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The prekernel of cooperative games with α-excess

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Abstract

In this paper, we introduce a new approach to measure the dissatisfaction for coalitions of players in cooperative transferable utility games. This is done by considering affine (and convex) combinations of the classical excess and the proportional excess. Based on this so-called α-excess, we define new solution concepts for cooperative games, such as the α-prenucleolus and the α-prekernel. The classical prenucleolus and prekernel are a special case. We characterize the α-prekernel by strong stability and the α-balanced surplus property. Also, we show that the payoff vector generated by the α-prenucleolus belongs to the α-prekernel.

Keywords: α-excess; α-prenucleolus; α-prekernel; α-balanced surplus

1. Introduction

The *excess* of a coalition at a given payoff vector in transferable utility (TU) games represents the gain or loss of the coalition if its members withdraw from the game in order to form their own coalition. Usually, the excess of a coalition can be viewed as the dissatisfaction of the coalition at the proposed payoff vector. The classical excess is defined by the difference between the worth of a coalition and the payoff assigned to the coalition members. The most popular solutions such as the core [2], the Shapley value [12], the nucleolus [11], the prenucleolus [15] and the (pre)kernel [6, 5] can be characterized on the basis of this classical excess. Especially, Peleg [9] provided an axiomatization of the prekernel, which avoids any reference to interpersonal comparison of utilities. He verified that there is a unique solution on the set of all TU-games that satisfies *nonemptiness, Pareto optimality, covariance under strategic equivalence, the equal treatment property, a reduced game property*, and *the converse reduced game property.* In view of the stability of a preimputation, which means that no player has incentives to move from the preimputation, Calvo and Gutiérrez [1] first defined the *strong stability property.* The least core of a TU-game is characterized using this property. They also proposed the *balanced surplus*

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property similar to the balanced contributions property of Myerson [7]. By means of these two properties, they gave a new characterization of the prekernel of a TU-game.

Thereafter, Lemarire [4] presented the *relative excess* to measure the dissatisfaction of any coalition as the quotient of the usual excess and the coalitional value, and defined the proportional nucleolus. In addition, Yanovskaya [16] defined proportional solutions for the class of positive TU-games with all nonempty coalitional values strictly positive, depending only on the *proportional excess*, which is defined as the quotient of the coalitional value and the coalitional payoff. Actually, the relative excess is ordinally equivalent to the proportional excess. Successively, the proportional prenucleolus and nucleolus were characterized by Naumova [8].

Which definition of excess is most appropriate, depends on the application one has in mind. To avoid ignoring some player's benefit for the general case, our aim in this paper is to define a more general excess (called *α-excess*) by considering affine combinations of the classical excess and the proportional excess for the class of positive TU-games. Based on this α-excess, we modify solutions like the core, ε-core, least core, (pre)nucleolus and prekernel for positive TU-games. In this way, corresponding α-solutions for positive TU-games are obtained. First, we show that the core and the α-core coincide for positive TU-games. However, we will see that this is not the case for the modifications of the prekernel, the least core, and the prenucleolus. Second, we prove that the α-prenucleolus is always contained in the α-prekernel. Third, we characterize the α-prekernel by strong stability and an α-balanced surplus property.

The rest of this paper is organized as follows. In Section 2, we recall some related preliminaries about cooperative game theory. Section 3 introduces the α-excess of a coalition, defines modifications of solutions using this modified excess, and characterizes the α-least core and α-prekernel by strong stability and α-balanced surplus properties. In Section 4, we define the α-prenucleolus and α-nucleolus. Also, we verify that the α-prenucleolus is contained in the α-prekernel. Section 5 concludes with a brief summary.

2. Preliminaries

A *cooperative game with transferable utility* (TU-game) is a pair (N, v) consisting of a finite set $N = \{1, 2, \cdots, n\}$ of n players, and a *characteristic function* $v : 2^N \to \mathbb{R}$, where 2^N denotes the family of all subsets or *coalitions* of N, such that $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ represents the *worth* that coalition S achieves when its members cooperate. The number of players in any coalition $S \subseteq N$ is denoted by s and the set of all TU-games with player set N is denoted by G^N. A vector $x \in \mathbb{R}^n$ will be called a *payoff vector*, and we denote $x(S) = \sum_{i \in S} x_i$ for any coalition S. Since the set of players is fixed, we often shortly write v instead of (N, v). For a game v, we say that a payoff vector $x \in \mathbb{R}^n$ is

- *efficient* if $x(N) = v(N)$;
- *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$;

- *coalitionally rational* if $x(S) \geq v(S)$ for all $S \subseteq N$.

A solution is a function φ that assigns to any game $v \in G^N$ a set of n-dimensional payoff vectors. A solution φ is *s*ingle-valued if $\varphi(N, v)$ consists of only one payoff vector for every game (N, v). In that case, we usually write it as a function $\varphi : G^N \to \mathbb{R}^n$ with $\varphi(N, v) \in \mathbb{R}^n$ being the unique payoff vector assigned to the game. Efficient payoff vectors are also called preimputations. The *preimputation set* of a game $v \in G^N$ is given by

$$\mathscr{I}^*(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N)\},$$

and consists of all efficient payoff vectors. The *imputation set* of a game $v \in G^N$ is given by

$$\mathscr{I}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \ \forall i \in N\},$$

and consists of all efficient and individually rational payoff vectors. The *core* of a game $v \in G^N$ is given by

$$\mathscr{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \ \forall S \subseteq N\},$$

and consists of all efficient and coalitionally stable payoff vectors. For any payoff vector $x \in \mathbb{R}^n$ and any nonempty coalition S, the *excess* of S at x is

$$e(S, x) = v(S) - x(S). \tag{1}$$

The excess $e(S, x)$ can be viewed as the gain (or loss, if it is negative) experienced by coalition S if its members depart from an agreement that yields x as payoff vector, and form their own coalition. The core of a game $v \in G^N$ can be written as

$$\mathscr{C}(N, v) = \{x \in \mathscr{I}^*(N, v) \mid \ e(S, x) \leq 0 \ \forall S \subseteq N\}.$$

The core is stable in the sense that each of its elements cannot be blocked by any coalition.

For any $\varepsilon \in \mathbb{R}$, Shapley and Shubik [13, 14] introduced the *strong ε-core* of a game $v \in G^N$ given by

$$\mathscr{C}_\varepsilon(N, v) = \{x \in \mathscr{I}^*(N, v) \mid e(S, x) - \varepsilon \leq 0 \ \forall S \in 2^N \setminus \{\emptyset\}\},$$

and thus allows coalitions to get 'a bit' less than their worth. Using this, the *least core* of a game $v \in G^N$ is defined as $\mathscr{C}_\lambda(N, v)$, where $\lambda = \min\{\varepsilon \in \mathbb{R} \mid \mathscr{C}_\varepsilon(N, v) \neq \emptyset\}$. We denote the least core of game v by $\mathscr{LC}(N, v)$.

