The risk-adjusted carbon price

Rick van der Ploeg[1]
Ton van den Bremer[2]

[1] Department of Economics, University of Oxford and Faculty of Economics and Business, University of Amsterdam
[2] Faculty of Civil Engineering and Geosciences, Delft University and Department of Engineering Science, University of Oxford

THE RISK-ADJUSTED CARBON PRICE*

Ton S. van den Bremer and Frederick van der Ploeg

The social cost of carbon is the expected present value of damages from emitting one ton of carbon today. We use perturbation theory to derive an approximate tractable expression for this cost adjusted for climatic and economic risk. We allow for different aversion to risk and intertemporal fluctuations, skewness and dynamics in the risk distributions of climate sensitivity and the damage ratio, and correlated shocks. We identify prudence, insurance, and exposure effects, reproduce earlier analytical results, and offer analytical insights into numerical results on the effects of economic and damage ratio uncertainty and convex damages on the optimal carbon price.

Keywords: precaution; insurance; exposure; economic, climatic and damage uncertainties; skewness; mean reversion; correlated risks; risk aversion; intergenerational inequality aversion; convex damages

JEL codes: H21, Q51, Q54

Final revision April 2021

* Van den Bremer: Faculty of Civil Engineering and Geosciences, 2628 CN Delft, The Netherlands and Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK (email: ton.vandenbremer@eng.ox.ac.uk). Van der Ploeg: Department of Economics, University of Oxford, OX1 3UQ, UK (email: rick.vanderploeg@economics.ox.ac.uk) and Faculty of Economics and Business, University of Amsterdam, 1001 NB Amsterdam, The Netherlands.

The social cost of carbon (SCC) is the expected present discounted value of all future marginal damages resulting from emitting one ton of carbon today.[1] The risk-adjusted SCC incorporates uncertainties[2] associated with climate and the economy. If global warming is the only market failure, it is optimal in a decentralized economy to set the price of carbon emissions (e.g., a specific carbon tax or the price in a competitive permit market) to the SCC. To evaluate the SCC, one must know how much of one ton of carbon emitted today is still left in the atmosphere at each future time; the effect of the atmospheric carbon stock on temperature; the effect of temperature on damages to aggregate output and consumption; and the marginal utility of consumption at all instants of time. All of these effects are subject to uncertainty.

Our aim is to derive a closed-form expression for the optimal risk-adjusted SCC from a simple analytic yet quantitatively calibrated integrated assessment model of climate and the economy, where attitudes to risk aversion differ from attitudes to intertemporal fluctuations in consumption. We allow for a wide range of uncertainties, regarding macroeconomic growth, the carbon stock, the climate sensitivity, the damage ratio (global warming damages as fraction of GDP) and the correlations between these uncertainties. We highlight the effects of time-varying and skewed distributions and the convexity of damages in temperature on the SCC.

We derive the SCC as the social optimum of a Dynamic Stochastic General Equilibrium (DSGE) model with recursive preferences, which seperate aversion to risk from aversion to intertemporal fluctuations (Kreps and Porteus, 1978; Epstein and Zin, 1989; Duffie and Epstein, 1992). Our closed-form expression for the optimal SCC is in the spirit of the rule derived by Golosov et al. (2014).[3] It has the

---

[1] Along an optimal allocation path the SCC corresponds to the Pigouvian tax on emissions, but the SCC can also be evaluated along other (e.g., business as usual) paths.

[2] We use the terms risk and uncertainty interchangeably.

[3] Our solution is an approximate closed-form expression derived from a DSGE model with endogenous growth, whereas Golosov et al. (2014) derive an exact closed-form expression from a DSGE model with convex growth and multiple fuel sectors. If the damage ratio is proportional to the stock of carbon as in Result 1 below, this is similar in spirit to the assumptions made about damages in Golosov et al. (2014). Result 2 extends this to more general damages.

usual precautionary, insurance and risk-exposure determinants of the risk-adjusted social discount rate and the SCC resulting from macroecomic uncertainty but adds multiplicative adjustment factors to allow for the uncertainties regarding the carbon stock, climate sensitivity, and the damage ratio.

Our methodological contribution is to show how perturbation methods can be used to solve DSGE models with more than a few states (four states plus time, here). The so-called small parameter of our perturbation, which measures the size of the perturbation, is the damage ratio.[4] As our small parameter goes to zero, we return to a known solution, in this case the endogenous growth model with investment adjustment costs of Pindyck and Wang (2013), which has a closed-form solution. Our full DSGE model extends this to allow for fossil fuel use, climate change and damages. We make use of power functions to capture a damage ratio that is convex in temperature and power-function transformations of normally distributed shocks to capture the right-skew of the long-run climate sensitivity. We use mean-reverting processes to capture that uncertainty in climate sensitivity is larger and more right-skewed on long than short horizons.[5] Our perturbation method offers a powerful alternative to both numerical methods (reviewed below), which are computationally intensive for many states and do not lend analytical insight, and other perturbation methods that rely on a high-order multi-variate Taylor-series expansion in the states, which can be prohibitively complex (see Appendix A.5 for a comparison).

We derive two main results.[6] By focusing on the leading-order effects of uncertainty, that is, those that dominate in the limit of small uncertainty relative to the mean, Result 1 gives a closed-form solution for the optimal SCC if the damage ratio is proportional to the atmospheric carbon stock and there is no delay in the

---

[4] We will show in Fig. 3 that this damage ratio is typically only a few percentage points and can rise to 10% at most.
[5] Specifically, we use Ornstein-Uhlenbeck processes. We abstract from fat tails, so that Weitzman's (2009) 'dismal theorem' does not apply.
[6] Our third and most general result, Result A in Appendix A, gives a general expression for the optimal SCC without the assumption of leading-order uncertainty and can be evaluated by numerical solution of a multi-dimensional integral. The assumptions leading to Result A and, from there, to Results 1 and 2 are laid out precisely in Appendix A.

deterministic temperature response to emissions. Result 2 generalizes this result for convex dependence of damages on the carbon stock and for a delayed deterministic temperature response.

Results 1 and 2 imply that policy makers should employ a lower discount rate and a higher SCC in case of uncertainty about future economic growth if aversion to intertemporal fluctuations exceeds one (cf. Gollier, 2002, 2010; Jensen and Traeger (2014)). In asset pricing theory, the opposite assumption is made (i.e., the elasticity of intertemporal substitution exceeds one) in which case macroeconomic uncertainty depresses the SCC, just like it depresses share prices.

Results 1 and 2 also imply that temperature uncertainty and damage ratio uncertainty call for a higher SCC, where the adjustment to the SCC is larger if damages are more convex, the distribution of uncertainty is wider and more right-skewed, uncertainty arises on shorter horizons, and the risk-adjusted social discount rate and carbon decay rate are smaller. Finally, Results 1 and 2 imply that, if positive shocks to the economy are associated with positive (negative) shocks to temperature, the optimal SCC is lower (higher) if relative risk aversion exceeds one (cf. Lemoine, 2020). If shocks to future damages are negatively (positively) correlated with future shocks to asset returns, the optimal SCC is higher (lower), if relative risk aversion exceeds one.

Most previous approaches to the optimal SCC have either used models where intergenerational consumption smoothing and risk aversion coincide, or where they are separated, but the intertemporal elasticity is assumed to be one. Jensen and Traeger (2014), who also combine analytic formula and quantitative assessment, only deal with economic uncertainty, and Cai and Lontzek (2019) combine economic and climate uncertainty but do not yield analytic insight. We offer a closed-form expressionfor the optimal SCC for a range of economic, climate and damage uncertainties.Moreover, our analysis has the potential to go beyond the correlation analysis of Dietz et al. (2018) and Lemoine (2020). Dietz et al. (2018)

examine the effects of the elasticity of damages with respect to output in the DICE integrated assessment model but without recursive preferences.

Different authors have performed numerical calculations of the optimal SCC under multiple sources of uncertainty, first with Monte-Carlo simulations (e.g., Nordhaus, 1994; Nordhaus and Popp, 1997; Ackerman and Stanton, 2012; Dietz and Stern, 2015). Others have used stochastic dynamic optimization methods from macroeconomics (e.g., Kelly and Kohlstad, 1999; Pizer, 1999) or advanced numerical methods (e.g., Crost and Traeger, 2013; Traeger, 2014a; Jensen and Traeger, 2014; Hambel et al., 2017; Lemoine and Traeger, 2014, 2016a; Lontzek et al., 2015; Cai et al., 2016; Cai and Lontzek, 2019).[7] Cai and Lontzek (2019) represent the state of the art in advanced numerical methods and also allow for recursive preferences. Our objective is complementary: we offer an approximate closed-form solution for the optimal SCC under a range of economic, climatic and damage uncertainties that may be correlated. In contrast to Cai and Lontzek (2019) we do not allow for tipping points, but we do allow for skewed distributions.

Our contribution is also related to the analytical literature on discounting under uncertainty and simple rules for the optimal carbon price, which typically deals with one uncertainty at a time, for example about future economic growth (e.g., Gollier, 2002, 2012; Traeger, 2014b). Golosov et al. (2014) obtained a simple rule for the optimal SCC reacting to world GDP only, making bold assumptions including logarithmic utility[8], which imply that economic growth uncertainty does not affect the SCC (cf. Traeger, 2017). Gerlagh and Liski (2016) also derive a simple rule and allow for learning about uncertain impacts. Jensen and Traeger (2019) show how the effect of climate sensitivity on the risk premium in the SCC depends on prudence and convexity of marginal damages. Lemoine (2020) decomposes the

---

[7] Lemoine and Trager (2016b) deal with ambiguity aversion and find only small effects on the optimal carbon tax.
[8] They have a discrete-time (decadal) model, assume logarithmic utility, Cobb-Douglas production, 100% depreciation of capital each period, and total factor productivity as an exponential function of the atmospheric carbon stock.

SCC into different components due to uncertain warming, damages and economic growth. He shows that the sign of the effect of the normalized covariances of different climatic uncertainties with the rate of economic growth on the SCC depends on whether relative risk aversion is greater than one or not.[9] In both the decompositions by Jensen and Traeger (2019) and Lemoine (2020) consumption is set exogenously. Recently, two important complementary studies to ours have also obtained a simple analytical rule for the risk-adjusted SCC in a general equilibrium model. Traeger (2017) develops an integrated assessment model with a range of climate uncertainties, in which consumption is determined endogenously, less than full capital depreciation in each period and the restriction that the model is linear in the states with additively separable controls. Bretschger and Vinogradova (2019) extend an endogenous macroeconomic growth model to allow for Poisson shocks in the capital stock in their analysis of optimal carbon pricing.

Finally, we make the proviso that we do not allow for (Bayesian) learning about economic and climatic uncertainty. Kelly and Kolstad (1999) find that learning of climate sensitivity takes a very long time (90 years). Kelly and Tan (2005) confirm this but find that "tail learning" can be fast as observations near the mean provide evidence against fat tails. The damage ratio could also be learned (Nordhaus and Popp, 1997). Lemoine and Rudik (2017) give a comprehensive overview of uncertainty and learning in climate policy. They highlight that policy makers learn expectations of future temperature increase better if temperature has been observed to rise, how temperature changes affect the ability to smooth welfare in response to the signal that is received about the climate sensitivity, and how active learning affects mean and precision of beliefs. Lemoine and Traeger (2014) and Cai and Lontzek (2020) allow for learning about irreversible changes in climate sensitivity after passing an unknown temperature threshold. If there is learning about "tail"

---

[9] Alternatively, it depends on whether the risk-insurance effect dominates the offsetting risk-exposure effect resulting from damages being proportional to GDP (cf. Lemoine, 2020).

uncertainty, then our results might suggest that the optimal SCC will be substantially reduced once learning has taken place.

Section I presents our model. Section II discusses our perturbation method for deriving the optimal SCC. Section III presents Results 1 and 2. After discussing our calibration in Section IV, Section V estimates the optimal SCC and discusses the effects of the different uncertainties. Section VI concludes.

