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Sign properties and axiomatizations of the weighted division values

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Abstract

In this paper, we study axiomatic foundations of the class of weighted division values. Firstly, while keeping efficiency, additivity and the nullifying player property from the original axiomatization of the equal division value, we consider relaxations of symmetry in line with Casajus (2019) to characterize the class of (positively) weighted division values. Secondly, we show that the class of weighted division values can also be characterized by replacing linearity in three axiomatizations of Béal et al. (2016) with additivity. Finally, we show how strengthening an axiom regarding null, non-negative, respectively nullified players in these three axiomatizations, provides three axiomatizations of the class of positively weighted division values.

Keywords: cooperative game; weighted division values; axiomatization; sign properties

1. Introduction

A cooperative game with transferable utility (or TU-game) describes a situation where players can achieve a specified amount of worth by cooperating. A central issue is to find a method to distribute the benefits of cooperation among these players. A (one-point) solution for TU-games is a function that assigns to every TU-game a vector with the same dimension as the size of the player set, where each component of the vector represents the payoff assigned to the corresponding player.

The Shapley value (Shapley 1953, [22]) and the equal division value are basic solutions for TU-games. In order to interpret asymmetries among players beyond the game, Shapley (1953, [21]) proposed a weighted version of the Shapley value, namely the positively weighted Shapley values, where these asymmetries are modelled by strictly positive weights for

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the players. Subsequently, Kalai and Samet (1987, [17]) extended the positively weighted Shapley values by taking into account a weight system that allows for zero weights of the players. There exists a number of axiomatic foundations for the class of weighted Shapley values in the literature (see, e.g., Besner 2020, [4]; Calvo and Santos 2000, [5]; Casajus 2018, [7]; Casajus 2019, [9]; Casajus 2021, [10]; Chun 1991, [13]; Hart and Mas-Colell 1989, [15]; Kalai and Samet 1987, [17]; Nowak and Radzik 1995, [20]; Yokote 2015, [26]). Similar to the Shapley value, the equal division value is also generalized by considering asymmetries between players. Given non-negative exogenous player weights, the corresponding weighted division value allocates the worth of the grand coalition (consisting of all players) proportional to these weights. If all weights are positive, we call it a positively weighted division value. In a sense, the weighted division values generalize the equal division value similar as the weighted Shapley values generalize the Shapley value.

A major purpose of axiomatization in TU-games is to motivate their use by characterizing them using desirable axioms. A well-known axiomatization of the Shapley value involves efficiency, additivity, the null player property and symmetry. Symmetry requires that equally productive players should get the same payoff. Casajus (2019, [9]) suggested a relaxation of symmetry, called sign symmetry, that relaxes symmetry and requires equally productive players' payoffs to have the same sign. Though sign symmetry is a considerable weakening of symmetry, he showed that replacing symmetry by sign symmetry in the original axiomatization of the Shapley value still characterizes this value.[1] In van den Brink (2007, [24]) an axiomatization of the equal division value by using efficiency, additivity, the nullifying player property and symmetry is proposed. This triggers the question whether sign symmetry can serve as a substitute for symmetry in this axiomatization of the equal division value. Though this is not possible, interestingly, we can characterize the class of positively weighted division values by replacing symmetry in van den Brink's axiomatization with sign symmetry. Furthermore, we also suggest a weak version of sign symmetry, called weak sign symmetry, that together with efficiency, additivity and the nullifying player property characterizes the class of weighted division values.

There exist several axiomatic characterizations for the class of weighted division values in the literature (see, e.g., Béal et al. 2016, [2]; Béal et al. 2015, [3]; Kongo 2019, [18] and van den Brink 2009, [25]). Béal et al. (2016, [2]) introduced three different axiomatizations of the class of weighted division values. The first axiomatization involves efficiency, lineari-

[1]Casajus and Yokote (2017, [12]) showed that the fairness, or differential marginality, axiom in the axiomatization of the Shapley value given by van den Brink (2002, [23]), respectively Casajus (2011, [6]), can be replaced by a weaker sign version to characterize the Shapley value.

ty, the nullifying player property and the null player in a productive environment property. The second axiomatization involves efficiency, linearity and the non-negative player property. The third axiomatization involves efficiency, linearity and nullified solidarity. Two common axioms used in these axiomatizations are efficiency and linearity. In their concluding remarks, Béal et al. (2016, [2]) state the claim that linearity can not be weakened to additivity in these axiomatizations. In this paper, we show that the class of weighted division values can also be characterized by replacing linearity in the axiomatizations of Béal et al. (2016, [2]) with additivity.

Inspired by the work of Béal et al. (2016, [2]), we suggest stronger versions of the null player in a productive environment property, the non-negative player property and nullified solidarity, called the sign null player in a productive environment property, the sign non-negative player property and sign nullified solidarity respectively. The sign null player in a productive environment property requires that a null player is rewarded (by a positive payoff) or punished (by a negative payoff) depending on whether the worth of the grand coalition is positive or negative. The sign non-negative player property requires that a non-negative player will get a positive payoff if the worth of the grand coalition is positive, and will get nothing otherwise. Sign nullified solidarity requires that the payoffs for all players change in the same direction in case a specified player becomes a null player. We show that the positively weighted division values can be characterized by using these stronger axioms instead of the corresponding axioms in the axiomatizations of Béal et al. (2016, [2]).

The rest of the paper is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, we characterize the classes of positively weighted division values and weighted division values by using relaxations of symmetry. In Section 4, we replace linearity in the axiomatizations of Béal et al. (2016, [2]) with additivity to characterize the class of weighted division values, and provide three axiomatizations of the class of positively weighted division values by strengthening one of the axioms in each of these axiomatizations. In Section 5, we give a brief summary.

