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Daniel Dimitrov*†

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Abstract

This paper examines the optimal allocation of risk across generations whose savings mix is subject to illiquidity in the form of uncertain trading costs. We use a stylized two-period OLG framework, where each generation makes a portfolio allocation decision for retirement, and show that illiquidity reduces the range of transferable shocks between generations and thus lowers the benefits of risk-sharing. Higher illiquidity then may justify higher levels of risk sharing to compensate for the trading friction. We still find that a contingent transfers policy based on a reasonably parametrized savings portfolio with liquid and illiquid assets increases aggregate welfare.

JEL codes: G11,G23,E21,H55

Keywords: intergenerational risk sharing, (il)liquidity, stochastic overlapping generations, funded pension plan

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1 Introduction

There are well-known hurdles in a free market economy to sharing risk between generations that are born over distinct periods and thus are subject to different economic prosperity over their lifetimes. Due to the natural physical limitation of a finite lifetime, individuals cannot directly participate in risk that materializes before or after they become economically active. Combined with a lack of a strong bequest motive this creates a classical incomplete market inefficiency. A policy intervention that sets contingent transfers between young and old generations can improve social welfare by widening the risk-bearing pool in the economy and thus increasing its capacity to bear risk[1]

However, any illiquidity that comes in the form of uncertain transaction cost to be paid when selling an asset from the lifetime savings mix of households has the potential to lower the benefits of such transfers. An increase in illiquidity, first of all, discourages individuals from holding risky assets, as the transaction cost that could be incurred when exiting the investment lowers the expected return from the asset. Second, it compresses the distribution of returns and thus lowers the range of asset returns that can be realized.[2]. As a result, there is less financial risk to be shared between generations, and the potential benefits from sharing the risk between generations is lower. We find that higher illiquidity, measured as higher expected transaction costs in selling an asset, requires higher levels of risk sharing to compensate for the loss of sensitivity of the risk transfers to the asset variance. Nevertheless, the total improvement in welfare compared to a situation of no risk-sharing decreases with the level of illiquidity.

The fields of intergenerational risk sharing (IRS) and asset illiquidity overlap naturally when we consider the typical structure of lifetime savings and investments, and the institutions that manage them. First, long-term investors are often seen as well-poised to bear liquidity risk[3]. In the search for diversification and return potential, pension funds in the developed world tend to allocate significant portions of their portfolios to alternative asset classes such as hedge funds, infrastructure, real estate, and private equity funds[4]. These alternative investments impose a liquidity cost that can be a substantial source of investment risk. Second, housing wealth tends to account for a significant share of the retirement wealth of individuals worldwide and the marketability of housing is found to be a significant factor affecting the well-being of retirees[5].

Nevertheless, there is currently, to the best of our knowledge, no other study that explores the intersection between risk sharing (intergenerational or otherwise) and market illiquidity. This is a significant gap in the literature, given that the implications of illiquidity are well known for portfolio choice (Ang et al., 2014; Constantinides, 1986; Acharya and Pedersen, 2005), as well as for the conduct of fiscal (Kaplan and Violante, 2014) and monetary (Sousa, 2010; Chatziantoniou et al., 2017) policy. In this study, we

[1] See Merton (1981); Gordon and Varian (1988); Shiller (1999); Ball and Mankiw (2007); Gottardi and Kubler (2011); Lancia et al. (2020). Beetsma and Romp (2016) provide an overview of the growing literate of intergenerational risk sharing, its policy relevance, and institutional arrangements.

[2] Assume a fixed proportional liquidation cost $\bar{l} \in (0,1)$ exists such that the realized returns net of liquidation costs are measured as $R(1-\bar{l})$. The variance of the realized returns will be $(1-\bar{l})^2 \mathbb{Var}(R) < \mathbb{Var}(R)$. The treatment of illiquidity here is more nuanced (see Section (3.2)) but follows this line of thinking.

[3] Academically, the point has been made for example by Ang (2014); Amihud and Mendelson (1991); Gârleanu (2009)

[4] See, for example, data from OECD (2019); PensionsEurope (2018)

[5] Refer to Lusardi and Mitchell (2007); Crawford and O'Dea (2020); Shao et al. (2019); Nakajima and Telyukova (2020); Munk (2020)

connect the personal finance aspect of illiquidity to its policy relevance.

We define illiquidity in an *ex-ante* sense as the expected proportional cost that needs to be paid to liquidate a risky investment. In terms of modeling the risk of illiquidity, we use a simple on and off shock that materializes with a given probability each period. This relates to the approach of Acharya and Pedersen (2005) who define illiquidity as a latent factor with a time-varying cost component in order to establish testable hypotheses of the way liquidity risk affects asset prices. The binary liquidity cost assumed in this paper can then be seen as a wedge between the fair value of the asset and its realizable market value in the spirit of Brunnermeier and Pedersen (2009).[6]

Our modeling approach relates to the fact that investments in alternative asset classes, such as private equity, often suffer large haircuts on their NAV when taken to the secondary market (Nadauld et al., 2019; Bollen and Sensoy, 2015; Albuquerque et al., 2018). It can also be seen as an expression of the latent costs associated with illiquidity, such as foregone earnings or diversification loss due to trading delays. Note that we look at the liquidity of the investor's portfolio around the time of switching from old age to young age. We ignore any illiquidity effects occurring before that. Expanding the model with more granular time periodicity, however, could also take that into account.

We develop a stylized framework with two overlapping generations (OLG) to consider the outlined problem. Wealth shocks arise from the returns of risky assets in the savings portfolio of individuals and from liquidation costs when the portfolio is sold to fund retirement consumption. Shocks each period occur before the current young have accessed the capital markets, and before they have made any investment decisions. The young start with labor endowment that is not affected by the current shock while the old bear financial risk on their savings. In a fully decentralized market economy, the young are making consumption, savings, and allocation decisions that optimize their lifetime utility, while the old consume from the accumulated retirement wealth.

This arrangement leaves room for an institutional designer to intervene and enforce transfers between the young and the old, which are contingent on the accumulated return of the old generation's savings portfolio. The transfers are designed from an ex-ante point of view and welfare in the economy is evaluated before any shocks materialize. The transfers are linear in the realized return on the individuals' retirement portfolio after paying out any liquidation costs and act as a partial insurance on retirement wealth, covered by the young. Once aware of the policy implementation, utility-optimizing individuals adjust their savings mix by factoring in the regulated transfer policy, giving rise also to indirect welfare effects.

We show that the link between optimal risk-sharing and risk itself can be split into two opposing effects. On one hand, a policy that engages the young in the shock that otherwise affects only the old widens the pool of people who can participate in that shock and increases the risk-bearing capacity of the economy. At the same time, this imports additional risk in the youth's labor endowment, thus extending the horizon over which individuals bear risk. Cumulatively, the later effect also leads to more risk in their old age. The larger the variance of the asset is, the more the second effect dominates, and thus the lower optimal risk sharing needs to be. Similarly, the lower the asset variance,

[6]Brunnermeier and Pedersen (2009) distinguish between *market liquidity*, the ease with which an asset can be placed on the market, and *funding liquidity*, the ease with which outside funds can be accessed once a liability shock hits on an agent's balance sheet. In this paper, liquidity is of the first kind as it is concerns only the marketability of accumulated assets at a particular point in the lifetime of agents associated with retirement age.

the more the first effect dominates, and the higher the optimal risk sharing should be.

We abstract from the particularities of the institutions through which IRS occurs. In reality, the contingent transfers between young and old, as modeled here, could be the result of several arrangements. First, risk sharing rules could be embedded in a collective pension system, for example, through indexation of the benefits received and contributions paid based on the funding ratio of the pension plan (Cai et al., 2013; Gollier, 2008). Alternatively, they could be implemented through counter-cyclical adjustments in the tax code in combination with adjustments to the public debt, through the pay-as-you-go pension system, or by some combination of each of these (Chen et al., 2016).[7]

As a benchmark risk-sharing case, we look at a planner solution, where the planner invests on behalf of the young and allocates consumption centrally between all young and old generations. In the spirit of Gollier (2008) this allows the clever use of wealth buffers to spread risk between generations that do not necessarily live in the same time periods.

Overall, we find that IRS mechanism increases the young's capacity to bear liquidity risk and allows them to allocate more wealth to illiquid assets compared to the case when those individuals are saving in isolation from the shocks that other cohorts are experiencing. We extend the results from earlier models which show that IRS increases the demand for risky assets (Gollier, 2008; Campbell and Nosbusch, 2007) by showing that the same effect holds for illiquid assets as well.

Quantitatively, we show that contingent transfers between two generations, as a second-best implementation of intergenerational risk-sharing (IRS) to what a central planner can do, can achieve a welfare improvement relative to the no-risk sharing case that is not too far from the benchmark first-best solution. For a reasonable parametrization based on global asset returns, we find that when the young can borrow, a policymaker should set risk sharing to 5% of the asset returns variation for risky liquid assets (and 2.1% for illiquid risky assets if agents are constrained), achieving 36% welfare improvement (17% improvement in the constrained case) relative to the no-risk-sharing case, when welfare is measured in the ex ante sense, i.e. before the realization of any shocks. Illiquid risky holdings by individuals increase by 61% on average after they adjust their portfolio to the policy. The benchmark planner case, on the other hand, realizes a welfare improvement of 48% by being able to spread risk among infinitely many generations.[8]

The paper continues as follows. Section (2) provides a short literature overview of two separate fields relating to the current paper, IRS and portfolio choice with illiquidity frictions, and puts the current paper in perspective. Section (3) provides the basic structure of the overlapping generations in the economy, defines the social welfare function, constructs the illiquid asset and discusses its properties. Section (4) defines the benchmark model with an infinitely lived planner. Section (5) redefines the problem for a decentralized economy where each generation solves its own savings-consumption-allocation optimization, while a policymaker determines the transfer policy between young and old. Section (6) explores the main mechanisms of risk sharing by exploring the analytical

[7] One can refer to the existing literature on details about the optimal institutional arrangements of IRS. Beetsma and Romp (2016) provide an overview of the institutional side; Bovenberg and Mehlkopf (2014) review the literature on funded pension schemes, exploring the overlap between life-cycle investing and IRS, elaborating on commitment issues, problems of intergenerational fairness, and sustainability of the pension contract. Gollier (2008) (revisited by Schumacher (2020)) looks at a collective pension plan which allocates funds between a risky and a riskless asset and pays out benefits on a rolling-window basis. Similarly, Cui et al. (2011) look at risk sharing within funded plans with defined-benefit and hybrid structures, where IRS occurs through adjustments in the contribution and benefit levels.

[8] Refer to tables (3) and (2) for further details.

solutions to several simplified cases which illustrate the benefits of pooling risks versus the costs of compounding risky returns over time. This provides a rationale for role of asset variance and illiquidity in the context of IRS. Finally, section (7) provides a welfare analysis and discusses the quantitative results.

2 Relation to the Literature

Several pioneering studies provide the theoretical backing for this paper. Gordon and Varian (1988) show the main mechanisms behind IRS and illustrate the possibility for welfare improvements for all generations together with the constraints that such a policy needs to handle. Even though their argument was initially developed to provide a non-Keynesian justification for the use of debt and social security transfers towards unlucky generations as a counter-cyclical policy, it also provided economic intuition for the existence of social security systems as risk-sharing mechanisms. We embed their arguments in a more formal OLG set-up, introducing a clear-cut welfare rule for the policymaker to set optimal policy, and add illiquidity risk to the investment asset.

Shiller (1999) argues that designing a social security system is a problem of creating a tool for optimal risk management, placing it naturally in the realm of theoretical finance and asset pricing. The planner problem is non-trivial compared to the standard problem of designing individual optimal asset allocation under risk. The risk-sharing system has to be implemented in a way that generations that are either not born yet or are not economically active, can participate in shocks currently occurring. We extend that point of view, arguing that the risk-sharing properties of the social security system should also be able to consider illiquidity of savings.

Within a general equilibrium framework, Ball and Mankiw (2007) develop the rationale for a funded social security trust that is sensitive to equity shocks in order to achieve an efficient allocation of risk across generations. Lancia et al. (2020) look at risk sharing between generations when the social planner policy cannot be enforced and needs to ensure that the participation constraints of each cohort are satisfied. In that case, a trade-off emerges between the efficiency of the policy and its sustainability over time. The current paper draws from their formulation of the policymaker welfare function while keeping the social policy mandatory and embedding it in a richer asset allocation context.

Merton (1981) develops the rationale that social security, if appropriately designed, can indirectly allow people to trade some of their human capital for partial old-age market-risk insurance. IRS policies thus allow agents to participate early in their lifetime in lotteries that otherwise materialize in old age. In aggregate, this widens the pool of risk participants each period and expands the risk-bearing capacity of the economy. The modeling framework in the current paper is different, but we also come to the same conclusion and extend the known results to the case where market illiquidity is present.

We relate also to the literature of portfolio choice with illiquidity and with transaction costs. In Ang et al. (2014), the illiquid asset is marketable only when liquidity materializes with the arrival of a Poisson shock. We relax this assumption by allowing access to a secondary market by accepting a price discount on the fair value of the asset, rather than barring trading altogether. This makes the properties of the asset more suitable to a two-period model of lifetime dynamics. Calibrating a period to 30-years, an individual should be able to always sell the illiquid asset within that time frame, what will vary is only whether a liquidation cost is paid or not. At the same, we keep the risk component that illiquidity has, deviating from the common assumption of fixed proportional cost in the transaction cost literature (Magill and Constantinides, 1976; Cai et al., 2013).

We also loosely relate to the macro literature of durable investments with life-cycle portfolio choice, and their policy implications. Kaplan and Violante (2014) in particular look at consumers' response to a fiscal stimulus when holding non-durable assets, and find that the ratio of housing to total wealth has significant implications on the effectiveness and timely response consumption demand to fiscal expansions. We add in the discussion the risk properties of liquidity and provide another regulatory perspective.

3 The Model

3.1 Assumptions

Time is discrete and indexed by $t \in \{0, 1, 2, 3 \ldots\}$. There is a small open overlapping generations (OLG) economy, where each generation lives for a fixed duration of two periods (youth and old age), the two cohorts are of equal sizes, each cohort has homogeneous preferences and receives the same fixed endowment, there is no population growth, and there is no technological progress. All stochastic variables are defined by the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and all variables indexed by t are measurable w.r.t. the filtration $\mathcal{F}_t$ which defines all public information. Agents form expectations conditional on current information and there is no information asymmetry between agents and policymakers.

Individuals have time-separable discounted lifetime utility of consumption which can be written as:

$$u_y(C_{y,t}) + \beta \mathbb{E}_t u_o(C_{o,t+1})$$

where $u_y(.)$ and $u_o(.)$ stand for the utility of consumption of the young and the old, with positive and diminishing marginal utility of consumption, and $\beta \in (0, 1]$ is a subjective lifetime discount factor for the agent. In the general case, we assume that the Inada conditions for utility hold, even though in Section 6, we break this assumption for illustrative purposes.

At the beginning of their lifetime, individuals receive a fixed endowment Y. The endowment which is not consumed can be saved and transferred for consumption in the next period trough several investment opportunities. First, agents can invest in a risk-free asset with fixed gross return of $R_f > 1$. Second, a frictionless market for N risky assets exists, where the price of each asset i follows a stochastic process $P_t^{s,i}$, and its gross return is defined as

$$\frac{P_t^{s,i}}{P_{t-1}^{s,i}} = R_t^{s,i} = \mu_{s,i} + \epsilon_t^{s,i}$$

Third, an illiquid market exists where an asset can be bought at a price P_t^x but can only be sold at the price $P_t^x(1 - l_t)$ where the liquidity cost l_t evolves independently of any asset shocks $\epsilon_t^{s,i}$ and ϵ_t^x, and follows an i.i.d. stochastic jump process such that

$$l_t = \begin{cases} 0 & \text{with probab.} \quad p \\ \bar{l} & \text{with probab.} \quad 1 - p \end{cases} \tag{1}$$

The proportional liquidity cost $\bar{l} \in (0,1)$ stands for the price discount over the fair value of the asset at the time of the sale in case an illiquidity shock hits. Illiquidity is thus asymmetric and presents only downside risk.

Short-selling of risky assets is not allowed.

3.2 Properties of the Illiquid Return

The gross return of the illiquid asset, excluding the effect ot illiquidity itself, is defined as $R_t^x = \mu_x + \epsilon_t^x$ and we have $\left[\epsilon_t^1, \ldots, \epsilon_t^N, \epsilon_t^x\right]' \sim IID(0, \Sigma)$. In a two period setting, assuming that agents buy the illiquid asset when young and sell it when old, we can write the after-liquidation return as:

$$\begin{aligned} \tilde{R}_t^x &= \frac{P_t^x(1-l_t)}{P_{t-1}^x} \\ &= R_t^x(1-l_t) = \mu_x - \mu_x l_t + \epsilon_t^x(1-l_t) \end{aligned} \tag{2}$$

The illiquidity component is thus first of all a drag on the expected return of the asset. At the same time, the illiquidity shock interacts with the asset-specific risk component ϵ_t^x, and whenever a liquidity shock hits, it lowers the magnitude of the asset specific return.