I. A DSGE Model of Global Warming and the Economy

We start from the DSGE model with endogenous AK-growth of Pindyck and Wang (2013) and add fossil fuel use as a production factor. Fossil fuel use gives rise to global warming and damages to output. The coefficient of relative risk aversion, η = CRRA ≥ 0, may differ from the coefficient of relative intergenerational inequality aversion, IIA = 1/EIS = $\gamma \geq 0$, where EIS is the elasticity of intertemporal substitution. We use the continuous-time version of recursive preferences (Duffie and Epstein, 1992), where the recursive aggregator $f(C,J)$ depends on consumption C and the value function

$$(1) \quad J = \mathrm{E}_t\left[\int_t^\infty f\left(C(s),J(s)\right)\mathrm{d}s\right] \text{ with } f\left(C,J\right) = \frac{1}{1-\gamma}\frac{C^{1-\gamma} - \rho\left((1-\eta)J\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)J\right)^{\frac{1-\gamma}{1-\eta}-1}}.$$

The dynamics of the aggregate capital stock follow from

$$(2) \quad \mathrm{d}K = \Phi(I,K)\mathrm{d}t + \sigma_K K\mathrm{d}W_1 \text{ with } \Phi(I,K) = I - \frac{1}{2}\omega\frac{I^2}{K} - \delta K,$$

where K denotes the capital stock, I investment, $\delta \geq 0$ the depreciation rate of

physical capital, and $\omega > 0$ the adjustment cost parameter. [10,11] Adjustment costs are quadratic and homogenous of degree one in capital and investment. Capital is subject to continuous geometric shocks with relative volatility σ_K, and W_1 is a Wiener process, representing both economic growth and asset return uncertainty in the context of the AK-model considered. Investment is $I = Y - C - bF$, where Y is aggregate production, F fossil fuel use, and b the production cost of fossil fuel. Fossil fuel is supplied inelastically at fixed cost. The final goods production function is $Y = AK^{\alpha}F^{1-\alpha}$ with $0 < \alpha < 1$ and $A \equiv A^*(1-D)$ is total factor productivity. Since we focus on endogenous growth, we abstract from labour-augmenting technical progress and population growth and thus omit time indices. Damages as share of pre-damage aggregate output D increase in global mean temperature relative to the preindustrial T. We use the power-function specification

(3) $\qquad D(T, \lambda) = T^{1+\theta_T} \lambda^{1+\theta_\lambda}$ with $\theta_T \geq -1$ and $\theta_\lambda \geq -1,$

where the (positive) stochastic damage ratio parameter λ captures the uncertain nature of the damage ratio at given temperature T. Convexity of the damage ratio (3) with respect to temperature corresponds to $\theta_T \equiv TD_{TT} / D_T > 0.$ [12] To allow for potential skewness in the damage distribution assuming λ has a symmetric distribution, we raise λ to the power $1 + \theta_\lambda$.

The part of atmospheric carbon, S, associated with man-made emissions is $E \equiv S - S_{PI}$, where S_{PI} is the preindustrial carbon stock. This is often referred to as the concentration or pollution stock above preindustrial. The rate of carbon

---

[10] With AK-growth, shocks to the capital stock and productivity are equivalent. To avoid an extra state, we introduce volatility directly in the capital dynamics (cf. Pindyck and Wang, 2013).
[11] For ease of presentation, we first introduce the separate evolution equations for the four stochastic variables before introducing the covariance matrix of these four state variables.
[12] Subscripts of functions denote partial derivatives.

emissions is $F\exp(-gt)$, where F is fossil fuel use and $\exp(-gt)$ the emission intensity which declines at the endogenous economic growth rate g. A proportion $0 < \mu < 1$ of fossil fuel emissions ends up in the atmosphere. Atmospheric carbon decays at the rate $\varphi \geq 0$. The carbon stock dynamics is

(4) $\qquad dE = (\mu F e^{-gt} - \varphi E)dt + \sigma_E dW_2,$

where W_2 denotes a second Wiener process, so the carbon stock is described by a Arithmetic Brownian motion with absolute volatility $\sigma_E \geq 0$.[13, 14] This specification ensures that the expected value of the carbon stock returns to its preindustrial value when emissions cease. We have for temperature

(5) $\qquad T(E, \chi) = \chi^{1+\theta_\chi} (E/S_{\mathrm{PI}})^{1+\theta_E} \quad \text{with} \quad \theta_E \geq -1 \quad \text{and} \quad \theta_\chi \geq -1,$

where the (positive) stochastic variable χ captures the uncertain nature of temperature for a given carbon stock. We will use a negative value of θ_E to capture the concave dependence of temperature on the carbon stock. The parameter θ_χ captures skewness of the climate sensitivity distribution assuming χ has a symmetric distribution (see (8a) below). The climate sensitivity is the temperature increase from doubling the carbon stock from its preindustrial level, i.e. $T_2 \equiv T(E = S_{PI}, \chi) = \chi^{1+\theta_\chi}$. It is a stochastic variable and depends on the stochastic climate sensitivity parameter χ. Its leading-order mean is

---

[13] One can allow for a permanent and one (Golosov et al., 2014), two (Gerlagh and Liski, 2018) or three (Millar et al., 2017) temporary basins of atmospheric carbon. Appendix F3 shows that our 1-box model reproduces historical atmospheric carbon stocks well, and section IV illustrates how it captures all the key features of future projections, although a value of μ substantially smaller than one may lead to too rapid initial decay of the carbon stock due to a marginal emission. Millar et al. (2017) allow the speed at which oceans absorb atmospheric carbon (akin to our φ) to fall with warming. We ignore such positive feedback effects and associated multiplicative uncertainty.

[14] Although E, χ and λ in (4), (8a) and (8b) can formally become zero or negative with finite probabilities due to their Gaussian distributions, we will show in section IV that these probabilities are negligibly small. Formally, all three variables are truncated in our model, so that they can only take positive values, and the model is well posed. For simplicity of presentation, we avoid additional notation to describe this truncation, which we do apply.

$\mathrm{E}[T_2] = \mu_\chi^{1+\theta_\chi} \left(1 + \theta_\chi (1+\theta_\chi)(\Sigma_\chi / \mu_\chi)^2 / 2\right)$ (see Appendix E.5) and skewness is

$\mathrm{skew}[T_2] = 3\theta_\chi (1+\theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)} (\Sigma_\chi / \mu_\chi)^4$, where μ_χ and Σ_χ are the mean and

volatility of the equilibrium value of χ. Both increase in the skewness parameter θ_χ

and the coefficient of variation Σ_χ / μ_χ. Combining equations (3) and (5), the

reduced-form damage ratio becomes

(6) $\qquad D(E,\chi,\lambda) = \chi^{1+\theta_{\chi T}} \lambda^{1+\theta_\lambda} (E/S_{PI})^{1+\theta_{ET}}$ with $\theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi \theta_T$.

The parameter $\theta_{ET} \equiv \theta_E + \theta_T + \theta_E \theta_T$ captures the combined effect of the concave

relationship between temperature and the carbon stock $(-1 \leq \theta_E < 0)$ and the

convex relationship between damages and temperature $(\theta_T > 0)$. It is positive or

negative depending on which effect dominates. We refer to $\theta_{ET} = 0$ as *proportional*

damages and $\theta_{ET} > 0$ as *convex damages*, reflecting the reduced-form dependence

of damages on the carbon stock. The parameter $\theta_{\chi T}$ captures the joint effect of

skewness of climate sensitivity $(\theta_\chi > 0)$ and convexity of the damage function

with respect to temperature $(\theta_T > 0)$. From (6), total factor productivity and

aggregate output are a decreasing function of the carbon stock and the climate

sensitivity and damage ratio parameters:

(7) $\quad Y = A(E,\chi,\lambda)K^\alpha F^{1-\alpha}$ with $A(E,\chi,\lambda) \equiv A^* \left(1 - (E/S_{PI})^{1+\theta_{ET}} \chi^{1+\theta_{\chi T}} \lambda^{1+\theta_\lambda}\right)$.

Uncertainties in the climate sensitivity and the damage ratio are driven by

truncated (see footnote 14) mean-reverting stochastic Ornstein-Uhlenbeck

processes with means $\bar{\chi}, \bar{\lambda}$, mean reversion coefficients v_χ, v_λ, and volatilities

$\sigma_\chi, \sigma_\lambda$,

(8a) $\qquad d\chi = \nu_\chi(\bar{\chi} - \chi)dt + \sigma_\chi dW_3,$

(8b) $\qquad d\lambda = \nu_\lambda(\bar{\lambda} - \lambda)dt + \sigma_\lambda dW_4,$

where W_3 and W_4 are two Wiener processes.[15] Together with $T \propto \chi^{1+\theta_\chi}$ in (5), the process (8a) captures two features of the actual climate sensitivity distribution. First, the expected response of temperature to increases in the carbon stock is delayed in time, from the (lower) transient climate response (TCR) at initial times to a steady state associated with the (higher) equilibrium climate sensitivity (ECS)[16] We allow for a delayed response of temperature to increases in the carbon stock via the time-varying dynamics of the stochastic process for the random variable χ. If $\chi_0 < \bar{\chi}$, then temperature (cf. (5)) will start low . Over time and in the absence of uncertainty, mean reversion ensures that χ gradually increases to its steady-state value, and this leads to gradual increase in temperature. Second, the uncertainty and skewness of the climate sensitivity distribution grow with time from the narrow and symmetric TCR to the wide and skew ECS in steady state, which like the steady state of the mean is reached as $t \gg 1/\nu_\chi$ with $1/\nu_\chi$ the e-folding time.[17] Note that our formulation implies that temperature increases are independent of when carbon is added to the atmosphere. In our model, temperature responds immediately to changes in the atmospheric carbon stock (cf. (5)); it does not suffer from the inertia in response of temperature to marginal emissions for which a number of integrated

---

[15] Equation (8a) has solution $\chi(t) = \chi_0 e^{-\nu_\chi t} + \bar{\chi}(1 - e^{-\nu_\chi t}) + \sigma_\chi \int_0^t \exp(-\nu_\chi(t-s))dW_3(s)$, and similarly for (8b). The variables $\chi(t)$ and $\lambda(t)$ have distributions $\chi(t) \sim N(\mu_\chi, \Sigma_\chi^2)$ and $\lambda(t) \sim N(\mu_\lambda, \Sigma_\lambda^2)$. Mean and variance of $\chi(t)$ are $\mu_\chi = \chi_0 e^{-\nu_\chi t} + \bar{\chi}(1 - e^{-\nu_\chi t})$ and $\Sigma_\chi^2 = \sigma_\chi^2(1 - \exp(-2\nu_\chi t))/2\nu_\chi$ with steady-state limits $\mu_\chi \to \bar{\chi}$ and $\Sigma_\chi^2 \to \sigma_\chi^2/2\nu_\chi$.

[16] We thus include potential effects of temperature lags from ocean heating, which affect estimates of the long-run climate sensitivity (e.g., Roe and Bauman, 2011). In reality, the response to small emissions is much faster and on a decadal scale (Ricke and Caldeira, 2014) than the response to larger emissions (Zickfeld and Herrington, 2015), reflecting nonlinearity in the system, which is not captured by our Ornstein-Uhlenbeck process (8a). Clearly, our parsimonious climate model cannot capture all features of state-of-the-art climate models, and we discuss its limitations in section VI.

[17] The e-folding time is how long it takes for an exponentially growing quantity to rise by a factor 2.27.

assessment models have recently been criticized (Mattauch et al., 2020), although the magnitude of the response is allowed to increase slowly with time. For all three uncertain climatic processes E, χ and λ, the uncertainties are exogenously given and cannot be learned in our model. Fundamentally, both statistical (or aleatoric) uncertainty and systemic (or epistemological) uncertainty play a role in reality, but their contributions cannot always be separated.[18] For all three processes, we use in our calibration the most high-level or 'consensus' range of uncertainty estimates available, which also do not make this distinction (see section IV). For example, the 'consensus' uncertainty range for the climate sensitivity (e.g., IPCC, 2014, AR5, Chapter 12, Box 12.2) captures both statistical uncertainty in individual climate models and (some) epistemological uncertainty arising from considering different climate models. The carbon stock, climate sensitivity and damage ratio uncertainty we examine are aggregate measures of uncertainty that capture present-day disagreement in the scientific literature.

Equations (2), (4) and (8) are part of a multi-variate Ornstein-Uhlenbeck process:

$$(9) \qquad d\mathbf{x} = \boldsymbol{\alpha} dt - \mathbf{v} \circ \left(\mathbf{x} - \overline{\boldsymbol{\mu}} \right) dt + \mathbf{S} d\mathbf{W}_t,$$

where the states are $\mathbf{x} \equiv (k, E, \chi, \lambda)^T$, with $k \equiv \log\left(K/K_0 \right)$, and $\circ$ denotes the elementwise product. The growth rates of this process are

$$(10) \qquad \boldsymbol{\alpha} \equiv \left(\frac{1}{dt} \frac{\mathrm{E}_t\left[dK \right]}{K} - \frac{1}{2} \sigma_K^2, \mu F e^{-gt}, 0, 0 \right)^T.$$

The vector of mean reversion rates and the vector of means of this process are

---

[18] Statistical uncertainty describes genuinely stochastic and continuously fluctuating processes, whereas systemic uncertainty is potentially learnable. Climate sensitivity is not learnable in our model. There are likely aspects of climate sensitivity that are difficult or impossible to learn (cf. Roe and Baker, 2007).