2. Preliminaries

A cooperative game with transferable utility, or simply a TU-game, is a pair $\langle N, v\rangle$, where $N \subseteq \mathbb{N}$ is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function assigning to each coalition $S \in 2^N$, the worth $v(S)$ with $v(\emptyset) = 0$. Denote the set of all TU-games on player set N by $\mathcal{G}^N$. The cardinality of a finite set S is denoted by s. For all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, $\langle N, av + bw\rangle$ is given by $(av + bw)(S) = av(S) + bw(S)$

for all $S \subseteq N$. Players $i, j \in N$ are symmetric players in $\langle N, v\rangle$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Player $i \in N$ is a null player in $\langle N, v\rangle$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. Player $i \in N$ is a nullifying player in $\langle N, v\rangle$ if $v(S) = 0$ for all $S \subseteq N$ with $S \ni i$. Player $i \in N$ is a non-negative player in $\langle N, v\rangle$ if $v(S) \geq 0$ for all $S \subseteq N$ with $S \ni i$.

For all $\emptyset \neq T \subseteq N$, the standard game, $\langle N, e_T\rangle$, is defined by

$$e_T(S) = \begin{cases} 1, & \text{if } S = T; \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that the class of standard games $\{\langle N, e_T\rangle\}_{\emptyset \neq T \subseteq N}$ is a basis of $\mathcal{G}^N$, specifically $v = \sum_{\emptyset \neq T \subseteq N} v(T) e_T$ for every $\langle N, v\rangle \in \mathcal{G}^N$.

A payoff vector for TU-game $\langle N, v\rangle \in \mathcal{G}^N$ is an n-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. A solution on $\mathcal{G}^N$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to every TU-game $\langle N, v\rangle \in \mathcal{G}^N$. The equal division value distributes the worth of the grand coalition equally among all players. For all $\langle N, v\rangle \in \mathcal{G}^N$, the equal division value is defined by

$$ED_i(N, v) = \frac{v(N)}{n}, \text{ for all } i \in N. \tag{2.1}$$

Let $\Delta_+^N = \{\omega \in \mathbb{R}^N | \sum_{i \in N} \omega_i = 1 \text{ and } \omega_i \geq 0 \text{ for all } i \in N\}$ and $\Delta_{++}^N = \Delta_+^N \cap \mathbb{R}_{++}^N = \{\omega \in \mathbb{R}^N | \sum_{i \in N} \omega_i = 1 \text{ and } \omega_i > 0 \text{ for all } i \in N\}$ be the sets of nonnegative, respectively positive, (player) weight vectors. For $\omega \in \Delta_+^N$, the ω-weighted division value distributes the worth of the grand coalition in proportion to the weights given by ω. For all $\langle N, v\rangle \in \mathcal{G}^N$ and $\omega \in \Delta_+^N$, the ω-weighted division value is defined by

$$WD_i^\omega(N, v) = \omega_i v(N), \text{ for all } i \in N. \tag{2.2}$$

These solutions are called weighted division values. The solutions WD^ω, $w \in \Delta_{++}^N$, are called positively weighted division values.

Next we recall several axioms of solutions for TU-games. Efficiency, symmetry, additivity and linearity are four standard axioms introduced by Shapley (1953, [22]), and they are often used to characterize various solutions of TU-games.

- **Efficiency.** For all $\langle N, v\rangle \in \mathcal{G}^N$, it holds that $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- **Symmetry.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i, j \in N$ are symmetric players in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = \varphi_j(N, v)$.
- **Additivity.** For all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$, it holds that $\varphi(N, v) + \varphi(N, w) = \varphi(N, v + w)$.

- **Linearity.** For all $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, it holds that $\varphi(N, av) + \varphi(N, bw) = \varphi(N, av + bw)$.

Obviously, a solution that satisfies linearity also satisfies additivity. The nullifying player property, introduced by Deegan and Packel (1978, [14])[2], requires that a nullifying player should receive a zero payoff. In van den Brink (2007, [24]) the equal division value is characterized by efficiency, symmetry, additivity and the nullifying player property.

- **Nullifying player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i \in N$ is a nullifying player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) = 0$.

The null player in a productive environment property, introduced by Casajus and Huettner (2013, [11]) to characterize the egalitarian Shapley values of Joosten (1996, [16]), requires that if the grand coalition generates a non-negative worth, then a null player should receive a non-negative payoff. Béal et al. (2016, [2]) used this axiom to characterize the class of weighted division values.

- **Null player in a productive environment property.** For all $\langle N, v\rangle \in \mathcal{G}^N$ with $v(N) \geq 0$, whenever $i \in N$ is a null player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) \geq 0$.

Theorem 2.1 (Béal et al. 2016, [2])**.** *A solution φ on $\mathcal{G}^N$ satisfies efficiency, linearity, the nullifying player property and the null player in a productive environment property if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

The non-negative player property is introduced and used by Béal et al. (2016, [2]) to characterize the class of weighted division values, and requires that for a given player, if the worths of all coalitions including him are non-negative, then this player should get a non-negative payoff.

- **Non-negative player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i \in N$ is a non-negative player in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) \geq 0$.

Theorem 2.2 (Béal et al. 2016, [2])**.** *A solution φ on $\mathcal{G}^N$ satisfies efficiency, linearity and the non-negative player property if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

[2] Deegan and Packel (1978, [14]) refer to nullifying players as zero players and use this property to characterize their (non-efficient) Deegan-Packel value.

Nullified solidarity, proposed by Béal et al. (2014, [1]), compares a TU-game before and after a specified player becomes a null player. Nullified solidarity requires uniformity in the direction of the payoff variation for all players when a player is nullified. For each $\langle N, v\rangle \in \mathcal{G}^N$ and each $i \in N$, the associated game in which i is nullified, denoted by $\langle N, v^{\mathbf{N}i}\rangle \in \mathcal{G}^N$, is defined by

$$v^{\mathbf{N}i}(S) = v(S\backslash\{i\}), \text{ for all } S \subseteq N. \tag{2.3}$$

- **Nullified solidarity.** For all $\langle N, v\rangle \in \mathcal{G}^N$ and $i, j \in N$, it holds that $\varphi_i(N, v) \geq \varphi_i(N, v^{\mathbf{N}i})$ implies $\varphi_j(N, v) \geq \varphi_j(N, v^{\mathbf{N}i})$.