Another well-known solution is the prekernel [6], which tries to balance the payoffs of players in a pairwise comparison. We denote by $\Gamma_{ij}(N)$ the set of all coalitions containing player i but not player j, that is, $\Gamma_{ij}(N) = \{S \subseteq N \mid i \in S, j \notin S\}$. If there is no confusion about the player set, we will shortly write Γ_{ij} instead of $\Gamma_{ij}(N)$. To formally define the prekernel of a game, we first need to calculate the maximal surplus of player i over another player j at $x \in \mathbb{R}^n$ in the game $v \in G^N$:

$$s^v_{ij}(x) = \max_{S \in \Gamma_{ij}} e(S, x) \tag{2}$$

is the maximal surplus (in terms of excess) that player i can obtain in a coalition without player j. The prekernel $\mathscr{PK}(N,v)$ of a game $v \in G^N$ balances, within the preimputation set, the surpluses by equalizing for every pair of players the maximal surplus of one player over the other. Formally,

$$\mathscr{PK}(N,v) = \{x \in \mathscr{I}^*(N,v) \mid s^v_{ij}(x) = s^v_{ji}(x) \text{ for all } i,j \in N, i \neq j\}. \tag{3}$$

Before defining the prenucleolus and nucleolus of a game, we need to introduce some concepts. Consider the 2^n-dimensional vector $\theta(x)$ whose components, correspond to x and are arranged in nonincreasing order, that is, $\theta_k(x) \geq \theta_l(x)$, $1 \leq k \leq l \leq 2^n$. The *lexicographic order* $\leq_L$ on $\mathbb{R}^{2^n}$ is used to compare $\theta(x)$ and $\theta(y)$ by taking into account their largest components or, if such should be the case, their second largest components and so on. More precisely, for any $x, y \in \mathbb{R}^n$,

(i) $\theta(x) <_L \theta(y)$ if there exists an integer $1 \leq l \leq 2^n$ such that $\theta_k(x) = \theta_l(y)$ for $1 \leq k < l$, and $\theta_l(x) < \theta_l(y)$.
(ii) $\theta(x) \leq_L \theta(y)$ if either $\theta(x) = \theta(y)$ or $\theta(x) <_L \theta(y)$.

Finally, for any game $v \in G^N$, the prenucleolus $\mathscr{PN}(N,v)$, respectively the nucleolus $\mathscr{N}(N,v)$, minimizes the excess $e(S,x)$ of any coalition over the preimputation set, respectively the imputation set. Formally,

$$\mathscr{PN}(N,v) = \{x \in \mathscr{I}^*(N,v) | \theta(e(S,x)_{S\subseteq N}) \leq_L \theta(e(S,y)_{S\subseteq N}) \ \forall y \in \mathscr{I}^*(N,v)\}.$$

and

$$\mathscr{N}(N,v) = \{x \in \mathscr{I}(N,v) | \theta(e(S,x)_{S\subseteq N}) \leq_L \theta(e(S,y)_{S\subseteq N}) \ \forall y \in \mathscr{I}(N,v)\},$$

where $e(S,x)_{S\subseteq N}$ denotes the $2^n - 1$-dimensional vector which components are the excesses of the coalitions at payoff vector x. In the remaining, we consider the class G^{N_+} of positive TU-games where all worths of nonempty coalitions are positive:

$$G^{N_+} = \{(N,v) \mid v(S) > 0 \ \forall S \subseteq N, S \neq \emptyset\}.$$

Positive versions of solutions are defined in such a way that they only consider positive payoff vectors. Specifically, the positive *preimputation set* of a game $v \in G^N$ is given by

$$\mathscr{I}^*_{++}(N,v) = \{x \in \mathbb{R}^n_{++} \mid x(N) = v(N)\}.$$

The positive *imputation set* of a game $v \in G^N$ is given by

$$\mathscr{I}_{++}(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) \mid x_i \geq v(\{i\}) \ \forall i \in N\}.$$

The positive *core* of a game $v \in G^N$ is given by

$$\mathscr{C}_{++}(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) \mid x(S) \geq v(S) \ \forall S \subseteq N\}.$$

Yanovskaya [16] considers a proportional excess function, where the dissatisfaction of a coalition is measured as the ratio between the worth of a coalition and the assigned payoff. Formally, for $v \in G^{N_+}$, $x \in \mathbb{R}^n_{++}$ and $S \subseteq N$, the *proportional excess* of S at x is

$$\overline{e}(S, x) = \frac{v(S)}{x(S)}.$$

Whereas the classical excess (see (1)) of a coalition that exactly gets its worth is equal to 0, for such a coalition the proportional excess is equal to 1.

3. The α-prekernel in TU-games

We begin with an example that illustrates the difference between the classical and proportional excesses introduced before.

Example 1. Let $N = \{1, 2, 3\}$ be three companies. Assume that these three companies lost money in cooperation. Let v be defined by $v(\{1, 2, 3\}) = 1194$, $v(\{1\}) = 10$, $v(\{2\}) = v(\{3\}) = 1000$, and $v(S) = \rho$ otherwise, ρ being a sufficiently small positive number. For the given payoff vector $(2, 992, 200)$, it holds that $e(\{1\}, x) = e(\{2\}, x) = 8$, $e(\{3\}, x) = 800$, however, $\overline{e}(\{1\}, x) = \overline{e}(\{3\}, x) = 5$, $\overline{e}(\{2\}, x) = \frac{125}{124}$.

Considering Example 1, now comes the question, which excess is better to measure the dissatisfaction of the companies at the payoff vector $(2, 992, 200)$? The rich company can "tolerate" a moderate or small loss more than the poor company. However, it does not "tolerate" a very large loss either. From our perspective, it is not obvious that one should consider either the classical excess or the proportional excess. In such cases, an affine (or convex) combination of these two excesses might be more reasonable. Consequently, also variations of solutions, such as the prekernel and the prenucleolus, based on an affine or convex combination of these two excesses, might be reasonable solution concepts.

Definition 1. Given $\alpha \in \mathbb{R}$, a game $v \in G^{N_+}$, a positive payoff vector $x \in \mathbb{R}^n_{++}$ and a coalition $S \subseteq N$, $S \neq \emptyset$, the α-excess of coalition S with respect to x is given by

$$e^\alpha_v(S, x) = \alpha \frac{v(S)}{x(S)} + (1 - \alpha)(v(S) - x(S)). \tag{4}$$

For $S = \emptyset$, we define $e^\alpha_v(\emptyset, x) = 0$ for all $v \in G^{N_+}$ and $x \in \mathbb{R}^n_{++}$.

If there is no confusion about the game $v \in G^{N_+}$, we will shortly write $e^\alpha(S, x)$ instead of $e^\alpha_v(S, x)$. Specifically, when $\alpha \in [0, 1]$, we speak about a convex combination of the classical and proportional excess. Observe that we obtain the classical excess as a special case of α-excess by taking $\alpha = 0$, and the proportional excess as special case when taking $\alpha = 1$. Similar as the classical and proportional excess, the α-excess represents the gain (or loss, if it is less than 1) to the coalition S if its members depart from an agreement that yields x in order to form their own coalition, but allow a trade-off between the classical and proportional excess.