(11) $$\mathbf{v} \equiv (0, \varphi, v_\chi, v_\lambda)^T \quad \text{and} \quad \overline{\mathbf{\mu}} \equiv (0, 0, \overline{\chi}, \overline{\lambda})^T.$$

The covariance matrix $\mathbf{SS}^T$ of the components of this multivariate process is

(12) $$\frac{1}{dt}\mathrm{E}_t\left[d\tilde{\mathbf{x}}d\tilde{\mathbf{x}}^T\right] = \mathbf{SS}^T = \begin{pmatrix} \sigma_K^2 & \rho_{KE}\sigma_K\sigma_E & \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{K\lambda}\sigma_K\sigma_\lambda \\ \rho_{KE}\sigma_K\sigma_E & \sigma_E^2 & \rho_{E\chi}\sigma_E\sigma_\chi & \rho_{E\lambda}\sigma_E\sigma_\lambda \\ \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{E\chi}\sigma_E\sigma_\chi & \sigma_\chi^2 & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda \\ \rho_{K\lambda}\sigma_K\sigma_\lambda & \rho_{E\lambda}\sigma_E\sigma_\lambda & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda & \sigma_\lambda^2 \end{pmatrix},$$

where $\rho_{ij}, i \neq j, i, j = K, E, \chi, \lambda$ denote partial correlation coefficients. The covariances imply that unexpected shocks to, for example, climate sensitivity (or temperature) may lead to unexpected shocks in the growth of the economy (or asset returns), which occurs on top of the direct effect of temperature shocks on economic activity via the damage ratio.[19]

The optimal solution from a social planner's perspective must satisfy the Hamilton-Jacobi-Bellman (HJB) equation

(13) $$\max_{C,F}\left[f(C,J) + \frac{1}{dt}\mathrm{E}_t\left[dJ\left(t, K, \tilde{E}, \tilde{\chi}, \tilde{\lambda}\right)\right]\right] = 0,$$

where $(1/dt)\mathrm{E}_t[dJ]$ is Ito's differential operator applied to J. Using $I(C, F, K, E, \chi, \lambda) = A(E, \chi, \lambda)K^\alpha F^{1-\alpha} - C - bF$ and Ito's lemma gives[20]

---

[19] An alternative interpretation of the covariances is that human behavioural bias (e.g., overconfidence or publication bias) may tilt prediction errors in the same direction for climatic and economic uncertainty.

[20] Strictly, (13) is not continuously differentiable, due to the truncation discussed in footnote 14, but we will ignore the (negligibly small) probability atoms at zero values of the states here (see section IV).

12

$$\max_{C,F} \left[f(C,J) + J_K \Phi(I(C,F,K,E,\chi,\lambda),K) + J_{\tilde{E}}(\mu F e^{-gt} - \varphi \tilde{E}) \right] + J_t$$

(14)

$$+ J_{\tilde{\chi}} \nu_\chi (\bar{\chi} - \tilde{\chi}) + J_{\tilde{\lambda}} \nu_\lambda (\bar{\lambda} - \tilde{\lambda}) + \frac{1}{2} J_{KK} K^2 \sigma_K^2 + \frac{1}{2} J_{\tilde{E}\tilde{E}} \sigma_E^2 + \frac{1}{2} J_{\tilde{\chi}\tilde{\chi}} \sigma_\chi^2 + \frac{1}{2} J_{\tilde{\lambda}\tilde{\lambda}} \sigma_\lambda^2$$

$$+ J_{K\tilde{E}} K \rho_{KE} \sigma_K \sigma_E + J_{K\tilde{\chi}} K \rho_{K\chi} \sigma_K \sigma_\chi + J_{K\tilde{\lambda}} K \rho_{K\lambda} \sigma_K \sigma_\lambda + J_{\tilde{E}\tilde{\chi}} \rho_{E\chi} \sigma_E \sigma_\chi$$

$$+ J_{\tilde{E}\tilde{\lambda}} \rho_{E\lambda} \sigma_E \sigma_\lambda + J_{\tilde{\chi}\tilde{\lambda}} \rho_{\chi\lambda} \sigma_\chi \sigma_\lambda = 0.$$

The optimality conditions with respect to C and F imply that the marginal value of investment and consumption must be the same, i.e. $f_C = C^{-\gamma} \left((1-\eta) J \right)^{(\gamma-\eta)/(\eta-1)} = J_K \Phi_I(I,K)$, and that the marginal product of energy equals its social cost, $(1-\alpha)Y / F = b + Pe^{-gt}$, where the optimal SCC is defined as the marginal disvalue of emitting an additional ton of carbon divided by the marginal value of consumption, i.e. $P \equiv -\mu J_{\tilde{E}} / f_C > 0.$ Our command optimum corresponds to the outcome in a decentralized market economy if emissions are priced at the SCC that results from the optimal solution, revenues are rebated in lump-sum manner, and no other externalities or market failures exist. We thus use the terms 'carbon price' and SCC interchangeably.

II. Perturbation Theory Solutions for the Optimal Risk-Adjusted SCC

A closed-form solution to the stochastic dynamic optimal control problem (14) does not exist.[21] Our approach to solving the HJB equation (14) is to use perturbation theory. Perturbation theory is a method for finding an approximate solution to a complicated problem by starting with the exact solution of a related, simpler problem, which in our case is that of the stochastic AK-model of Pindyck and Wang (2013). The complicated problem is thus not solved exactly, but instead

---

[21] Solving (14) numerically by approximating the value function and its derivatives in 5-dimensional space (time and the four states) is challenging due to the curse of dimensionality and does not yield analytical insight into the stochastic drivers of the optimal SCC.

so-called 'small' terms are added to adjust the solution of the simpler, exactly solvable problem. Perturbation theory provides a formal framework to control how small these adjustment terms are. A so-called small parameter ϵ must be defined so that we return to the simpler, exactly solvable problem in the limit $\epsilon \to 0$.

In Appendix A, we lay out in detail and from first principles how perturbation theory can be applied to (14). Crucially, we identify a single small parameter corresponding to the initial damage ratio (from (6)):

(15) $\qquad \epsilon \equiv D_0 = \chi_0^{1+\theta_{\chi T}} \lambda_0^{1+\theta_\lambda} (E_0 / S_{PI})^{1+\theta_{ET}}$,

where the subscript 0 denote values at $t = 0$. As we will show in Section IV, where we calibrate the model, damages only ever make up a few percent of GDP and are typically less than 10% even in the worst scenario (Nordhaus and Moffat (2017)), which justifies our choice of ϵ as small parameter. The solution for the value function takes the form of a series with terms of increasing order in ϵ, i.e.

(16) $\qquad J = J^{(0)} + \epsilon J^{(1)} + \mathcal{O}(\epsilon^2)$,

where we only evaluate the zeroth- and first-order terms, and the error between the unknown complicated problem J and our approximation $J^{(0)} + \epsilon J^{(1)}$ is said to be $\mathcal{O}(\epsilon^2)$. The first two terms are sufficient to evaluate a so-called leading-order estimate of the optimal SCC known as Result A in Appendix A.[22]

III. A Closed-Form Solution for The Optimal Risk-Adjusted SCC

Although Result A is amenable to rapid numerical evalation, we can obtain closed-form solutions for the optimal SCC if we make three additional

---

[22] This result involves evaluation of a high-dimensional integral, which is much less computationally demanding than using numerical methods to solve the Hamilton-Jacobi-Bellman equation (14) directly. In Section III, we further assume that the relative uncertainty in the three climate variables can be modelled as small parameters in order to get a tractable closed-form expression for the SCC.

assumptions[23], as explained in detail in Appendix A. These closed-form solutions are known as Results 1 and 2, and their accuracy compared to Result A is less than 2% even in the most demanding case we consider in Appendix F

We first assume *proportional damages* ($\theta_{ET} = 0$), so that marginal damages do not depend on the carbon stock, and no delayed deterministic response of temperature to increases in the carbon stock ($\chi_0 = \bar{\chi}$).

Result 1: If $\theta_{ET} = 0$ and $\chi_0 = \bar{\chi}$, the optimal SCC is

$$(17) \qquad P = \frac{\mu \, \Theta Y\big|_{P=0}}{r^* + \varphi}(1 + \Delta_\chi + \Delta_\lambda + \Delta_{CK} + \Delta_{CC}) \text{ with } \Theta \equiv \frac{D_E}{1 - D},$$

where $r^* = \rho + (\gamma - 1)(g^{(0)} - \frac{1}{2}\eta\sigma_K^2), \; \Delta_\chi = \frac{1}{2}\theta_{\chi T}(1 + \theta_{\chi T})\frac{(\sigma_\chi/\bar{\chi})^2}{r^* + 2v_\chi + \varphi},$

$$\Delta_\lambda = \frac{1}{2}\theta_\lambda(1 + \theta_\lambda)\frac{(\sigma_\lambda/\bar{\lambda})^2}{r^* + 2v_\lambda + \varphi}, \; \Delta_{CC} = (1 + \theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\bar{\chi}\bar{\lambda}}{r^* + v_\chi + v_\lambda + \varphi} \text{ and}$$

$$\Delta_{CK} = -(\eta - 1)\sigma_K\left((1 + \theta_{\chi T})\frac{\rho_{K\chi}\sigma_\chi/\bar{\chi}}{r^* + v_\chi + \varphi} + (1 + \theta_\lambda)\frac{\rho_{K\lambda}\sigma_\lambda/\bar{\lambda}}{r^* + v_\lambda + \varphi}\right). \qquad \square$$

Without uncertainty, $P = \mu \, \Theta Y\big|_{P=0}\big/(r^* + \varphi)$ with $r^* = \rho + (\gamma - 1)g^{(0)}$. This expression shows the well-known geophysical (μ and φ), economic (Y and $g^{(0)}$), damage (Θ) and ethical (ρ and γ) determinants of the optimal deterministic SCC.

---

[23] First, we assume the future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through its dependence on the stochastic capital stock (*Assumption I*). Second, we will include only the leading-order effects of uncertainty (*Assumption II*) by performing an additional perturbation expansion. Third, we set the initial and steady-state values of the damage ratio parameter λ_0 and $\bar{\lambda}$ to be equal, so deterministic damages are not subject to a delay (*Assumption III*), but do not make the same assumption for the climate sensitivity parameter χ .

More patience (lower ρ) boosts the SCC. [24] Rising affluence (higher $g^{(0)}$) pushes up the discount rate if intergenerational inequality aversion exceeds one, and thus curbs the appetite of current generations for ambitious climate policy (the $\gamma g^{(0)}$ term in r^*). Higher economic growth also implies growing damages and a lower (growth-corrected) discount rate (the $-g^{(0)}$ term in r^*), which increases the optimal SCC. Economic growth thus depresses the SCC if $\gamma > 1$. Higher economic activity (Y) and normalized marginal damage ratio (Θ) also push up the SCC. A small fraction of emissions that stays in the atmosphere (μ) and fast decay of atmospheric carbon (higher φ) curb the SCC.

A. Effects of economic growth uncertainty on the optimal SCC

Including economic, but not climatic uncertainty, Result 1 boils down to $P = \mu \, \Theta Y|_{P=0} / (r^* + \varphi)$ where the risk-adjusted discount rate can be written as

$$(18) \qquad r^* = \underbrace{\rho}_{\text{impatience}} + \underbrace{\gamma g^{(0)}}_{\text{rising affluence}} - \underbrace{g^{(0)}}_{\text{growing damages}} \underbrace{-\frac{1}{2}(1+\gamma)\eta\sigma_K^2}_{\text{prudence}} + \underbrace{\eta\sigma_K^2}_{\text{insurance}}.$$

The first two terms are the familiar Keynes-Ramsey terms, the third term corrects for damages growing in line with output, the fourth term is the prudence term,

---

[24] In contrast to exogenous Ramsey growth models such as Golosov et al. (2014) and Nordhaus (2017), our rate of economic growth $g^{(0)}$ is endogenous. Hence, there are indirect effects on the optimal SCC via the growth rate $g^{(0)}$. For example, the direct effect of a higher rate of pure time preference ρ is to lower the SCC and the indirect effect is to raise the SCC as economic growth is lowered (for $\gamma \geq 1$). Together, the effect of a higher rate of pure time preference on the discount rate is always positive $\partial r^*/\partial \rho = 1 + (\gamma - 1)\partial g/\partial \rho = 1/\gamma$ with $\partial g/\partial \rho = -1/\gamma$ (and thus always negative on the SCC). Although the optimal SCC does not depend directly on the share of fossil fuel in value added, the cost of fossil fuel, adjustment costs or the depreciation rate of physical capital, it does depend on adjustment costs and the depreciation rate via their effect on the endogenous rate of economic growth, which we treat as fixed in the analysis below. Furthermore, Ramsey growth models with an exogenous long-run growth rate include a second time scale associated with economic convergence, which will typically be faster than the climatic time scales. We conjecture that our formula for the optimal SCC derived in an AK-growth model will therefore be a good approximation to the optimal SCC for a Ramsey growth model.

which increases in the coefficient of relative prudence $1+\gamma$ and risk aversion η (cf. Leland, 1968; Kimball, 1990), and the insurance term stems from perfect correlation between damages and output. Impatience, rising affluence and insurance depress the SCC but growing damage and prudence boost the SCC. For $\gamma > 1$, the prudence effect dominates the insurance effect so growth uncertainty curbs the discount rate and boosts the SCC (cf. Nordhaus, 2017; Gollier, 2018). [25] If EIS $= 1/\gamma > 1$, growth uncertainty depresses the SCC; as in the asset pricing literature, it depresses the price-dividend ratio.[26] Our equation for r^{*} in Result 1 corresponds to equation (13) in Barro (2009) for the dividend-price ratio (abstracting from the risk of macroeconomic disasters). Result 1 shows that this equation is still relevant with climate uncertainties.