Note that we can also write the gross return of the asset in a way that isolates the expected return from the noise term, where each of the two take into account the effect of illiquidity:

$$\tilde{R}_t^x = \tilde{\mu}_x + \tilde{\epsilon^x}_t \tag{3}$$

with

$$\begin{aligned} \tilde{\mu}_x &\equiv \mathbb{E}(\tilde{R}_t^x) = \mu_x(1 - \mathbb{E}(l_t)) \\ &= \mu_x\left(p + (1-\bar{l})(1-p)\right) \\ \tilde{\epsilon^x}_t &\equiv R_t(1-l_t) - \tilde{\mu}_x \end{aligned} \tag{4}$$

The liquid asset then will be a special case with either p or $\bar{l}$ set to zero.

Formally, we define *ex-ante* illiquidity as the expected proportional cost that needs to be paid when selling the illiquid asset:

$$\mathbb{E}(l_t) = \bar{l}(1-p) \tag{5}$$

As a result, *ex-ante* illiquidity will be increasing in the liquidation cost of the asset $\bar{l}$ and will be decreasing in the probability of incurring this cost p. *Ex-post* illiquidity, on the other hand, is quantified as $\bar{l}$ and measures the proportional transaction cost that needs to be paid given that the illiquidity risk has materialized. Going forward, unless specified otherwise, the illiquidity considered here refers to the ex-ante type.

We can then isolate several properties of the illiquid asset return. First of all, the expected return of the illiquid asset is monotonously decreasing with the severity of the liquidity friction (as $\bar{l}$ increases or p decreases). This is clear from (4). Taking first derivatives, we get $\frac{\partial \tilde{\mu}_x}{\partial \bar{l}} = \bar{l} < 0$ and $\frac{\partial \tilde{\mu}_x}{\partial p} = p - 1 > 0$.

Second, as the asset becomes more illiquid, the expected quadratic variations in the asset returns become smaller. To see that, note that the independence between the illiquidity shock l_t and the shock ϵ_t implies that

$$\begin{aligned}\mathbb{E}\left((\tilde{R}^x_{t+1})^2\right) &= \mathbb{E}\left((R^x_{t+1})^2(1-l_{t+1})^2\right) = \mathbb{E}\left((R^x_{t+1})^2\right)\mathbb{E}\left((1-l_{t+1})^2\right) \\ &= \left(\mu_x^2+\sigma_x^2\right)\left(p+(1-\bar{l})^2(1-p)\right)\end{aligned} \tag{6}$$

Then, the partial derivatives of $\mathbb{E}(\tilde{R}_{t+1})^2$ are:

$$\begin{aligned}\frac{\partial\mathbb{E}\left((\tilde{R}^x_{t+1})^2\right)}{\partial p} &= \left(\mu_x^2+\sigma_x^2\right)\bar{l}(2-\bar{l}) > 0 \\ \frac{\partial\mathbb{E}\left((\tilde{R}^x_{t+1})^2\right)}{\partial \bar{l}} &= 2\left(\mu_x^2+\sigma_x^2\right)(1-\bar{l})(p-1) < 0\end{aligned} \tag{7}$$

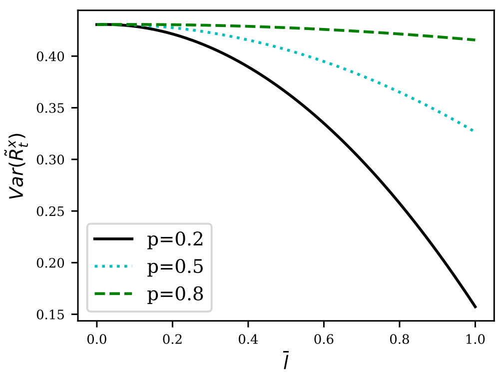
.

The effect of illiquidity on the variance of the asset returns is unclear in general, as both the expectation and the expected variation of asset returns are decreasing. To see that formally, note that

$$\begin{aligned}\tilde{\sigma}_x^2 \equiv \mathbb{Var}\left(\tilde{R}^x_t\right) &= \mathbb{E}\left((\tilde{R}^x_t)^2\right) - \left(\mathbb{E}\tilde{R}^x_t\right)^2 \\ &= \mathbb{E}\left((R^x_t)^2(1-l_t)^2\right) - \left(\mathbb{E}R^x_t\right)^2\left(\mathbb{E}(1-l_t)\right)^2 \\ &= (\sigma_x^2+\mu_x^2)(p+(1-p)(1-\bar{l})^2) - \mu_x^2(p+(1-p)(1-\bar{l}))^2\end{aligned}$$

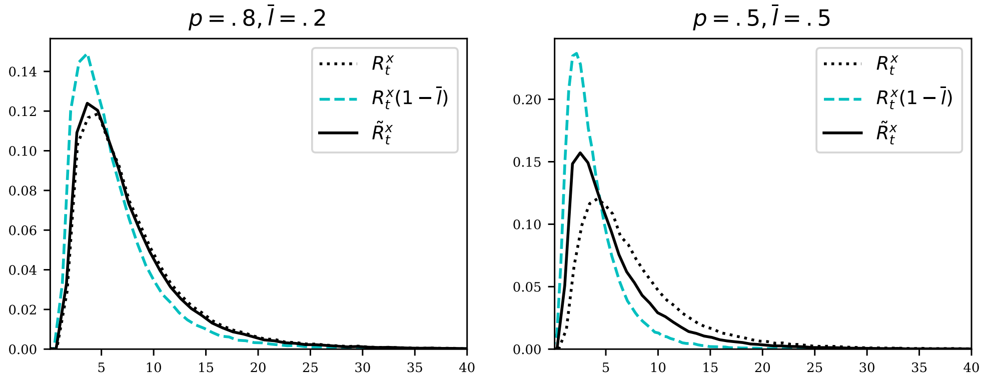
Even though in general the effect is ambiguous, figure (1) illustrates that for a reasonable parametrization the variance will be monotonously decreasing in illiquidity. Figure (2) illustrates how the distribution of the illiquid asset return is formed by mixing the distribution of the *ex-post* liquid returns R^x_t and the distribution of the *ex-post* illiquid returns $R^x_t(1-\bar{l})$. The higher the liquidation cost $\bar{l}$, the more the distribution of $\tilde{R}^x_t$ (illustrated with the dashed-line distribution) is shifted to the left, and the lower is the resulting range of possible returns. At the same time, when the trading probability p is low, the *ex-post* illiquid returns distribution dominates when forming the distribution for $\tilde{R}^x_t$ and in the extreme case of p approaching one, the ex-post and ex ante distributions will merge.

Figure 1: Illiquid Return Variance



Note. This plot shows the effect of varying the liquidity parameters to the variance of asset return $\tilde{R}^x_t$ if R^x_t is log-normally distributed with annualized mean .061 and variance of .156, and the asset is held for 30 years.

Figure 2: Gross Return with Illiquidity



Note. This figure illustrates the effect of the stochastic illiquidity cost l_t on the gross return distribution of the illiquid asset. The dotted line shows the distribution of the ex-post liquid returns. The dashed line shows the ex-post illiquid returns, where the cost $l_t = \bar{l}$ is paid in all scenarios. The solid line in each case shows the ex-ante illiquid returns $\tilde{R}_t = R_t(1 - l_t)$, where l_t is unknown in advance. Returns are log-normally distributed with $\mu_x = .061$ and $\sigma^x = .156$, and the asset is held for 30 years.

Finally, note that the combined term $\tilde{\epsilon}^x$, satisfies the same properties that are otherwise natural for a liquid asset, regardless of the liquidity parameters:

$$\begin{aligned} \mathbb{E}(\tilde{\epsilon}^x) &= 0 \\ \mathbb{E}((\tilde{\epsilon}^x)^2) &= \mathbb{E}(\tilde{\epsilon}^x \tilde{R}_t^x) = \mathbb{V}\mathrm{ar}(\tilde{\epsilon}_t^x) \equiv \tilde{\sigma}_x^2 \end{aligned} \tag{8}$$

3.3 Social Welfare

Welfare is quantified *ex-ante* through the unconditional expectation with respect to all generations' lifetime utilities in all possible states of the world, over all future time periods. The resulting social welfare is the discounted sum of the weighted expected utilities of all future young and old generations:

$$V_0 = \mathbb{E}\left(\sum_{t=1}^{\infty} \delta^{t-1} \left(\frac{\beta}{\delta} u_o(C_{o,t}) + u_y(C_{y,t})\right)\right) \tag{9}$$

where $\delta < 1$ is a policy-relevant discount factor and $\frac{\beta}{\delta}$ keeps the relative social weights between young and old utility fixed between time periods[9][10].

It is worth noting that in order to make the problem tractable, we abstract from some real-world complexity. First of all, we look at a partial equilibrium setting, justified by the assumption of a small open economy, such that world market returns are left unaffected by investment or consumption decisions within the home country. Thus, asset market

[9]Equivalently, we can also write the expectation as conditional on all information that the policymaker has available in period $t = 0$, as only consumption happening after period zero is policy relevant, and shocks happening in period one are independent from the realizations in period zero.

[10]This welfare specification is similar to Lancia et al. (2020) who use it in a social planner setting to develop an optimal intergenerational insurance rule under a voluntary scheme. The approach relates back to Ball and Mankiw (2007).

returns are assumed to be exogenous and any possible general equilibrium effects on asset prices and on economic growth once the risk sharing system is implemented are ignored. Also, we ignore any spillover effects from the investment. In reality, potential investments in illiquid assets which finance for example infrastructure projects could have a positive spillover on social welfare. Second, we concentrate on risk coming from asset holdings and ignore labor income risk and possible correlations between labor and financial market earnings[11]. Third, we focus on a purely utilitarian approach and ignore any political risks on keeping the policy. It is well known that risk sharing is welfare improving for all future generations on an ex-ante basis (before shocks materialize), and not necessarily beneficial for a particular generation on an ex-post basis, as paying compensation after the shock has materialized will make a particular generation worse off (Ball and Mankiw, 2007). Here, generations pre-commit to the scheme before they are born, and participation is mandatory. In reality, there is an incentive for the young to walk away from the arrangement if a negative asset return shock occurs, or the old to walk away if a positive asset return shock occurs. Finally, following the standard approach of a representative agent, we abstract also from any heterogeneity within cohorts.

4 Planner Problem

First, we consider a mechanism for optimizing social welfare, defined through an infinitely lived planner. The planner is taking over the young generation's labor endowment and is providing consumption for the young and the old every period. Any residual is invested on the market with allocations optimally set across the available assets. The resulting problem is in line with Gollier (2008), whose planner simultaneously optimizes over retirement benefits and the investment allocation for multiple overlapping generations. Two generations are used to illustrate the dynamics of the problem, even though in theory the model can be extended to cover multiple generations. In contrast to Gollier, we model consumption for the young in addition to retirement (old-age) consumption to capture more completely lifetime motives of investment. The illiquid asset here extends the investment universe of Gollier's problem and provides an additional dimension for asset allocation.

Using liquid and illiquid wealth as separate state variables is common in the portfolio choice literature whenever there is a transaction cost (Cai et al., 2013) or a liquidity friction (Ang et al., 2014) associated with one of the assets. Wefollow the same convention here. The planner allocates aggregate savings between liquid wealth W_t (consisting of a risk-free liquid investment and a risky liquid investment S_t) and illiquid wealth X_t, which is managed by withdrawing amounts D_t^+ and investing amounts D_t^-. Note that modeling the flows into and out of illiquid wealth as separate choice variables allows the introduction of asymmetric liquidity costs while also allowing for differentiability of the objective with respect to all decision variables. Here whenever some amount is withdrawn from illiquid wealth, the stochastic proportional cost l_t has to be paid for being able to access the market.

This gives rise to the intertemporal wealth constraints:

[11] Models relating labor income risks and investment shocks have been developed, for example, by Hemert (2005); Krueger and Kubler (2006); Boelaars and Mehlkopf (2018)

$$
\begin{aligned}
W_{t+1} &= \left(W_t + Y - C_{y,t} - C_{o,t} - D_t^+ + D_t^-(1-l_t)\right) R^f + S_t' r_{t+1}^s \\
X_{t+1} &= \left(X_t + D_t^+ - D_t^-\right) R_{t+1}^x
\end{aligned} \tag{10}
$$

where $r_{t+1}^s = R_{t+1}^s - R^f \mathbb{1}$ is the vector of excess returns on the liquid risky assets, and $\mathbb{1}$ is a vector of ones.

Denoting $r_{t+1}^x = R_{t+1}^x - R^f$ as the excess return on the illiquid asset, we can see that total wealth $Q_{t+1} = W_{t+1} + X_{t+1}$ evolves as:

$$
Q_{t+1} = (W_t + Y - C_{y,t} - C_{o,t})R^f + (D_t^+ - D_t^-)r_{t+1}^x + S_t' r_{t+1}^s + X_t R_t^x - D_t^- l_t R^f
$$

Wealth is thus being destroyed each period when the liquidity shock hits through the term $D_t^- l_t R^f$, as the planner needs to pay the costs of withdrawing from illiquid wealth instead of earning the risk-free rate on this investment.

The solvency region $\mathcal{A}$ is defined by several constraints. First, borrowing is allowed up to a limited amount $L \geq 0$, so that aggregate consumption and investment do not exceed the available liquid wealth and income by more than the limit amount. Since the planner needs to stay solvent in all states of nature, the withdrawal amount is corrected by the maximum liquidity costs $\bar{l}$ that can be paid:

$$
C_{y,t} + C_{o,t} + D_t^+ - D_t^-(1-\bar{l}) + S_t'\mathbb{1} \leq W_t + Y + L \tag{11}
$$

Second, the illiquid asset cannot be set up as collateral, indicating that the amount withdrawn from illiquid wealth cannot be larger than illiquid wealth itself. The liquid risky asset cannot be short as well. These lead to the following constraints, respectively:

$$
\begin{aligned}
&D_t^- \leq X_t \\
&D_t^-, S_t, D_t^+ \geq 0
\end{aligned} \tag{12}
$$

The planner is maximizing the ex-ante social welfare defined in equation (9). Following Bellman's principle of optimality, we can re-write it in recursive form as:

$$
V(W_t, X_t) = \max_{C_{y,t}, C_{o,t}, S_t, D_t^+, D_t^- \in \mathcal{A}} \left\{ \tilde{u}(C_{y,t}, C_{o,t}) + \delta \mathbb{E} V(W_{t+1}, X_{t+1}) \right\} \tag{13}
$$

with $\tilde{u}(C_{y,t}, C_{o,t}) = \frac{\beta}{\delta} u_o(C_{o,t}) + u_y(C_{y,t})$.

In optimality, as shown in the appendix (A.1) the planner will then set the consumption of the young and the old such that

$$
u_y'(C_{y,t}) = \frac{\beta}{\delta} u_o'(C_{o,t}) = V_W(W_t, X_t) \tag{14}
$$

The appendix derives also the first-order relations with respect to the investments in each risky asset.

5 Intergenerational Transfer Scheme

Now, we transition from an economy fully governed by a planner to one where generations make independent savings and asset allocation decisions. In the process, we introduce a policymaker, operating in that environment, who decides on welfare-improving transfers between the young and the old.

5.1 The Individuals' Problem

In a decentralized framework, agents decide how much of their endowment Y to save and how to allocate it across liquid and illiquid wealth (W_t and X_t respectively). Individuals are solving a similar problem to the planner, with the difference that now they face a limited horizon and have to liquidate all holdings before retirement, paying any liquidity cost if such arise. Unlike the planner, who can take advantage of illiquid wealth buffers over time, a single generation has to liquidate all wealth in the second period of their life to finance retirement consumption.

Between periods zero and one, transfer policy T_t is introduced between the young and the old. The policy is not anticipated before its introduction[12]. All future cohorts are obliged to participate without a walk-out option. The transfers can be either positive or negative for each cohort depending on the realization of the risky returns. The evolution of wealth from young age to retirement can then be written as

$$\begin{aligned} W_{t+1} &= (Y - C_{y,t} - D_t^+ - T_t)R^f + S_t' r_{t+1}^s \\ X_{t+1} &= D_t^+ R_{t+1}^x \end{aligned}$$

In old age, agents sell all accumulated assets paying any liquidation fees, and consume their retirement wealth net of the transfers T_{t+1} with the new-born cohort:

$$C_{o,t+1} = W_{t+1} + X_{t+1}(1 - l_{t+1}) + T_{t+1}$$

I ignore any bequests in the utility specification. This keeps the model tractable, avoiding any time path dependencies across generations. On an intuitive level, it can be expected that the stronger the bequest motive, the closer the decentralized solution will get to the planner solution defined earlier.