In view of the concept of α-excess, the definition of the core of a positive game $v \in G^{N_+}$ could be modified by considering those imputations which α-excess is at most equal to α, i.e. one could consider

$$\mathscr{C}^{\alpha}(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) \mid e^{\alpha}(S,x) \leq \alpha,\ \forall S \subseteq N\}.$$

However, it turns out that for any $\alpha \in [0,1]$ this coincides with the classical core as long as we consider only positive payoff vectors.

Proposition 1. *For every $\alpha \in [0,1]$ and $v \in G^{N_+}$, we have $\mathscr{C}^{\alpha}(N,v) = \mathscr{C}(N,v)$.*

Proof. Notice that for $v \in G^{N_+}$, $\mathscr{C}(N,v) = \mathscr{C}_{++}(N,v)$ since $v(\{i\}) > 0$ $\forall i \in N$. For every $\alpha \in [0,1]$, $v \in G^{N_+}$, and $x \in \mathbb{R}^n_{++}$, we have

$$\begin{aligned}
e^{\alpha}(S,x) \leq \alpha \quad & \Leftrightarrow \quad \alpha\frac{v(S)}{x(S)} + (1-\alpha)(v(S)-x(S)) \leq \alpha \\
& \Leftrightarrow \quad \frac{\alpha v(S) + (1-\alpha)(v(S)-x(S))x(S)}{x(S)} \leq \alpha \\
& \Leftrightarrow \quad \alpha v(S) + (1-\alpha)(v(S)-x(S))x(S) \leq \alpha x(S) \\
& \Leftrightarrow \quad \alpha(v(S)-x(S)) + (1-\alpha)(v(S)-x(S))x(S) \leq 0 \\
& \Leftrightarrow \quad (v(S)-x(S))(\alpha + (1-\alpha)x(S)) \leq 0 \\
& \Leftrightarrow \quad v(S) - x(S) \leq 0 \Leftrightarrow e(S,x) \leq 0
\end{aligned}$$

where the last but one equivalence follows since $\alpha + (1-\alpha)x(S) > 0$ for all $x \in \mathbb{R}^n_{++}$. $\square$

From this proposition, we can conclude that considering different α-excesses from our class to measure the dissatisfaction of coalitions, has no effect on the definition of the core. However, we will see that it does affect the definition of the prekernel, the least core, and the prenucleolus.

First, modifying the ε-core of a game $v \in G^{N_+}$, we obtain the *$\alpha\varepsilon$-core* given by

$$\mathscr{C}^{\alpha}_{\varepsilon}(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) \mid e^{\alpha}(S,x) - \varepsilon \leq \alpha,\ \forall S \subseteq N, S \neq \emptyset\},$$

and the *α-least core* of a game $v \in G^{N_+}$ being $\mathscr{C}^{\alpha}_{\lambda}(N,v)$, where $\lambda = \lambda^{v,\alpha} = \min\{\varepsilon \in \mathbb{R} \mid \mathscr{C}^{\alpha}_{\varepsilon}(N,v) \neq \emptyset\}$. We denote the α-least core of game v by $\mathscr{LC}^{\alpha}(N,v)$. If there does not exist a minimal $\varepsilon \in \mathbb{R}$ such that $\mathscr{C}^{\alpha}_{\varepsilon}(N,v) \neq \emptyset$, then $\mathscr{LC}^{\alpha}(N,v) = \emptyset$.

Example 2. Consider the 3-person game v defined as $v(\{1,2,3\}) = 1, v(\{1,2\}) = v(\{1,3\}) = \frac{1}{2}$, and $v(S) = \rho$ otherwise, ρ being a sufficiently small positive number. The symbol $\underline{\text{conv}}$ indicates the convex hull excluding the boundary points with a component of 0. We find out that

$$\begin{aligned}
\mathscr{C}_{++}(N,v) \quad & = \quad \left\{x \in \mathbb{R}^3_{++} \mid x_1 + x_2 + x_3 = 1, x_2 \leq \frac{1}{2}, x_3 \leq \frac{1}{2}\right\} \\
& = \quad \underline{\text{conv}}\left\{(1,0,0), (\frac{1}{2},\frac{1}{2},0), (\frac{1}{2},0,\frac{1}{2}), (0,\frac{1}{2},\frac{1}{2})\right\},
\end{aligned}$$

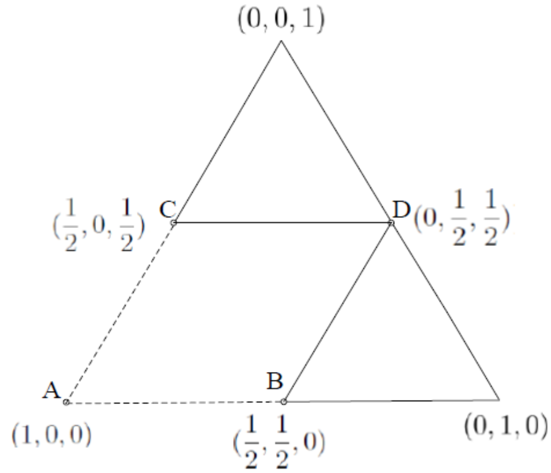

Fig. 1. The positive core $\mathscr{C}_+(N, v)$ of Example 2

see the area ABCD in Fig.1. We show that the $\alpha\varepsilon$-cores are different for the classical and proportional excess. For $\alpha = 0$, we have

$$\begin{aligned} \mathscr{C}^0_\varepsilon(N, v) &= \Big\{ x \in \mathbb{R}^3_{++} \mid x_1 + x_2 + x_3 = 1, x_1 \geq -\varepsilon, -\varepsilon \leq x_2 \leq \frac{1}{2} + \varepsilon \,, \\ &\quad -\varepsilon \leq x_3 \leq \frac{1}{2} + \varepsilon \Big\}. \end{aligned}$$

Then, for $\varepsilon \geq 0$, we have

$$\begin{aligned} \mathscr{C}^0_\varepsilon(N, v) &= \underline{\text{conv}} \Big\{ (1, 0, 0), (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon, 0), (\frac{1}{2} - \varepsilon, 0, \frac{1}{2} + \varepsilon), (0, \frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon) \,, \\ &\quad (0, \frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon) \Big\}, \end{aligned}$$

see the area ABCDE in Fig.2. If $\varepsilon < 0$,

$$\begin{aligned} \mathscr{C}^0_\varepsilon(N, v) &= \underline{\text{conv}} \Big\{ (1 + 2\varepsilon, -\varepsilon, -\varepsilon), (\frac{1}{2}, \frac{1}{2} + \varepsilon, -\varepsilon), (\frac{1}{2}, -\varepsilon, \frac{1}{2} + \varepsilon) \,, \\ &\quad (-2\varepsilon, \frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon) \Big\}, \end{aligned}$$

see the area ABCD in Fig.3. Moreover, $\mathscr{C}^0_\varepsilon(N, v) \neq \emptyset$ iff $\varepsilon \geq -\frac{1}{4}$. Thus, $\mathscr{L}\mathscr{C}^0(N, v) = \{(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\}$.