Dietz et al. (2018) allow the elasticity of damages with respect to output denoted by $0 \leq \theta_D \leq 1$ to differ from one.[27] If $\eta = \gamma$, equation (18) becomes[28]

$$(18') \qquad r^{*} = \rho + \gamma g^{(0)} - \underbrace{\theta_D \left(g^{(0)} - \frac{1}{2}(1-\theta_D)\sigma_K^{2} \right)}_{\text{growing damages}} - \underbrace{\frac{1}{2}(1+\gamma)\gamma\sigma_K^{2}}_{\text{prudence}} + \underbrace{\theta_D \gamma \sigma_K^{2}}_{\text{insurance}}.$$

---

[25] With logarithmic preferences ($\gamma=1$) and proportional damages, $\Delta_{CK} = 0$ and (17) simplifies to $P = \mu \Theta Y|_{P=0} (1 + \Delta_{\chi} + \Delta_{\lambda} + \Delta_{CC}) / (\rho + \varphi)$. Economic growth uncertainty and the covariance of climate sensitivity and the damage ratio with respect to the economy do not affect the optimal SCC, but climate sensitivity and damage ratio uncertainty and their correlation do. The simple rule put forward by Golosov et al. (2014) does not consider these uncertainties and in our case reduces to $P = \mu \Theta Y|_{P=0} / (\rho + \varphi)$. When we use a 2-box carbon cycle with a permanent and a temporary reservoir, we get in this case that $P = \Theta Y|_{P=0} \left[(1-\mu) / \rho + \mu / (\rho + \varphi) \right]$ with μ is the fraction of emissions that goes into the temporary reservoir.

[26] Bansal et al. (2012) argue that values of EIS < 1 give rise to the wrong sign of several risk premia in asset markts.

[27] Dietz et al. (2018) use Monte Carlo simulations of DICE (Nordhaus, 2008) and find that, with emissions-neutral technical change, future states with rapid technical progress imply more emissions, more warming and a greater benefit from curbing emissions. The positive correlation between consumption and the benefits of mitigation implies a positive climate beta. This beta is close to one if damages are proportional to GDP, but closer to zero if damages are additive. Our section III.C analyses the effects of correlations between temperature, damage and economic shocks more generally.

[28] A similar expression is derived by Svenssen and Traeger (2014) and Dietz et al. (2018). Rewriting (18′), the risk-adjusted discount rate becomes $r^{*} = r_{\text{rf}}^{(0)} - \theta_D(g - \sigma_K^2/2 - \gamma\sigma_K^2) - \theta_D^2\sigma_K^2/2$ with $r_{\text{rf}}^{(0)} = \rho + \gamma g^{(0)} - \gamma(1+\gamma)\sigma_K^2/2$ the risk-free interest rate, corresponding to Proposition 1 in Dietz et al. (2018). We only show (18′) for illustrative purposes and do not derive it as part our model.

A value of θ_D lower than 1 curbs the negative effect of growing damages on the discount rate and raises the SCC by less than if $\theta_D = 1$. The insurance term is smaller for $\theta_D < 1$, so this pushes up the SCC relative to when $\theta_D = 1$.[29]

B. Climate and damage uncertainties

The term $\Delta_\chi = (1/2)\theta_{\chi T}(1+\theta_{\chi T})(\sigma_\chi / \bar{\chi})^2 / (r^* + 2\nu_\chi + \varphi)$ in (17) is the *climate sensitivity risk adjustment* and depends on $\theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi \theta_T$, which combines positive skewness of the (equilibrium) climate sensitivity distribution ($\theta_\chi > 0$) and convex dependence of damages on temperature ($\theta_T > 0$). This adjustment is positive and larger for a more convex damage function, a more skewed climate sensitivity distribution with higher uncertainty (σ_χ), a smaller discount rate (r^*), and slower carbon decay rate (φ). The *damage ratio risk adjustment* $\Delta_\lambda = (1/2)\theta_\lambda(1+\theta_\lambda)(\sigma_\lambda / \bar{\lambda})^2 / (r^* + 2\nu_\lambda + \varphi)$ in (17) is zero if the distribution of the damage ratio is not skewed ($\theta_\lambda = 0$). A right-skewed distribution requires an increase in the SCC, more so if damages are more uncertain. When keeping the steady-state uncertainties $\Sigma_\chi^\infty \equiv \sigma_\chi / \sqrt{2\nu_\chi}$ and $\Sigma_\lambda^\infty \equiv \sigma_\lambda / \sqrt{2\nu_\lambda}$ fixed, higher rates of mean reversion ν_χ and ν_λ increases the risk adjustments as the near future becomes more uncertain.

C. Risk-insurance and risk-exposure effects

---

[29] The SCC increases if θ_D is decreased depending on the sign of $\partial r^* / \partial \theta_D = -g^{(0)} + (1 - 2\theta_D + 2\gamma)\sigma_K^2 / 2$.

We can rewrite the term in Result 1 that adjusts for correlations between climate and damage ratio risks, on the one hand, and economic risks, on the other hand, as

$$(19) \quad \Delta_{CK} = -(\eta-1)\Sigma^2_{K,\chi,\lambda} \quad \text{with } \Sigma^2_{K,\chi,\lambda} \equiv \sigma^2_K \left(\frac{(1+\theta_{\chi T})\beta_{K\chi}}{r^* + v_\chi + \varphi} + \frac{(1+\theta_\lambda)\beta_{K\lambda}}{r^* + v_\lambda + \varphi} \right),$$

where $\beta_{K\chi} \equiv \rho_{K\chi}\sigma_\chi/(\bar{\chi}\sigma_K)$ and $\beta_{K\lambda} \equiv \rho_{K\lambda}\sigma_\lambda/(\bar{\lambda}\sigma_K)$ denote the normalized covariances of the climate sensitivity and damage ratio shocks, respectively, with shocks to the rate of economic growth.[30] The sign of (19) depends on whether relative risk aversion η exceeds one, i.e. on whether the *risk-insurance* effect ($\eta\Sigma^2_{K,\chi,\lambda}$) dominates the *risk-exposure* effect ($\Sigma^2_{K,\chi,\lambda}$) due to growing damages (cf. Lemoine, 2020).

Turning to the *risk-insurance* ($\eta\Sigma^2_{K,\chi,\lambda}$) effect first, we note that a negative correlation between climate sensitivity and economic shocks ($\beta_{K\chi} < 0$) implies that asset returns are low in future states of nature in which temperature is high. It is then optimal to insure these investments more by raising the SCC. If the world economy benefits from higher temperature in future states, this correlation is positive ($\beta_{K\chi} > 0$), so the SCC is lower. An example of such a positive correlation may be volcanic eruptions, which can be seen as a combination of a negative shock to the climate sensitivity through particulate emissions and a negative shock to the economy. Another example could be innovation leading to increased installation of solar panel, which boosts both the economy and temperature (due to albedo effect of dark panels). The adjustment is large if risk aversion is high, climate sensitivity is more uncertain and skew, damages are more convex, and the normalized covariance for the climate sensitivity is large (high η, σ_χ, θ_χ, θ_T, $\beta_{K\chi}$) and is non-

---

[30] Consistent with our perturbation scheme, the volatility of GDP is given to leading order by the volatility of the capital stock neglecting the effect of climate damages and thus the carbon stock, climate sensitivity and damage uncertainties on this volatility. These betas are defined analogously to the betas in asset pricing theory (e.g., Lucas, 1978; Breeden, 1979), but they are related to 'hedging' of the climate asset against the only risky 'financial asset' in our model (capital).

zero even for a symmetric climate sensitivity distribution and a damage ratio that depends linearly on temperature ($\theta_{\chi T} = 0$).

A negative correlation between damage ratio and economic shocks ($\beta_{K\lambda} < 0$) implies that asset returns will be low in future states of nature in which the damage ratio is high (over and above the effect of damages being proportional to GDP). This justifies a higher SCC. One example of this negative correlation may be the Covid-19 pandemic, which was unexpected and may lower asset returns and make more areas of the economy vulnerable to climate change (e.g., food supply chains or health systems). In the hypothetical case that the world economy benefits from climate damage (e.g., through ingenious water engineering in response to damages that improves living conditions), there is a positive correlation ($\beta_{K\lambda} > 0$) and carbon should be priced less vigorously. The adjustment is large if risk aversion is high, the damage ratio has high uncertainty and skewness (high $\eta, \sigma_{\lambda}, \theta_{\lambda}$) and is non-zero even for a symmetric damage ratio distribution ($\theta_{\lambda} = 0$).

The offsetting *risk-exposure* effects ($\Sigma^2_{K,\chi,\lambda}$) in (19) occur because future states of nature that are associated with high asset returns are associated with large damages (as damages are proportional to GDP). E.g., if $\beta_{K\chi} < 0,$ future states of nature with negative GDP shocks are associated with lower damages, which requires a lower SCC. *Risk-insurance* effects dominate *risk-exposure* effects if risk aversion is large enough, i.e. $\eta > 1$.[31]

---

[31] Sandsmark and Vennemo (2007) have one stochastic parameter (the loss of GDP for given temperature) and additive damages ($\theta_D = 0$). High future damages are then associated with low future aggregate consumption, so the corresponding beta is negative. It relies on the product of the change in marginal utility due to damages and marginal damages themselves, which is $\mathcal{O}(\epsilon^2)$ in our perturbation scheme and too small to be included. Nordhaus (2011) argues that "those states in which the global temperature increase is particularly high are also ones in which we are on average richer in the future", suggesting a positive beta. In our perturbation theory approach this effect does not feature in our correction factors, since it requires the integration of a Geometric Brownian Motion (for *K*), when solving the differential equation for the carbon stock, which cannot conveniently be done in closed form. If $\theta_{ET} = 0,$ this effect is zero as marginal damages are no longer proportional to the carbon stock *E* and enhanced uncertainty of this term due to uncertain new emissions does not contribute to the SCC.

D. Correlation between temperature and damage ratio risks

The term $\Delta_{CC} = (1+\theta_{\chi T}) \rho_{\chi\lambda}(\sigma_\chi / \bar{\chi})(\sigma_\lambda / \bar{\lambda})/(r^* + v_\chi + v_\lambda + \varphi)$ in (19) of Result 1 captures the effect of correlation between temperature and damage ratio uncertainty on the SCC. This is positive if high temperature shocks are associated with a disproportionally high damage ratio (e.g., extreme weather/climate events such as hurricanes and fires as far as they are not captured by the convex dependence of damages on temperature), in which case the optimal SCC is higher. Risk aversion η plays no role, since there is no possibility of self-insuring.

E. Result 2

Result 2 relaxes the two assumption underlying Result 1 ($\theta_{ET} = 0$ and $\chi_0 = \bar{\chi}$), while still only considering the leading-order effects of uncertainty.[32]

Result 2: The optimal SCC if damages are not proportional to the carbon stock ($\theta_{ET} \neq 0$) and with a delayed deterministic temperature response ($\chi_0 \neq \bar{\chi}$) is

(20)
$$P = \frac{\mu \, \Theta(E, \chi_0) Y|_{P=0}}{r^\star} \left(1 + \theta_{ET} \frac{\mu F^{(0)}}{E} \frac{\Upsilon_{\theta_{ET} \neq 0}}{r^{\star\star}} \right.$$
$$\left. + (1+\theta_{\chi T}) \frac{v_\chi}{r^\star} \frac{\bar{\chi} - \chi}{\chi} \Upsilon_{\chi_0 \neq \bar{\chi}} + \Delta_{EE} + \Delta_{\chi\chi} + \Delta_{\lambda\lambda} + \Delta_{\chi\times\lambda} + \Delta_{CK} + \Delta_{CC} \right),$$

where $r^\star \equiv r^* + (1+\theta_{ET})\varphi$, $r^{\star\star} \equiv r^* + (\eta-1)\sigma_K^2 - \varphi$ and $F^{(0)} = ((1-\alpha)/b)^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} K$ is optimal fossil fuel use without climate policy (to zeroth order of approximation). Like in (17), the Δ-terms in (20) are the uncertainty adjustments and are given by

---

For the case $\theta_{ET} > 0$, we examine this effect by numerically solving the stochastic differential equations and the integral in Result A and find it to be small (see Appendix F).