Theorem 2.3 (Béal et al., 2016 [2])**.** *A solution φ on $\mathcal{G}^N$ satisfies efficiency, linearity and nullified solidarity if and only if there exists a weight vector $\omega \in \Delta^N_+$ such that $\varphi = WD^\omega$.*

3. Relaxations of symmetry and the weighted division values

In van den Brink (2007, [24]), the equal division value is characterized by efficiency, symmetry, additivity and the nullifying player property. It is clear that the weighted division values, except the equal division value, fail symmetry. Casajus (2018, [8]) introduced a relaxation of symmetry called sign symmetry, and showed that replacing symmetry by sign symmetry in the original axiomatization of the Shapley value still characterizes the Shapley value. Sign symmetry is a qualitative version of symmetry that is weaker than symmetry. Instead of equating payoffs for symmetric players, it just fixes a common reference point, the zero utility, and requires that symmetric players are either rewarded simultaneously (positive payoff) or punished simultaneously (negative payoff). Recall the sign function, sign: $\mathbb{R} \to \{-1, 0, 1\}$ given by $\text{sign}(t) = 1$ for $t > 0$, $\text{sign}(0) = 0$, and $\text{sign}(t) = -1$ for $t < 0$.

- **Sign symmetry.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i, j \in N$ are symmetric players in $\langle N, v\rangle$, it holds that $\text{sign}(\varphi_i(N, v)) = \text{sign}(\varphi_j(N, v))$.

Obviously, sign symmetry is a considerable weakening of symmetry. One easily checks that the positively weighted division values satisfy sign symmetry. Next we provide a characterization of the class of positively weighted division values by using sign symmetry instead of symmetry appearing in van den Brink's characterization (van den Brink 2007, [24]) for the equal division value.

Theorem 3.1. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and sign symmetry if and only if there exists a weight vector $\omega \in \Delta^N_{++}$ such that $\varphi = WD^\omega$.*

Proof. It is straightforward to verify that the positively weighted division values satisfy efficiency, sign symmetry, additivity and the nullifying player property. It is left to show that the axioms determine that φ is a positively weighted division value.

Let φ be a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. We will show that for some weight vector $\omega \in \Delta^N_{++}$, $\varphi = WD^\omega$. Let $c \in \mathbb{R}$. First, for the null game $\langle N, \mathbf{0}\rangle$ given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$, efficiency and sign symmetry imply that $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$.[3]

Second, we consider $\langle N, ce_T\rangle$ for $\emptyset \neq T \subsetneq N$. By the nullifying player property, we have $\varphi_i(N, ce_T) = 0$ for all $i \in N\backslash T$. Then, by efficiency, we have $\sum_{i\in T} \varphi_i(N, ce_T) = 0$. Since players $i, j \in T$ are symmetric players in $\langle N, ce_T\rangle$, by sign symmetry, we have $\varphi_i(N, ce_T) = 0$ for all $i \in T$. Thus, $\varphi_i(N, ce_T) = 0$ for all $\emptyset \neq T \subsetneq N$ and $i \in N$.

Third, we consider $\langle N, ce_N\rangle$. Set $\omega_i = \varphi_i(N, e_N)$ for all $i \in N$. By efficiency and sign symmetry, we have $\sum_{i\in N} \omega_i = 1$ and $\omega_i > 0$ for all $i \in N$, showing that $\omega \in \Delta^N_{++}$. Now, we show that $\varphi(N, ce_N) = c\varphi(N, e_N)$. Choose two sequences of rationals $\{r_k\}^\infty_{k=1}$ and $\{s_k\}^\infty_{k=1}$ which converge to c from above and below, respectively. We obtain that, for all $i \in N$ and for all $k = 1, ..., \infty$,

$$\begin{aligned} \varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) =& \varphi_i(N, (r_k - c)e_N) \geq 0, \text{ and} \\ \varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) =& \varphi_i(N, (c - s_k)e_N) \geq 0, \end{aligned} \tag{3.1}$$

where in both cases the equality follows from additivity and the inequality follows from efficiency and sign symmetry. Notice that additivity also implies that for all $i \in N$, $\varphi_i(N, r_k e_N) - \varphi_i(N, s_k e_N) = \varphi_i(N, (r_k - s_k)e_N) \to 0$ as $k \to \infty$, since $(r_k - s_k) \to 0$ as $k \to \infty$, and $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$ as shown above. Then, we have $\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) + \varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) \to 0$ as $k \to \infty$. Since, both $\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) \geq 0$ and $\varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) \geq 0$ by Eq.(3.1), this implies that $\varphi(N, r_k e_N) \to \varphi(N, ce_N)$ as $k \to \infty$. Since $\varphi(N, r_k e_N) = r_k\varphi(N, e_N) \to c\varphi(N, e_N)$ (where the equality follows by additivity) [4] and $\varphi(N, r_k e_N) \to \varphi(N, ce_N)$ as $k \to \infty$, we have that $\varphi(N, ce_N) = c\varphi(N, e_N)$ for constant $c \in \mathbb{R}$.

Therefore, for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, with additivity it holds that

$$\varphi_i(N, v) = \sum_{T\subseteq N} \varphi_i(N, v(T)e_T) = \varphi(N, v(N)e_N)$$

[3] Notice that this also follows from additivity since $\varphi_i(N, \mathbf{0}) + \varphi_i(N, \mathbf{0}) = \varphi_i(N, \mathbf{0})$ implies that $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$.