Denoting $M_t \equiv Y - C_{y,t} - S_t'\mathbb{1} - D_t^+ - T_t$ as the investment in the risk-free asset and combining the equations above, we get a simpler formulation of the problem. Individuals optimize consumption, taking the current state of the world and any transfer policy at the time they are born as given. This gives rise to the following optimization problem:

$$\begin{aligned} &\max_{M_t, S_t, D_t^+} \{u_y(C_{y,t}) + \beta \mathbb{E}_t u_o(C_{o,t+1})\} \\ \text{s.t.} \quad & C_{y,t} = Y - I_t'\mathbb{1} - T_t \\ & C_{o,t+1} = I_t' R_{t+1} + T_{t+1} \end{aligned} \tag{15}$$

where $R_{t+1} = \left[R_f, (R_{t+1}^s)', R_{t+1}^x(1 - l_{t+1})\right]'$ is a vector of asset returns net of any liquidation fees, and $I_t = \left[M_t, S_t', D_t^+\right]'$ is a vector of investment amounts allocated across all available assets.

The transfers between generations that we consider are driven purely by risk sharing and thus are designed to be neutral in expectation. As the policymaker does not have a re-distributive objective, there is no sharing in the expected asset returns. This is ensured by considering transfers which are linear in the shock.[13] The transfers are thus given by

[12]A more granular multi-period specification can incorporate anticipation effects. In a two-period setting, however, where each period represents 30 years the implementation of an unanticipated policy seems more realistic.

[13]While linear transfers are to some extent restrictive, this does capture first-order effects and significantly simplifies the consequent optimization problems. Non-linear transfers which will further increase welfare are possible, but the added insight relative to the added modeling complexity is likely to be low.

$T_{t\geq 1} = T(\tau, R_t)$ where τ is a vector of parameters standing for the portion of the risk in each asset that is transferred from the old to the young.

To ensure that there is no systematic component and expected transfers are zero, the transfer function is constrained to be linear in the deviations of asset returns, net of any liquidation costs:

$$T(\tau, R_t) = Y\tau' \left(\mathbb{E}(R_t) - R_t\right) \tag{16}$$

To illustrate, assume that there is only one risky asset and that τ is set to be positive. A negative shock ($\epsilon_t = R_t - \mu < 0$) then implies that returns fall bellow the expectation, which in turn implies that the transfer $T_t = -\tau\epsilon_t$ will have a positive value for the old, so the young will partially reimburse the old for their losses. If the shock is positive on the other hand, this implies that the old will share a proportion of their excess return with the young.

We can then write the individual's first-order optimizing conditions with respect to each asset in the portfolio as a vector equation:

$$\begin{aligned} & \mathbb{1}u'_y(C_{y,t}) = \beta\mathbb{E}_t R_{t+1} u'_o(C_{o,t+1}) \\ \implies & \mathbb{1}u'_y(Y - I'_t\mathbb{1} + T(\tau, R_t)) = \beta\mathbb{E}_t R_{t+1} u'_o(I'_t R_{t+1} - T(\tau, R_{t+1})) \end{aligned} \tag{17}$$

where $\mathbb{1}$ is a vector of ones, $u'_i(.)$ stands for the marginal utility of young-age or respectively old-age consumption with $i = y, o$.

The system (17) implicitly defines the optimal investment amounts as a function of the policy instruments and of the realized random shocks, which can be written as $I_t = I(\tau, R_t)$. Substituting into the budget constraints of (15) we get the resulting optimal consumption policies:

$$\begin{aligned} C_y(\tau, R_t) &= Y - I(\tau, R_t)'\mathbb{1} - T(\tau, R_t) \\ C_o(\tau, R_t) &= I(\tau, R_t)' R_{t+1} + T(\tau, R_{t+1}) \end{aligned} \tag{18}$$

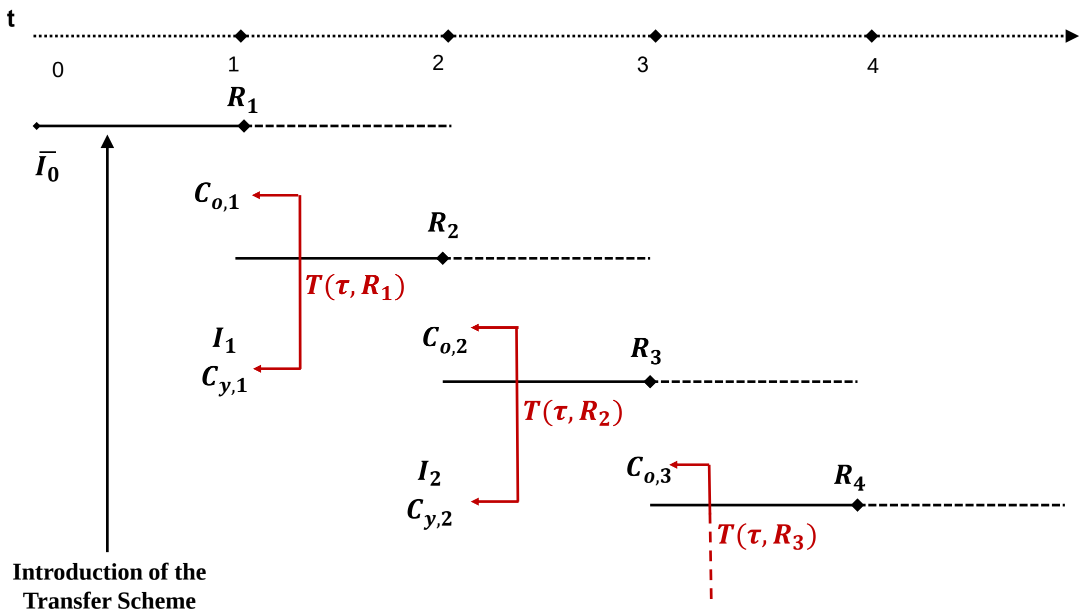
Note that, when a policymaker sets the risk-sharing parameters τ, this affects optimal consumption in two ways: first, through the transfers that are directly dependent on the policy parameters, and second, through the adjustment that individuals make on their savings and asset allocations mix in anticipation of the policy. This will then guide the marginal effect on the individuals' utility from changing the policy parameters.

As policy anticipation effects have been ruled out, the generation born in period zero will not factor in the possibility of a transfer in its optimal investment-consumption decision, but still will get to participate in the risk-sharing scheme once it is old:

$$\overline{C}_o(\tau, R_1) = \overline{I}' R_1 + T(\tau, R_1) \tag{19}$$

where the investment amounts in the vector $\overline{I}$ are fixed before the risk-sharing policy is implemented. This implies that while $\overline{I}$ is set without anticipating the ensuing installment of a transfer scheme when the generation born in period zero reaches old age it gets to participate in the risk-sharing scheme, and they are compensated by the young born in period one if a negative shock is realized or get to transfer to the young some of the accumulated wealth if the shock is positive. Thus, only the direct channel of transfers will affect their lifetime utility.

Figure 3: Two-period OLG Model



This figure illustrates the timing of the intergenerational transfers in the presented model, the policy introduction and the overlapping structure of the generations.

5.2 Policymaker's Problem

The policymaker maximizes the welfare for current and future generations by implementing the IRS policy between periods zero and period one as illustrated in Figure (3). She fine-tunes the transfers, knowing that the young will take the transfer policy is into account when choosing consumption and asset allocation. In this context, the welfare function in (9) becomes an indirect utility that arises from summing up and weighting each generation's optimization problem as a function of the policy instrument τ.

To illustrate that, first note that we can write the social welfare function from (9) as:

$$V_0 = \mathbb{E}\frac{\beta}{\delta}u(C_{o,1}) + \mathbb{E}\left[u_y(C_{y,1}) + \beta\mathbb{E}_1 u_o(C_{o,2})\right] + \delta\mathbb{E}\left[u_y(C_{y,2}) + \beta\mathbb{E}_2 u_o(C_{o,3})\right] + ...$$

We can then substitute in the individuals' optimal consumption from (15). Denote the optimal lifetime utility of a generation as $v(\tau, R_t)$. Since asset returns are independent and identically distributed, and the optimal decision of each generation born after period zero is equivalent to the optimal decision of each consequent generation, the problem is stationary, and when looked at from period zero, $v(\tau, R_t)$ is identical in expectation for any t. We can factor it out of the sum, such that:

$$V(\tau) = \mathbb{E}\frac{\beta}{\delta}u\left(\overline{C}_o(\tau, R_1)\right) + \sum_{j=1}^{\infty}\delta^{j-1}\mathbb{E}v(\tau, R_t) = \mathbb{E}\frac{\beta}{\delta}u(\overline{C}_o(\tau, R_1)) + \mathbb{E}v(\tau, R_t)\sum_{j=1}^{\infty}\delta^{j-1}$$

[13] The planner and the policymaker problems defined here fall under a Ramsey planner macro treatment where the policymaker has a restricted set of policy instruments at her disposal. The pension finance literature is relatively loose in defining both as social planner problems, even though in macro context there is a strict distinction between the two. Ball and Mankiw (2007) provide a link between the social planner and the Ramsey planner problems in the context of risk sharing with conditions on the social planner weights which ensure equivalence to a Ramsey solution.

which results in the indirect utility of the policymaker as a function of the policy instruments

$$V(\tau) = \mathbb{E}\frac{\beta}{\delta}u\left(\overline{C}_o(\tau, R_1)\right) + \frac{1}{1-\delta}\mathbb{E}v(\tau, R_t) \tag{20}$$

The policymaker then solves for the optimal transfer parameters τ^*:

$$\tau^* = \arg\max_{\tau}\left\{\frac{\beta}{\delta}\mathbb{E}u_o\left(\overline{C}_o(\tau, R_1)\right) + \frac{1}{1-\delta}\mathbb{E}u_y\left(C_y(\tau, R_t)\right) + \frac{\beta}{1-\delta}\mathbb{E}u_o(C_o(\tau, R_t))\right\} \tag{21}$$

The degree of risk sharing will then naturally depend on the individuals' optimal investments as determined in (17), or equivalently, the resulting optimal consumption from (18-19). This implies that the government needs to balance the utility of the old generation present immediately after the scheme is implemented with young age and old age utilities of future cohorts, weighted appropriately through the discount factors of the policymaker and the individuals. To reduce the notation overload going forward, we write $C_{o,1}$,$C_{y,t}$ and $C_{o,t}$ while keeping in mind that each of these satisfy the forms of (18) and (19).

Assuming that the expectation operator and the derivative can be interchanged, the optimality condition with respect to one of the instruments i can be written as:

$$\frac{\partial V(\tau)}{\partial \tau_i}: \quad \frac{\beta}{\delta}\frac{\partial \mathbb{E}u_o\left(C_{o,1}\right)}{\partial \tau_i} + \frac{1}{1-\delta}\frac{\partial \mathbb{E}v(\tau, R_t)}{\partial \tau_i} = 0 \tag{22}$$

As all consumption terms are assumed to satisfy the individual optimality conditions, relying on the Envelope Theorem for the individuals' optimal consumption sensitivity to τ, we can write:

$$\frac{\beta}{\delta}\mathbb{E}\left[u_o'(C_{o,1})\frac{\partial C_{o,1}}{\partial \tau_i}\right] + \frac{1}{1-\delta}\mathbb{E}\left[u_y'(C_{y,t})\frac{\partial C_{y,t}}{\partial \tau_i} + \beta u_o'(C_{o,t})\frac{\partial C_{o,t}}{\partial \tau_i}\right] = 0 \tag{23}$$

This can further be expanded by splitting the expectation-of-product terms into expectations and covariances, and noting that for generation zero $\mathbb{E}\frac{\partial C_{o,1}}{\partial \tau_i} = 0$, we have :

$$\begin{aligned}\frac{\beta}{\delta}\mathbb{Cov}\left(u_o'(C_{o,1}), \frac{\partial C_{o,1}}{\partial \tau_i}\right) + \frac{1}{1-\delta}\left[\mathbb{E}u_y'(C_{y,t})\mathbb{E}\frac{\partial C_{y,t}}{\partial \tau_i} + \mathbb{Cov}\left(u_y'(C_{y,t}), \frac{\partial C_{y,t}}{\partial \tau_i}\right)\right] \\ + \frac{\beta}{1-\delta}\left[\mathbb{E}u_o'(C_{o,t})\mathbb{E}\frac{\partial C_{o,t}}{\partial \tau_i} + \mathbb{Cov}\left(u_o'(C_{o,t}), \frac{\partial C_{o,t}}{\partial \tau_i}\right)\right] = 0\end{aligned} \tag{24}$$

6 Main Mechanism with Quadratic Utility

In the general set-up, in optimality the individuals' condition (17) and the policymaker's condition (22) are simultaneously fulfilled. Now, we look at several special cases, which keep this set up but simplify the optimization conditions to make the resulting problem analytically tractable and to provide intuition in the dynamics of the model.

Assume for now that agents have the same quadratic period utility of the form $u(C) = C - \frac{\gamma}{2}C^2$ implying that their expected utility is a function of the mean and the variance of the random payoff C such that

$$\mathbb{E}u(C) = \mathbb{E}C - \frac{\gamma}{2}\left(\mathbb{V}\mathrm{ar}C + (\mathbb{E}C)^2\right) \tag{25}$$

where $\gamma > 0$ defines the degree of risk aversion, with higher γ implying higher aversion. Once the shock is realized, and consumption is deterministic, it also defines the marginal utility of consumption, evaluated as $u'(C) = 1 - \gamma C$. Consumption is assumed to stay below the statiation level of utility, such that $C < \frac{1}{\gamma}$ almost surely.[14].

Next, section (6.1) looks at a simple setting when agents do not adjust their savings levels in response to the risk-sharing policy, section (6.2) introduces endogenous savings, and section (6.3) summarizes the main mechanisms at play and clarifies the intuition behind the observed mechanism. Appendix (A.4 looks at a setup with a risky and risk-free asset as individuals can consume in their youth and retirement. The analytical expressions become very complex, so we examine further the case in detail in the numerical section of 7 where additional complexity is added.

6.1 Exogenous Savings

Assume now that the savings are fixed to some $\bar{S}$ where $\bar{S} \in (0, Y)$. All shocks imported in the young age endowment wealth through the transfer scheme are completely absorbed through the consumption of the young. Assume for simplicity that only one risky asset is available, such that $T_t = \tau Y(\tilde{\mu} - \tilde{R}_t^x) = -\tau Y \tilde{\epsilon}_t^x$ where the risk-sharing parameter is τ. Optimal consumption defined in (18) then evolves as follows:

$$\begin{aligned} C_{y,t} &= Y - \overline{S} + \tau\tilde{\epsilon}_t^x Y \\ C_{o,t+1} &= \overline{S}\tilde{R}_{t+1}^x - \tau\tilde{\epsilon}_{t+1}^x Y \end{aligned} \tag{26}$$

Now, investments are fixed and do not react to changes in τ, so we have that $\mathbb{E}\frac{\partial C_{y,t}}{\partial \tau} = \mathbb{E}\frac{\partial C_{o,t+1}}{\partial \tau} = 0$, and the policymaker's optimality condition (24) simplifies. Furthermore, the policy surprise effect becomes irrelevant for the generation born in period zero, as neither they, nor by construction any future generation adjust their investments to the policy parameters. Also, The policy parameter τ is set once and for all before the realization of any shocks. As a result, the consumption streams for the old in period one and in any other period differ only in the realization of the shock. As a result, we can write condition (24) as

$$\begin{aligned} &\frac{\beta}{\delta}\mathbb{C}\mathrm{ov}\left(u'(C_{o,t}), \frac{\partial C_{o,t}}{\partial \tau}\right) + \left[\frac{1}{1-\delta}\mathbb{C}\mathrm{ov}\left(u'(C_{y,t}), \frac{\partial C_{y,t}}{\partial \tau}\right) + \frac{\beta}{1-\delta}\mathbb{C}\mathrm{ov}\left(u'(C_{o,t}), \frac{\partial C_{o,t}}{\partial \tau}\right)\right] \\ &\equiv \underbrace{\delta\mathbb{C}\mathrm{ov}\left(u'(C_{y,t}), \frac{\partial C_{y,t}}{\partial \tau}\right)}_{<0} + \underbrace{\beta\mathbb{C}\mathrm{ov}\left(u'(C_{o,t}), \frac{\partial C_{o,t}}{\partial \tau}\right)}_{>0} \stackrel{!}{=} 0 \end{aligned} \tag{27}$$

[14]Loosely speaking, the quadratic utility assumption can be seen as a second-order approximation of the expected utility of a more complex utility function (Levy and Markowtiz, 1979; Buccola, 1982; Sharpe, 2007). For details on the use of quadratic utility in portfolio choice models see Brandimarte (2006), Černý (2009), and D'Amato and Galasso (2010) who use it within an IRS context with a political game determining the optimal level of risk sharing with voting.