Next, consider the case that $\alpha = 1$.

$$\mathscr{C}^1_\varepsilon(N, v) = \Big\{ x \in \mathbb{R}^3_{++} \mid x_1 + x_2 + x_3 = 1, x_1 \geq \frac{-\varepsilon}{\varepsilon + 1}, x_2 \leq \frac{2\varepsilon + 1}{2(\varepsilon + 1)}, x_3 \leq \frac{2\varepsilon + 1}{2(\varepsilon + 1)} \Big\}.$$

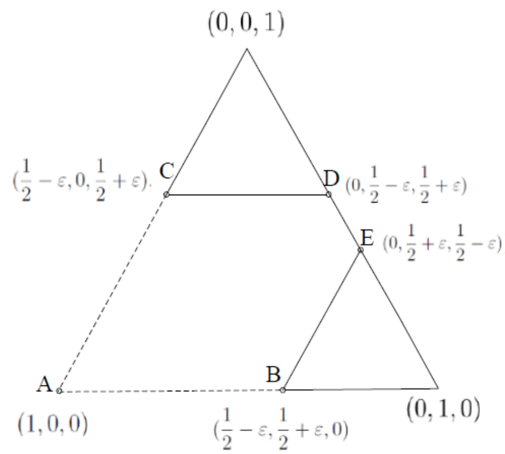

Fig. 2. The 0ε-core, $\mathscr{C}^0_\varepsilon(N,v)$, when $\varepsilon \geq 0$

If $\varepsilon \geq 0$,

$$\begin{aligned}\mathscr{C}^1_\varepsilon(N,v) \quad &= \quad \underline{\text{conv}}\left\{(1,0,0),(\frac{1}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)},0),(\frac{1}{2(\varepsilon+1)},0,\frac{2\varepsilon+1}{2(\varepsilon+1)})\right.,\\ &\left.(0,\frac{1}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)}),(0,\frac{2\varepsilon+1}{2(\varepsilon+1)},\frac{1}{2(\varepsilon+1)})\right\}.\end{aligned}$$

Now, the shape of $\mathscr{C}^1_\varepsilon(N,v)$ is the same as that of $\mathscr{C}^0_\varepsilon(N,v)$ when $\varepsilon \geq 0$ in Fig.2, but it is determined by different extreme points. If $-1 < \varepsilon < 0$,

$$\begin{aligned}\mathscr{C}^1_\varepsilon(N,v) \quad &= \quad \underline{\text{conv}}\left\{(1,0,0),(\frac{1}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)},0),(\frac{1}{2(\varepsilon+1)},0,\frac{2\varepsilon+1}{2(\varepsilon+1)})\right.,\\ &\left.(\frac{-2\varepsilon}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)})\right\},\end{aligned}$$

see the area ABCD in Fig.4. Hence, $\mathscr{C}^1_\varepsilon(N,v) \neq \mathscr{C}^0_\varepsilon(N,v)$ as long as $\varepsilon \neq 0$ or $\varepsilon \neq -\frac{1}{2}$. Also, $\mathscr{C}^1_\varepsilon(N,v) \neq \emptyset$ iff $\varepsilon > -\frac{1}{2}$. Thus, $\mathscr{L}\mathscr{C}^1(N,v) = \emptyset \neq \mathscr{L}\mathscr{C}^0(N,v)$. $\square$

Example 2 indicates that the $\alpha\varepsilon$-core and the α-least core are different for different α. To define the α-prekernel, we first adapt the definition of maximal surplus.

Recall that $\Gamma_{ij} = \{S \subseteq N \mid i \in S, j \notin S\}$.

Definition 2. Given $\alpha \in [0,1]$, $v \in G^{N_+}$, and $x \in \mathbb{R}^n_{++}$, the maximal α-surplus of player i over another player j at x in the game v is given by

$$s^{v;\alpha}_{ij}(x) = \max_{S\in\Gamma_{ij}} e^\alpha(S,x). \tag{5}$$

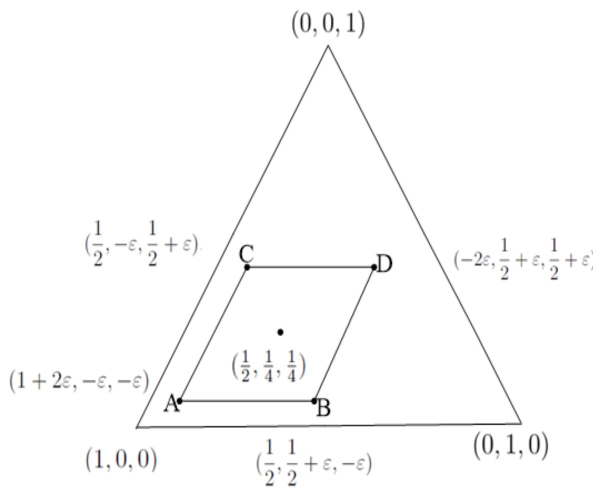

Fig. 3. The 1ε-core, $\mathscr{C}^0_\varepsilon(N, v)$, when $\varepsilon < 0$

Notice that a maximal surplus not less than (respectively not greater than) 1 of i over j at a payoff vector x can be interpreted as the maximal (respectively minimal) amount that player i can gain (respectively lose) without cooperation with j. Consequently, the maximum α-surplus can be regarded as another measure of the power of player i to threaten player j at the preimputation x.

Definition 3. Given $\alpha \in [0, 1]$, and $x \in \mathscr{I}^*_{++}(N, v)$, the α-prekernel $\mathscr{PK}^\alpha(N, v)$ of the game $v \in G^{N_+}$ is the set of preimputations x given by

$$\mathscr{PK}^\alpha(N, v) = \{x \in \mathscr{I}^*_{++}(N, v) \mid s^{v,\alpha}_{ij}(x) = s^{v,\alpha}_{ji}(x) \text{ for all } i, j \in N, i \neq j\}. \tag{6}$$

Obviously, for $\alpha = 0$, the closure of the α-prekernel coincides with the traditional prekernel. Similar as the prekernel, for any $\alpha \in [0, 1]$, the corresponding α-prekernel balances the surpluses pairwise, but using the modified α-excess where dissatisfaction is measured by a mix of the difference and the ratio of potential and realized payoffs $v(S)$, respectively $x(S)$.

We illustrate that the α-prekernel is different for different α with the following example.