[4] Given any rational r_k, there must exist two integers $a, b \in \mathbb{N}$, $a \neq 0$, such that $r_k = \frac{b}{a}$. Then by additivity, we have $\varphi(N, r_k e_N) = \varphi(N, \frac{b}{a} e_N) = b\varphi(N, \frac{1}{a} e_N) = \frac{b}{a} \cdot a\varphi(N, \frac{1}{a} e_N) = \frac{b}{a}\varphi(N, \frac{a}{a} e_N) = r_k\varphi(N, e_N)$.

$$=v(N)\varphi_i(N, e_N) = \omega_i v(N).$$

□

Logical independence of the axioms used in Theorem 3.1 can be shown by the following alternative solutions on $\mathcal{G}^N$.

(i) The solution φ, defined by $\varphi_i(N, v) = 0$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The solution φ, defined by

$$\varphi_i(N, v) = \begin{cases} \frac{v(\{i\})^2}{\sum_{j\in N} v(\{j\})^2} v(N), & \text{if } \sum_{j\in N} v(\{j\})^2 \neq 0; \\ \frac{1}{n} v(N), & \text{if } \sum_{j\in N} v(\{j\})^2 = 0, \end{cases} \tag{3.2}$$

for all $\langle N, v\rangle \in \mathcal{G}^N$, satisfies all axioms except additivity.

(iii) The Shapley value Sh (Shapley 1953, [22]), defined by

$$Sh_i(N, v) = \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S\backslash\{i\})), \tag{3.3}$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except the nullifying player property.

(iv) The weighted division values that are not positively weighted division values satisfy all axioms except sign symmetry.

The fourth example also shows that, although the positively weighted division values satisfy sign symmetry, the other weighted division values fail sign symmetry. Next we introduce a further relaxation of (sign) symmetry, called weak sign symmetry, that is satisfied by all weighted division values.[5]

- **Weak sign symmetry.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i, j \in N$ are symmetric players in $\langle N, v\rangle$, it holds that $\varphi_i(N, v) > 0$ implies $\varphi_j(N, v) \geq 0$.

Since players i and j are interchangeable, the contraposition of the implication in weak sign symmetry entails that $\varphi_i(N, v) < 0$ implies $\varphi_j(N, v) \leq 0$. Weak sign symmetry relaxes sign symmetry: instead of requiring equal signs, it only rules out opposite signs. Weakening sign symmetry in this way in Theorem 3.1, we obtain a characterization of the class of all weighted division values.

[5]This is different than weak sign symmetry as defined in Casajus (2019, [9]) to characterize the class of weighted Shapley values (Shapley 1953, [21]), which requires that the payoffs of mutually dependent players have the same sign.

Theorem 3.2. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and weak sign symmetry if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

Proof. It is clear that all the weighted division values satisfy efficiency, weak sign symmetry, additivity and the nullifying player property. To show that the axioms determine that the solution is a weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. We will show that for some weight vector $\omega \in \Delta_+^N$, $\varphi = WD^\omega$. By the nullifying player property (or additivity, see Footnote 1), we have $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$, where $\langle N, \mathbf{0} \rangle$ is the null game given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. Let $c \in \mathbb{R}$. Similar as in the proof of Theorem 3.1, for $\emptyset \neq T \subsetneq N$, (i) by the nullifying player property, $\varphi_i(N, ce_T) = 0$ for all $i \in N \backslash T$, and (ii) by efficiency and weak sign symmetry, $\varphi_i(N, ce_T) = 0$ for all $i \in T$. Set $\omega_i = \varphi_i(N, e_N)$ for all $i \in N$. By efficiency and weak sign symmetry, we have $\sum_{i \in N} \omega_i = 1$ and $\omega_i \geq 0$ for all $i \in N$, showing that $\omega \in \Delta_+^N$. Again similar as in the proof of Theorem 3.1, by efficiency, weak sign symmetry and additivity, we can also show that $\varphi(N, ce_N) = c\varphi(N, e_N)$. Finally, by additivity, for all $\langle N, v \rangle \in \mathcal{G}^N$ and $i \in N$, we have $\varphi_i(N, v) = \sum_{T \subseteq N} \varphi_i(N, v(T)e_T) = \omega_i v(N)$. □

Notice that the proof of Theorem 3.2 is almost the same as that of Theorem 3.1, where weak sign symmetry implies the same conclusions as sign symmetry, except that weak sign symmetry does not imply the weights (i.e. payoffs in $\langle N, e_N \rangle$) to be strictly positive.

Logical independence of the axioms in Theorem 3.2 can be shown by the same alternative solutions (i), (ii) and (iii) as those used to show logical independence of the axioms in Theorem 3.1, and replacing alternative solution (iv) by WD^ω with $\omega \in \{\omega \in \mathbb{R}^N | \sum_{i \in N} \omega_i = 1 \text{ and } \omega_i < 0 \text{ for at least one } i \in N\}$.

4. Axiomatizations using null player related axioms

In the preliminaries, we revisited three axiomatizations of the class of weighted division values (see Theorem 2.1, 2.2, 2.3) proposed by Béal et al. (2016, [2]). In this section, we show that the class of weighted division values can also be characterized by replacing linearity in the axiomatizations of Béal et al. (2016, [2]) with additivity. Moreover, we also characterize the class of positively weighted division values by introducing stronger versions of the null player in a productive environment property, the non-negative player property and nullified solidarity.

4.1. Sign null player in a productive environment property

Firstly, we give an alternative axiomatization of the class of weighted division values by replacing linearity with additivity in Theorem 2.1.

Theorem 4.1. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and the null player in a productive environment property if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

Proof. Since linearity implies additivity, by Theorem 2.1, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity, the nullifying player property and the null player in a productive environment property. Also, by Theorem 2.1, it suffices to show that φ is homogeneous, that is, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$. By the nullifying player property this is obviously satisfied for $c = 0$. Notice that

$$\varphi(N, -cv) = \varphi(N, \mathbf{0}) - \varphi(N, cv) = -\varphi(N, cv),$$

where the first equality follows from additivity and the second equality follows from the nullifying player property. Then it suffices to show that $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and positive scalar $c \in \mathbb{R}_{++}$.

