5.3.[23] ■

Theorem 5.5. *The ExES method is the only method satisfying additivity, efficiency, the necessary segment property, the downstream inessential segment property, externality fairness, and segment monotonicity.*

Proof. From Theorems 5.1 and 5.3, we know that the ExES method satisfies efficiency, the downstream inessential segment property, and externality fairness[24]. We only show that the ExES method satisfies segment monotonicity. Let $T \in \mathcal{T}^N$ and $i \in N$. For $m = 0$, we have

$$f_i^{Ex}(N^{+m}, T^{+m}) = \sum_{k=i}^{n} \frac{\sum_{h=0}^{k} t_{hk}^{+m}}{k} = \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = f_i^{Ex}(N, T)$$

where the second equality follows from $t_{ij}^{+m} = t_{ij}$ for all $i, j \in N$, and $t_{ij}^{+m} = 0$ if $m \in \{i, j\}$.

For $m = n + 1$, we have

$$f_i^{Ex}(N^{+m}, T^{+m}) = \sum_{k=i}^{n+1} \frac{\sum_{h=1}^{k} t_{hk}^{+m}}{k} = \frac{\sum_{h=1}^{n+1} t_{h,n+1}^{+m}}{n+1} + \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}^{+m}}{k} = \sum_{k=i}^{n} \frac{\sum_{h=1}^{k} t_{hk}}{k} = f_i^{Ex}(N, T)$$

where the third equality follows from $t_{ij}^{+m} = t_{ij}$ for all $i, j \in N$, and $t_{ij}^{+m} = 0$ if $m \in \{i, j\}$. Thus, f^{Ex} satisfies segment monotonicity.

The proof of uniqueness follows similar steps as those of Theorem 5.3. The main difference is in Case 4 ($h > 1$ and $k = n$), where we have

$$f_i(N^{+(n+1)}, T^{hk+(n+1)}) = \frac{t_{hk}^{+(n+1)}}{k} = \frac{t_{hk}}{k} \text{ for all } i \in N.$$

which follows from the results in Case 2. By segment monotonicity, we have

$$f_i(N, T^{hk}) \geq f_i(N^{+(n+1)}, T^{hk+(n+1)}) = \frac{t_{hk}}{k} \text{ for all } i \in N.$$

By efficiency, we have $f_i(N, T^{hk}) = \frac{t_{hk}}{k}$ for all $i \in N$. ■

We show logical independence of the axioms stated in Theorem 5.5 by presenting six alternative methods.

[23] The proof of Theorem 5.4 and logical independence of the axioms can be obtained from the authors on request.
[24] Notice that externality fairness is weaker than externality fairness I.

1. Let $N^1 = \{j \in N \mid t_{1j} > 0\}$ be the set of exits $j \in N$ with $t_{1j} > 0$. Let f^9 be the modified version of f^{Ex}, in which the total toll is distributed based on f^{Ex}, except that, if there is at least one exit that has positive toll revenues from users entering at the first segment, (i) the segments j in $[minN^1 + 1, maxN^1]$ only get a fraction $\frac{1}{k+1}$ of the toll revenues earned from users entering at segment 1 and leaving at a segment after j, and (ii) the part of those tolls that are not allocated in this way, are equally allocated over the segments in $[1, minN^1]$. Thus, the method f^9 is defined as follows.
If $N^1 \neq \emptyset$ and $maxN^1 \neq maxN$, then[25]
$$f_i^9(N,T) = \begin{cases} \sum\limits_{k\geq i} \frac{t_{1k}}{k} + \sum\limits_{k>minN^1} \frac{t_{1k}(k-minN^1)}{k(k+1)minN^1} + \sum\limits_{k=2}^{n} \frac{\sum_{h=2}^{k} t_{hk}}{k} & \text{if } i = 1, \\ \sum\limits_{k\geq i} \frac{t_{1k}}{k} + \sum\limits_{k>minN^1} \frac{t_{1k}(k-minN^1)}{k(k+1)minN^1} + \sum\limits_{k=i}^{n} \frac{\sum_{h=2}^{k} t_{hk}}{k} & \text{if } 1 < i \leq minN^1, \\ \sum\limits_{k\geq i} \frac{t_{1k}}{k+1} + \sum\limits_{k=i}^{n} \frac{\sum_{h=2}^{k} t_{hk}}{k} & \text{if } minN^1 < i \leq maxN^1, \\ \sum\limits_{k=i}^{n} \frac{\sum_{h=2}^{k} t_{hk}}{k} & \text{if } i > maxN^1. \end{cases} \tag{15}$$
Otherwise, $f_i^9(N,T) = f_i^{Ex}(N,T)$.
This method satisfies all axioms in Theorem 5.5 except additivity.