[32] These two assumptions are known as *Assumptions IV and V* in Appendix A. We could also allow for the initial damage ratio not to be at its steady state, but this would lead to a more cumbersome expression. We allow for this in Result A.

(A4.1)-(A4.5) in Appendix A. We distinguish two types of multiplicative correction factors, for $\theta_{ET} \neq 0$ and for $\chi_0 \neq \bar{\chi}$, which can be linearly combined, for example: $\Upsilon_{\chi\chi} \equiv \Upsilon_{\chi\chi,\theta_{ET} \neq 0} + \Upsilon_{\chi\chi,\chi_0 \neq \bar{\chi}}$, where $\Upsilon_{\chi\chi}$ is the correction factor by which $\Delta_{\chi\chi}$ in Result 1 has to be multiplied to obtain $\Delta_{\chi\chi}$ in Result 2. These correction factors are given in (D3.4)-(D3.5) in Appendix D.

The effects of convexity of reduced-form damages ($\theta_{ET} > 0$) in Result 2 are fourfold. First, the normalized marginal damage ratio rises with the stock of atmospheric carbon so that the time path for the carbon price is steeper than of world GDP. The correction factor $\Upsilon_{\theta_{ET} \neq 0} > 0$ reflects the more harmful effect of future emissions when the stock is higher. Second, convex damages boost the effective discount rate, since the marginal damage of a unit of carbon decays more quickly than the unit itself, depressing the SCC. Third, if damages are not too convex $(0 < \theta_{ET} < 1)$, the adjustment for carbon stock uncertainty is negative instead of zero as in Result 1. Fourth, the adjustments for the other two climatic uncertainties are multiplied by correction factors that are greater than one, reflecting rising marginal damages.

Result 2 also captures the delay in the deterministic temperature response $(\chi_0 \neq \bar{\chi})$) by the multiplicative correction factors. If Result 1 is evaluated with the generally higher value $\bar{\chi}$, it ignores this delay and overestimates the SCC. Further discussion of the effects captured by Result 2 can be found in Appendix A.4.2.

IV. Calibration

Table 1 summarizes our calibration starting from base year 2015 with further details in Appendix E. To calibrate the non-climatic part of our model to match historical asset returns, we follow Pindyck and Wang (2013) but abstract from catastrophic shocks to economic growth (see Appendices E.1 and E.2). This gives

a coefficient of relative risk aversion of $\eta = 4.3$, intergenerational inequality aversion of $\gamma = 1.5$, pure time preference of $\rho = 5.8\%$ per year, trend growth of $g^{(0)} = 2.0\%$ per year, volatility of asset returns of $\sigma_K = 12\%$ per year$^{1/2}$ and a risk premium of $r^{(0)} - r_{\rm rf}^{(0)} = \eta\sigma_K^2 = 6.4\%$ per year (with $r^{(0)}$ the risky and $r_{\rm rf}^{(0)}$ the risk-free rate). In line with the specification in (4), we assume the global ratio of CO_2 emissions to GDP declines at a rate of 2.0% per year, which matches recent data.[33] Following Nordhaus (2017), we use world GDP at PPP of 116 trillion US dollars in 2015. Table 1 gives details for investment, depreciation and fossil fuel costsil fuel.

TABLE 1 – SUMMARY OF BASE CASE CALIBRATION

Impatience and aversion to intergenerational inequality and risk	$\rho = 5.8\%$/year, IIA = 1/EIS = γ = 1.5, RRA = η = 4.3
World economy	$A^* = 0.113$ /year, GDP PPP= 116\$T/year, $g^{(0)} = 2.0$ %/year
Investment, depreciation and adjustment cost	$i^{(0)} = 2.8\%$/year, $\delta = 0.33\%$/year, $\omega = 12.5$ year
Asset volatility and returns	$\sigma_K = 12\%$/year$^{1/2}$, $r^{(0)} = 7.2\%$/year, $r_{\rm rf}^{(0)} = 0.80\%$/year, $r^{(0)} - r_{\rm rf}^{(0)} = \eta\sigma_K^2 = 6.4\%$/year
Share of fossil fuel and production cost	$1 - \alpha = 4.3\%$, $b = \$5.4 \times 10^2$ /tC
Preindustrial and 2015 ($t = 0$) carbon stocks Concavity of Arrhenius'law & stochastic carbon stock dynamics	$S_{\rm PI} = 596$ GtC, $S_0 = 854$ GtC, $E_0 = 258$ GtC, $\theta_E = -0.36$, $\mu = 0.65$, $\varphi = 0.35\%$/year, $\sigma_E = 13$ ppmv/year$^{1/2}$
Distribution of the climate sensitivity	$\chi_0 = 1.11$, $\bar{\chi} = 1.26$, $\sigma_\chi = 2.0\%$/year$^{1/2}$ $\nu_\chi = 0.86\%$/year, $\theta_\chi = 3.0$
Distribution of the damage ratio	$\theta_T = 0.56$ ($\theta_{ET} = 0$), $\bar{\lambda} = 0.21$, $\sigma_\lambda = 2.3\%$/year$^{1/2}$, $\theta_\lambda = 2.7$, $\nu_\lambda = 0.20$/year
Initial damage ratio and initial normalized marginal damage ratio	$D_0 = 0.29\%$, $\Theta_0 = 2.07\%$ GDP/TtC
Conversion factors	1 ppmv CO_2 = 2.13 GtC, 1 tC = 3.664 tCO_2

A. Carbon stock uncertainty

To calibrate our 1-box model for carbon stock dynamics (4), we use the 17 linear impulse response functions from the survey in Joos et al. (2013) and find $\mu = 0.65$ and $\varphi = 0.35\%$/year.[34] We use the 90% confidence range 794-1149 ppmv in 2100

---

[33] The global ratio of CO_2 emissions to GDP ratio declined at 2.1% per year during 2000-2015 versus a decline of 0.8% per year in the decade before. Nordhaus (2017) uses a decline of 1.5% per year.
[34] It is possible to estimate these values from historical data too (see Appendix E3).

predicted by simulations for the high-temperature scenario RCP 8.5 (Chapter 12.4.8.1, IPCC, 2014 AR5) to calibrate $\sigma_E = 13$ ppmv/year$^{1/2}$.Fig. 1a shows the impulse response function for our 1-box model and Fig. 1b shows the stock of atmospheric carbon, including 95%-confidence bounds.[35] Fig. 1 shows that our simple 1-box model compares well with the 4-box model fitted to the same data by Aengenheyster et al. (2018) and the 2-box model of Golosov et al. (2014).[36,37,38] Our confidence bands are much wider than those in Joos et al. (2013)[39] and still much wider than the uncertainty range obtained from historical data,[40] suggesting that model uncertainty may far exceed any inherent variability.

---

[35] From $\Sigma_E(t) = (1149\text{-}794)/3.29 = 108$ ppmv, $\sigma_E = \Sigma_E(t)\sqrt{2\varphi/\left(1-\exp(-2\varphi t)\right)} = 13$ ppmv CO_2 /year$^{1/2}$ with $t = 2100$ — 2005 = 95 years and using $\varphi = 0.35\%$/year, which corresponds to a steady-state uncertainty of $\Sigma_E^\infty = \sigma_E/\sqrt{2\varphi} = 155$ ppmv CO_2. The confidence band from IPCC (2014, AR5) is shown centred around the (different) mean of our prediction and translated in time to 2110 to reflect different initial times. Note that the RCP 8.5 scenario is associated with higher emissions than in our base case as can be gauged from Fig. 1b. We nevertheless use the standard devation from the RCP 8.5 scenario to act as an upper bound on atmospheric stock uncertainty. The probability of a value of $E \leq 0$ is indeed negligibly small, as previously assumed, and we formally have a negligibly small atom at $E = 0$.

[36] For a linear N-box carbon cycle $\tilde{S} = \sum_{i=0}^N \tilde{S}_i$ by $\mathrm{d}\tilde{S}_i/\mathrm{d}t = \mu_i F e^{-gt} - \varphi_i \tilde{S}_i$, Aengenheyster et al. (2018) obtain $\mu = \{0.2173, 0.2240, 0.2824, 0.2763\}$, $\varphi = \{0, 0.25, 2.74, 23.23\}\%$/year with $\tilde{S}(t=0) = \{328, 40, 27, 5\}$ppmv scaled so $\tilde{S}(t=0) = 401$ ppm. We adapt Golosov et al. (2014) to continuous time and get $\mu = \{0.2, 0.3215\}$, $\varphi = \{0, 0.23\}\%$/year and $\tilde{S}(t=0) = \{0.85, 0.15\}^\chi 401$ ppm, ignoring its third box for carbon that decays within the first decadal period.

[37] We set the initial atmospheric carbon concentration to $S_0 = 401$ ppm of CO_2 (May 2015), corresponding to 0.854 TtC or 3.13 TtCO$_2$, and the preindustrial atmospheric carbon concentration to 280 ppm CO_2, 0.596 TtC or 2.19 TtCO$_2$, so that $E_0 = 121$ ppm CO_2, 0.258 TtC or 0.94 TtCO$_2$. Updated and historical values can be found online at http://www.esrl.noaa.gov/gmd/ccgg/trends/global.html.

[38] Although our calibration for μ and $\emptyset$ captures the long-term impulse response (Fig. 1a) and stock build-up (Fig. 1b), the resulting underestimation of the impulse response for small time (Fig. 1a could lead to an underestimate of the SCC, especially for high discount rates. Although the impulse response function is less well captured by our 1-box model, this must be time integrated (after discounting) to evaluate the SCC. Agreement of the time path of the atmospheric stock (Fig. 1b) is thus more important, especially if $\theta_{ET} \neq 0$ and the dependence on the stock is nonlinear.

[39] Using the distribution at $t = 95$ years and $\varphi = 0.35\%$/year, we get $\sigma_E = 3.7$ ppm/year$^{1/2}$, which is much higher than the value of $\sigma_E = 0.65$ ppm/year$^{1/2}$ obtained by Aengenheyster et al. (2018) based on Joos et al. (2013).

[40] Based on the historical Law Dome Ice Core 2000-year dataset for emissions and concentrations, we estimate $\sigma_E = 0.1$-0.15 ppmv CO_2/year$^{1/2}$ (see Appendix E3). Using the same dataset but fitting a Geometric Brownian Motion, Hambel et al. (2017) find a much larger volatility of 0.78 %/year$^{1/2}$. Estimating this volatility, we find 1.4, 0.5 and 0.2 %/year$^{1/2}$ for the periods 1800-2004, 1900-2004 and 1959-2004. This large variation of volatility with time suggest that historical volatility in the atmospheric carbon concentrations is better described by an Arithmetic Brownian Motion, as in (6).
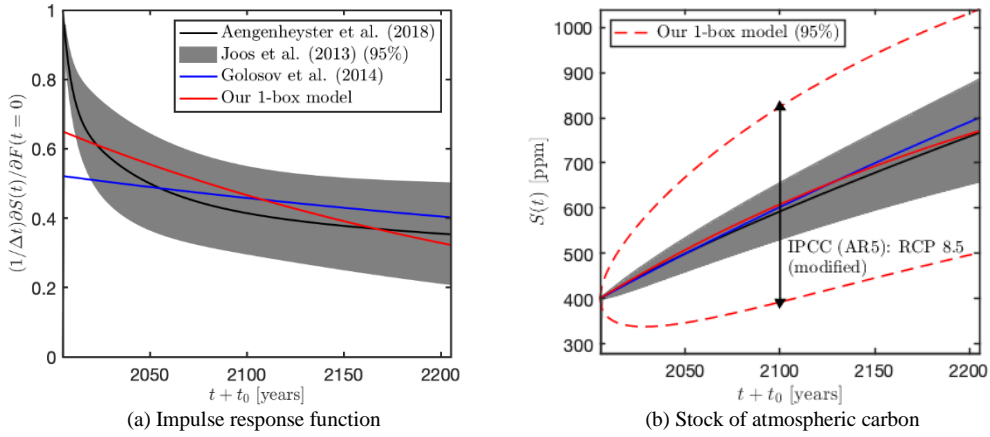
(a) Impulse response function   (b) Stock of atmospheric carbon

FIGURE 1. ATMOSPERHIC CARBON CYCLE AND UNCERTAINTY

Nevertheless, we will show in section V that even with our high value of σ_E, the adjustment to the optimal SCC is small for $\theta_{ET} \neq 0$ (it is zero for $\theta_{ET} = 0$).[41]

B. Climate sensitivity uncertainty

We calibrate our temperature model (5) and (8a) to capture the key features of *both* the transient climate response (TCR) and the equilibrium climate response (ECS).[42] The ECS is the equilibrium or long-term change in annual mean global temperature following a gradual doubling of the atmospheric carbon stock relative to pre-industrial levels. The TCR is the change in temperature following an increase of 1% in the atmospheric stock of carbon each year at the time of doubling (i.e., 70 years). The distributions of the ECS and the TCR capture both statistical and modelling uncertainties, but are in our view the best characterized measures of the

---

[41] The adjustment to the SCC is potentially larger than we calculate here, since there is a risk that as global warming continues (sudden) releases of greenhouse gases (e.g., from thawing permafrost) and reductions in the capacity of oceans to absorb CO_2 cause additional global warming. The existing modelling of such positive feedbacks "do not yield coherent results beyond the fact that present-day permafrost might become a net emitter of carbon during the 21st century under plausible future warming scenarios (low confidence)" (IPCC, 2014, AR5, Chapter 12.4.8.1) and we thus exclude it here.