The underbrackets show the signs of the covariance terms assuming[15] that $0 < \tau < \frac{\overline{S}}{Y}$. They hold for any utility function with a decreasing marginal utility of consumption. To illustrate why this is happening, imagine that there is a negative financial shock ($\epsilon_t < 0$). The IRS mechanism then transfers wealth from the young to the old. This has two effects. First, the marginal consumption of the old ($\frac{\partial C_{o,t}}{\partial \tau}$) increases and their marginal utility decreases for higher τ as they get compensated for the resource loss on their retirement savings. Second, the transfers induce a resource loss to the young as any compensation for the old is subtracted from their initial endowment, driving down the young's marginal consumption ($\frac{\partial C_{y,t}}{\partial \tau}$) and driving up their marginal utility. Since the transfers are linear, exactly the opposite effect occurs with a positive financial shock.

Overall, the IRS policy allows the old to trade negative retirement-wealth shocks with the currently young, who in turn in good times gain from the additional accumulated wealth of the old. The risk-sharing parameter τ drives the sizes of the trade-offs for each generation, so it needs to balance out the willingness of one generation to get protection in bad states of nature in lieu of smaller gain in good states with the willingness of the other generation to forego current consumption in bad states in lieu of higher consumption in good states. Overall, optimal τ needs to be set such that the two effects, as captured by the covariance terms, balance out.

In particular, for quadratic utility, we have

$$\mathbb{Cov}\left(u'(C_{y,t}), \frac{\partial C_{y,t}}{\partial \tau}\right) = \mathbb{Cov}(-\gamma\tau\tilde{\epsilon}_t^x, \tilde{\epsilon}_t^x) = -\gamma\tau\mathbb{Var}(\tilde{\epsilon}_t^x) \leq 0$$

$$\mathbb{Cov}\left(u'(C_{o,t+1}), \frac{\partial C_{o,t+1}}{\partial \tau}\right) = \mathbb{Cov}\left(-\gamma\left(\frac{\overline{S}}{Y} - \tau\right)\tilde{\epsilon}_{t+1}^x, -\tilde{\epsilon}_{t+1}^x\right) = \gamma\left(\frac{\overline{S}}{Y} - \tau\right)\mathbb{Var}\left(\tilde{\epsilon}_{t+1}^x\right) \geq 0$$

Substituting in (27), the asset return variance and risk preferences cancel out of the policymaker condition and do not play a role in determining optimality. The optimal proportion of the shock that will be shared across generations is proportional to the savings rate and depends on the discount rates:

$$\tau^* = \left(\frac{\beta}{\beta + \delta}\right)\frac{\overline{S}}{Y} \tag{28}$$

The more a generation values old-age relative to young-age utility (higher personal discount factor β), the higher the optimal level of IRS should be in order to allow generations to hedge the negative states of nature they could experience in retirement. Similarly, by construction a lower value for δ decreases the relative weight of the young in the welfare function (9) driving down the need for IRS.

6.2 Utility of Old-age Consumption Only

Now, assume that agents derive utility from old-age consumption only.[16] By construction, they will save all their young-age endowment and consume it when old. In contrast to

[15]It can also be shown that in optimality τ cannot be negative when savings are positive, as then both covariances will be positive and the first-order condition would never hold. If $\tau > \bar{S}/Y$ on the other hand, both covariances are negative, and again the optimality condition cannot hold.

[16]This set-up is common in the pension literature where agents derive utility only from pension income and retirement consumption.

the previous case, where young-age wealth shocks resulting from the transfers were fully absorbed by consumption, now the transfer shocks are fully absorbed by savings.[17].

Formally, the indirect utility of consumption for each generation becomes

$$v(\tau, \tilde{\epsilon}_t^x) = \beta \mathbb{E}_t u_o(C_{o,t+1}) \tag{29}$$

where

$$C_{o,t+1} = S_t \tilde{R}_{t+1}^x - \tau Y \tilde{\epsilon}_{t+1}^x \tag{30}$$

For each generation born after implementation of the policy we have $S_t = Y + \tau Y \tilde{\epsilon}_t^x$, while in absence of anticipation effects generation zero has fixed savings $S_0 = Y$.

Substituting in the policymaker's optimizing condition (23) and simplifying we get:

$$\frac{\beta}{\delta}\mathbb{E}\left(u'(C_{o,1}) \cdot \frac{\partial C_{o,1}}{\partial \tau}\right) + \frac{\beta}{1-\delta}\mathbb{E}\left(u'(C_{o,t+1}) \cdot \frac{\partial C_{o,t}}{\partial \tau}\right) = 0$$

Simplifying further (Appendix (A.2) shows the derivation details), Y, γ and $\tilde{\sigma}^2$ cancel out, and we can solve for the optimal level of risk-sharing

$$\tau^* = \frac{1}{\delta \mathbb{E}\left((\tilde{R}_t^x)^2\right) + 1} = \frac{1}{\delta \mathbb{E}\left(\tilde{\mu}^2 + \tilde{\sigma}^2\right) + 1} \tag{31}$$

In contrast to the case with fixed savings, τ^* is now a decreasing function of the riskiness of the asset, quantified as the expected quadratic variation in the return of the asset after the risk of liquidity costs is covered. The discount rates do not appear here in the optimal term, as young-age consumption is not modeled and risk is not discounted over the lifetime of individuals.

The expected quadratic variation $\mathbb{E}\left((\tilde{R}_t^x)^2\right)$ is positively related to the probability to trade and negatively related to the liquidity cost, as shown in section (3.2). Then it follows that as the illiquidity friction becomes more severe, the quadratic variations in the asset returns become smaller and an increase in the risk-sharing parameter is needed to ensure that enough risk is transferred across generations. Formally, we can show that

$$\frac{\partial \tau^*}{\partial p} < 0, \frac{\partial \tau^*}{\partial \bar{l}} > 0 \tag{32}$$

6.3 Risk Pooling vs. Compounding of Risk

Now, we explore the relationship between the level of IRS and the variance of the risky asset. We decompose the relationship into two counterbalancing effects. First, in aggregate, IRS expands the pool of people who can participate in a shock occurring in a given period by including the individuals who are not economically active in the risk-bearing pool. Second, it extends the time window over which individuals bear risk by forcing them to participate earlier in their lifetime in the realization of financial shocks, which

[17] A similar risk-sharing set up appears within the context of a political game for example in D'Amato and Galasso (2010); Ciurila and Romp (2015)

are otherwise only affecting the wealth of the old. We know that the first effect enhances the overall risk-bearing capacity of the population. The second effect, however, on its own produces a welfare loss for the young and needs to be explored further.

It is well-known that the uncertainty in a risky asset's returns does not diversify with time, and that longer investment horizons do not lead to lower variance of the accumulated wealth. In essence, this is the fallacy of time diversification which states that aggregating shocks over time increases their cumulative variability (Samuelson, 1963; Ross, 1999). Gordon and Varian (1988) also refer to the *compounding of lotteries* and suggest that the time accumulation of variance, due to the random shocks transferred from one generation to the next, embeds a cost in the IRS mechanism. In our setting, as well, compounding of uncertainty makes it expensive in utility terms to transfer risks over to the young as the risk they will start bearing when young will accumulate through their savings and will lead to higher consumption variability in retirement. It is then natural that the larger the variance of the savings portfolio is, the more costly it is to transfer risk across generations.

Consider again the set-up of section (6.2). The multiplicative shock which will appear in the old-age consumption equation (30) is the key driver of the inverse relationship between the level of optimal IRS and the magnitude of the asset variance. To illustrate, assume for the sake of argument that the savings asset is liquid. Then, old-age consumption is

$$C_{o,t+1} = S_t R_{t+1} + T_{t+1} = Y\left((1+\tau\epsilon_t)R_{t+1} - \tau\epsilon_{t+1}\right)$$

This means that the policy ($\tau > 0$) imports additional uncertainty into old-age consumption, having made the young-age starting wealth uncertain as $Y(1+\tau\epsilon_t)$. In old age, the variance of endowment is translated into additional variance of savings and in old age gets magnified by the variance of the accumulated asset return. Old-age wealth as a fraction of Y then becomes $(1+\tau\epsilon_t)R_{t+1} = \mu + \epsilon_{t+1} + \tau\mu\epsilon_t + \tau\epsilon_t\epsilon_{t+1}$. As a result, the variance of consumption becomes function of the risk sharing parameter.

We can decompose the total variance of old age consumption with IRS into the following two effects[18]:

$$\begin{aligned} \mathbb{Var}(C_{o,t+1}) &= \mathbb{Var}\left(S_t R_{t+1} - Y\tau\epsilon_{t+1}\right) \\ &= Y^2\left(\underbrace{\sigma^2(1-\tau)^2}_{\text{Pooling Effect}} + \underbrace{\mu^2\sigma^2\tau^2 + \tau^2\sigma^4}_{\text{Risk-Compoinding Effect}}\right) \end{aligned} \tag{33}$$

First of all, the IRS mechanism expands the pool of individuals that can participate in the risk, which is about to materialize in a given period. So, the old will bear only $1-\tau$ proportion of the risk occurring in their retirement, while the rest is transferred to the newly born. This drives the *pooling effect* of the variance.

The young consume their wealth when they retire. The risk that they have to participate in while young, proportional to τ, is then reinvested until retirement, when a new shock occurs and this amplifies the initial one. This results in the *risk-compounding effect.*

As the variance of consumption is convex in the risk-sharing parameter, there will be a point after which the uncertainty imported in old age consumption through risk

[18] See Appendix (A.3) for details on the derivation

sharing will dominate over the reduction in uncertainty coming from the increased risk-bearing pool. The policymaker's problem then is to set τ such that the two effects are balanced. The variance of consumption is thus minimized for $\tau^* = \frac{1}{1+\sigma^2+\mu^2}$. Note that as the variance of the asset increases, the risk-compounding effect, having a higher order, dominates over the pooling effect, and to achieve optimal variance of consumption, the policymaker needs to reduce the level of risk sharing.

The individuals born in period zero, before the policy is implemented, get to experience solely the pooling effect, without being subject to the compounding cost. In particular, the variance of their consumption is

$$\mathbb{Var}(C_{o,1}) = Y^2\mathbb{Var}(R_1 - \tau\epsilon_1) = Y^2\mathbb{Var}(\mu + (1-\tau)\epsilon_1) = Y^2 \underbrace{(1-\tau)^2\sigma^2}_{\text{Pooling benefit}} \tag{34}$$

These individuals are thus privileged from an ex-ante point of view, as they benefit for each $\tau \in (0, 1]$ and in optimality will want to have it set to unity. With no risk sharing, they bear the full risk of their old-age consumption, and with complete sharing their old-age consumption risk is reduced to zero and any negative shocks are shifted to the newly born young generation at period one. For this generation, the risk reduction occurs free of the compounding cost, that other generations need to bear.

Note that sometimes the literature uses time-additive shocks to illustrates the pooling benefits of risk sharing. Crucially, this misses the time aggregation component of risk risk. To illustrate how additive shocks can mislead, assume first that the young bear τ portion of the shock while the old take proportion $1-\tau$. Additive shocks would imply that the young save $S_t = Y + \tau\epsilon_t Y$ and the old consume $C_{o,t} = S_{t-1} + (1-\tau)\epsilon_t Y = Y(1+\tau\epsilon_{t-1}+(1-\tau)\epsilon_t)$. It is clear that with *i.i.d.* shocks, we then have $\mathbb{Var}(C_{o,t}) = \left((1-\tau)^2 + (\tau)^2\right)\sigma^2 Y^2$. The sharing parameter which minimizes the variance of consumption then is $\tau^*_{add.} = 1/2$ and it is clearly independent of the variance of the asset returns.

From that point of view, the argument can be made that averaging n independent shocks additive shocks, such that each generation gets a portion $\frac{1}{n}$ of each shock, leads to $\mathbb{Var}\left(\frac{1}{n}\sum_{i=0}^{n}\epsilon_{t-i}\right) = \frac{1}{n}\sigma^2$, and $n \to \infty$, the shocks will diversify away. Using this argument within the context of risk sharing can mislead that splitting shocks over many generations can make the risk disappear. When aggregating stochastic *i.i.d.* returns over time, however, which is in a way done by multiplying out the gross returns over time, the variance of the accumulated return will grow linearly with time:

$$\mathbb{Var}(R_t \cdot R_{t-1} \dots R_{t-n}) = \mathbb{Var}(\epsilon_t \cdot \epsilon_{t-1} \dots \epsilon_{t-n}) = \mathbb{E}\left(\epsilon_t^2 \cdot \epsilon_{t-1}^2 \dots \epsilon_{t-n}^2\right) + \mathbb{E}(\epsilon_t) \dots \mathbb{E}(\epsilon_{t-n}) = n\sigma^2$$

7 Quantitative Evaluation and Welfare Analysis with CRRA Utility

7.1 Set-up, Parameters and Initial Conditions

Now, assume that there are three assets available for investment: a risk-free liquid, a risky liquid, and a risky illiquid asset. The risky assets' gross returns follow a log-normal distribution, such that $R_t = \begin{bmatrix} R_t^s \\ R_t^x \end{bmatrix}$ where $\log(R_t) \sim N(\mu, \Sigma)$ with $\mu = \begin{bmatrix} \mu_s \\ \mu_x \end{bmatrix}$ and

$\Sigma = \begin{bmatrix} \sigma_s^2 & \rho\sigma_s\sigma_x \\ \rho\sigma_s\sigma_x & \sigma_x^2 \end{bmatrix}$ and returns are scaled over a thirty-year holding period in line with the two-period OLG setting.

The risk-free asset is calibrated to the expected return of medium-term world government bonds. The risky liquid asset matches global equity's risk and returns characteristics. The risky illiquid asset is calibrated to match the properties of a portfolio of representative illiquid asset classes, where the weights are based on the relative sizes in a typical pension fund of private loans, equity, hedge funds, real estate and infrastructure holdings. Aggregated data on pension fund allocations are gathered from OECD (2019) for global funds and (PensionsEurope, 2018) for European funds. The data is summarized in Table (1). Expected asset returns, volatility, and correlations across the asset classes are based on the long term capital market forecasts in JP Morgan (2020).

The costless trading probability (1) follows from a Poisson specification as the probability of having at least one trading opportunity during a time period Δt, such that[19]:

$$p = 1 - \mathrm{e}^{-\eta\Delta t} \tag{35}$$

Note that in this set-up $(1/\eta)$ is the average time one needs to wait for a costless trading opportunity to arrive. In calibrating η, we rely on data from Ang et al. (2014), who provide estimates of the average holding times and turnover of the illiquid asset classes considered here. Table (1) summarizes the data and the corresponding probability estimates for the variety of illiquid assets considered here. Further, Δt represents the period over which individuals would seek to sell their illiquid holdings. Weassume five years as an approximation of the time before or after retirement during which individuals liquidate their asset holdings in order to fund retirement consumption. Overall, based on these considerations and based on the data, we assume $p = .8$ for the base scenario as a reasonable ballpark figure corresponding to the representative illiquid asset considered here.

The liquidity cost parameter $\bar{l}$ is calibrated through the average trading discount on the Net Asset Value (NAV) which needs to be accepted when selling private investments on the secondary market. Nadauld et al. (2019) explore the secondary market for private equity funds and find that they trade on average at a discount of 13.8%. The number varies significantly, typically in a range between 5% and 30%, depending on market conditions, fund age, and fund type (e.g., Buyout, Venture Capital, Real Estate). we assume $\bar{l} = 20\%$ as a ballpark base figure. Further on, we explore the solution's sensitivity by varying both p and $\bar{l}$ parameters. We set $\gamma = 5$ and $\beta = \delta = e^{-.03 \cdot 30}$.

[19]The costless trading opportunity for the illiquid asset arrives as a Poisson event with an expectation η. The probability of having n such trading opportunities over a period of time Δt then is

$$P(n) = \mathrm{e}^{-\eta\Delta t}\frac{(\eta\Delta t)^n}{n!}, \quad n = 0, 1, ...$$

Denoting N_t as the cumulative number of trading events which occurred up to time t, the probability p of being able to trade costlessly in the illiquid asset at least once over a given period can then be derived as

$$p = P(N_{t+\Delta t} - N_t \geq 1) = 1 - P(N_{t+\Delta t} - N_t = 0) = 1 - \mathrm{e}^{-\eta\Delta t}$$

Formally, this is equivalent to modeling liquidity events as a Poisson process, as in e.g. Ang et al. (2014).