2. f^2 given by (6) satisfies all axioms in Theorem 5.5 except efficiency.

3. For $T \in \mathcal{T}^N$, let $T^{hk} \in \mathcal{T}^N$ be such that $t^{hk}_{hk} = t_{hk}$ and $t^{hk}_{ij} = 0$ if $(i,j) \neq (h,k)$. Let
$$f^{10}(N,T) = \sum_{k=1}^{n} \sum_{h=1}^{k} f''(N, T^{hk})$$
where $f''(N,T^{hk})$ is the combined method given by
$$f_i''(N,T^{hk}) = \begin{cases} f_i^{Ex}(N,T^{hk}) & \text{if } h > 1 \text{ and } i \in N \\ \frac{2t_{hk}}{k+1} & \text{if } h = 1 \text{ and } i = 1 \\ \frac{t_{hk}}{k+1} & \text{if } h = 1 \text{ and } h < i \leq k \\ 0 & \text{if } h = 1 \text{ and } i > k. \end{cases}$$
f^{10} satisfies all axioms in Theorem 5.5 except the necessary segment property.

4. f^{En} satisfies all axioms in Theorem 5.5 except the downstream inessential segment property.

5. f^{Se} satisfies all axioms in Theorem 5.5 except externality fairness.

[25] If $minN^1 = 1$, then the second case of equation (15) has no meaning. If $minN^1 = maxN^1$, the third case of equation (15) has no meaning.

6. For $T \in \mathcal{T}^N$, let $T^{hk} \in \mathcal{T}^N$ be such that $t^{hk}_{hk} = t_{hk}$ and $t^{hk}_{ij} = 0$ if $(i,j) \neq (h,k)$. Let

$$f^{11}(N,T) = \sum_{k=1}^{n} \sum_{h=1}^{k} f'''(N,T^{hk})$$

where $f'''(N,T^{hk})$ is the combined method where T^{hk} is distributed by f^{Se} if $k = n$, and T^{hk} is distributed by f^{Ex} if $k < n$:

$$f_i'''(N,T^{hk}) = \begin{cases} f^{Se}(N,T^{hk}) & \text{if } k = n, \\ f^{Ex}(N,T^{hk}) & \text{if } k < n. \end{cases}$$

f^{11} satisfies all axioms in Theorem 5.5 except segment monotonicity.

Theorem 5.6. *The EnES method is the only method satisfying additivity, efficiency, the necessary segment property, the upstream inessential segment property, externality fairness, and segment monotonicity.*

Proof. The proof follows similar lines as those of Theorem 5.5 and is therefore omitted. Also, logical independence can be shown with similar alternative methods as mentioned after the proof of Theorem 5.5.[26] ■

Proposition 5.1. *Let $T \in \mathcal{T}^N$. Then, $f^{Ex}(N,T) = \mathcal{S}h(\nu^{Ex})$.*

Proof. Let $T \in \mathcal{T}^N$ and ν^{Ex} be given by equation (2). Consider the series of games ω^k, $1 \leq k \leq n$, where

$$\omega^k(E) = \begin{cases} 1 & \text{if } E \cap \{1,\ldots,k\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\nu^{Ex}(E) = \sum\limits_{k=1}^{n} \sum\limits_{h=1}^{k} t_{hk}\omega^k(E)$. Moreover, the marginal contribution of player $i > k$ in game ω^k is always zero[27], and the marginal contribution of player $i \leq k$ is

$$\omega^k(E \cup \{i\}) - \omega^k(E) = \begin{cases} 1 & \text{if } E \subseteq \{k+1,\ldots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of the Shapley value, we have

$$\mathcal{S}h_i(\omega^k) = \begin{cases} \frac{1}{k} & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$

Thus, by linearity of the Shapley value, for $g \in N$, $\mathcal{S}h_g(\nu^{Ex}) = \mathcal{S}h_g(\sum\limits_{j=1}^{n} \sum\limits_{i=1}^{j} t_{ij}\omega^j) = \sum\limits_{j=1}^{n} \sum\limits_{i=1}^{j} t_{ij}\mathcal{S}h_g(\omega^j) = \sum\limits_{j=g}^{n} \sum\limits_{i=1}^{j} t_{ij}\frac{1}{j} = f_g^{Ex}(N,T)$. ■

[26] The proof of Theorem 5.6 and logical independence of the axioms can be obtained from the authors on request.