Example 3. Let $N = \{1, 2, 3, 4\}$ and let v be defined by $v(\{1, 2, 3, 4\}) = v(\{1, 2\}) = v(\{3, 4\}) = 1$, $v(\{2\}) = v(4) = 2\rho$ and $v(S) = \rho$ otherwise, ρ being a sufficiently small positive number. For $\alpha = 0$, we find that $e(\{12\}, x) = 1 - x_1 - x_2$, $e(\{34\}, x) = 1 - x_3 - x_4$, $e(\{2\}, x) = 2\rho - x_2$, $e(\{4\}, x) = 2\rho - x_4$, $e(S, x) = v(S) - x(S)$ otherwise. Thus,

$$s^{v,0}_{13}(x) = s^{v,0}_{14}(x) = \max\{\rho - x_1, 1 - x_1 - x_2\},$$

$$s^{v,0}_{23}(x) = s^{v,0}_{24}(x) = \max\{2\rho - x_2, 1 - x_1 - x_2\},$$

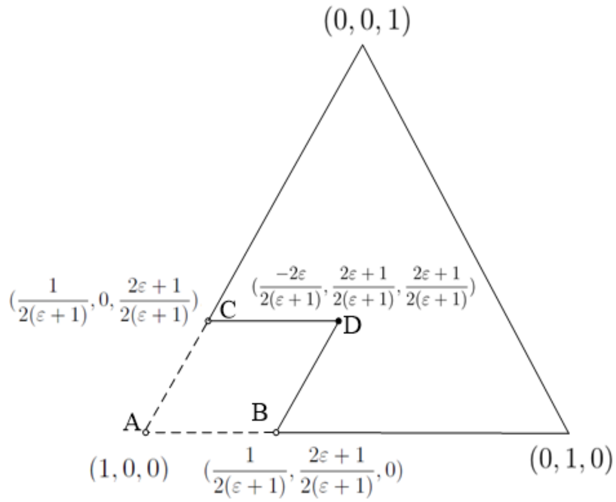

Fig. 4. The 1ε-core, $\mathscr{C}_\varepsilon^1(N,v)$, when $-1 < \varepsilon < 0$

$$s_{31}^{v,0}(x) = s_{32}^{v,0}(x) = \max\{\rho - x_3, 1 - x_3 - x_4\},$$
$$s_{41}^{v,0}(x) = s_{42}^{v,0}(x) = \max\{2\rho - x_4, 1 - x_3 - x_4\},$$
$$s_{34}^{v,0}(x) = \rho - x_3, s_{43}^{v,0}(x) = 2\rho - x_4,$$
$$s_{12}^{v,0}(x) = \rho - x_1, s_{21}^{v,0}(x) = 2\rho - x_2.$$

Following from the definition of the 0-prekernel, i.e., $s_{ij}^{v,0}(x) = s_{ji}^{v,0}(x)$ for all $i, j \in N, i \neq j$, and ρ being a sufficiently small positive number, we know that the 0-prekernel is the set

$$\{x \in \mathscr{I}_{++}(N,v) \mid x_1 = x_3 = \frac{1}{4} - \frac{1}{2}\rho \text{ and } x_2 = x_4 = \frac{1}{4} + \frac{1}{2}\rho\}.$$

For $\alpha = 1$, it is found that $\bar{e}(\{12\}, x) = \frac{1}{x_1+x_2}, \bar{e}(\{34\}, x) = \frac{1}{x_3+x_4}$, $\bar{e}(\{2\}, x) = \frac{2\rho}{x_2}$, $\bar{e}(\{4\}, x) = \frac{2\rho}{x_4}$, $\bar{e}(S, x) = \frac{\rho}{x(S)}$ otherwise. Thus,

$$s_{13}^{v,1}(x) = s_{14}^{v,1}(x) = \max\{\frac{\rho}{x_1}, \frac{1}{x_1 + x_2}\},$$

$$s_{23}^{v,1}(x) = s_{24}^{v,1}(x) = \max\{\frac{2\rho}{x_2}, \frac{1}{x_1 + x_2}\},$$

$$s_{31}^{v,1}(x) = s_{32}^{v,1}(x) = \max\{\frac{\rho}{x_3}, \frac{1}{x_3 + x_4}\},$$

$$s_{41}^{v,1}(x) = s_{42}^{v,1}(x) = \max\{\frac{2\rho}{x_4}, \frac{1}{x_3 + x_4}\},$$

$$s_{34}^{v,1}(x) = \frac{\rho}{x_3}, s_{43}^{v,1}(x) = \frac{2\rho}{x_4},$$

$$s_{12}^{v,1}(x) = \frac{\rho}{x_1}, s_{21}^{v,1}(x) = \frac{2\rho}{x_2}.$$

By the definition of the 1-prekernel, $s_{ij}^{v,1}(x) = s_{ji}^{v,1}(x)$ for all $i, j \in N, i \neq j$, and ρ being a sufficiently small positive number, it holds that $x_4 = 2x_3$, $x_2 = 2x_1$ and $x_1 = x_3$. Therefore, the 1-prekernel is $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3})$, which is different from the 0-prekernel as long as $\rho \neq \frac{1}{6}$. $\square$

Now, given $v \in G^{N_+}$ and $\alpha \in \mathbb{R}$, we define the α-surplus function of player i as the map $s_i^{v,\alpha} : \mathscr{I}_{++}^*(N, v) \to \mathbb{R}^n$ with

$$s_i^{v,\alpha}(x) = \max\{e^\alpha(S, x) \mid S \subset N, i \in S\} \text{ for all } i \in N, x \in \mathscr{I}_{++}^*(N, v),$$

i.e., $s_i^{v,\alpha}(x)$ is the maximum α-surplus that player i can obtain by cooperation given payoff vector x. Recall that $s_{ij}^{v,\alpha}(x)$ gives the potential gain of player i with respect to player j (see Definition 2). Thus, $s_{ij}^{v,\alpha}(x)$ is a relational surplus comparing the positions of two players in a game, whereas the α-surplus of player i, $s_i^{v,\alpha}(x)$, is an individual measure for player i's position in the game. We call a preimputation α-strongly stable if the individual α-surplusses are equal for all players.

Definition 4. Given $\alpha \in [0, 1]$, the preimputation $x \in \mathscr{I}_{++}^*(N, v)$ is said to be α-*strongly stable* for game $v \in G^{N_+}$ if $s_i^{v,\alpha}(x) = s_j^{v,\alpha}(x)$ for all $i, j \in N$.

For any $\alpha \in [0, 1]$, we now provide another characterization of the α-least core $\mathscr{LC}^\alpha(N, v)$ for a positive game, by showing that it consists of all α-strongly stable payoff vectors. For notational convenience, we often write λ instead of $\lambda^{v,\alpha}$ if there is no confusion about v and α.

Theorem 2. *For any $\alpha \in [0, 1]$ and $v \in G^{N_+}$, $x \in \mathscr{LC}^\alpha(N, v)$ if and only if $x \in \mathscr{I}_{++}^*(N, v)$ and $\lambda^{v,\alpha} = s_i^{v,\alpha}(x) - \alpha, \forall i \in N$.*

Proof. Take $\alpha \in [0, 1]$ and $v \in G^{N_+}$.

'Only if': Assume that $x \in \mathscr{LC}^\alpha(N, v)$. Then, by definition $x(N) = v(N)$ and $e^\alpha(S, x) \leq \alpha + \lambda$, for any $S \in 2^N \setminus \{\emptyset, N\}$. In addition, owing to the definition of $\lambda = \lambda^{v,\alpha}$, there is a coalition $T \in 2^N \setminus \{\emptyset, N\}$ such that $e^\alpha(T, x) = \alpha + \lambda$. Denote $\mathcal{T} = \{T \in 2^N \setminus \{\emptyset, N\} \mid e^\alpha(T, x) = \alpha + \lambda\}$. We assert that for any $i \in N$, there exists $T \in \mathcal{T}$, such that $i \in T$. On the contrary, assume that $\exists i \in N$ such that $e^\alpha(S, x) < \alpha + \lambda$, for every $S \subset N$, $S \neq N$ with $i \in S$. Let $\beta_1 = \max\{e^\alpha(S, x) \mid i \in S, S \neq N\} < \alpha + \lambda$, and let $y \in \mathbb{R}_{++}^n$ be defined by

$$y_k = \begin{cases} x_k - \beta_2, & \text{if } k = i, \\ x_k + \frac{\beta_2}{n-1} & \text{if } k \neq i, \end{cases} \tag{7}$$

where $0 < \beta_2 < \min\limits_{S \in 2^N \setminus \{\emptyset, N\}} \{\frac{\lambda - \beta_1}{1-\alpha}, \frac{(n-1)\underline{x}x(S)}{(n-s)(\overline{v}+\underline{x})}\}$, $\underline{x} = \min\limits_{S \in 2^N \setminus \{\emptyset, N\}} x(S)$, and $\overline{v} = \max\limits_{S \in 2^N \setminus \{\emptyset, N\}} v(S)$.