Table 1: Parameters Calibration

	Holding Time	p	Weight	μ	σ	ρ
Liquid Risk-Free Asset						
- Mid-Term Gov Bonds			100%	0.002	-	
Liquid Risky Asset				-	-	1.000
- Global Equity			100%	0.061	0.156	
Illiquid Risky Asset				0.049	0.120	0.586
- Hedge Funds	1 - 2	0.92 - 0.99	16%	0.030	0.074	0.730
- Private Equity	4	0.71	23%	0.078	0.202	0.800
- Institutional Real Estate	8 - 10	0.39	39%	0.046	0.111	0.500
- Institutional Infrastructure	50 - 60	0.08	14%	0.047	0.105	0.550
- Private Loans	-	-	8%	0.017	0.045	0.150

This table shows the calibrated values for the return and risk properties of the three assets in the model. The average number of years it takes to trade on one of the illiquid assets is used as input to calculate the probability p that the asset can be sold over a five-year period. The Poisson probability formula links the two. The figures on μ and σ indicate the expected return and standard deviation respectively of the asset class, and ρ indicates the corresponding correlation to equity global. Data is used from JP Morgan (2020) adjusted for a fixed inflation rate of 2%. The weights indicate the asset proportions within the illiquid portfolio of a typical global pension fund as in OECD (2019) and they are used to aggregate the basket of illiquid asset into one representative illiquid asset.

The young and the old are assumed to have the same CRRA Utility of consumption, where $\gamma > 1$ is the usual parameter of risk aversion:

$$u(C) \equiv \frac{1}{1-\gamma} C^{1-\gamma}, \quad C > 0$$

In the planner case, after the system is initiated (for $t \geq 1$), due to (14) in optimality we have:

$$C^*_{y,t} = \left(\frac{\beta}{\delta}\right)^{-\frac{1}{\gamma}} C^*_{o,t}$$

As a result, one only needs to solve for the youth age consumption in the planner case, and optimal old-age consumption will be proportional.

Using the social welfare formulation of (9), we can translate the utility units into Certainty Equivalent Consumption (CEC) units, where the CEC measures the stream of fixed risk-free future consumption that the current and all future generations would be willing to accept for the stream of risky consumption leaving the indifferent utility-wise between the two options. In the case of a CRRA utility, as Appendix (A.5) shows, the CEC for the whole population can then be written as:

$$CEC_0 = \left[(1-\delta)(1-\gamma)\frac{\delta}{\beta+\delta} V_0\right]^{\frac{1}{1-\gamma}} \tag{36}$$

where V_0 is a function of the starting wealth values $V_0 = V(W_0, X_0)$ in the case of the planner problem, while in the case of intergenerational transfers, it is a function of the policy instruments such that $V_0 = V(\tau^*)$.

In the decentralized cases, either with or without risk sharing, all generations but generation zero are identical ex-ante and start with the same endowment and the same wealth, so this eliminates the need to establish initial conditions. In the autarky case, welfare is evaluated through equation (20), with all transfers set to zero, while in the case with transfers, the policymaker picks the optimal risk-sharing parameters.

In the planner economy, the initial conditions matter. To be consistent with the policymaker problem of section 5, we assume that the economy exists initially in a decentralized no-risk-sharing regime. Individuals from generation zero then freely pick their asset allocation and savings level ignorant of the following policy implementation. Between periods zero and one, the planner appropriates the accumulated savings in the economy. After period one, she starts optimally distributing consumption between the young and the old and starts picking the appropriate asset mix and the aggregate savings level, as indicated in section 7.2. The accumulated up to period one liquid and illiquid wealth then provide the initial conditions on which the planner relies. Consequently, we evaluate the welfare as the probability-weighted value-function level at accumulated liquid and illiquid wealth in period one, assuming that further on the planner is picking optimal values for the decision variables.

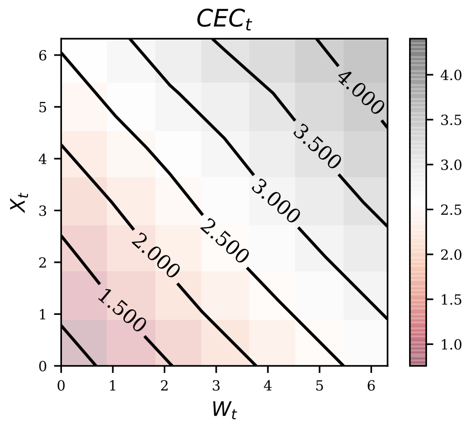
7.2 Planner Solution

First, in the planner economy, we look at how aggregate welfare, consumption and investment behave as functions of the stochastic state variables X_t and W_t. Figures (4a) and (4b) show that welfare and aggregate consumption, respectively, unambiguously increase if more liquid or illiquid wealth becomes available. The increase with respect to illiquid wealth follows from the fact that for $\bar{l} < 1$ and $p > 0$ the illiquidity friction is only setting a potential cost to withdrawing funds for consumption but is not barring withdrawals entirely. Note, however, that due to the risk of incurring withdrawal costs, a proportional increase in liquid or illiquid wealth does not lead to an equal increase in welfare, so the slope of the iso-lines is not unity.

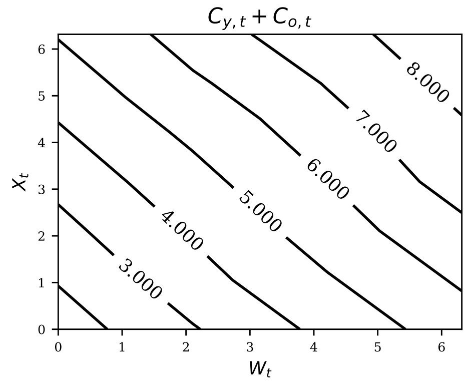
Figure (4c) shows the optimal levels of liquid risky asset investments S_t given the beginning of period illiquid and liquid wealth. First, it can be seen that if the current liquid wealth is already high (the right half of the chart), it is not optimal to allocate resources to the liquid risky assets, as the regular endowment income provides enough liquidity for the coming period, and it becomes more profitable to allocate to the illiquid asset and reap the illiquidity premium embedded in it. At the same time, if the start-of-period illiquid wealth is too high (in the upper half of the chart) it becomes optimal to shift resources from the illiquid risky asset towards liquid risky assets, so S_t starts picking up, again to secure diversification.

Figure (4d) shows the total amount of withdrawals D_t^- (if negative) and investments D_t^+ (if positive) in illiquid wealth. This chart complements the findings so far. If a certain proportion of illiquid-to-liquid wealth is breached, indicated with the zero diagonal line, the planner will reallocate, withdrawing or depositing into illiquid savings in order to avoid over-concentrated holdings in one type of wealth. That relocation will either be consumed or invested in risk-free liquid holdings or risky-liquid holdings.

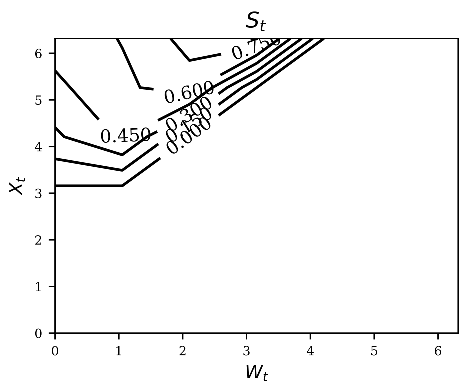
Figure 4: Planner Optimal Policies



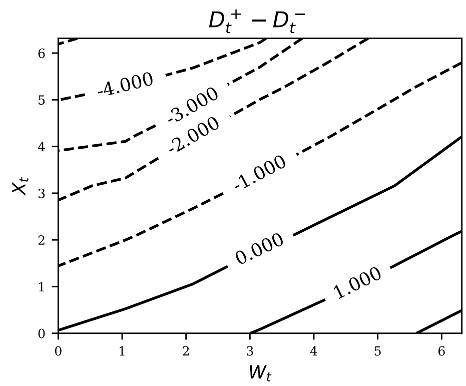
(a) Aggregate Welfare



(b) Aggregate Consumption



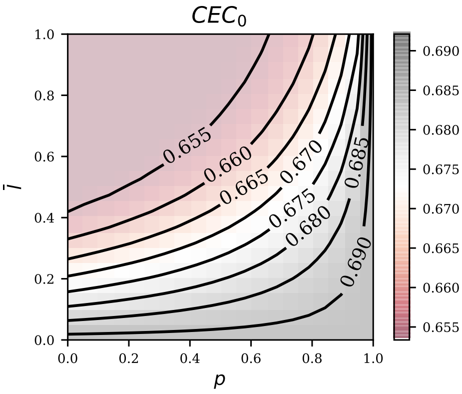
(c) Liquid Risky Investments



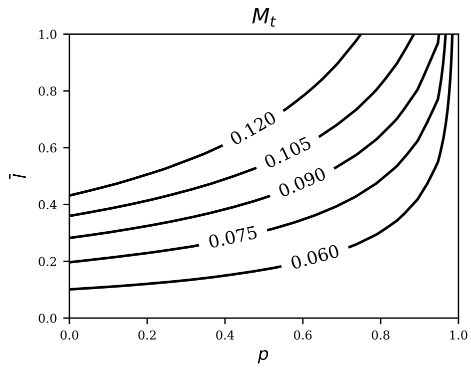
(d) Illiquid Wealth Withdrawals

This set of figures shows the solution of the planner dynamic programming problem. The charts show the value function (presented in CEC units), optimal consumption, optimal risky liquid investments, and the withdrawals (dashed, negative lines) and investments (solid, positive lines) into illiquid wealth, respectively. Borrowing is constrained with $L = 0$ in (11). The optimal solutions are presented as functions of the two stochastic state variables, W_t and X_t. In a dynamic setting, optimal consumption and investment for the coming period are set by knowing the two-state variables at the beginning of the period.

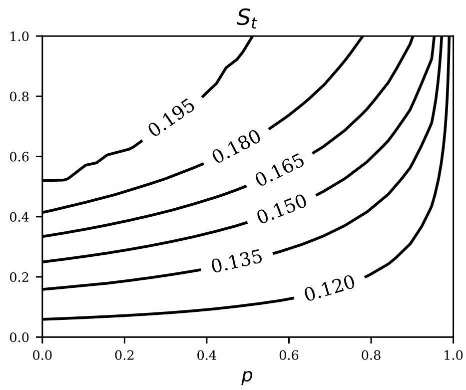
Figure 5: Illiquidity without Risk Sharing



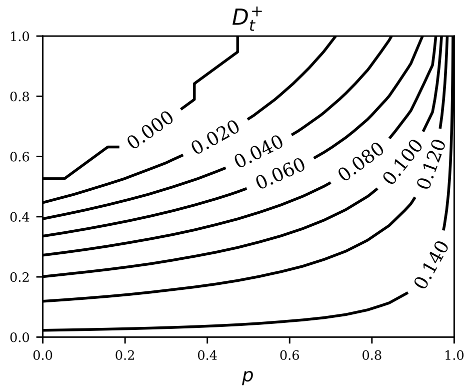
(a) Aggregate Welfare



(b) Risk-Free Investments



(c) Liquid Risky Investments



(d) Illiquid Risky Investments

This set of contour plots show the effect of the liquidity friction in the base case on the invested amounts in each of the assets. The lifetime CEC measures the certainty equivalent consumption of individuals over the two periods of their lifetime.

7.3 No Risk Sharing Solution

Now we look at an economy where generations themselves optimize the asset mix in their portfolio. First, they cannot share risks with other generations. we keep in line with the literature by referring to this as *autarky*, taking into account the fact that agents consume purely out of their endowments and do not have the technology to consume out of the endowment income of other generations (Beetsma and Romp, 2016; Gollier, 2008). Formally, the solution follows from the individuals' savings problem (15) where all transfers T_t are set to zero and where the policymaker does not play any role.

The set of figures (5) shows how welfare and investments are affected by illiquidity. It is clear that as the liquidity friction increases (either p increases or $\bar{l}$ decreases), the welfare in the economy (Figure (5a)) goes down monotonically. This is driven by several factors. First, investment in the illiquid asset decreases (Figure (5d)). Investment in the risky liquid asset increases (Figure (5c)) to compensate, but overall there is a drop in total risky asset holdings as illiquidity rises[20]. Precautionary risk-free savings increase (Figure 5b). In this case, as the illiquidity friction increases, agents suffer from reduced diversification in their investment mix. At the same time, their capacity to bear market risk is reduced and they are not in a position to exploit fully the market risk premia.

7.4 Risk Sharing Transfers

Agents now solve the allocation-savings problem in a decentralized economy, where a policymaker administers the risk-sharing instruments by optimizing aggregate ex-ante welfare. The set of figures (6) shows the effect that risk sharing has on welfare and investment. The parameters τ_s and τ_x stand for sharing in the liquid and in the illiquid risky asset respectively. Individuals are allowed to borrow in their youth.

Figure (6a) shows the aggregate welfare in the economy in period one as a function of the policy instruments. Welfare is measured in CEC units in line with (36). It is growing for the most part with the degree of risk-sharing, but it suddenly cuts off to zero at the chart's upper-right edge. This occurs when the risk-sharing parameters are set too high, which in combination with a large negative shock in either of the risky assets has the potential to produces scenarios where the endowment income of the young after transfers to the old leave nothing for young-age consumption and thus marginal utility becomes infinite.

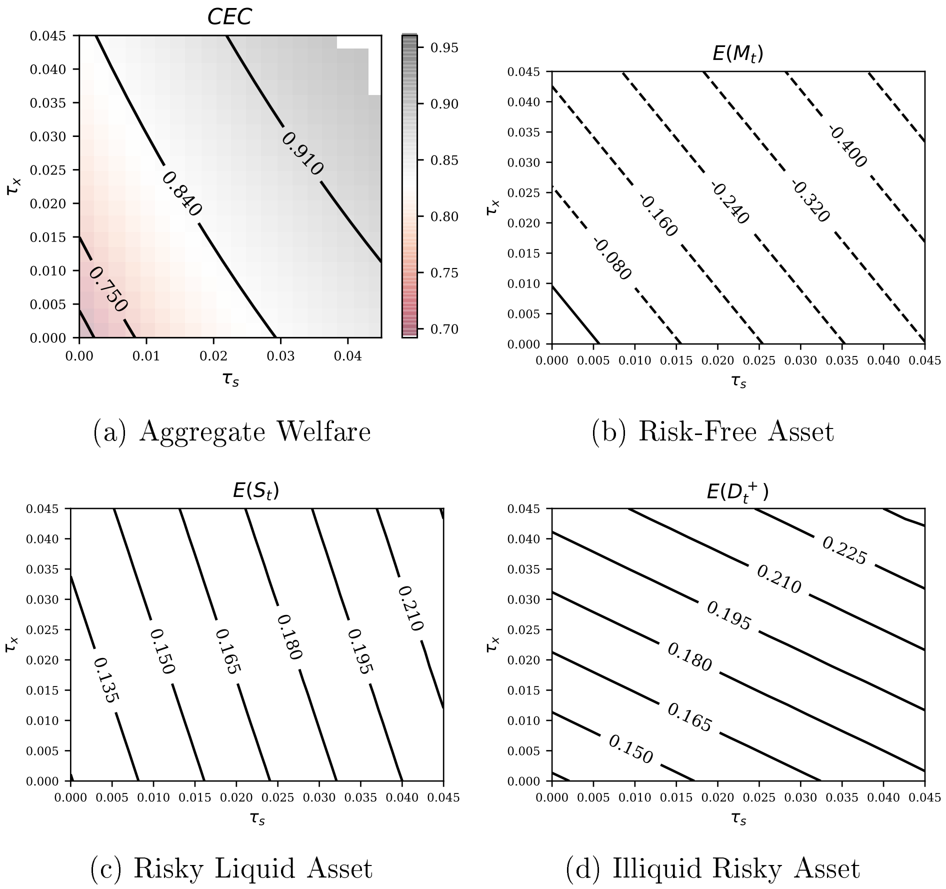
Figures (6b), (6c), (6d) illustrate how the allocation to each of the three assets changes when either of the risk sharing instruments is varied. The charts, thus, show the expected investments over time, since (as shown in Section (5)) risk sharing makes the amount available for investment random by being dependent on the realized financial shocks of the current period.

In line with studies done before (Gollier, 2008; Shiller, 1999; Campbell and Nosbusch, 2007), the increase in risk sharing between generations enhances the ability of individuals to bear investment risk. The average amounts allocated to the liquid (Figure (6c)) and the illiquid asset (Figure (6d)) increase with the degree of IRS, while the investment in the risk-free asset decreases (Figure (6b)) and even becomes negative, as the individual leverages up by borrowing. In line with the diversification principle, the increase in risk

[20] For example, in a high liquidity case (bottom right corner of each chart, where $\bar{l}$ is low and p is high) total risky investments are around .26. In the low-liquidity case (the upper right corner of each chart), total liquid and illiquid risky investments drop to about .195

sharing even in one of the assets increases the ability to bear risk in both of the risky assets. For example, an increase in τ_x from .009 to .016 increases the average optimal holdings in the liquid asset from .135 to .15. Still, an equivalent increase in one sharing parameter favors more the risky asset that it targets.