[27] Notice that such a player does not exist if $k = n$.

Proposition 5.2. *Let $T \in \mathcal{T}^N$. Then, $f^{En}(N,T) = \mathcal{S}h(\nu^{En})$.*

Proof. The proof follows similar lines as those of Proposition 5.1 and is therefore omitted.[28] ■

Appendix B: Proof of the statement in Remark 2

Toll fairness. For every $T, T' \in \mathcal{T}^N$ and $i \in N \setminus \{n\}$, such that $t'_{hk} = t_{hk}$ if $h \leq k \leq i$ or $i+1 \leq h \leq k$, and $t'_{hk} = 0$ if $h \leq i < k$, it holds that $f_i(N,T) - f_i(N,T') = f_{i+1}(N,T) - f_{i+1}(N,T')$.

Definition. *A coalition of consecutive segments $E \subseteq N$ is a* sub-highway *if $t_{ij} = 0$ for every $i, j \in N$ with $\mathcal{P}([i,j]) \cap E \neq \emptyset$ and $\{i,j\} \not\subseteq E$.*[29]

Sub-highway efficiency. For every $T \in \mathcal{T}^N$ and sub-highway $E \subseteq N$, $\sum\limits_{i \in E} f_i(N,T) = \sum\limits_{\substack{h,k \in N \\ \mathcal{P}([h,k]) \subseteq E}} t_{hk}$.

Theorem A1. *The SES method is the only method satisfying toll fairness and sub-highway efficiency.*

Proof. First, we prove that f^{Se} satisfies the two axioms.

To show that f^{Se} satisfies toll fairness, let $T, T' \in \mathcal{T}^N$ and $i \in N \setminus \{n\}$, be such that $t'_{hk} = t_{hk}$ if $h \leq k \leq i$ or $i+1 \leq h \leq k$, and $t'_{hk} = 0$ if $h \leq i < k$. We have

$$\begin{aligned}
f_i^{Se}(N,T) - f_i^{Se}(N,T') &= \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{t_{hk}}{k-h+1} - \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{t'_{hk}}{k-h+1} \\
&= \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{t_{hk}}{k-h+1} - \left(\sum_{h=1}^{i} \frac{t'_{hi}}{i-h+1} + \sum_{h=1}^{i} \sum_{k=i+1}^{n} \frac{t'_{hk}}{k-h+1} \right) \\
&= \sum_{h=1}^{i} \sum_{k=i}^{n} \frac{t_{hk}}{k-h+1} - \sum_{h=1}^{i} \frac{t_{hi}}{i-h+1} \\
&= \sum_{h=1}^{i} \sum_{k=i+1}^{n} \frac{t_{hk}}{k-h+1},
\end{aligned}$$

[28] The proof of Proposition 5.2 can be obtained from the authors on request.

[29] N is a sub-highway of itself. Also notice that a sub-highway does not correspond to a component in the line-graph, but to a union of components.

and

$$\begin{aligned}
f^{Se}_{i+1}(N,T) - f^{Se}_{i+1}(N,T') &= \sum_{h=1}^{i+1}\sum_{k=i+1}^{n}\frac{t_{hk}}{k-h+1} - \sum_{h=1}^{i+1}\sum_{k=i+1}^{n}\frac{t'_{hk}}{k-h+1}\\
&= \sum_{h=1}^{i+1}\sum_{k=i+1}^{n}\frac{t_{hk}}{k-h+1} - \left(\sum_{h=1}^{i}\sum_{k=i+1}^{n}\frac{t'_{hk}}{k-h+1} + \sum_{k=i+1}^{n}\frac{t'_{i+1,k}}{k-(i+1)+1}\right)\\
&= \sum_{h=1}^{i+1}\sum_{k=i+1}^{n}\frac{t_{hk}}{k-h+1} - \sum_{k=i+1}^{n}\frac{t'_{i+1,k}}{k-(i+1)+1}\\
&= \sum_{h=1}^{i}\sum_{k=i+1}^{n}\frac{t_{hk}}{k-h+1},
\end{aligned}$$

showing that f^{Se} satisfies toll fairness.