Let $c \in \mathbb{R}_{++}$. First, we consider $\langle N, ce_T\rangle$ for $\emptyset \neq T \subsetneq N$. By the nullifying player property, we have $\varphi_i(N, ce_T) = 0$ for all $i \in N\backslash T$. Now we show that $\varphi_i(N, ce_T) = 0$ for all $i \in T$. If $T = \{i\}$, then by the nullifying player property we have $\varphi_j(N, ce_{\{i\}}) = 0$ for all $j \in N \setminus \{i\}$, and consequently by efficiency $\varphi_i(N, ce_{\{i\}}) = 0$. Now, suppose that $T \supsetneq \{i\}$. Set $w_T^i = e_T + e_{T\backslash\{i\}}$.[6] Since player i is a null player in $\langle N, cw_T^i\rangle$ and $cw_T^i(N) = 0$, we have

$$\varphi_i(N, ce_T) = \varphi_i(N, cw_T^i) - \varphi_i(N, ce_{T\backslash\{i\}}) \geq -\varphi_i(N, ce_{T\backslash\{i\}}),$$

where the equality follows from additivity and the inequality from the null player in a productive environment property. By the nullifying player property, $\varphi_i(N, ce_{T\backslash\{i\}}) = 0$, and thus $\varphi_i(N, ce_T) \geq 0$ for all $i \in T$. Since we already showed that $\varphi_i(N, ce_T) = 0$ for all $i \in N\backslash T$, and $ce_T(N) = 0$ for $T \subsetneq N$, efficiency implies that $\varphi_i(N, ce_T) = 0$ for all $i \in T$.

Second, we show that $\varphi(N, ce_N) = c\varphi(N, e_N)$ for constant $c \in \mathbb{R}_{++}$. For all $i \in N$, we have

$$\varphi_i(N, ce_N) = \varphi_i(N, ce_N) + \varphi_i(N, ce_{N\backslash\{i\}}) = \varphi_i(ce_N + ce_{N\backslash\{i\}}) \geq 0, \tag{4.1}$$

[6]These are the same games that are used by Béal et al. (2016, [2]).

where the first equality follows from the nullifying player property, the second equality follows from additivity, and the inequality follows from the null player in a productive environment property. Similar as in the proof of Theorem 3.1, choose two sequences of rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to c from above and below, respectively. By additivity, efficiency and Eq.(4.1), we obtain the same inequalities (3.1) as in the proof of Theorem 3.1, that is, for all $i \in N$ and for all $k = 1, ..., \infty$,

$$\begin{aligned}\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) =& \varphi_i(N, (r_k - c)e_N) \geq 0, \text{ and} \\ \varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) =& \varphi_i(N, (c - s_k)e_N) \geq 0.\end{aligned}$$

Similar as in the proof of Theorem 3.1[7], it follows that $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$, which concludes the proof. □

Logical independence of the axioms used in Theorem 4.1 can be shown by the alternative solutions mentioned in Béal et al. (2016, [2]) to show logical independence of the axioms in Theorem 2.1, since their example that is used to show that linearity is independent of the other axioms also does not satisfy additivity.

In Theorem 4.1, the null player in a productive environment property is used to characterize the class of weighted division values. In a sense, the null player in a productive environment property is not strong enough to generate only positively weighted division values. Next we strengthen the null player in a productive environment property to characterize the class of positively weighted division values.[8]

- **Sign null player in a productive environment property.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i \in N$ is a null player in $\langle N, v\rangle$, it holds that $\text{sign}(\varphi_i(N, v)) = \text{sign}(v(N))$.

It is clear that the sign null player in a productive environment property is stronger than the null player in a productive environment property. Instead of only the non-negativity restrictions, the sign null player in a productive environment property requires that a null

[7]This is shown identical as in the proof of Theorem 3.1 as follows: For all $i \in N$, $\varphi_i(N, r_k e_N) - \varphi_i(N, s_k e_N) = \varphi_i(N, (r_k - s_k)e_N) \to 0$ as $k \to \infty$, since $(r_k - s_k) \to 0$ as $k \to \infty$, and $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$. Then $\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) + \varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) \to 0$ as $k \to \infty$. Since, both $\varphi_i(N, r_k e_N) - \varphi_i(N, ce_N) \geq 0$ and $\varphi_i(N, ce_N) - \varphi_i(N, s_k e_N) \geq 0$, this implies that $\varphi(N, r_k e_N) \to \varphi(N, ce_N)$ and $\varphi(N, r_k e_N) = r_k \varphi(N, e_N) \to c\varphi(N, e_N)$ as $k \to \infty$, which proves that $\varphi(N, ce_N) = c\varphi(N, e_N)$ for constant $c \in \mathbb{R}_{++}$. Hence, by additivity, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$, which concludes the proof.

[8]Notice that the 'sign' axioms in this section strengthen known null player related axioms, while sign symmetry weakened symmetry.

player is rewarded (positive payoff) or punished (negative payoff) depending on whether the worth of the grand coalition is positive or negative. One easily checks that the positively weighted division values satisfy the sign null player in a productive environment property, but the other weighted division values do not. Next we provide a characterization of the class of positively weighted division values by replacing the null player in a productive environment property in Theorem 4.1 with the sign null player in a productive environment property.

Theorem 4.2. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity, the nullifying player property and the sign null player in a productive environment property if and only if there exists a weight vector $\omega \in \Delta^N_{++}$ such that $\varphi = WD^\omega$.*

Proof. It is straightforward to verify that the positively weighted division values satisfy the sign null player in a productive environment property.