Figure 6: Illiquidity with Risk-Sharing Transfers



(a) Aggregate Welfare

(b) Risk-Free Asset

(c) Risky Liquid Asset

(d) Illiquid Risky Asset

This plot shows the effect of varying the risk-sharing parameters on welfare and the average levels of the optimal asset holdings. The white space in the upper-right edge of Figure (6a) shows the values of τ_s and τ_x where the CEC cuts off to zero as scenarios appear in which transfers become larger than the endowment of the young.

Table (2) presents the values for consumption and optimal investment in an economy under optimal risk sharing. Two cases are considered for robustness: when agents are either able to borrow and when borrowing is restricted. As observed before, compared to autarky, risk sharing effectively increases the capacity to invest in the risky assets in both cases, even though the effect is more substantial without a borrowing constraint. Even in the case of restricted borrowing, the portfolio is on average fully invested in risky assets.

Going forward, we quantify the degree of welfare improvement in line with Beetsma and Romp (2016):

$$\left(\frac{CEC^i}{CEC^a} - 1\right) \cdot 100\%$$

where CEC^i stands either for the specific policy to be evaluated, and CEC^a stands for the aggregate welfare in the benchmark autarky economy.

Table (3) compares the welfare improvement from the two IRS mechanisms considered so far - from the intergenerational transfers and from the planner case. Naturally, the welfare achieved through a planner is higher, as a policymaker can share risk between two generations only, while a planner can share risk with infinitely many future generations. In addition, policymaker is restricted to apply only linear transfers of risk. Still, whether borrowing is restricted or not, the model projects that a policymaker is already able to realize a lot of the benefits possible through risk sharing, and the difference to what is achievable by the planner is not large. If borrowing is possible, the model projects a welfare improvement of 36% in the decentralized case with transfers vs. 48% for in the policymaker case. With borrowing, the relation is 17% vs. 21%.

Table 2: Risk Sharing vs. Base Case

	No Risk Sharing	Risk Sharing with borrowing	Risk Sharing without borrowing
$\mathbb{E}Cy$	0.694	1.008	0.805
$\mathbb{E}Co$	1.359	1.374	1.379
$\mathbb{E}M$	0.054	-0.446	0.000
$\mathbb{E}S$	0.119	0.224	0.110
$\mathbb{E}D^+$	0.133	0.214	0.085
τ_s^*		0.050	0.018
τ_x^*		0.021	0.030

Note. This table compares the policymaker risk-sharing and the autarky (no risk sharing) solution, looking at the case where a lower limit of zero on borrowing (investment M_t) is applied and when no such limit is applied. There is an internal optimal solution in the no sharing solution, so a constraint on borrowing does not change anything. The expected values are calculated for generations born in $t > 0$.

Table 3: Risk Sharing

	No Risk Sharing	Policymaker	Planner
		With borrowing	
CEC	0.687	0.932	1.014
Improvement	-	36%	48%
		Without borrowing	
CEC	0.687	0.805	0.830
Improvement	-	17%	21%

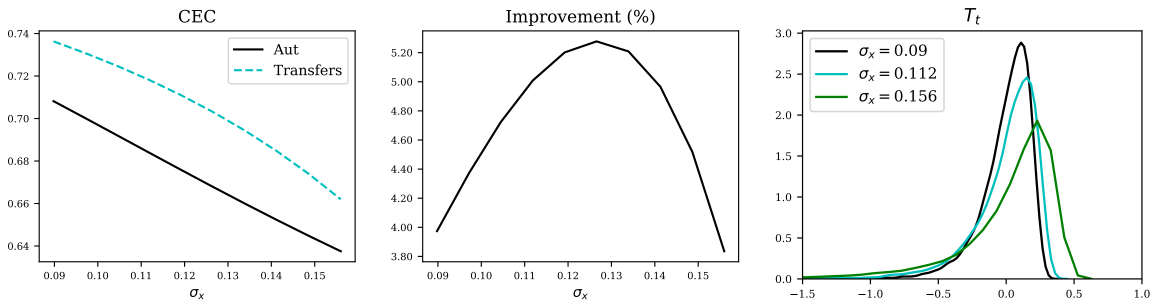
Note. This table shows the welfare improvement over autarky when risk sharing is introduced either through a policymaker or through a planner approach. In the former case, individuals are allowed to borrow. In the later case, borrowing is allowed only up to a level comparable to the one observed in the policymaker optimal solution.

A question of interest is to what degree the introduction of risk-sharing between the young and the old can help lower the welfare losses from increased asset risk. Next, we look at how changes in the risk profile of the illiquid asset, either in terms of increased variance or increased liquidity cost, affect welfare.

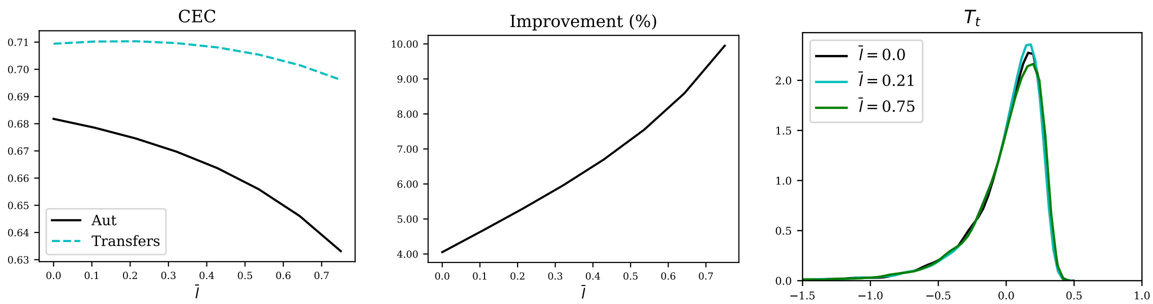
First, holding investment and risk-sharing fixed as in Figure (7a), we can see that for increased risk, there is, at least initially, more potential for welfare improvement in transferring part of it from the old to the young. The linear intergenerational transfers relocate wealth from the old to the young in states of nature where the marginal utility of the additional consumption of the old is low, and when asset returns are above their expected values. Furthermore, wealth is relocated in the oposite direction, from the young to the old, in states of nature when asset returns are low, consequently savings of the old are depreciated, and the old's marginal utility is currently high.

However, when the variance of the investment portfolio and consequently the variance of transfers increases, at some point the transfers may become too much for the young to bear. As scenarios of larger transfers from the young become more frequent, this may leave them with too little endowment to cover their young-age consumption, causing more scenarios where young-age utility drops more than the corresponding increase in the utility of the old, and thus also lowering the overall expected lifetime utility of the generation. As a result, at some point, this second push-back effect prevails and the welfare improvement starts declining with further variance. If τ_x is set too large, it may even happen that the risk-sharing policy comes at a disadvantage compared to the initial situation. Allowing τ_x to be adjusted avoids the problem of transferring too much risk from one generation to another, as shown in Figures (8).

Figure 7: Fixed Allocation and Risk Sharing



(a) Increased Asset Variance



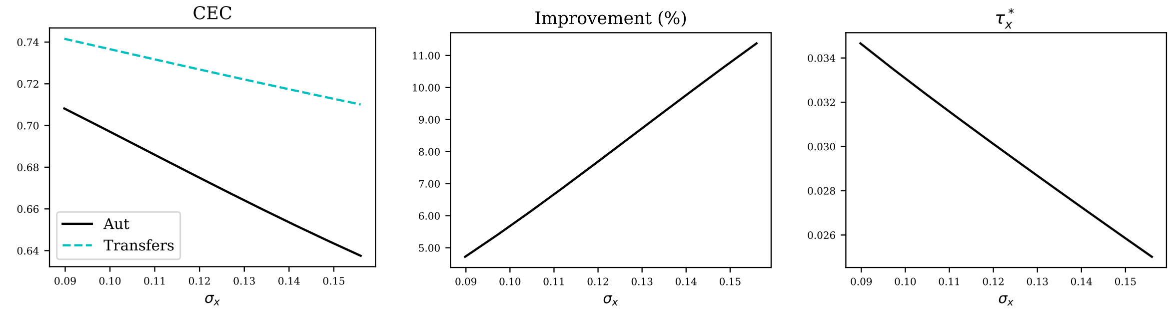
(b) Increased Liquidity Cost

Note. This set of charts shows the effect of varying the volatility on the illiquid asset and respectively its liquidity cost. The invetments are fixed in all states of nature to $M_t = S_t = D_t^+ = .1$, and risk sharing is fixed such that $\tau_s = 0$, $\tau_x = .05$

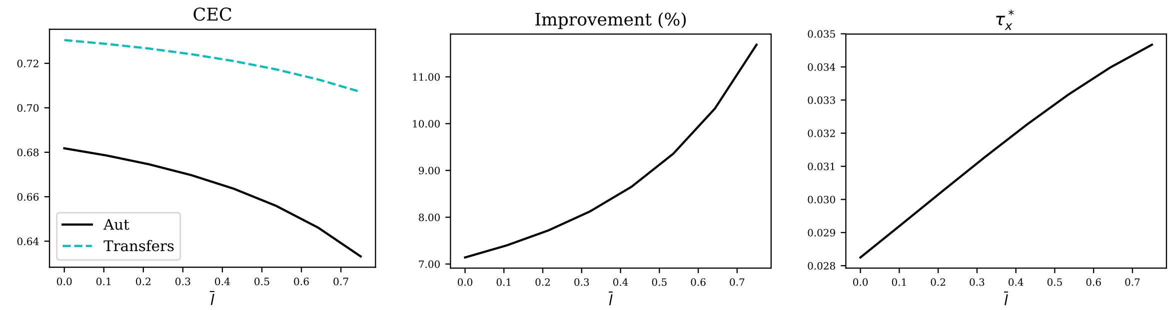
Note however that increased liquidity risk does not increase the variance of transfers (7b). This is in line with the way illiquidity affects asset returns as shown in Section (3.2): higher illiquidity lowers the variance of wealth, thus lowering also the chance that more variance is transferred to the young than they could bear. As a result, improvement in welfare is monotonously increasing with rising illiquidity cost, in contrast to the effect observed for higher variance.

Second, as the risk of the illiquid asset increases, an investment substitution effect occurs, where individuals reduce holdings in that asset and increase their holdings in the

Figure 8: Fixed Allocation and Optimized Risk Sharing



(a) Increased Asset Variance



(b) Increased Liquidity Cost

Note. This set of charts shows the effect of varying the volatility on the illiquid asset and respectively its liquidity cost. The invetments are fixed in all states of nature to $M_t = S_t = D_t^+ = .1$, and risk sharing for the illiquid asset is optimized by the policymaker, while $\tau_s = 0$.

risky liquid asset. This is clearly seen in Figures (9) where allocations are optimized while the risk sharing instruments stay fixed. The implementation of risk-sharing transfers shifts up the investments in each of the two risky assets, the increase being financed through lower holdings in the risk-free asset.

The plots in Figure (10) combine the effects of individuals optimizing their asset holdings and the policymaker simultaneously setting the optimal degree of risk sharing. First, as variance increases, the amount of optimal risk-sharing τ_x^* decreases to stabilize the variance of transfers. Second, risk sharing increases the desire of individuals to hold risky assets compared to autarky. Still, with higher risk, individuals hold less of the riskier illiquid asset and more of the substitute liquid asset. The investment in risky assets, in total, is diminishing with the increase in risk. In end effect, as the variance of the asset increases, the reduction of diversification in the savings portfolio held by the individuals results in lower welfare improvements as the variance of the illiquid asset increases. IRS cannot compensate for this effect, so overall, as the risk of the illiquid asset increases, the improvement in welfare is going down.

There is one notable difference in how an increase in σ_x versus an increase in $\bar{l}$ is affecting optimal risk sharing. Higher volatility justifies sharing a lower degree of IRS τ_x, while higher illiquidity calls for increased risk-sharing. The decrease in the variance of transfers caused by illiquidity needs to be compensated by a higher sensitivity of the transfers to the variance of the asset. This is in line with the arguments made earlier that with growing illiquidity, the variance of the savings portfolio decreases and sharing in the illiquid asset needs to be increased to compensate.

Figure (11) summarizes all cases considered and illustrates that the welfare increases with each relaxation of the constraints from the base case, defined as an individual with fixed investment shares without access to a risk sharing technology. The highest welfare occurs with case C5 when both IRS parameters are optimized by the policymaker while simultaneously individuals optimize their asset holdings.

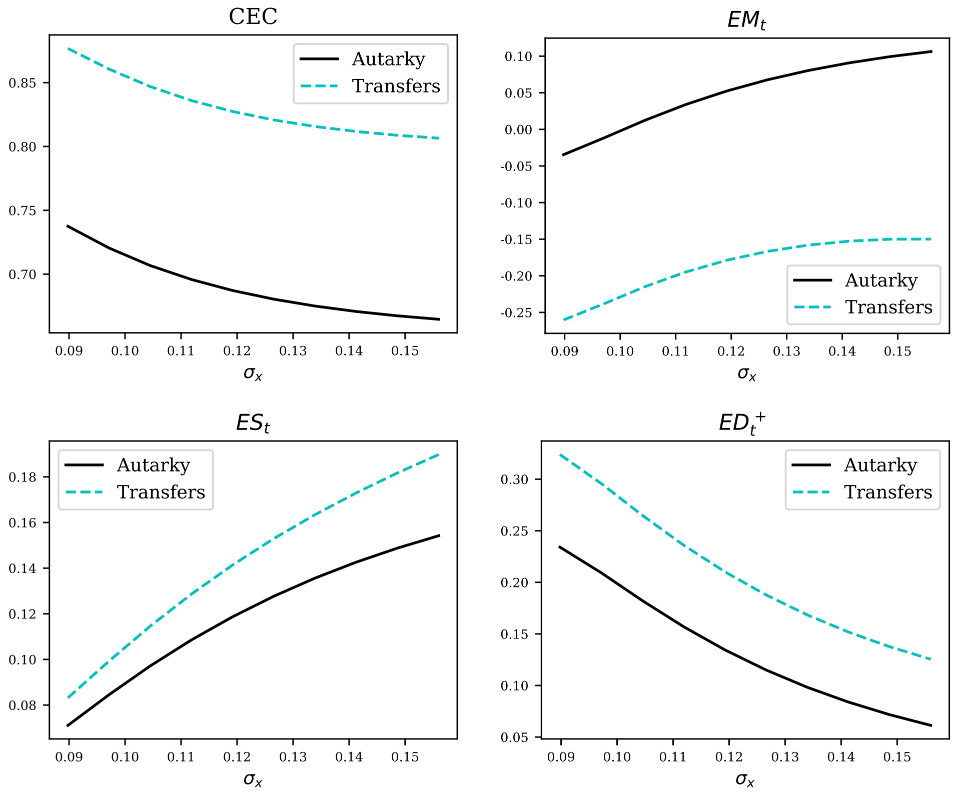
8 Conclusion

This paper examined the problem of optimally allocating risks across generations in the presence of market illiquidity within the asset mix of individuals' savings. we show in a stylized two-period overlapping generations framework that a contract of risk transfers between coexisting young and old cohorts enforced by a policymaker can improve welfare.

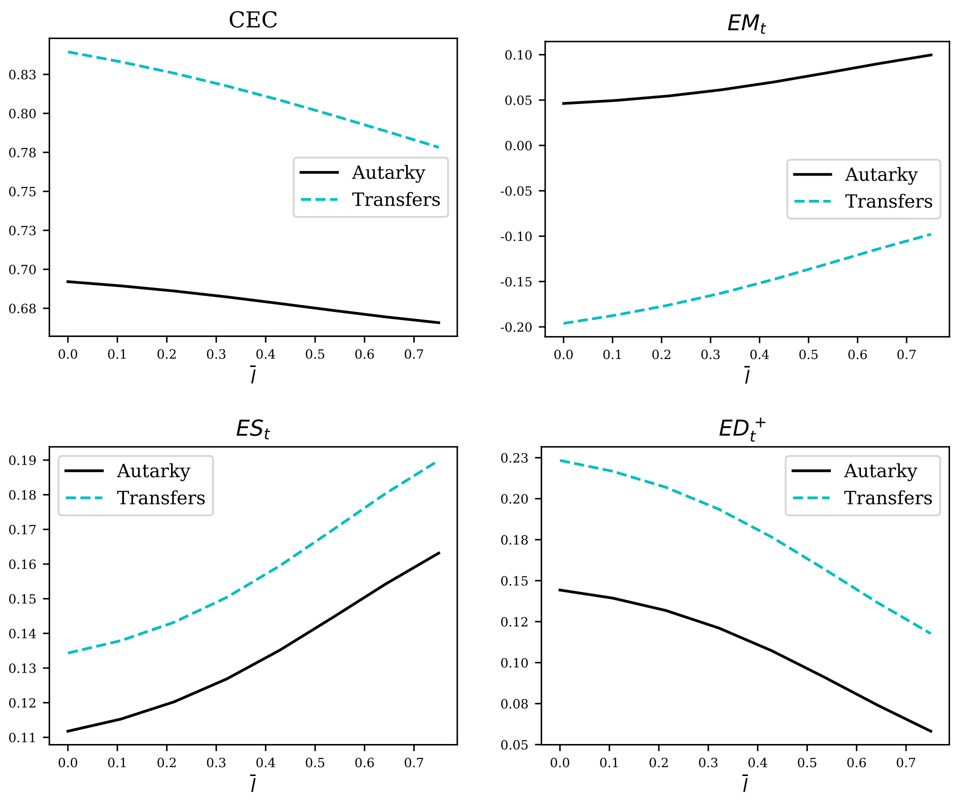
First, we show that optimal IRS is dependent on the variance of the savings portfolio of individuals. On one hand, the policymaker can create a mechanism that expands the pools of individuals who can bear the variance risk by including the young in the risk-sharing pool (pooling effect). On the other hand, introducing additional risk early on in individuals' lifetime savings accumulates higher variance in their old-age consumption (risk-compounding effect). The higher the variance, the more the latter dominates, and the lower the risk-sharing parameter should be.