To show that f^{Se} satisfies sub-highway efficiency, let $E \subseteq N$ be a sub-highway in $T \in \mathcal{T}^N$, i.e. $t_{hk} = 0$ for every $h, k \in N$ with $\mathcal{P}([h,k]) \cap E \neq \emptyset$ and $\{h,k\} \not\subseteq E$. Then, we have

$$\begin{aligned}
\sum_{i\in E} f^{Se}_i(N,T) &= \sum_{i\in E}\sum_{h=1}^{i}\sum_{k=i}^{n}\frac{t_{hk}}{k-h+1}\\
&= \sum_{i\in E}\sum_{\substack{h,k\in N\\ \mathcal{P}([h,k])\ni i}}\frac{t_{hk}}{k-h+1}\\
&= \sum_{i\in E}\Big(\sum_{\substack{h,k\in N\\ \mathcal{P}([h,k])\ni i\\ \{h,k\}\not\subseteq E}}\frac{t_{hk}}{k-h+1} + \sum_{\substack{h,k\in N\\ \mathcal{P}([h,k])\ni i\\ \{h,k\}\subseteq E}}\frac{t_{hk}}{k-h+1}\Big)\\
&= \sum_{i\in E}\sum_{\substack{h,k\in N\\ \mathcal{P}([h,k])\ni i\\ \{h,k\}\subseteq E}}\frac{t_{hk}}{k-h+1}\\
&= \sum_{\substack{h,k\in N\\ \mathcal{P}([h,k])\subseteq E}}(k-h+1)\frac{t_{hk}}{k-h+1} = \sum_{\substack{h,k\in N\\ \mathcal{P}([h,k])\subseteq E}} t_{hk}
\end{aligned}$$

where the fourth equality follows from E being a sub-highway in T. Thus, f^{Se} satisfies sub-highway efficiency.

Second, we show that f^{Se} is the only method satisfying the two axioms. Let $T \in \mathcal{T}^N$, $i \in \{1, \ldots n-1\}$, and let the toll matrix $T^{-i} \in \mathcal{T}^N$ be given by

$$t^{-i}_{hk} = \begin{cases} 0 & \text{if } h \leq i < k,\\ t_{hk} & \text{otherwise,}\end{cases}$$

which represents the tolls that are collected by users entering after segment i or exiting before segment i.

Suppose that method f satisfies the two axioms, and consider $T \in \mathcal{T}^N$. Let $\mathcal{H}(N,T) = \{(i,j) \in N \times N \mid t_{ij} > 0\}$ be the set of (entrance, exit)-pairs with a positive toll revenue (i.e. positive entries in the toll matrix). We prove uniqueness by induction on the number of $|\mathcal{H}(N,T)|$.

Let $|\mathcal{H}(N,T)| = 0$, then $T = T^0$ is the null matrix with $t^0_{ij} = 0$ for all $i,j \in N$. In that case every singleton $\{i\}$, $i \in N$, is a sub-highway, and by sub-highway efficiency, we have $f_i(N,T^0) = 0$ for all $i \in N$. Proceeding by induction, suppose that uniqueness holds for every $T' \in \mathcal{T}^N$ with $|\mathcal{H}(N,T')| < |\mathcal{H}(N,T)|$.

Notice that $|\mathcal{H}(N,T)| > 0$. We first observe the following two facts.

(i) Since N is a sub-highway of itself, there exists at least one partition of N into sub-highways.

(ii) If $\{E_j \mid j \in \{1,\ldots,r\}\}$ is a partition of N into sub-highways with r maximal, then there exists no sub-highway being a subset of E_j, $1 \leq j \leq r$.

Let $\{E_j \mid j \in \{1,\ldots,r\}\}$ be as in (ii) above. Fix $j \in \{1,\ldots,r\}$. We consider two cases based on the number of segments in a sub-highway.

Case 1: Let $|E_j| = 1$, i.e. $E_j = \{h\}$. Then sub-highway efficiency determines that $f_h(N,T) = t_{hh}$.