[42] From (7), $T_2 \equiv T(E = E_{PI}, \chi) = \chi^{1+\theta_\chi}$ with χ normally distributed with time-varying mean $\mu_\chi = \chi_0 \exp(-\nu_\chi t)$ $+ \bar{\chi}\left(1 - \exp(-\nu_\chi t)\right)$ and standard deviation $\Sigma_\chi = \sigma_\chi \sqrt{\left(1 - \exp(-2\nu_\chi t)\right)/2\nu_\chi}$, and its skewness is given to leading-order by skew$\left[T_2\right] \equiv \mathrm{E}\left[\left(T_2 - \mathrm{E}[T_2]\right)^3\right] = 3\theta_\chi (1 + \theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)} (\Sigma_\chi / \mu_\chi)^4 + \mathcal{O}(\Sigma_\chi^6)$.

uncertainty associated with predicted temperature increase in the climate science literature.

(a) Transient climate response   (b) Equilibrium climate sensitivity
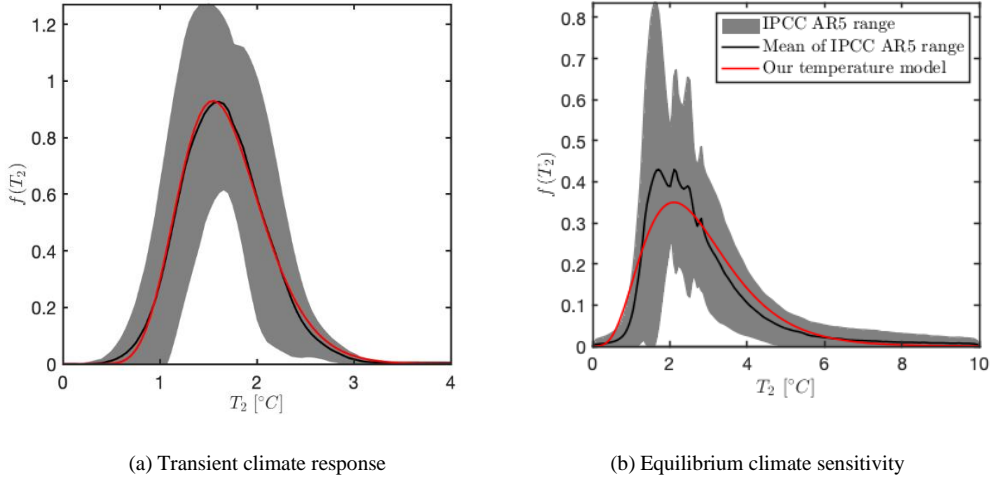
FIGURE 2. CLIMATE SENSITIVITY (PROBABILITY DENSITY FUNCTION)

Fig. 2 shows the range of probability density functions proposed for the TCR and ECS in IPCC (2014, AR5).[43] We take the mean of these distributions and fit our model to the first two moments of the TCR (mean and variance) and the first three moments of the ECS (mean, variance and skewness), as well as an initial temperature of $T_0 = 0.89°C$ above preindustrial. Reflecting our different initial CO_2 concentration from the TCR scenario, we ensure that our model matches the uncertainty of the TCR at the time the concentration reaches twice preindustrial, i.e., at $t = 17$ years from initial time (2015), rather than the original definition of 70

---

[43] We take the 7 distributions for the TCR and the 13 distributions for the ECS from Fig. 10.20 in Chapter 10.8 of IPCC (2014, AR5). The grey area in Fig. 2 corresponds to one standard deviation either side of the mean of these different distributions (negative values not shown). More recent climate model simulations find a higher range for the ECS which is due to cloud feedback and the interaction with aerosol forcing (Meehl et al., 2020). These effects are not captured by our calibration.

years from preindustrial.[44,45] Tables 2 and 3 show that we match these moments well, and are in line with the consensus likelihood ranges in IPCC (2014, AR5). Fig. E2 in Appendix E plots the distribution of temperature as function of time.

TABLE 2. CLIMATE SENSITIVITY UNCERTAINTY

	TCR		ECS	
	IPCC (2014, AR5)	Our calibration	IPCC (2014, AR5)	Our calibration
$E[T_2]$	1.7°C	1.7°C	2.8°C	2.8°C
$var[T_2]$	0.19°C^2	0.20°C^2	1.5°C^2	1.7°C^2
$skew[T_2]$	0.16°C^3	0.054°C^3	2.4°C^3	2.5°C^3

TABLE 3. CLIMATE SENSITIVITY LIKELIHOOD

		IPCC (2014, AR5)	Our calibration
TCR	1-2.5°C	'very likely' (90-100%)	91%
	> 3°C	'extremely unlikely' (0-5%)	0.72%
ECS	1.5-4.5°C	'likely' (66-100%)	75%
	< 1°C	'extremely unlikely' (0-5%)	4.2%
	> 6°C	'very unlikely' (0-10%)	2.3%

C. Damage ratio uncertainty

To calibrate the damage ratio and its uncertainty given in (3), we use the survey by Nordhaus and Moffat (2017) (henceforth NM17) including their subjective weights to reflect the reliability of different estimate shown in Fig. 3.[46] From these data, we estimate a mean $\lambda_0 = \bar{\lambda} = 0.21$, standard deviation $.\Sigma_\lambda^\infty = .036$, damage convexity $\theta_T = 0.56$ and skewness parameter $\theta_\lambda = 2.7$ of the damage ratio,[47] which we take to correspond to the steady state, setting the mean-reversion coefficient ν_λ

---

[44] To capture these and initial temperature, we match the TCR at $t = \ln(2S_{PI}/S_0)/0.02 = 17$ years from 2015 (instead of 70 years from preindustrial). We thus deviate from the formal definition of TCR, but argue this is justified as the high-level uncertainties in TCR and ECS are by far the best characterized of all summary statistics. This gives $\chi_0 = 1.11$, $\bar{\chi} = 1.26$, $T_0 = 0.89$°C, $\sigma_\chi = 2.0\%/\text{year}^{1/2}$, $\theta_\chi = 3.0$ and $\nu_\chi = 0.86\%$ per year corresponding to an e-folding scale of $1/(2\nu_\chi) = 58$ years. Climate sensitivity (as a proxy for temperature) is initially below its long-run value ($\chi_0 \le \bar{\chi}$).

[45] The probability of $\chi \le 0$ is indeed negligibly small, as previously assumed. Since the truncated variable $\max[\chi,0]$ cannot take negative values, we formally have a negligibly small atom at $\chi = 0$ (and $T_2 = 0$).

[46] Since our formulation does not allow for negative damages, we omit these estimates, which were given low weights of 0.1 by NM17. Fig. 3a shows omitted estimates in open circles and included estimates in closed circles. Since λ cannot take negative values, there is a negligibly small atom at $\lambda = 0$ (and $D = 0$).

[47] Ackerman and Stanton (2012) and Weitzman (2012) used a damage function which is even more convex at high temperatures. NM17 examines evidence for thresholds or large convexities in the damage function, but did not find any.

to a large value of 20%/year (so $\sigma_\lambda = \Sigma_\lambda^\infty \sqrt{2\nu_\lambda} = 2.3\%/\text{year}^{1/2}$). The distribution has a positive standardized skewness $\text{skew}^*[D|T] = 3\theta_\lambda \Sigma_\lambda^\infty / \bar{\lambda} = 0.29$.

The continuous red line in Fig. 3a denotes the expected damage ratio with the red shaded area corresponding to the 90% confidence band.[48] Fig. 3a also shows NM17's preferred regression $(D = 0.0018T^2)$, which agrees closely with our expected damage ratio. Finally, following Nordhaus and Sztorc (2013) and NM17, we adjust damages shown in Fig. 3a upwards by 25% at all temperatures to reflect damages not included in current estimates. Combined with our calibrated value of $\theta_E = -0.36$ (see Appendix E4), we obtain *proportional* damages $(\theta_{ET} = 0)$ from this calibration.

(a) Proportional damages ($\theta_T = 0.56$, $\theta_{ET} = 0$) (b) Convex damages ($\theta_T = 1$, $\theta_{ET} = 0.28$)
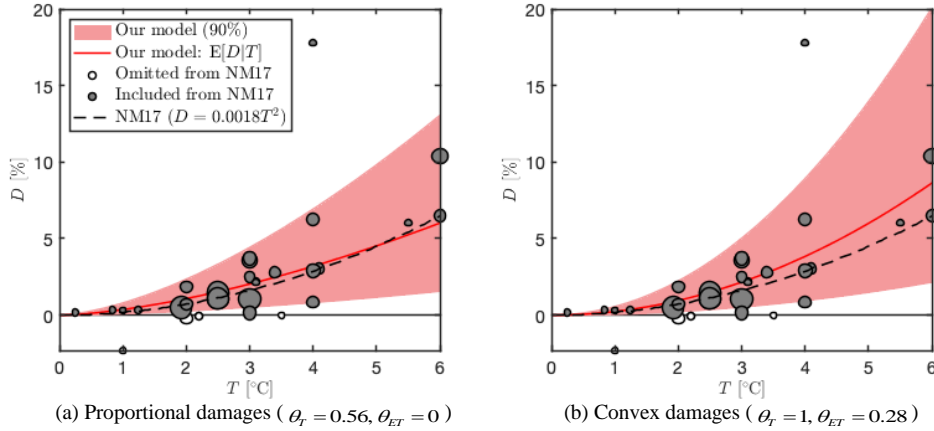
FIGURE 3. DAMAGE RATIO UNCERTAINTY

Fig. 3b also gives a calibration in which the damage ratio is constrained to be quadratic in temperature, so *convex* damages ($\theta_T = 1$, $\theta_{ET} = 0.28$).[49] Our estimates imply normalized marginal damage ratio $\Theta(E_0, \chi_0, \bar{\lambda})$ of 2.1% and 1.8% GDP/TtC

---

[48] Further details are given in Appendix E6.

[49] Setting $\theta_T = 1$, we obtain $\theta_\lambda = 6.3$, $\lambda_0 = \bar{\lambda} = 0.43$, $\Sigma_\lambda^\infty = 0.039$, $\sigma_\lambda = 2.5\%/\text{year}^{1/2}$, $\nu_\lambda = 0.20/\text{year}$ and $\theta_{ET} = 0.28$. This corresponds to a standardized skewness $\text{skew}^*[D|T] = 0.27$ (similar to the unconstrained case). See Fig. 3b.

for *proportional* and *convex* damages, respectively. The normalized marginal damage ratio and the optimal SCC rise as the atmospheric carbon stock rises with continued emissions (for $\theta_{ET} > 0$) and as the expected climate sensitivity rises to equilibrium, as captured by the two correction factors in Result 2. Golosov et al. (2014, p. 67-68) have a constant $\Theta = 2.4\%$ GDP/TtC, which includes an upward adjustment for tipping risk.

V. Estimates of the Optimal Risk-Adjusted SCC

A. Market- versus. ethics-based calibration

Using Result 2 and the calibration in Table 1, Table 4 reports estimates of the optimal SCC derived from the *market-based* calibration (base case, with *proportional* damages), where all risk mark-ups in this and the other tables below are a percentage of the deterministic SCC.[50] The table shows the important role of the initial value of the climate sensitivity parameter χ_0: if it is mistakenly set to the higher steady-state value $\bar{\chi}$, the optimal SCC roughly doubles. This replicates a result found in Traeger (2017), who models temperature delay explicitly; hence our simplified modelling of temperature delay in (5) and (8a) seems to work well. Similarly, if one does not allow for the lags in reaching the ECS and its distribution (by setting $\nu_\chi \to \infty$), the optimal risk-adjusted SCC is considerably increased (cf. column 3), as the large uncertainties associated with the ECS are then experienced instantly. The SCC of $6.6/tCO_2$ is low, since it is based on market rates of return.