From that point of view, illiquidity poses a friction to risk sharing between generations, as it reduces the variance over which the illiquid asset in the portfolio can be traded. To compensate for the loss of sensitivity of the IRS transfers to movements in the fair value of the asset, a policymaker needs to increase the level of risk sharing. In contrast, increases in the variance of a liquid asset, ceteris paribus, justify lower levels of risk sharing, as otherwise resources of the young will be destabilized, pushing them towards states of

Figure 9: Optimized Allocation and Fixed Risk Sharing



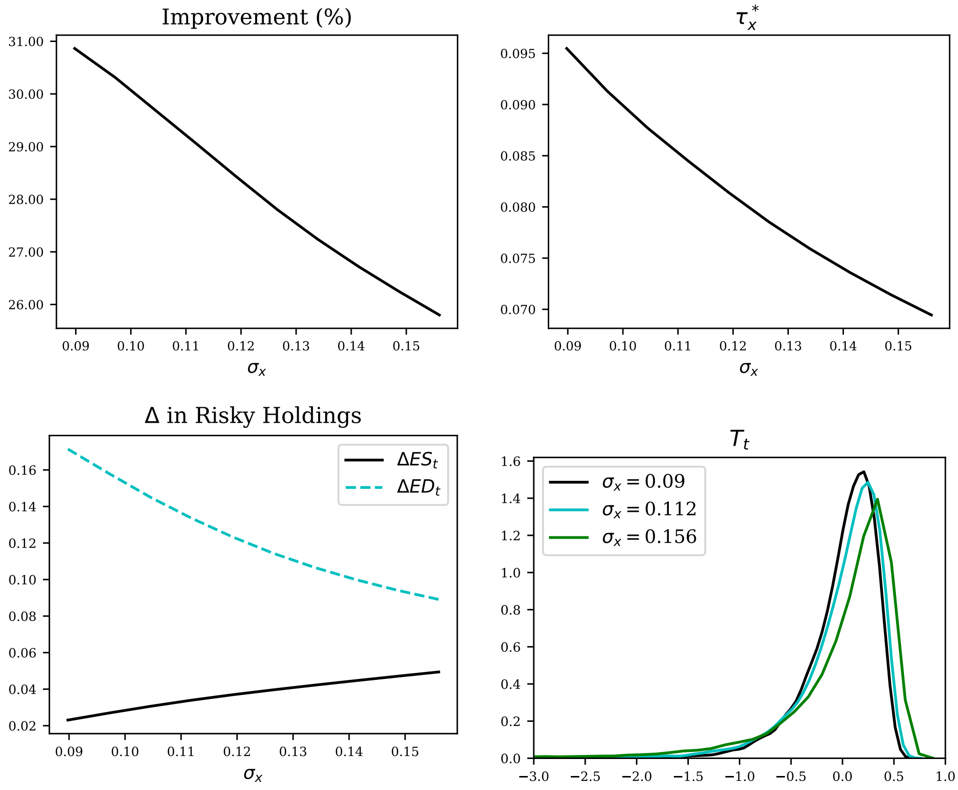
(a) Increased Asset Variance



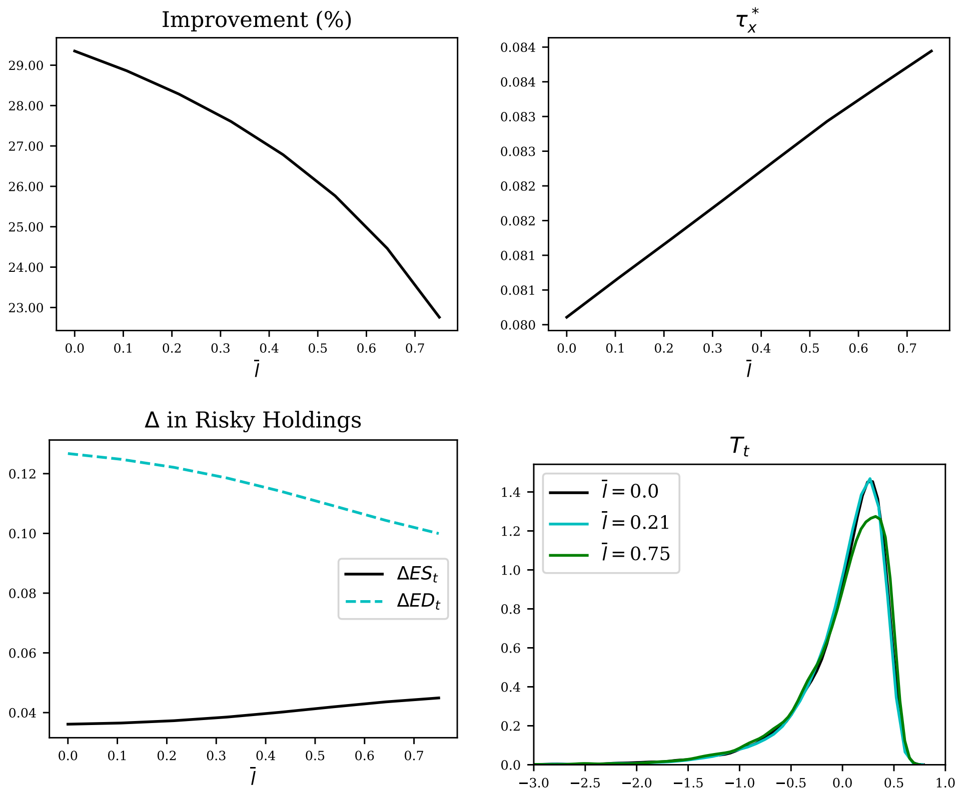
(b) Increased Liquidity Cost

Note. This set of charts shows the effect of varying the volatility on the illiquid asset and respectively its liquidity cost. The investments are determined based on the individual optimality conditions while the risk sharing is fixed such that $\tau_s = 0$, $\tau_x = .05$.

Figure 10: The Effect of Risk on Welfare Improvements. Optimal Solution



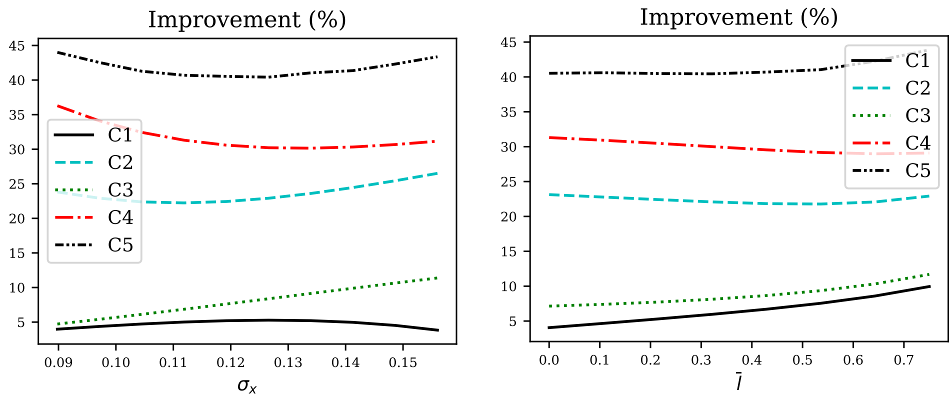
(a) Increased Asset Variance



(b) Increased Liquidity Cost

Note. This set of charts shows the effect of varying the volatility on the illiquid asset and respectively its liquidity cost. The level of inter-generational risk sharing is optimized over τ_x while $\tau_s = 0$. The model is parameterized for the base case, and in the first column the variance is varied, while in the second column the liquidation cost is varied.

Figure 11: Welfare Improvements vs. the Constrained No-Risk-Sharing Benchmark



Note. This set of charts shows the resulting welfare improvement vs. an autarky benchmark economy, in which agents cannot adjust their savings and $M_t = S_t = D_t^+ = 0.1$ in all states of nature. The lines represent the percentage CEC improvement resulting from switching to an intergenerational risk-sharing economy where C1: $\tau_s = 0$, $\tau_x = .05$, while the investments that individuals hold are fixed; C2: $\tau_s = 0$, $\tau_x = .05$, while individuals can adjust their asset holdings; C3: investments are fixed, $\tau_s = 0$ and τ_x is optimally adjusted; C4: investments are optimized, $\tau_s = 0$ and τ_x is optimized; C5: investments, τ_s and τ_x are optimized.

nature where the decline in young age utility of consumption is higher than the benefit of the elderly, thus lowering their lifetime utility and lowering the overall welfare in the aggregate economy.

Second, we find that risk sharing allows individuals to invest more into illiquid assets compared to the case when they are holding personal savings accounts without a mechanism to shift risks across generations. This is also in line with the literature which explores intergenerational risk-sharing mechanisms within funded pension plans (Gollier, 2008; Cui et al., 2011; Shiller, 1999).

The analytical results were justified by a quantitative welfare analysis with a realistic asset composition and utility specification. The framework highlights several policy-relevant implications. The tendency of pension funds to invest in illiquid asset classes, such as private equity, infrastructure projects, etc., may call for increased risk-sharing mechanisms between pension fund participants of different generations. Alternatively, increasing the liquidity of otherwise illiquid assets, for example with the development of attractive secondary markets for OTC traded assets, has the potential to also increase the benefits of intergenerational risk sharing, while also allowing for lower levels of risk sharing between cohorts.

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A Appendix: Derivations

A.1 Planner Problem Derivations

Taking into account the constraints of the Bellman equation (13), we can write the optimization problem in Lagrangian form:

$$\mathcal{L} = \tilde{u}(C_{y,t}, C_{o,t}) + \delta \mathbb{E} V(W_{t+1}, X_{t+1}) - \sum_j \lambda_j g_j(W_t, X_t, C_{y,t}, C_{o,y}, S_t, D_t^+, D_t^-)$$

where each of the $g_j(.)$ functions, $j = 1, \dots 6$, represent one of the constraints written in a form such that $g_j(.) \leq 0$

$$\begin{aligned}
g_1(.) &= C_{y,t} + C_{o,t} + D_t^+ - D_t^-(1 - \bar{l}) + S_t \mathbb{1} - W_t - Y - L; \\
g_2(.) &= D_t^- - D_t^+ - X_t; \\
g_3(.) &= D_t^- - X_t; \\
g_4(.) &= -D_t^-; g_5(.) = -D_t^+; g_6(.) = -S_t
\end{aligned}$$

and λ_j are the non-negative KKT multipliers subject to the standard interpretation as sensitivity of the optimal solution to relaxing the corresponding constraint. The standard first-order conditions apply together with complementary slackness and non-negativity:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 0 \quad x \in \{C_{y,t}, C_{o,y}, S_t, D_t^+, D_t^-\} \\
\lambda_j g_j(.) &= 0 \\
g_j(.) &\leq 0 \\
\lambda_j &\leq 0, \quad j = 1, ...6
\end{aligned}$$

In particular, the first-order conditions with respect to consumption can be written as

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_{y,t}} : \quad \frac{\partial \tilde{u}(C_{y,t}, C_{o,t})}{\partial C_{y,t}} &= \delta R_f \mathbb{E} V_W(W_{t+1}, X_{t+1}) + \lambda_1 \\
\implies u'_y(C_{y,t}) &= \delta R_f \mathbb{E} V_W(W_{t+1}, X_{t+1}) + \lambda_1 \\
\frac{\partial \mathcal{L}}{\partial C_{o,t}} : \quad \frac{\partial \tilde{u}(C_{y,t}, C_{o,t})}{\partial C_{o,t}} &= \delta R_f \mathbb{E} V_W(W_{t+1}, X_{t+1}) + \lambda_1 \\
\implies \frac{\beta}{\delta} u'_o(C_{o,t}) &= \delta R_f \mathbb{E} V_W(W_{t+1}, X_{t+1}) + \lambda_1
\end{aligned} \tag{37}$$

As the optimal condition is symmetric w.r.t. the consumption of the young and the old, from (37) we have:

$$u'_y(C_{y,t}) = \frac{\beta}{\delta} u'_o(C_{o,t}) \tag{38}$$

Applying the Envelope Theorem on the constrained problem, we get:

$$\frac{\partial V(W_t, X_t)}{\partial W_t} : \quad V_W(W_t, X_t) = \delta R_f \mathbb{E} V_W(W_{t+1}, X_{t+1}) + \lambda_1 \tag{39}$$

and combining (39) with (37), we get this planner's version of the standard Ramsey equivalence of the marginal utility and the (liquid) wealth derivative of the value function:

$$\begin{aligned} u_y'(C_{y,t}) &= V_W(W_t, X_t) \\ \frac{\beta}{\delta} u_o'(C_{o,t}) &= V_W(W_t, X_t) \end{aligned} \tag{40}$$

Substituting (40) forward in (37), we can get the Euler relation, now corrected for a possible breach of the non-negativity borrowing constraint:

$$\begin{aligned} u_y'(C_{y,t}) &= \delta R_f \mathbb{E} \tilde{u}'(C_{i,t+1}) + \lambda_1 \\ \frac{\beta}{\delta} u_o'(C_{o,t}) &= \delta R_f \mathbb{E} \tilde{u}'(C_{i,t+1}) + \lambda_1 \end{aligned} \tag{41}$$

The first-order condition w.r.t. the liquid risky investment in asset i is

$$\frac{\partial \mathcal{L}}{\partial S_t^i} : \mathbb{E} V_W(W_{t+1}, X_{t+1}) r_{t+1}^{s,i} - \frac{1}{\delta} (\lambda_1 - \lambda_6) = 0 \tag{42}$$

The first-order condition w.r.t. new investments in the illiquid asset D_t^+ and withdrawals from illiquid wealth D_t^-, respectively, can be written as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial D_t^+} &: R_f \mathbb{E} V_W(W_{t+1}, X_{t+1}) + \frac{1}{\delta} (\lambda_1 - \lambda_2 - \lambda_5) = \mathbb{E} V_X(W_{t+1}, X_{t+1}) R_{t+1}^x \\ \frac{\partial \mathcal{L}}{\partial D_t^-} &: R_f \mathbb{E} V_W(W_{t+1}, X_{t+1})(1 - l_t) + \frac{1}{\delta} \left(\lambda_1(1 - \bar{l}) - \lambda_2 - \lambda_3 + \lambda_4\right) = \mathbb{E} V_X(W_{t+1}, X_{t+1}) R_{t+1}^x \end{aligned}$$

A.2 Utility of Old-Age Consumption

The generation born before implementation of the policy saves $S_0 = Y$, and consumes $C_{o,1} = Y \left(\tilde{R}_1 - \tau\tilde{\epsilon}_1\right)$ which implies the sensitivity of consumption to the risk-sharing parameter of $\frac{\partial C_{o,1}}{\partial \tau} = -Y\tilde{\epsilon}_1$.

Using the quadratic utility assumption, we get

$$\begin{aligned} \mathbb{E}\left(u_o'(C_{o,1}) \cdot \frac{\partial C_{o,1}}{\partial \tau}\right) &= \mathbb{E}\left((1 - \gamma C_1)(-Y\tilde{\epsilon}_1)\right) \\ &= \mathbb{E}\left(\left(1 - \gamma Y(\tilde{R}_1 - \tau\tilde{\epsilon}_1)\right)(-Y\tilde{\epsilon}_1)\right) \\ &= -Y\mathbb{E}(\tilde{\epsilon}_1) + \gamma Y \mathbb{E}\left((\tilde{R}_1 - \tau\tilde{\epsilon}_1)\tilde{\epsilon}_1)\right) \\ &= \gamma Y^2 \tilde{\sigma}^2 (1 - \tau) \end{aligned}$$

where we use the fact that $\mathbb{E}(\tilde{\epsilon}_t) = 0$, $\mathbb{E}(\tilde{R}_t^x \tilde{\epsilon}_t) = \mathbb{E}(\tilde{\epsilon}_t) = \tilde{\sigma}^2$.