To show that the axioms determine that the solution is a positively weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the four mentioned axioms. Since the sign null player in a productive environment property is stronger than the null player in a productive environment property, by Theorem 4.1, φ is a weighted division value WD^ω for some $\omega \in \Delta^N_+$. We are left to prove that $\omega_i > 0$ for all $i \in N$. For all $i \in N$, let $\omega_i = \varphi_i(N, e_N)$. Then, for all $i \in N$, we have

$$\omega_i = \varphi_i(N, e_N) = \varphi_i(N, e_N) + \varphi_i(N, e_{N\setminus\{i\}}) = \varphi_i(N, e_N + e_{N\setminus\{i\}}) > 0,$$

where the second equality follows from the nullifying player property, the third equality follows from additivity, and the inequality follows from the sign null player in a productive environment property. Thus, $\omega \in \Delta^N_{++}$. □

Logical independence of the axioms used in Theorem 4.2 can be shown by the following alternative solutions on $\mathcal{G}^N$.

(i) The solution φ, defined by $\varphi_i(N, v) = v(N)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The solution φ, defined by Eq.(3.2), satisfies all axioms except additivity.

(iii) For $\alpha \in [0, 1]$, the corresponding α-egalitarian Shapley value Sh^α, introduced by Joosten (1996, [16]), is defined by

$$Sh^\alpha_i(N, v) = \alpha Sh_i(N, v) + (1 - \alpha)\frac{v(N)}{n}, \tag{4.2}$$

for all $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$. The α-egalitarian Shapley values with $0 < \alpha < 1$ satisfy all axioms except the nullifying player property.

(iv) The weighted division values that are not positively weighted division values satisfy all axioms except the sign null player in a productive environment property.

4.2. Non-negative player property

Similar as in the previous subsection, we first give an alternative axiomatization of the class of weighted division values by replacing linearity in Theorem 2.2 with additivity, and after that strengthen one of the axioms (related to null players) to obtain the class of positively weighted division values.

Theorem 4.3. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and the non-negative player property if and only if there exists a weight vector $\omega \in \Delta_+^N$ such that $\varphi = WD^\omega$.*

Sketch of proof. Since the proof follows similar lines as previous proofs, here we only sketch the proof.[9] By Theorem 2.2, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity and the non-negative player property. Similar as before, by Theorem 2.2, it suffices to show that φ is homogeneous, that is, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$. This can be shown in a similar way as in the proof of Theorem 4.1, replacing the role of the null player in a productive environment property by the nonnegative player property. □

Logical independence of the axioms can be shown using the same alternative solutions used to show logical independence of the axioms in Theorem 4 (that is Theorem 2.2 in this paper) in Béal et al. (2016, [2]), since their example that does not satisfy linearity, also does not satisfy additivity.

The non-negative player property is not strong enough to generate only positively weighted division values. To characterize the class of positively weighted division values, we strengthen the non-negative player property requiring the payoff of a player to be positive (respectively zero) if the worth of the grand coalition is positive (respectively zero).

- **Sign non-negative player property.** For all $\langle N, v\rangle \in \mathcal{G}^N$, whenever $i \in N$ is a non-negative player in $\langle N, v\rangle$, it holds that $\text{sign}(\varphi_i(N, v)) = \text{sign}(v(N))$.

One easily checks that the positively weighted division values satisfy the sign non-negative player property. Next we provide a characterization of the class of positively weighted division values by using the sign non-negative player property.

[9] For completeness, we give the full proof in the appendix.

Theorem 4.4. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and the sign non-negative player property if and only if there exists a weight vector $\omega \in \Delta^N_{++}$ such that $\varphi = WD^\omega$.*

Proof. It is straightforward to verify that the positively weighted division values satisfy the sign non-negative player property.

To show that the axioms determine that the solution is a positively weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the three mentioned axioms. Since the sign non-negative player property is stronger than the non-negative player property, by Theorem 4.3, φ is a weighted division value WD^ω for some $\omega \in \Delta^N_+$ and, by its proof, $\omega_i = \varphi_i(N, e_N)$. We are left to prove that $\omega_i > 0$ for all $i \in N$. By the sign non-negative player property, we have $\omega_i = \varphi_i(N, e_N) > 0$ for all $i \in N$, showing that $\omega \in \Delta^N_{++}$. □

Logical independence of the axioms used in Theorem 4.4 can be shown by the same alternative solutions (i), (ii) and (iii) (or (iv)) showing logical independence of the axioms in Theorem 4.2, where the first two examples also satisfy the stronger sign non-negative player property.

4.3. Nullified solidarity

Finally, we replace linearity in Theorem 2.3 with additivity to give an alternative axiomatization of the class of weighted division values.

Theorem 4.5. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and nullified solidarity if and only if there exists a weight vector $\omega \in \Delta^N_+$ such that $\varphi = WD^\omega$.*

Proof. By Theorem 2.3, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity and nullified solidarity. Similar as before, by Theorem 2.3, it suffices to show that φ is homogeneous, that is, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$. By additivity, this is satisfied for $c = 0$, since $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$ (see Footnote 1). Also by additivity, we have

$$\varphi(N, -cv) = \varphi(N, \mathbf{0}) - \varphi(N, cv) = -\varphi(N, cv). \tag{4.3}$$

Then it suffices to show that $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$.