Our calibration has EIS = $1/1.5 < 1$. Asset pricing theory (ATP) typically assumes EIS > 1 to ensure that macroeconomic uncertainty depresses share prices, in which case macroeconomic uncertainty also lowers the SCC. To match the same risky and

---

[50] To assess the accuracy of the approximations made in Result 1 and 2 used in Tables 4-8 relative to that of Result A, we evaluate Result A numerically and show that the error is small (see Appendix F for details).

risk-free financial rates of return, the rate of impatience ρ drops to 4.8% per year if we set EIS = 1.5. The ATP column in Table 4 then confirms that the adjustment for macroeconomic uncertainty is -1.5 \$/tCO$_2$ (negative), and the deterministic SCC is higher at 6.9 \$/tCO$_2$ (cf. Results 1 and 2 and Jensen and Traeger (2014) and Cai and Lontzek (2019).

TABLE 4. ESTIMATES OF THE SCC: MARKET-AND ETHICS-BASED

	Market-based calibration				Ethics-based calibration		
	base case	$\chi_0 = \bar{\chi}$	$v_\chi = \infty$	ATP: EIS > 1	base case	$\chi_0 = \bar{\chi}$	$v_\chi = \infty$
Deterministic SCC (\$/tCO$_2$)	4.1	8.4	8.4	6.9	11.5	20.8	20.8
due to economic uncertainty (\$/tCO$_2$)	1.3	2.4	2.4	-1.5	18.7	26.2	26.2
due to carbon stock uncertainty	0	0	0	0	0	0	0
due to climate sensitivity uncertainty	0.4	0.6	2.6	0.4	4.7	6.4	11.2
due to damage ratio uncertainty	0.7	1.4	1.7	0.7	4.9	7.5	8.1
Risk-adjusted SCC (\$/tCO$_2$)	**6.6**	**12.8**	**15.0**	**6.6**	**39.8**	**61.0**	**66.3**
Economic risk mark-up	32%	29%	29%	-21%	163%	126%	126%
Climate sensitivity risk mark-up	9%	7%	31%	7%	41%	31%	54%
Damage ratio risk mark-up	18%	17%	20%	13%	43%	36%	39%
Total risk mark-up	59%	53%	80%	-4.9%	247%	193%	219%
Discount rate $r^{(0)}$ (per year)	7.2%	7.2%	7.2%	7.2%	2.9%	2.9%	2.9%
Estimates are for *proportional* damages (θ_{ET} = 0), asset return volatility (σ_K = 12%/year$^{1/2}$), and ρ = 5.8%/year (*market-based*) or ρ = 1.5%/year (*ethics-based*), except for APT (asset pricing theory), which has EIS = 1.5 and ρ = 4.8%/year.							

The two market-based calibrations (the base and APT)imply a growth-corrected discount rate of 5.2% per year, which is very high; this is why the SCC is very low. This is the inevitable result of deriving policy-maker preferences from decisions made in financial markets. This is not necessarily a consensus view. For example, Drupp et al. (2018) found that three quarters of climate experts found a social discount rate of 2% per year acceptable. We therefore continue with what we call an *ethics-based* calibration, in which we use a much lower rate of pure time preference (1.5% instead of 5.8% or 4.8% per year), which corresponds to a risk-adjusted (not growth-corrected) discount rate $r^{(0)} = r^* + g^{(0)}$ of only 2.9% (instead of 7.2% per year in the market-based cases).[51] Although not modelled here, this

---

[51] Appendix G also shows outcomes for the optimal SCC under common alternative calibrations.

would require policy makers to also implement a capital subsidy attain the socially optimal outcome and correct for the fact that households save too little from a social perspective (Belfiori, 2018; Barrage, 2018).[52] As shown in Table 4, this pushes up the deterministic SCC to \$11.5/tCO$_2$ and the risk-adjusted SCC to \$39.8/tCO$_2$. Using market-based volatility, the mark-up for asset price risk is 163%, which exceeds that for climate sensitivity (41%) and damage ratio risk (43%). Starting off at the long-run value of the climate sensitivity parameter $(\chi_0 = \overline{\chi})$ boosts the deterministic SCC considerably as before but lowers all risk mark-ups. Ignoring both deterministic stochastic temperature delays so that the ECS and its distribution is reached instantaneously (setting $\nu_\chi \rightarrow \infty$), the risk-adjusted carbon price rises to \$66.3/tCO$_2$. In our calibration, the large uncertainty and skewness of the ECS (vs. the TCR) only arise in the relatively long run (with an e-folding time of 58 years). From comparing the *market-* and *ethics-based* calibrations, we find that the ECS plays a more significant role for lower *ethics-based* discount rates, as is clear from the case in which the distribution of the ECS is achieved instantenously $(\nu_\chi \rightarrow \infty)$.

B. Volatility from asset returns vs. GDP

The most important drawback of our AK-model is that asset returns (capital growth) and GDP growth have the same volatility (see also the discussion in Pindyck and Wang (2013)), while the former is empirically much greater. Ideally, we would have a model general enough to calibrate asset returns and GDP growth separately but it is not trivial to extend Result 1 and 2 for this.[53] However, if we calibrate to GDP volatility, Table 5 shows that the mark-up for economic risk drops

---

[52] E.g., Gollier (2018) relies on ethical arguments to use a zero or much lower discount rates as an alternative to discount rates derived from asset market returns. To analyse this problem, the government should maximize expected welfare using low ethically-motivated discount rates subject to the constraints of the decentralized market economy, which is characterized by a higher discount rate. An alternative is to adjust γ when lowering ρ to ensure that the risky and safe interest rates are still matched to the data, but this would not affect the optimal SCC given in Results 1 or 2.

[53] One way of obtaining a higher asset price volatility than GDP volatility is to assume that dividends are a leveraged function of output or consumption (e.g., Bansal and Yaron, 2004; Wachter, 2013).

dramatically.[54] Due to the higher risk-adjusted discount rate $r^{(0)}$, the mark-ups for climate sensitivity and damage ratio uncertainty and the risk-adjusted SCC are also considerably reduced.

TABLE 5. ESTIMATES OF THE SCC: ASSET RETURN VS. GDP VOLATILITY

| | Asset return volatility | | | GDP growth volatility | | | |
| | (σ_K = 12%/ year$^{1/2}$) | | | (σ_K = 1.5%/ year$^{1/2}$) | | | |
	base case	$\eta = 6.0$	$\gamma = 2.0$	base case	$\eta = 6.0$	$\gamma = 2.0$	ρ = 0.1%/year
Deterministic SCC ($/tCO$_2$)	11.5	11.5	8.1	11.5	11.5	8.1	25.5
Risk-adjusted SCC ($/tCO$_2$)	**39.8**	**92.2**	**87.2**	**14.6**	**14.6**	**10.2**	**34.1**
Economic risk mark-up	163%	492%	691%	1%	1%	1%	2%
Climate sensitivity risk mark-up	41%	112%	149%	11%	11%	9%	15%
Damage ratio risk mark-up	43%	101%	134%	15%	15%	15%	16%
Total risk mark-up	247%	705%	974%	27%	28%	25%	34%
Discount rate $r^{(0)}$ (per year)	2.9%	2.3%	2.3%	4.5%	4.5%	5.5%	3.1%
Estimates in this table are for *proportional* damages (θ_{ET} = 0) and ρ = 1.5%/year (*ethics-based* calibration), except for the last column, which considers a lower rate of impatience.							

With asset return volatility, an increase in RRA[55] from 4.3 to 6.0 depresses the discount rate $r^{(0)}$ from 2.9% to 2.3% per year and pushes up the risk-adjusted SCC to $92.2/tCO$_2$, corresponding to a total risk mark-up of 705%, whereas with GDP volatility this effect is negligibly small. With asset return volatility, an increase in IIA from 1.5 to 2.0 also pushes down the discount rate $r^{(0)}$ to 2.3% per year and the risk-adjusted SCC up to $87.2/tCO$_2$. With GDP volatility, a similar increase in IIA instead increases the discount rate $r^{(0)}$ (from 4.5% to 5.5% per year), pushes down the deterministic SCC from $11.5 to $8.1/tCO$_2$ and the risk-adjusted SCC from $14.6 to $10.2/tCO$_2$.

Summarizing, the effect of RRA on the risk-adjusted SCC depends crucially on the magnitude of economic volatility and is very substantial for asset return volatility but negligibly small for GDP growth volatility. More IIA substantially

---

[54] Historical data for the growth rate of world GDP for 1961-2015 imply σ_K = 1.5 %/year$^{1/2}$, which we use here.

[55] We use the short-hands RRA and IIA to denote relative risk aversion (RRA = η) and intergenerational inequality aversion (IIA = γ), respectively.

boosts the risk-adjusted SCC for asset return volatility,[56] but decreases for GDP growth volatility. This accords with Crost and Traeger (2013), Ackerman et al. (2013) and Hambel et al. (2017), who all use uncertainty based on GDP.[57]

TABLE 6. ESTIMATES OF THE SCC: CONVEXITY OF THE DAMAGE FUNCTION

	Proportional damages ($\theta_{ET}=0$)	Convex damages ($\theta_{ET}=0.28$)	Highly convex damages (AS12, $\theta_{ET}=0.63$)
Deterministic SCC ($/tCO₂)	25.5	26.8	45.9
Risk-adjusted SCC ($/tCO₂)	**34.1**	**41.9**	**87.6**
Economic risk mark-up	2%	1%	-1%
Carbon stock risk mark-up	0%	-1%	-1%
Climate sensitivity risk mark-up	15%	30%	61%
Damage ratio risk mark-up	16%	26%	21%
Total risk mark-up	34%	56%	91%
Discount rate $r^{(0)}$ (per year)	3.1%	3.1%	3.1%
Estimates in this table are for $\rho=0.1\%$/year (*ethics-based* calibration) and GDP growth volatility ($\sigma_K=1.5\%$/ year$^{1/2}$).			

C. Convexity of the damage function

Table 6 considers the effect of our *convex* damage function ($\theta_{ET}=0.28$) on the SCC. Generally, the SCC is larger due to larger damages for higher temperatures (cf. Fig. 3b), which is felt more strongly for lower discount rates.[58] There is now a small *negative* adjustment for carbon stock uncertainty due to the convexity of marginal damages for $\theta_{ET}=0.28$ (cf. (A4.1), Result 2).[59] The climate sensitivity risk mark-up increases considerably due to the more convex damages-temperature

---

[56] As $g-\eta\sigma_K^2<0$ (cf. (18), when written as $r^*=\rho+(\gamma-1)(g^{(0)}-\eta\sigma_K^2/2)$).

[57] With GDP growth volatility, it is possible to use an even lower *ethics-based* value of impatience of $\rho=0.1\%$/year without negative discount rates and unbounded value of the SCC, which we will use below.

[58] This effect more than compensates the higher effective discount rate due to atmospheric decay of carbon in the case of *convex* damages (cf. $r^\star\equiv r^*+(1+\theta_{ET})\varphi$ in Result 2).

[59] IPCC (2014, AR5) suggests that carbon cycle uncertainty and climate sensitivity uncertainty contribute about equally to uncertainty in temperature. From our temperature model (5), we obtain $\Sigma_T/E[T]=(1+\theta_\chi)\Sigma_\chi/\mu_\chi+(1+\theta_E)\Sigma_E/E[E]$ and from Table 1 we estimate the first term to be 48% and the second term to be 36% at a typical value of the future carbon stock of $E[E]=550$ ppmv (see Fig. 1b), thus also of comparable magnitude. However, this does not mean that both terms contribute equally to the SCC. With proportional damages, carbon cycle uncertainty has zero effect (Result 1). With convex damages, climate sensitivity has a large and positive effect on the SCC, whilst carbon cycle uncertainty has only a small negative effect.

relationship ($\theta_T = 1.0$ vs. 0.56 for *proportional damages*). If we consider the highly convex damage function of Ackerman and Stanton (2012) (i.e. AS12) with damages rapidly rising above 1°C, we get an even larger deterministic SCC of $45.9/tCO$_2$, a climate sensitivity risk mark-up of 61% and a risk-adjusted SCC of $87.6/tCO$_2$.[60,61]

D. *Correlation of climate sensitivity and damage risks with the economy*

Table 7 shows that, if the elasticity of damages with respect to world GDP is cut from 1 to $\theta_D = 0.8$, two opposing effects occur: damage shocks are no longer fully insured, depressing the risk-adjusted discount rate (the insurance term in (18′)) and pushing up the SCC, and damages now grow less rapidly than GDP, pushing up the discount rate (the growing-damages term in (18′)) and depressing the SCC. Table 7 shows that the insurance-effect dominates if economic volatility is based on asset returns, and the growing-damages effect if it is based on GDP growth.