Generations born in periods $t \geq 1$, after the policy has been implemented, save $S_t = Y + \tau Y \epsilon_t$ and consume in old age $C_{o,t+1} = S_t \tilde{R}_{t+1} - \tau Y \epsilon_{t+1}$ with $\frac{\partial C_{o,t+1}}{\partial \tau} =$

$Y\left(\tilde{\epsilon}_t\tilde{R}_{t+1}-\tilde{\epsilon}_{t+1}\right)$. As a result:

$$\begin{aligned}\mathbb{E}\left(u_o'(C_{o,t+1})\cdot\frac{\partial C_{o,t+1}}{\partial\tau}\right)&=\mathbb{E}\left((1-\gamma C_{t+1})\cdot\frac{\partial C_{o,t+1}}{\partial\tau}\right)\\&=\mathbb{E}\left(\frac{\partial C_{o,t+1}}{\partial\tau}\right)-\gamma\mathbb{E}\left(C_{t+1}\frac{\partial C_{o,t+1}}{\partial\tau}\right)\\&=-\gamma Y\mathbb{E}\left(C_{t+1}\left(\tilde{\epsilon}_t\tilde{R}_{t+1}-\tilde{\epsilon}_{t+1}\right)\right)\\&=-\gamma Y^2\mathbb{E}\left((1+\tau\tilde{\epsilon}_t)\tilde{R}_{t+1}-\tau\tilde{\epsilon}_{t+1}\right)\left(\tilde{\epsilon}_t\tilde{R}_{t+1}-\tilde{\epsilon}_{t+1}\right)\end{aligned}$$

Opening up the brackets and noting that due to time independence

$$\begin{aligned}\mathbb{E}(\tilde{R}_{t+1}^2\epsilon_t)&=\mathbb{E}\tilde{R}_{t+1}^2\mathbb{E}\epsilon_t=0\\\mathbb{E}\tilde{\epsilon}_t^2\tilde{R}_{t+1}^2&=\mathbb{E}\tilde{\epsilon}_t^2\mathbb{E}\tilde{R}_{t+1}^2=\tilde{\sigma}^2(\tilde{\mu}^2+\tilde{\sigma}^2)\\\mathbb{E}(\tilde{\epsilon}_t^2\tilde{R}_{t+1}^2)&=\mathbb{E}\tilde{\epsilon}_t^2\mathbb{E}\tilde{R}_{t+1}^2=\tilde{\sigma}^2\end{aligned}$$

we get

$$\mathbb{E}\left(u_o'(C_{o,t+1})\cdot\frac{\partial C_{o,t+1}}{\partial\tau}\right)=\gamma Y^2\tilde{\sigma}^2\left(1-\tilde{\mu}^2\tau-\tilde{\sigma}^2\tau-\tau\right)$$

The policymaker maximizes welfare by reconciling the marginal expected benefits for all generations by applying condition (23).

$$\frac{\beta}{\delta}\mathbb{E}\left(u_o'(C_{o,1})\cdot\frac{\partial C_{o,1}}{\partial\tau}\right)+\frac{\beta}{1-\delta}\mathbb{E}\left(u_o'(C_{o,t+1})\cdot\frac{\partial C_{o,t+1}}{\partial\tau}\right)=0$$

Substituting in the derived terms for the generations born at period zero and after that, and canceling out γ, Y, β we get

$$\begin{aligned}\frac{1}{\delta}\tilde{\sigma}^2(1-\tau)+\frac{1}{1-\delta}\tilde{\sigma}^2\left(1-(\tilde{\mu}^2+\tilde{\sigma}^2)\tau-\tau\right)&=0\\\implies\tau^*=\frac{1}{\delta\mathbb{E}\tilde{R}_t^2+1}&=\frac{1}{\delta(\tilde{\mu}^2+\tilde{\sigma}^2)+1}\end{aligned}$$

A.3 Variance of Old-Age Consumption

In (33) we have:

$$\begin{aligned}\mathbb{Var}\left(\frac{C_{o,t+1}}{Y}\right)&=\mathbb{Var}\left(\frac{S_t}{Y}R_{t+1}-\tau\epsilon_{t+1}\right)\\&=\mathbb{Var}\left((1+\tau\epsilon_t)(\mu+\epsilon_{t+1})-\tau\epsilon_{t+1}\right)\\&=\mathbb{Var}\left(\mu+\epsilon_{t+1}+\tau\mu\epsilon_t+\tau\epsilon_t\epsilon_{t+1}-\tau\epsilon_{t+1}\right)\\&=\mathbb{Var}\left((1-\tau)\epsilon_{t+1}+\tau\mu\epsilon_t+\tau\epsilon_t\epsilon_{t+1}\right)\\&=\mathbb{Var}((1-\tau)\epsilon_{t+1})+\mathbb{Var}(\tau\mu\epsilon_t)+\mathbb{Var}(\tau\epsilon_t\epsilon_{t+1})\\&=(1-\tau)^2\sigma^2+(\tau)^2\mu^2\sigma^2+(\tau)^2\sigma^4\end{aligned}$$

where due to the zero-mean and *i.i.d.* properties of the shock we have $\mathbb{Cov}(\epsilon_t, \epsilon_{t+1}) = \mathbb{Cov}(\epsilon_t, \epsilon_t\epsilon_{t+1}) = \mathbb{Cov}(\epsilon_{t+1}, \epsilon_t\epsilon_{t+1}) = 0$ and $Var(\epsilon_t\epsilon_{t+1}) = \sigma^4$ or in more details:

$$\mathbb{Cov}(\epsilon_t, \epsilon_t\epsilon_{t+1}) = \mathbb{E}(\epsilon_t^2\epsilon_{t+1}) + \mathbb{E}(\epsilon_t)\mathbb{E}(\epsilon_t\epsilon_{t+1}) \overset{i.i.d.}{=} \mathbb{E}(\epsilon_t^2)\mathbb{E}(\epsilon_{t+1}) = 0$$

$$\mathbb{Var}(\epsilon_t\epsilon_{t+1}) = \mathbb{E}(\epsilon_t^2\epsilon_{t+1}^2) - \mathbb{E}(\epsilon_t)\mathbb{E}(\epsilon_{t+1}) \overset{i.i.d.}{=} \mathbb{E}(\epsilon_t^2)\mathbb{E}(\epsilon_{t+1}^2) = \sigma^4$$

Minimizing the variance of consumption for the generation born after $t > 0$, we get the first-order condition with respect to τ

$$\frac{\partial \mathbb{Var}(C_{o,t+1})}{\partial \tau} = -(1-\tau)\sigma^2 + \tau\mu^2 + \tau\sigma^2 = 0$$
$$\implies \tau(\mu^2 + \sigma^2 + 1) = 1$$

This illustrates again that for an increase in the variance, τ needs to go down in order to keep the outcome optimal for a future generation.

A.4 Allocation Decision with Risk-Free and Risky Assset

Assume that generations have a nonzero utility of consumption when old and when young, such that $u_y(C) = u_o(C)$. Also, there is a risk-free asset with a fixed gross return of R_f and a risky asset. At the same time, the young have to decide how much to consume and save and how to allocate their savings between the risk-free asset and the risky asset. The two-period budget constraints of (15) simplify to

$$\begin{aligned} C_{y,t} &= Y - M_t - S_t + \tilde{\epsilon}_t\tau Y \\ C_{o,t+1} &= M_t R^f + S_t\tilde{R}^s_{t+1} - \tilde{\epsilon}_{t+1}\tau Y \end{aligned}$$

Applying the individuals' optimality conditions (17), it can be shown that

$$I(\tau, \epsilon_t) = \begin{bmatrix} M_t \\ S_t \end{bmatrix} = Y\left(\begin{bmatrix} -(a_1+b_1) \\ a_1 \end{bmatrix} + \begin{bmatrix} -(a_2+b_2) \\ a_2 \end{bmatrix}\tau + \begin{bmatrix} -(a_3+b_3) \\ a_3 \end{bmatrix}\tau\tilde{\epsilon}_t\right) \tag{43}$$

where all coefficients but b_1 are positive functions of the problem's primals. Optimal consumption for the young and the old then is

$$\begin{aligned} C_{y,t} &= Y\left[1 + b_1 + b_2\tau + (1-b_3)\tau\tilde{\epsilon}_t\right] \\ C_{o,t} &= Y\left[-\left((a_1+b_1) + (a_2+b_2)\tau - (a_3+b_3)\tau\tilde{\epsilon}_{t-1}\right)R_f + \left(a_1 + a_2\tau - a_3\tau\tilde{\epsilon}_{t-1}\right)(\mu + \tilde{\epsilon}_t) - \tau\tilde{\epsilon}_t\right] \end{aligned} \tag{44}$$

where $\overline{\mu} = \tilde{\mu} - R_f$ is the excess return on the risky asset the coefficients in equation (43) are

$$a_1 = \frac{\overline{\mu}\left(-R_f Y\gamma + R_f + 1\right)}{Y\gamma\left(R_f^2\beta\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2\right)}; \quad a_2 = \frac{\tilde{\sigma}^2\left(R_f^2\beta + 1\right)}{R_f^2\beta\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2}; \quad a_3 = \frac{R_f\overline{\mu}}{R_f^2\beta\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2}$$

$$b_1 = \frac{-R_f\beta\tilde{\sigma}^2 - Y\overline{\mu}^2\gamma - Y\gamma\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2}{Y\gamma\left(R_f^2\beta\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2\right)}; \quad b_2 = \frac{R_f\overline{\mu}\beta\tilde{\sigma}^2}{R_f^2\beta\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2}; \quad b_3 = \frac{\overline{\mu}^2 + \tilde{\sigma}^2}{R_f^2\beta\tilde{\sigma}^2 + \overline{\mu}^2 + \tilde{\sigma}^2}$$

The parameters have an appealing interpretation: a_1 captures the autonomous level of savings, independent of the realized shocks; a_2 is the increase in the share of (risky) savings which varies with the risk sharing; a_3 is the variation in savings due to shocks; b_1 and b_2 capture the autonomous change in young-age consumption after the policy is implemented while $1 - b_3$ captures consumption variation with the realization of the shock.

As a result, with risk sharing, on average the individual starts investing more into risky assets and less into risk-free assets. So, the standard conjecture that the capacity of individuals to bear risk increases with risk sharing, holds here as well. In addition, now we can see that the realization of a positive shock leads to a decrease in risky asset holdings and either increase in risk-free assets or an increase in young-age consumption.

A.5 Certainty Equivalent Consumption

Equating the cumulative utility for the two, we get:

$$\sum_{j=1}^{\infty} \delta^{t-1} \left(u(CEC) + \frac{\beta}{\delta} u(CEC) \right) = \sum_{j=1}^{\infty} \delta^{t-1} \mathbb{E} \left(\frac{\beta}{\delta} u(C_{o,t}) + u(C_{y,t}) \right) = V_0$$

where in the case of the planner problem of Section (4), V_0 is a function of the starting wealth values $V_0 = V(W_0, X_0)$, while in the case of optimal transfers of Section (5.2), welfare is a determined by the optimal transfers, such that $V_0 = V(\tau^*)$.

The left-hand side can then be expanded so that we get

$$\frac{1}{1-\delta} \left(u(CEC) + \frac{\beta}{\delta} u(CEC) \right) = \frac{1}{1-\delta} \left(1 + \frac{\beta}{\delta} \right) u(CEC)$$

which produces the equation

$$CEC_0 = I_u \left[(1-\delta) \frac{\delta}{\beta + \delta} V_0 \right] \tag{45}$$

where I_u is the inverse of the utility function.

B Appendix: Numerical Algorithms

B.1 Solution Algorithm for the Planner Problem

I solve the problem numerically through value function iteration. Several authors among other examine the tools needed to set up the examination algorithm (Judd, 1998; Miranda and Fackler, 2002; Cai and Judd, 2014; Rust, 1996; Cai et al., 2013). We use the following approach:

0. **Initialize**
   - Set up a grid for the two state variables $\{W_t\}_{j_w}, \{X_t\}_{j_x}$
   - Set up an initial guess on the function values V_0 on the grid.
   - Set-up interpolating splines for $V(W,X)$ to approximate the value function. We use in particular cubic splines, which allows me to evaluate also the first-order derivatives of the function.
   - Get bivariate quadrature points and weights for R^S and R^X (see Section B.3)
1. **Maximization:** For each W_{j_w} and for each X_{j_x} find the consumpiton and investment vector which maximize the RHS of the Bellman equation (13) subject to the constraints. Sequential Quadratic Programming (SLSQP) is used to solve numerically the resulting problem.
2. **Approxmiation:** Find the left-hand side of (13) and fit a new splines approximation to the value function approximating
3. **Evaluation** Evaluate distance from previous-run value function $dist = ||V^i - V^{i-1}||^1$ and iterate until $dist < tol$ for some error tolerance level

B.2 Solution Algorithm for the Policymaker Problem

Solving the risk-sharing problem accounts to solving two optimization problems. We have to solve the unconstrained optimization of the policymaker (21) by providing as input the optimal solution of the individual portfolio choice problem (15). We use the Nelder-Mead method to find numerically the solution of the former and the SLSQP to find the optimum of the later. Again, a two-dimensional quadrature (Section B.3) is used to evaluate expectations.

Also, note that this specification of the liquidity factor allows me, using the law of iterated expectations, to evaluate individuals indirect utility function as

$$\mathbb{E}v(\tau, R_t) = p\mathbb{E}\left(v(\tau, R_t)|l_t = 0\right) + (1-p)\mathbb{E}\left(v(\tau, R_t)|l_t = \bar{l}\right)$$

I also split the expectation of the value function of the planner in the same way.

B.3 Quadrature

I use quadrature to evaluate the expectation terms in the numerical section of this paper. In low dimensions, the quadrature provides a fast and reliable approximation.[21]

[21] For more details see (Judd, 1998; Cai et al., 2013; Cai and Judd, 2014).

Multi-dimensional quadrature methods are less common, so I provide here explicitly the approach taken.

In the uni-variate space, the Gaussian quadrature performs the following approximation:

$$\int_a^b f(x)w(x)dx \approx \sum_{i=1}^m w_i f(x_i)$$

for some quadrature nodes x_i, and some positive quadrature weights w_i, and m is the number of quadrature points used.

When working with a normally distributed random variable, it is usefule to apply in particular the Gauss-Hermite (GH) quadrature which selects a weight function of the form $w(x) = e^{-x^2}$.

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx \approx \sum_{i=1}^m w_i f(x_i) \tag{46}$$

with x_i and w_i as the GH nodes and weights, respectively.

Assume for example that the random variable y is normally distributed with $y \sim N(\mu, \sigma)$. We can evaluate the expectation of $f(y)$ as

$$\mathbb{E}(f(y)) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(y)e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

To reconcile this with the Gauss-Hermite approach of (46), we use a change of variable $y = \sqrt{2}\sigma x + \mu$ such that

$$\begin{aligned}\mathbb{E}(f(\sqrt{2}\sigma x + \mu)) &= (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma x + \mu)e^{-x^2}\sqrt{2}\sigma dx \\ &\approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^m w_i f(\sqrt{2}\sigma x_i + \mu)\end{aligned}$$

We can apply this approximation in asset pricing context for an asset whose log returns R are normally distributed such that $\ln(R) \equiv y \sim N(\mu, \sigma)$. We can then find the expectation of a function of R as $\mathbb{E}f(R)$ as

$$\mathbb{E}f(e^y) = (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} f(e^y)e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^m w_i f(e^{\sqrt{2}\sigma x_i + \mu})$$

In the multi-dimensional space, we can use the quadrature product rule, which approximates

$$\int_{R^d} f(x)dx \approx \sum_{i_1=1}^m \cdots \sum_{i_d=1}^m w_{i_1} w_{i_2} \cdots w_{i_d} f(x_{i_1}, x_{i_2}, \cdots, x_{i_d})$$

For example, assuming that a two-dimensional vector process $Y \sim N(\mu, \Sigma)$ where μ is a 2×1 vector of expectations and Σ is the (positive semi-definite) covariance matrix. To reconcile this with the Hermite-Gauss approximation, we follow the same approach as before. We perform a Choleski decomposition $\Sigma = LL'$ and do a change of variable $Y = \sqrt{2}LX + \mu$ and as a result, we get

$$
\begin{aligned}
\mathbb{E}(f(x)) &= 2\pi|\Sigma|^{-1/2}\int_{R^2} f(y)e^{-\frac{(Y-\mu)'\Sigma(Y-\mu)}{2}}dY \\
&= 2\pi|\Sigma|^{-1/2}\int_{R^2} f(X)e^{-X'X}2|L|dX \\
&\approx \frac{1}{\pi}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m} w_{i_1}w_{i_2}f(\sqrt{2}L_{11}x_{i_1}+\mu_1, \sqrt{2}(L_{21}x_{i_1}+L_{22}x_{i_2})+\mu_2)
\end{aligned}
$$

where x_i and w_i are the nodes and weights from the Gaus-Hermite quadrature, μ_1,μ_2 are elements of the vector of expectations, and L_{11},L_{22} are elements of the L matrix.