Case 2: Let $|E_j| > 1$. For every $i \in E_j$, there is at least a pair (h,k), $h \leq i \leq k$, such that $t_{hk} > 0$ and $i \in \mathcal{P}([h,k])$.[30] Toll fairness implies that

$$f_i(N,T) - f_{i+1}(N,T) = f_i(N,T^{-i}) - f_{i+1}(N,T^{-i}) \text{ for every } i \in E_j \setminus \max E_j. \tag{16}$$

The right-hand side of (16) is uniquely determined by the induction hypothesis. By sub-highway efficiency, we have

$$\sum_{i \in E_j} f_i(N,T) = \sum_{\substack{h,k \in E_j \\ h \leq k}} t_{hk}. \tag{17}$$

Together, the $(|E_j| - 1)$ equations (16) and equation (17), form a system of $|E_j|$ linearly independent equations in the $|E_j|$ unknown payoffs $f_i(N,T)$, $i \in E_j$, which thus are uniquely determined.

Since $E_1,\ldots,E_r$ is a partition of N, with Case 1 and Case 2, $f_i(N,T)$ is uniquely determined for every $i \in N$. Since f^{Se} satisfies the two axioms, $f(N,T) = f^{Se}(N,T)$. ■

We show logical independence of the axioms stated in Theorem A1 by presenting two alternative methods.

1. f^2 given by (6) satisfies toll fairness, but does not satisfy sub-highway efficiency.

2. f^4 given by (8) satisfies sub-highway efficiency, but does not satisfy toll fairness.

[30] If there does not exist $t_{hk} > 0$ such that $i \in \mathcal{P}([h,k])$, then $\{i\}$ is a sub-highway, and thus is treated by Case 1.

Appendix C: Proof of the statement in Remark 5

Proposition A1. *Let* $T \in \mathcal{T}^N$*. Then,* ν^{Ex} *is the dual game of the conjunctive (or disjunctive) restricted game of* (N, ν^{Se}, D) *where* $D = \{(i, i+1) \mid i = 1, \ldots, n-1\}$*.*

Proof. Let $D = \{(i, i+1) \mid i = 1, \ldots, n-1\}$. By the definition of the conjunctive (or disjunctive) restricted game, see Gilles et al. (1992) and Gilles and Owen (1994), we have

$$r_{\nu^{Se},D} = \sum_{j=1}^{n} \sum_{i=1}^{j} t_{ij} u_{\mathcal{P}([1,j])}$$

where $u_{\mathcal{P}([1,j])}$ is the unanimity game of coalition $\mathcal{P}([1,j])$.

The dual game $r^*_{\nu^{Se},D}$ of $r_{\nu^{Se},D}$ is given by

$$\begin{aligned}
r^*_{\nu^{Se},D}(E) &= \sum_{j=1}^{n} \sum_{i=1}^{j} t_{ij} u_{\mathcal{P}([1,j])}(N) - \sum_{j=1}^{n} \sum_{i=1}^{j} t_{ij} u_{\mathcal{P}([1,j])}(N \setminus E) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{j} t_{ij} u_{\mathcal{P}([1,j])}(N) - \sum_{\substack{i,j \in N \\ \mathcal{P}([i,j]) \subseteq N \setminus E}} t_{ij} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{j} t_{ij} - \sum_{j=1}^{minE-1} \sum_{i=1}^{j} t_{ij} \\
&= \sum_{j=minE}^{n} \sum_{i=1}^{j} t_{ij} = \nu^{Ex}(E) \text{ for all } E \subseteq N.
\end{aligned}$$

■

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References

Alcalde-Unzu, J., Gomez-Rua, M. and Molis, E. (2015). Sharing the costs of cleaning a river: the upstream responsibility rule. *Games and Economic Behavior*, **90** 134–150.

Algaba, E., Fragnelli, V., Llorca, N. and Sánchez-Soriano, J. (2019). Horizontal cooperation in a multimodal public transport system: The profit allocation problem. *European Journal of Operational Research*, **275(2)** 659–665.

Ambec, S. and Sprumont, Y. (2002). Sharing a river. *Journal of Economic Theory*, **107(2)** 453–462.

Béal, S., E., R. and Solal, P. (2010). Rooted-tree solutions for tree games. *European Journal of Operational Research*, **203(2)** 404–408.

Chen, D., Ignatius, J., Sun, D., Goh, M. and Zhan, S. (2018). Impact of congestion pricing schemes on emissions and temporal shift of freight transport. *Transportation Research Part E: Logistics and Transportation Review*, **118** 77–105.