If $S = \{i\}$, then $0 < \beta_2 < \min\limits_{S \in 2^N \setminus \{\emptyset, N\}} \{\frac{\lambda - \beta_1}{1-\alpha}, \frac{\underline{x}x_i}{(\overline{v}+\underline{x})}\}$. We obtain that $x_i > \beta_2$ since $x_i > \frac{\underline{x}}{\overline{v}+\underline{x}} x_i$ and $\beta_2 < \frac{\underline{x}}{\overline{v}+\underline{x}} x_i$. Thus, $y \in \mathscr{I}_{++}^*(N, v)$. We show that $e^\alpha(S, y) < \alpha + \lambda$ for any $S \in 2^N \setminus \{\emptyset, N\}$ establishing a contradiction to the definition of λ.

In the case that $i \notin S \neq \emptyset$, it holds that

$$
\begin{aligned}
e^{\alpha}(S,y) =& \alpha\frac{v(S)}{x(S)+\frac{s\beta_2}{n-1}} + (1-\alpha)(v(S)-x(S)-\frac{s\beta_2}{n-1}) \\
<& \alpha\frac{v(S)}{x(S)} + (1-\alpha)(v(S)-x(S)) - (1-\alpha)\frac{s\beta_2}{n-1} \\
=& e^{\alpha}(S,x) - (1-\alpha)\frac{s\beta_2}{n-1} \\
<& e^{\alpha}(S,x) \leq \alpha+\lambda.
\end{aligned}
$$

In the case that $i \in S \neq N$, we obtain that

$$
\begin{aligned}
e^{\alpha}(S,y) &= \alpha\frac{v(S)}{x(S)-\beta_2+\frac{(s-1)\beta_2}{n-1}} + (1-\alpha)\left(v(S)-x(S)+\beta_2-\frac{(s-1)\beta_2}{n-1}\right) \\
&= \alpha\left(1+\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\right)\frac{v(S)}{x(S)} \\
&\quad +(1-\alpha)\left(v(S)-x(S)+\beta_2-\frac{(s-1)\beta_2}{n-1}\right) \\
&= \alpha\frac{v(S)}{x(S)} + \alpha\left(\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\right)\frac{v(S)}{x(S)} \\
&\quad +(1-\alpha)(v(S)-x(S)) + (1-\alpha)\left(\beta_2-\frac{(s-1)\beta_2}{n-1}\right) \\
&< e^{\alpha}(S,x) + \alpha\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\frac{\overline{v}}{\underline{x}} + (1-\alpha)\beta_2 \\
&\leq \beta_1+\alpha+(1-\alpha)\beta_2 < \alpha+\lambda,
\end{aligned}
$$

where the second inequality follows from $0 < \alpha\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\frac{\overline{v}}{\underline{x}} < 1$ (which follows since, by definition of β_2, $\left(x(S)-\frac{n-s}{n-1}\beta_2\right)\underline{x} > (x(S)-\frac{\underline{x}x(S)}{\overline{v}+\underline{x}})\underline{x} = \frac{\underline{x}\overline{v}x(S)}{\overline{v}+\underline{x}} > \frac{n-s}{n-1}\beta_2\overline{v}$), and the last inequality follows from $(1-\alpha)\beta_2 < \lambda-\beta_1$.

Hence, $e^{\alpha}(S,y) < \alpha+\lambda$ for any $S \in 2^N \setminus \{\emptyset, N\}$, which contradicts with the definition of λ. Therefore, $s_i^{v,\alpha}(x) = \alpha+\lambda$ for any $i \in N$.

'If': Assume that $x \in \mathscr{I}^*_{++}(N,v)$ and $s_i^{v,\alpha}(x) = \alpha+\lambda$ for any $i \in N$. Then, for any $S \in 2^N \setminus \{\emptyset, N\}$, there exists $i \in S$ such that $e^{\alpha}(S,x)-\lambda \leq s_i^{v,\alpha}(x)-\lambda = \alpha$. Therefore, $e^{\alpha}(S,x) \leq \lambda+\alpha$ for any $S \in 2^N \setminus \{\emptyset, N\}$. That is to say, $x \in \mathscr{LC}^{\alpha}(N,v)$. $\square$

Above, we considered two ways to evaluate the position of a player i in a game (N,v). First, with respect to every other player $j \neq i$, the surplus $s_{ij}^{v,\alpha}(x)$ compares the relative position of i with respect to every other player j. In the prekernel, these surpluses are α-balanced for every pair of players. Second, the α-surplus $s_i^{v,\alpha}(x)$ is a measure of the overall position of player i in the game. Instead of comparing payoff vectors by only the individual or pairwise surpluses separately, we combine these surpluses, and compare payoff vectors by balancing the differences between the pairwise and individual surpluses.

Definition 5. For a given $\alpha \in [0,1]$, $x \in \mathscr{I}^*_{++}(N,v)$ satisfies the *α-balanced surplus property* if

$$s_i^{v,\alpha}(x) - s_{ij}^{v,\alpha}(x) = s_j^{v,\alpha}(x) - s_{ji}^{v,\alpha}(x), \text{ for any } i,j \in N.$$

We can characterize the α-prekernel using α-strong stability and this α-balanced surplus property.

Theorem 3. *Given $\alpha \in [0,1]$, $x \in \mathscr{PK}^\alpha(N,v)$ if and only if $x \in \mathscr{I}^*_{++}(N,v)$ is α-strongly stable and satisfies the α-balanced surplus property.*

Proof. 'Only If:' Let $x \in \mathscr{PK}^\alpha(N,v)$ and take any $i,j \in N$, $i \neq j$. By the definition of the α-prekernel, it holds that $s_{ij}^{v,\alpha}(x) = s_{ji}^{v,\alpha}(x)$. Hence, we get that

$$\begin{aligned} s_i^{v,\alpha}(x) &= \max\{s_{ij}^{v,\alpha}(x), \max\{e^\alpha(T,x) \mid \{i,j\} \subseteq T \neq N\}\} \\ &= \max\{s_{ji}^{v,\alpha}(x), \max\{e^\alpha(T,x) \mid \{j,i\} \subseteq T \neq N\}\} \\ &= s_j^{v,\alpha}(x), \end{aligned}$$

which implies that x is α-strongly stable. Since, additionally $x \in \mathscr{PK}^\alpha(N,v)$, and thus $s_{ij}^{v,\alpha}(x) = s_{ji}^{v,\alpha}(x)$, x satisfies the α-balanced surplus property.