Let $c \in \mathbb{R}_{++}$. For all $S \subseteq N$ and $i \in S$, $(ce_S)^{\mathbf{N}i} = \mathbf{0}$, where $\langle N, (ce_S)^{\mathbf{N}i}\rangle$ is the TU-game where player i is nullified as defined by Eq.(2.3). Then, we have $\varphi_j(N, (ce_S)^{\mathbf{N}i}) = \varphi_j(N, \mathbf{0}) = 0$ for all $j \in N$. Now we show that $\varphi_i(N, ce_S) \geq 0$ for all $S \subseteq N$ and $i \in S$. On the contrary, suppose that there are $S \subseteq N$ and $i \in S$ such that $\varphi_i(N, ce_S) < 0$. By Eq.(4.3), we then have $\varphi_i(N, -ce_S) > 0$, and thus by $\varphi_i(N, (ce_S)^{\mathbf{N}i}) = \varphi_i(N, \mathbf{0}) = 0$, we have $\varphi_i(N, -ce_S) > \varphi_i(N, (-ce_S)^{\mathbf{N}i})$. Then, by nullified solidarity, $\varphi_j(N, -ce_S) \geq \varphi_j(N, (-ce_S)^{\mathbf{N}i}) = 0$ for all $j \in N$. Thus, we obtain $\sum_{j \in N} \varphi_j(N, -ce_S) > 0$, which is in contradiction with the fact that φ satisfies efficiency. So, we conclude that, $\varphi_i(N, ce_S) \geq 0$ for all $S \subseteq N$ and $i \in S$. Therefore, by nullified solidarity, $\varphi_i(N, ce_S) \geq 0 = \varphi_i(N, (ce_S)^{\mathbf{N}i})$ implies that $\varphi_j(N, ce_S) \geq \varphi_j(N, (ce_S)^{\mathbf{N}i}) = 0$ for all $S \subseteq N$, $i \in S$ and $j \in N$. That is, $\varphi_j(N, ce_S) \geq 0$ for all $j \in N$. Similar as in the proof of Theorem 4.3, choosing two sequences of rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to c from above and below, respectively, we can prove that $\varphi(N, ce_S) = c\varphi(N, e_S)$ for all $S \subseteq N$ and constant $c \in \mathbb{R}_{++}$. Hence, by additivity, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v \rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_{++}$, which concludes the proof. □

Logical independence of the axioms used in Theorem 4.5 can be shown by the same alternative solutions as used to show logical independence of the axioms in Theorem 2.3, in Béal et al. (2016, [2]).

Finally, we strengthen nullified solidarity to characterize the class of positively weighted division values in a similar way as the sign non-negative player property is stronger than the non-negative player property.

- **Sign nullified solidarity.** For all $\langle N, v \rangle \in \mathcal{G}^N$ and $i, j \in N$, it holds that $\text{sign}(\varphi_i(N, v) - \varphi_i(N, v^{\mathbf{N}i})) = \text{sign}(\varphi_j(N, v) - \varphi_j(N, v^{\mathbf{N}i}))$.

Next we provide a characterization of the class of positively weighted division values by using sign nullified solidarity.

Theorem 4.6. *A solution φ on $\mathcal{G}^N$ satisfies efficiency, additivity and sign nullified solidarity if and only if there exists a weight vector $\omega \in \Delta_{++}^N$ such that $\varphi = WD^{\omega}$.*

Proof. It is straightforward to verify that the positively weighted division values satisfy sign nullified solidarity.

To prove that the axioms determine that the solution is a positively weighted division value, let φ be a solution on $\mathcal{G}^N$ satisfying the three mentioned axioms. Since sign nullified solidarity is stronger than nullified solidarity, by Theorem 2.3, φ is a weighted division value WD^{ω} for some $\omega \in \Delta_{+}^N$. We are left to prove that $\omega_i > 0$ for all $i \in N$. Since $\varphi_i(N, (ce_N)^{\mathbf{N}i}) = \varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$, by sign nullified solidarity, we have $\text{sign}(\varphi_i(N, e_N)) = \text{sign}(\varphi_j(N, e_N))$ for all $i, j \in N$. Thus, by efficiency, we have $\omega_i = \varphi_i(N, e_N) > 0$ for all $i \in N$, showing that $\omega \in \Delta_{++}^N$. □

Logical independence of the axioms used in Theorem 4.6 can be shown by the following alternative solutions on $\mathcal{G}^N$.

(i) The solution φ, defined by $\varphi_i(N,v) = v(N)$ for all $\langle N,v\rangle \in \mathcal{G}^N$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The solution φ, defined by $\varphi_i(N,v) = ED_i(N,v) + t_i$ for all $\langle N,v\rangle \in \mathcal{G}^N$ and $i \in N$, where $t \in \mathbb{R}^N$ is such that $\sum_{i\in N} t_i = 0$ and $t_i \neq 0$ for some $i \in N$, satisfies all axioms except additivity.

(iii) The α-egalitarian Shapley value Sh^α, defined by Eq.(4.2), satisfies all axioms except sign nullified solidarity.

5. Summary

In this paper, we focus on studying axiomatic characterizations of the classes of weighted division values and positively weighted division values. We have shown that relaxing symmetry in van den Brink's characterization (van den Brink 2007, [24]) for the equal division value, by replacing it with sign symmetry of Casajus (2019, [9]), gives a characterization of the class of positively weighted division values (Theorem 3.1). Moreover, a weaker version of sign symmetry allows to characterize the class of all weighted division values (Theorem 3.2).

Somewhat surprising, whereas relaxing symmetry by sign symmetry in the traditional axiomatization of the Shapley value still gives the same Shapley value in Casajus (2019, [9]), applying this relaxation in the axiomatization of the equal division value results in a class of solutions, specifically the weighted division values. Li et al. (2021, [19]) have shown that Casajus (2019, [9]) result can be generalized to a subfamily of efficient, symmetric and linear values (for short, ESL values), in the sense that relaxing symmetry into sign symmetry in a specific axiomatization of such an ESL value still characterizes that ESL value.

Casajus (2018, [7]) replaces symmetry by sign symmetry in Young's axiomatization (Young 1985, [27]) of the Shapley value. In van den Brink (2007, [24]) another characterization of the equal division value is proposed by using efficiency, symmetry and coalitional monotonicity. In view of the former results, the question naturally arises whether the classes of weighted division values can be characterized by efficiency, coalitional monotonicity and sign symmetry or weak sign symmetry.