Curiel, I., Potters, J., Prasad, R., Tijs, S. and Veltman, B. (1993). Cooperation in one machine scheduling. *Zeitschrift für Operations Research*, **38(2)** 113–129.

Curiel, I., Potters, J., Prasad, R., Tijs, S. and Veltman, B. (1994). Sequencing and cooperation. *Operations Research*, **42(3)** 566–568.

Demange, G. (2004). On group stability in hierarchies and networks. *Journal of Political Economy*, **112(4)** 754–778.

Dong, B., Guo, G. and Wang, Y. (2012a). Highway toll pricing. *European Journal of Operational Research*, **220** 744–751.

Dong, B., Ni, D. and Wang, Y. (2012b). Sharing a polluted river network. *Environmental and Resource Economics*, **53(3)** 367–387.

Estañ, T., Llorca, N., Martínez, R. and Sánchez-Soriano, J. (2021). On how to allocate the fixed cost of transport systems. *Annals of Operations Research*, **301(1)** 81–105.

Gilles, R. and Owen, G. (1994). Games with permission structures: The disjunctive approach. *Mimeo* Department of Economics, Virginia Polytechnic Institute and State University.

Gilles, R., Owen, G. and van den Brink, R. (1992). Games with permission structures: the conjunctive approach. *International Journal of Game Theory*, **20(3)** 277–293.

Harsanyi, J. (1959). A bargaining model for cooperative n-person games. *In: Tucker AW, Luce RD (eds) Contributions to the theory of games IV. Princeton University Press, Princeton* 325–355.

Herings, P., van der Laan, G. and Talman, D. (2008). The average tree solution for cycle-free graph games. *Games and Economic Behavior*, **62(1)** 77–92.

Hou, D., Sun, P. and Guangjing, Y. (2020). Sharing the costs of cleanup polluted river: Upstream compensation method. *Economics Letters*, **195** 109473.

Jaarsma, C. and van Dijk, T. (2002). Financing local rural road maintenance. who should pay what share and why?some analytical results on spatial price differentiation in first-best congestion pricing schemes. *Transportation Research Part A: Policy and Practice*, **36(6)** 507–524.

Kuipers, J., Mosquera, M. A. and Zarzuelo, J. M. (2013). Sharing costs in highways: A game theoretic approach. *European Journal of Operational Research*, **228** 158–168.

Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of operations research*, **2(3)** 225–229.

Ni, D. and Wang, Y. (2007). Sharing a polluted river. *Games and Economic Behavior*, **60** 176–186.

Otaki, I., Imanishi, Y., Miyatake, K., Nemoto, T. and Uchiyama, N. (2017). Effects of the change of toll system on social surplus: A case study of distance-based toll in tokyo metropolitan expressway. *Transportation research procedia*, **25** 2923–2933.

Owen, G. (1986). Values of graph-restricted games. *SIAM Journal on Algebraic Discrete Methods*, **7(2)** 210–220.

Ren, T., Huang, H., Liu, T. and Nie, Y. (2020). Some analytical results on spatial price differentiation in first-best congestion pricing schemes. *Transportation Research Part C: Emerging Technologies*, **114** 425–445.

Rosenthal, E. C. (2017). A cooperative game approach to cost allocation in a rapid-transit network. *Transportation Research Part B: Methodological*, **97** 64–77.

Shapley, L. S. (1953). A value for n-person games. *In: Kuhn H.W., A.W. Tucker (eds). Contributions to the theory of games, vol. II Princeton University Press* 307–317.

Sudholter, P. and Zarzuelo, J. M. (2017). Characterizations of highway toll pricing methods. *European Journal of Operational Research*, **260** 161–170.

van den Brink, R. (1997). An axiomatization of the disjunctive permission value for games with a permission structure. *International Journal of Game Theory*, **26(1)** 27–43.

van den Brink, R. and Gilles, R. (1996). Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behavior*, **12(1)** 113–126.

van den Brink, R., He, S. and Huang, J.-P. (2018). Polluted river problems and games with a permission structure. *Games and Economic Behavior*, **108** 182–205.

van den Brink, R., van der Laan, G. and Vasil'ev, V. (2007). Component efficient solutions in line-graph games with applications. *Economic Theory*, **33(2)** 349–364.

Xiao, F., Shen, W. and Zhang, H. (2012). The morning commute under flat toll and tactical waiting. *Transport Research Part B: Methodological*, **46(10)** 1346–1359.