'If:' Let x be α-strongly stable and satisfy the α-balanced surplus property. By x being α-strongly stable, $s_i^{v,\alpha}(x) = s_j^{v,\alpha}(x)$ for any $i,j \in N$, $i \neq j$. Then, since x verifies the α-balanced surplus property, it holds that $s_{ij}^{v,\alpha}(x) = s_{ji}^{v,\alpha}(x)$ for any $i \neq j$, and therefore $x \in \mathscr{PK}^\alpha(N,v)$. □

4. The α-prenucleolus and the α-nucleolus

In this section, based on the lexicographical order $\leq_L$, considering the 2^n-dimensional vector $\theta(e^\alpha(S,x)_{S\subseteq N})$, whose components are arranged in nonincreasing order, we propose the α-prenucleolus and α-nucleolus of a cooperative game as follows.

Definition 6. Let $\alpha \in [0,1]$. For any game $v \in G^{N_+}$, the α-prenucleolus $\mathscr{PN}^\alpha(N,v)$ and the α-nucleolus $\mathscr{N}^\alpha(N,v)$ which minimize the excess $e^\alpha(S,x)$ of any coalition over the preimputation set, respectively, the imputation set are defined as follows

$$\mathscr{PN}^\alpha(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) | \theta(e^\alpha(S,x)_{S\subseteq N}) \leq_L \theta(e^\alpha(S,y)_{S\subseteq N}) \ \forall y \in \mathscr{I}^*_{++}(N,v)\}.$$

and

$$\mathscr{N}^\alpha(N,v) = \{x \in \mathscr{I}_{++}(N,v) | \theta(e^\alpha(S,x)_{S\subseteq N}) \leq_L \theta(e^\alpha(S,y)_{S\subseteq N}) \ \forall y \in \mathscr{I}_{++}(N,v)\}.$$

Remark 1. Owing to the results obtained by Justman [3], we have the following statements. For any given $\alpha \in [0,1]$,

(i) If $\mathscr{I}_{++}(N,v)$ is nonempty and compact and if all $e^\alpha(S,x)$, $S \subseteq N$, are continuous with respect to the second variable, then $\mathscr{N}^\alpha(N,v) \neq \emptyset$.

(ii) If $\mathscr{I}_{++}(N,v)$ is convex and all $e^\alpha(S,x)$, $S \subseteq N$, are convex with respect to the second variable, then $\mathscr{N}^\alpha(N,v)$ is convex and $e^\alpha(S,x) = e^\alpha(S,y)$ for all $S \subseteq N$ and all $x, y \in \mathscr{N}^\alpha(N,v)$.

Inspired by the method provided by Peleg and Sudhölter [10], define

$$\mathscr{I}'_{++}(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) | \max_{S\subseteq N} e^\alpha(S,x) \leq \max_{S\subseteq N} e^\alpha(S,y) \ \forall y \in \mathscr{I}^*_{++}(N,v)\}.$$

Following from Remark 1, we obtain that the α-nucleolus is a singleton. Also, for any $x \in \mathscr{I}^*_{++}(N,v)$, in the definition of $\mathscr{P}\mathscr{N}^\alpha(N,v)$, we may replace $\mathscr{I}^*_{++}(N,v)$ by the compact, nonempty, and convex set $\mathscr{I}'_{++}(N,v)$. Thus, the α-prenucleolus is also a singleton.

Theorem 4. *Given $\alpha \in [0,1]$, for every game $v \in G^{N_+}$: (i) the α-nucleolus is a singleton, and (ii) the α-prenucleolus is a singleton.*

From now on, we often write the α-prenucleolus of game v just as its unique element, and denote it by $\nu^\alpha(N,v) \in \mathbb{R}^n_{++}$. Next, we show that the α-prenucleolus is an element of the α-prekernel.

Theorem 5. *For every game $v \in G^{N_+}$ and for all $\alpha \in [0,1]$, $\nu^\alpha(N,v) \in \mathscr{P}\mathscr{K}^\alpha(N,v)$.*

Proof. Let $\alpha \in [0,1]$ and $x^\alpha = \nu^\alpha(N,v)$. We show that $x^\alpha \in \mathscr{P}\mathscr{K}^\alpha(N,v)$. On the contrary, assume that there exists $\bar{\alpha} \in [0,1]$ such that $x^{\bar{\alpha}} \notin \mathscr{P}\mathscr{K}^{\bar{\alpha}}(N,v)$. For easiness of notation, let $x = x^{\bar{\alpha}}$. Since $x \in \mathscr{I}^*_{++}(N,v)$, there exist two distinct players $i, j \in N$ with $s^{v,\bar{\alpha}}_{ij}(x) > s^{v,\bar{\alpha}}_{ji}(x)$.

First, we show that there exists δ with $0 < \delta < \hat{x} = \min\limits_{S\in\Gamma_{ji}(N)} x(S)$ such that

$$s^{v,\bar{\alpha}}_{ji}(x) = s^{v,\bar{\alpha}}_{ij}(x) - \delta - \frac{\bar{\alpha}\delta\hat{v}}{(\hat{x}-\delta)\hat{x}}, \tag{8}$$

where $\hat{v} = \max\limits_{S\in\Gamma_{ji}(N)} v(S)$. This is equivalent to showing that the second degree equation on δ

$$\hat{x}\delta^2 - [(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x} + \hat{x}^2 + \bar{\alpha}\hat{v}]\delta + (s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x}^2 = 0 \tag{9}$$

has at least one real solution. This is true when the discriminant of the equation is non-negative, i.e.,

$$[(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x} + \hat{x}^2 + \bar{\alpha}\hat{v}]^2 - 4\hat{x}^3(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) \geq 0,$$

or, equivalently after some algebra[1],

$$\hat{v}^2\bar{\alpha}^2 + 2[(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) + \hat{x}]\hat{x}\hat{v}\bar{\alpha} + [(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) - \hat{x}]^2\hat{x}^2 \geq 0. \tag{10}$$

[1] This follows since $[(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x} + \hat{x}^2 + \bar{\alpha}\hat{v}]^2 - 4\hat{x}^3(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) = (s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))^2\hat{x}^2 + \hat{x}^4 + \bar{\alpha}^2\hat{v}^2 + 2\hat{x}^3(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) + 2(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x}\hat{v}\bar{\alpha} + 2\hat{x}^2\hat{v}\bar{\alpha} - 4\hat{x}^3(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) = (s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))^2\hat{x}^2 - 2\hat{x}^3(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x)) + \hat{x}^4 + \bar{\alpha}^2\hat{v}^2 + 2(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x}\hat{v}\bar{\alpha} + 2\hat{x}^2\hat{v}\bar{\alpha} = [(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x} - \hat{x}^2]^2 + \bar{\alpha}^2\hat{v}^2 + 2[(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))]\hat{x}\hat{v}\bar{\alpha} + 2\hat{x}^2\hat{v}\bar{\alpha}$.