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Appendix

Proof of Theorem 4.3. By Theorem 2.2, we only need to prove that the axioms determine that the solution is a weighted division value. Therefore, let φ be a solution on $\mathcal{G}^N$ satisfying efficiency, additivity and the non-negative player property. Similar as before, by Theorem 2.2, it suffices to show that φ is homogeneous, that is, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}$. Similar as in previous proofs, choose two sequences of rationals $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ which converge to c from above and below, respectively. By additivity and the non-negative player property, we obtain that, for all $S \subseteq N$, $i \in N$ and for all $k = 1, ..., \infty$,

$$\begin{aligned}\varphi_i(N, r_k e_S) - \varphi_i(N, ce_S) =& \varphi_i(N, (r_k - c)e_S) \geq 0, \text{ and}\\ \varphi_i(N, ce_S) - \varphi_i(N, s_k e_S) =& \varphi_i(N, (c - s_k)e_S) \geq 0.\end{aligned}$$

Notice that $\varphi_i(N, r_k e_S) - \varphi_i(N, s_k e_S) = \varphi_i(N, (r_k - s_k)e_S) \to 0$ as $k \to \infty$, since $(r_k - s_k) \to 0$ as $k \to \infty$, and $\varphi_i(N, \mathbf{0}) = 0$ for all $i \in N$ by efficiency and the nonegative player property. Then $\varphi_i(N, r_k e_S) - \varphi_i(N, ce_S) + \varphi_i(N, ce_S) - \varphi_i(N, s_k e_S) \to 0$ as $k \to \infty$. Since, both $\varphi_i(N, r_k e_S) - \varphi_i(N, ce_S) \geq 0$ and $\varphi_i(N, ce_S) - \varphi_i(N, s_k e_S) \geq 0$, this implies that $\varphi(N, r_k e_S) \to \varphi(N, ce_S)$ and $\varphi(N, r_k e_S) = r_k\varphi(N, e_S) \to c\varphi(N, e_S)$ as $k \to \infty$, which proves that $\varphi(N, ce_S) = c\varphi(N, e_S)$ for constant $c \in \mathbb{R}_+$. Hence, similar as in previous proofs by additivity, $\varphi(N, cv) = c\varphi(N, v)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ and scalar $c \in \mathbb{R}_+$ follows similar as in previous proofs, which concludes the proof. □

References

[1] Béal S, Casajus A, Huettner F, Rémila E, Solal P. Solidarity within a fixed community. Economics Letters, 2014, 125(3): 440-443.

[2] Béal S, Casajus A, Huettner F, Rémila E, Solal P. Characterizations of weighted and equal division values. Theory and Decision, 2016, 80(4): 649-667.

[3] Béal S, Rémila E, Solal P. Preserving or removing special players: What keeps your payoff unchanged in TU-games?. Mathematical Social Sciences, 2015, 73: 23-31.

[4] Besner M. Parallel axiomatizations of weighted and multiweighted Shapley values, random order values, and the Harsanyi set. Social Choice and Welfare, 2020, 55(1): 193-212.

[5] Calvo E, Santos J C. Weighted weak semivalues. International Journal of Game Theory, 2000, 29(1): 1-9.

[6] Casajus A. Differential marginality, van den Brink fairness, and the Shapley value. Theory and Decision, 2011, 71(2): 163-174.

[7] Casajus A. Symmetry, mutual dependence, and the weighted Shapley values. Journal of Economic Theory, 2018, 178: 105-123.

[8] Casajus A. Sign symmetry vs symmetry: Young's characterization of the Shapley value revisited. Economics Letters, 2018, 169: 59-62.

[9] Casajus A. Relaxations of symmetry and the weighted Shapley values. Economics Letters, 2019, 176: 75-78.

[10] Casajus A. Weakly balanced contributions and the weighted Shapley values. Journal of Mathematical Economics, 2021, 94: 102459.

[11] Casajus A, Huettner F. Null players, solidarity, and the egalitarian Shapley values. Journal of Mathematical Economics, 2013, 49(1): 58-61.

[12] Casajus A, Yokote K. Weak differential marginality and the Shapley value. Journal of economic Theory, 2017, 167: 274-284.

[13] Chun Y. On the symmetric and weighted Shapley values. International Journal of Game Theory, 1991, 20(2): 183-190.

[14] Deegan J, Packel E W. A new index of power for simplen-person games. International Journal of Game Theory, 1978, 7(2): 113-123.

[15] Hart S, Mas-Colell A. Potential, value, and consistency. Econometrica, 1989, 57(3): 589-614.

[16] Joosten R. Dynamics, equilibria and values, dissertation. Maastricht University, The Netherlands, 1996.

[17] Kalai E, Samet D. On weighted Shapley values. International journal of game theory, 1987, 16(3): 205-222.

[18] Kongo T. Players' nullification and the weighted (surplus) division values. Economics Letters, 2019, 183: 108539.

[19] Li W, Xu G, van den Brink R. A note on sign symmetry for the efficient, symmetric and linear values. Discussion paper, 2021.

[20] Nowak A S, Radzik T. On axiomatizations of the weighted Shapley values. Games and Economic Behavior, 1995, 8(2): 389-405.

[21] Shapley L S. Additive and non-additive set functions (Ph.D. thesis), Department of Mathematics, Princeto University, 1953.

[22] Shapley L S. A value for n-person games, in: H.W. Kuhn, A.W. Tucker (Eds.). Contributions to the Theory of Games II, Princeton University Press, 1953, pp. 307-317.

[23] van den Brink R. An axiomatization of the Shapley value using a fairness property. International Journal of Game Theory, 2002, 30(3): 309-319.

[24] van den Brink R. Null or nullifying players: the difference between the Shapley value and equal division solutions. Journal of Economic Theory, 2007, 136(1): 767-775.

[25] van den Brink R. Efficiency and collusion neutrality of solutions for cooperative TU-games. Tinbergen Institute Discussion Paper 09-065/1, 2009. (Available at SSRN: https://ssrn.com/abstract=1437521.)

[26] Yokote K. Weak addition invariance and axiomatization of the weighted Shapley value. International Journal of Game Theory, 2015, 44(2): 275-293.

[27] Young H P. Monotonic solutions of cooperative games. International Journal of Game Theory, 1985, 14(2): 65-72.