The formula (10) holds since every term in the inequality is non-negative. Therefore, the quadratic equation (9) has at least one solution, and therefore there exists $\delta \in (0, \hat{x})$ for which (8) holds.

Second, we define $y \in \mathbb{R}^n_{++}$ as

$$y_k = \begin{cases} x_k + \delta, \text{ if } k = i, \\ x_k - \delta, \text{ if } k = j, \\ x_k, \text{ otherwise,} \end{cases} \tag{11}$$

and show that $y \in \mathscr{I}^*_{++}(N, v)$ with $\theta((e^\alpha(S, y))_{S\subseteq N}) <_L \theta((e^\alpha(S, x))_{S\subseteq N})$.

On the one hand, $x \in \mathscr{I}^*_{++}(N, v)$ implies

$$\sum_{k\in N} y_k = \sum_{k\in N} x_k = v(N),$$

while $x \in \mathscr{I}^*_{++}(N, v)$ and $0 < \delta < \hat{x} = \min_{S\in\Gamma_{ji}(N)} x(S) \leq x_j$ imply $y_k > 0$ for all $k \in N$. Thus, $y \in \mathscr{I}^*_{++}(N, v)$.

On the other hand, to show $\theta((e^\alpha(S, y))_{S\subseteq N}) <_L \theta((e^\alpha(S, x))_{S\subseteq N})$, we consider the following three cases. Denote

$$\mathcal{S} = \{S \in 2^N \setminus \Gamma_{ij}(N) \mid e^\alpha(S, x) \geq s^{v,\alpha}_{ij}(x)\} \text{ and } \tilde{s} = |\mathcal{S}|.$$

(i) First, if $S \in 2^N \setminus (\Gamma_{ij}(N) \cup \Gamma_{ji}(N))$ then $e^\alpha(S, y) = e^\alpha(S, x)$ in view of the form of y as in (11).

(ii) Second, if $S \in \Gamma_{ij}(N)$,

$$e^\alpha(S, y) = \alpha\frac{v(S)}{x(S) + \delta} + (1 - \alpha)(v(S) - x(S) - \delta) < e^\alpha(S, x).$$

(iii) Third, if $S \in \Gamma_{ji}(N)$, then

$$\begin{aligned} e^\alpha(S, y) =& \alpha\frac{v(S)}{x(S) - \delta} + (1 - \alpha)(v(S) - x(S) + \delta) \\ =& \alpha\frac{x(S)}{x(S) - \delta}\frac{v(S)}{x(S)} + (1 - \alpha)(v(S) - x(S)) + (1 - \alpha)\delta \\ =& \alpha\left(1 + \frac{\delta}{x(S) - \delta}\right)\frac{v(S)}{x(S)} + (1 - \alpha)(v(S) - x(S)) + (1 - \alpha)\delta \\ =& e^\alpha(S, x) + \alpha\frac{\delta}{x(S) - \delta}\frac{v(S)}{x(S)} + (1 - \alpha)\delta \\ \leq& s^{v,\alpha}_{ji}(x) + \left(\frac{\alpha}{x(S) - \delta}\frac{v(S)}{x(S)} + 1\right)\delta \\ \leq& s^{v,\alpha}_{ij}(x) - \frac{\alpha(v_{max} - v(S))}{(x_{min} - \delta)x_{min}}\delta \\ <& e^\alpha(S, x) \end{aligned}$$

where the first inequality follows by definition of $s_{ji}^{v,\alpha}(x)$, and the second from (8) and the definition of x_{min}.

Let $S^1, S^2 \in 2^N$ be such that $e^{\alpha}(S^1, x) \geq e^{\alpha}(S, x)$ for any $S \in 2^N$ and $e^{\alpha}(S^2, y) \geq e^{\alpha}(S, y)$ for any $S \in 2^N$. From the above three cases, it holds that $e^{\alpha}(S, y) \leq e^{\alpha}(S, x)$ for any $S \in 2^N$. Thus, $e^{\alpha}(S^2, y) \leq e^{\alpha}(S^2, x) \leq e^{\alpha}(S^1, x)$. Denoting $x = (e^{\alpha}(S, x)_{S\subseteq N})$ and $y = (e^{\alpha}(S, y)_{S\subseteq N})$, therefore, it holds that $\theta_t(y) \leq \theta_t(x)$ for all $t \leq \tilde{s}$, and $\theta_{\tilde{s}+1}(y) < s_{ij}^{v,\alpha}(x) = \theta_{\tilde{s}+1}(x)$ if $\theta_t(y) = \theta_t(x)$ for all $t \leq \tilde{s}$, for every $\alpha \in [0, 1]$. That is to say, $\theta(y) <_L \theta(x)$, and thus $x \neq \nu^{\alpha}(N, v)$ and the desired contradiction has been obtained. □

From Theorems 4 and 5, we obtain the following corollary.

Corollary 6. *For every game $v \in G^{N_+}$ and every $\alpha \in [0, 1]$, $\mathscr{PK}^{\alpha}(N, v) \neq \emptyset$.*

5. Conclusions

In this paper, we propose a family of excesses (α-excess) for positive TU-games that measure the dissatisfaction of any coalition and generalizes the classical and proportional excesses. Then, the corresponding solutions, such as the α-least core, the α-(pre)nucleolus, and the α-(prekernel) are defined based on the α-excess. We give a characterization of the α-prekernel by strong stability and the α-balanced surplus property. Meanwhile, the α-least core of a positive TU-game can be characterized in terms of strong stability. Finally, we introduce the α-prenucleolus and α-nucleolus, and showed that, for every game, these are singletons and the unique α-prenucleolus element belongs to the corresponding α-prekernel.

For future research, we intend to modify the famous Davis and Maschler reduced game (Davis and Maschler (1965)), taking account of the modified excess. There is a large literature on reduced game consistency. Reduced game consistency requires that, after some players leave the game with the payoffs assigned to them by a solution, applying the same solution on the reduced game on the remaining players gives these remaining players the same payoff as in the original game. Different solutions can be characterized by different reduced game properties, where the difference is in the way the reduced game is defined. For the α-prekernel, we might consider the following reduced game. Let $\alpha \in [0, 1]$. Given a game $v \in G^{N_+}$, a nonempty coalition S, and a positive payoff vector x, the α-reduced game on S at x, denoted $(S, v_{x,S}^{\alpha})$, is the game defined by

$$v_{x,S}^{\alpha}(T) = \begin{cases} 0, \text{ if } T = \emptyset, \\ v(N) - x(N \setminus T), \text{ if } T = S, \\ \frac{\alpha}{\alpha+(1-\alpha)x(T)} \max\limits_{Q\subseteq N\setminus S}\{\frac{v(T\cup Q)x(T)}{x(T\cup Q)}\} + \frac{(1-\alpha)x(T)}{\alpha+(1-\alpha)x(T)} \max\limits_{Q\subseteq N\setminus S}\{(v(T \cup Q) - x(Q))\}, \text{ if } T \subsetneq S. \end{cases}$$

It can be shown that the α-prekernel satisfies the corresponding reduced game property. However, a characterization of the α-prekernel using the α-reduced game property is still an open problem.

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