Taking economic volatility based on GDP growth, the SCC drops from $40.1 to $28.1/tCO$_2$ as the correlation coefficient between the climate sensitivity and GDP, is increased from its minimum to its maximum value ($\rho_{K\chi}$ from -1 to 1). The reason is that, as correlation between climate sensitivity shocks and GDP increases and flips from negative to positive, the scope for insurance increases, and carbon

---

[60] The damage function of AS12 is $D = 1 - (1 + 0.00245T^2 + 5.021 \times 10^{-6} T^{6.76})^{-1}$. As our formulation has power-law damage functions, we fit $D = T^{1+\theta_{T,AS}} (C_{AS}\lambda)^{1+\theta_\lambda}$ to the AS12 damage function over the range 0-4.0°C to obtain $\theta_{T,AS} = 1.54$ and $C_{AS} = 0.99$, as illustrated in Fig. E4a. We retain the distribution for λ and the value of θ_λ for *proportional* damages given in Table 1.

[61] As an alternative to our multiplicative uncertainty, Crost and Traeger (2014) have argued that the power-coefficient in the relationship between damages and temperature should be uncertain. To illustrate this, we calibrate $D = D_0 T^\lambda$ with $\lambda \sim N(\mu_\lambda, \Sigma_\lambda^2)$, to obtain $D_0 = 0.20$, $\mu_\lambda = 1.1$ and $\Sigma_\lambda = 0.59$, as shown in Fig. E4b. Since damages cannot be stochastic at 1.0°C, we only use damage estimates for which temperature exceeds 1.1°C. From a leading-order expansion in λ, we obtain a standardized skewness which rises with temperature, i.e. skew*$(D \mid T) = 3\Sigma_\lambda \log(T)$ (e.g., to 2.45 at 4°C), which is much higher than our (constant) value of 0.29, especially at higher temperatures. Fig. E4b indicates that this alternative gives a damage ratio distribution that is also more uncertain (wider confidence bands) at temperatures higher than 3°C or 4°C compared to *convex* damages. Both the higher skewness and higher uncertainty push up the optimal SCC for low discount rates, but this effect is like our case of *convex* damages (Fig. 3b).

can be priced less. Similarly, the SCC drops from \$36.5 to \$31.7/tCO$_2$ as the correlation coefficient between the damage ratio and GDP is increased from its minimum to its maximum value ($\rho_{K\lambda}$ from -1 to 1). Finally, if we vary $\rho_{\chi\lambda}$ from -1 to 1, the SCC increases from \$29.2 to \$39.0/tCO$_2$, with the largest value corresponding to the case when future climate sensitivity shocks are perfectly (positively) correlated with future damage ratio shocks.[62]

TABLE 7. ESTIMATES OF THE SCC: CORRELATED RISK

	Asset return volatility		GDP growth volatility (σ_K = 1.5%/ year$^{1/2}$)							
	base	θ_D	base	θ_D	$\rho_{K\chi}$		$\rho_{K\lambda}$		$\rho_{\chi\lambda}$	
		0.8		0.8	-1	1	-1	1	-1	1
Deterministic SCC (\$/tCO$_2$)	11.5	9.9	25.5	19.0	25.5	25.5	25.5	25.5	25.5	25.5
Risk-adjusted SCC (\$/tCO$_2$)	**39.8**	**122.9**	**34.1**	**25.3**	**40.1**	**28.1**	**36.5**	**31.7**	**29.2**	**39.0**
Economic risk mark-up	163%	811%	2%	3%	2%	2%	2%	2%	2%	2%
Climate sensitivity risk mark-up	41%	181%	15%	14%	15%	15%	15%	15%	15%	15%
Damage ratio risk mark-up	43%	156%	16%	16%	40%	-7%	26%	7%	-3%	36%
Total risk mark-up	247%	1147%	34%	33%	57%	10%	43%	24%	15%	53%
Discount rate $r^{(0)}$ (per year)	2.9%	2.2%	3.1%	3.5%	3.1%	3.1%	3.1%	3.1%	3.1%	3.1%

Estimates in this table are for *proportional* damages (θ_{ET} = 0), for ρ = 1.5%/year in the case of asset return volatility (σ_K = 12%/ year$^{1/2}$), and for ρ = 0.1%/year in the case of GDP growth volatility (σ_K = 1.5%/ year$^{1/2}$).

VI. Concluding Remarks

Using perturbation methods, we have derived a tractable closed-form approximate solution for the optimal risk-adjusted SCC under economic and climatic uncertainties allowing for skewed distributions and accounting for the time scales on which the uncertainties arise and their correlation. Our solution is a better approximation if damages are a small fraction of world GDP, which they are for most available estimates. Our solution offers new analytical insights and complements insights from numerical solutions of stochastic, dynamic, nonlinear systems. We have calibrated our uncertainties based on high-level surveys (IPCC

---

[62] The effects of $\rho_{EK}, \rho_{E\chi}$ and $\rho_{E\lambda}$ on the risk-adjusted SCC are very small in our calibration, so we omit these here.

(2014, AR5) for atmospheric carbon stock and climate sensitivity uncertainties and Nordhaus and Moffat (2017) for damage ratio uncertainty.

We confirm earlier results that the optimal SCC decreases with increasing intergenerational inequality aversion if economic growth adjusted for its uncertainty is positive, but increases in risk aversion if economic growth (or asset returns) are volatile provided the elasticity of intertemporal substitution is less than one. If damages are proportional to GDP, there is an insurance effect which curbs the optimal SCC. If the elasticity of damages with respect to GDP is below one, there is less insurance potential, which increases the SCC, but damages grow less rapidly, which reduces the SCC (cf. Dietz et al., 2018). In our simulations, the first effect dominates if economic volatility is derived from asset returns, but the second effect dominates if volatility is derived from GDP growth.

Uncertainty in atmospheric carbon stock dynamics only requires adjustments to the SCC if damages are convex, but these effects are negligible if based on historical uncertainty and negative and small if based on future projections. Uncertain climate sensitivity increases the SCC significantly, especially due to the skewness of the equilibrium climate sensitivity distribution, further enhanced by the convex dependence of damages on temperature. The magnitude of this mark-up depends crucially on the time scale on which it arises, and the much larger and more skew equilibrium climate sensitivity only plays a role for lower *ethics-based* discount rates. There is some evidence that the distribution damage ratio is right-skewed with an increase in the optimal SCC as a result. These results are complementary to those of Traeger (2017).

Our solution for the optimal SCC also allows for correlated risks. If relative risk aversion exceeds one, the *risk-insurance* effects dominate the offsetting *risk-exposure* effects resulting from damages being proportional to GDP. It is thus optimal to insure and raise the SCC if climate sensitivity and economic shocks are negatively correlated. If risk aversion exceeds one, we also show that the optimal

SCC is higher if damage ratio and economic shocks are negatively correlated. This occurs if asset returns are high in future states of nature in which the damage ratio is lower than expected. If risk aversion equals one, correlation of damage ratio or climate sensitivity risks with economic risks do not affect the SCC as in Lemoine (2020), but correlation between climate sensitivity and damage ratio risks does.

We have made two crucial simplifications in our analysis. First, we have assumed that all forms of climatic uncertainty that might exist are captured by present-day disagreement in the scientific literature about four key metrics: the atmospheric carbon stock in 2100 (given an emission scenario), the transient climate response, the equilibrium climate sensitivity and the climate damage ratio. We thus capture only 'known unknowns', and present-day disagreement may underestimate future uncertainty (cf. 'unknown unknowns'). [63] If, on the other hand, learning takes place, we may be overestimating uncertainty. Furthermore, through our use of stochastic processes of the Ornstein-Uhlenbeck type to model climatic uncertainty, we impose a particular shape of the (joint) probability distribution and its variation in time. In doing so, we have ruled out 'ab-normal' events associated with the (fat) tail of the probability distribution, including tipping. We have also ruled out climatic uncertainty that grows continuously with increasing time horizons (as for example captured by geometric vs. arithmetic Brownian motions or even more rapidly as in long-run risk models).

Second, our analysis is based on a very parsimonious representation of the climate, which captures some but not all features of state-of-the-art climate models. Our 1-box atmospheric carbon stock model (similar to Golsov et al. 2014) does not fully capture the slowing rate of carbon decay with time, potentially causing an underestimate of (short-term) temperature increase. Yet, our temperature response

---

[63] In a different context, Rich and Tracy (2010) find little evidence that disagreement among inflation forecasters is a useful proxy for uncertainty about forecasts.

does not suffer from the inertia for which many integrated assessment models have recently been criticized (Mattauch et al., 2020).

Future research should be aimed at models with ethics-based discounting for policy makers and market-based discounting for the private sector that are general enough to distinguish volatility of equity returns and GDP growth. We have abstracted from long-run risk in economic growth (Bansal and Yaron, 2004) and a downward-sloping term structure resulting from mean reversion in economic growth (Gollier and Mahul, 2017), both of which tend to boost the SCC.[64] We need robust empirical estimates and more structural underpinning of correlations between climate sensitivity and damage ratio shocks with the economy. Other challenges are to allow for compound Poisson shocks to temperature and damages (cf. Hambel et al., 2018; Bretschger and Vinogradova, 2019; Bansal et al., 2016), positive feedbacks such as the CO_2 absorption capacity of the oceans declining with temperature (Millar et al., 2016), the timing of climatic uncertainty, the risk of tipping points (e.g., Lemoine and Traeger, 2014, 2016a; Lontzek et al., 2016; Cai et al., 2016; van der Ploeg and de Zeeuw, 2018; Cai and Lontzek, 2019), which may further increase the optimal SCC. Finally, future research may attempt to distinguish between modelling disagreement and statistical uncertainty. Robust optimal control techniques may also be used to deal with modelling uncertainties in climate policy (e.g., Rudik, 2020).

References

Ackerman, F. and E.A. Stanton. 2012. "Climate Risks and Carbon Prices: Revising the Social Cost of Carbon", *Economics: The Open-Access, Open-Assessment E-Journal*, 6, 1-25.

---

[64] Epstein et al. (2014) argue that long-run risk and preference of early resolution of uncertainty implies that the timing premium needed to calibrate asset returns is implausibly high (20-30%). Bansal et al. (2016) show that this long-run risk pushes up the optimal SCC by a factor 2 or 3 if aversion to risk exceeds aversion to intertemporal fluctuations.

Ackerman, F., E.A. Stanton and R. Bueno. 2013. "Epstein-Zin Utility in DICE: Is Risk Aversion Irrelevant to Climate Policy", *Environmental and Resource Economics*, 56, 73-84.

Aengenheyster, M., Q.Y. Feng, F. van der Ploeg and H.A. Dijkstra. 2018. "The Point of No Return for Climate Action: Effects of Climate Uncertainty and Risk Tolerance", *Earth System Dynamics*, 9, 1085-1095.

Allen, M., D. Frame, C. Huntingford, C. Jones, J. Lowe, M. Meinshausen and N. Meinshausen. 2009. "Warming Caused by Cumulative Carbon Emissions Towards the Trillionth Tonne", *Nature*, 458, 7242, 1163-1166.

Arrhenius, S. 1896. "On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground", *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 5, 41, 237-276.

Bansal, R. and A. Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles", *Journal of Finance*, 59, 4, 1481-1509.

Bansal, R., D. Kiku and A. Yaron. 2012. "Empirical Evaluation of the Long-Run Model for Asset Prices", *Critical Finance Review*, 1, 183-221.

Bansal, R., M. Ochoa and D. Kiku. 2016. "Climate Change and Growth Risks", Working Paper 23009, NBER, Cambridge, MA.

Barro, R. 2017. "Rare Disasters, Asset Prices, and Welfare Costs", *American Economic Review*, 99, 1, 243-264.

Barrage, L. 2018. "Be Careful What You Calibrate For: Social Discounting in General Equilibrium", *Journal of Public Economics*, 160C, 33-49.

Belfiori, E. 2018. "Climate change and intergenerational equity: revisiting the uniform taxation principle on carbon energy inputs", *Energy Policy*, 121, 292-299.

Breeden, D.T. 1979. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities", *Journal of Financial Economics*, 7, 265-296.

Bretschger, L. and A. Vinogradova (2018). "Best Policy Response to Environmental Shocks: Building a Stochastic Framework", *Journal of Environmental Economics and Management*, 97, 23-41.

Cai, Y. and T.S. Lontzek 2019. "The Social Cost of Carbon with Economic and Climate Risks", *Journal of Political Economy*, 127, 6, 2684-1734.

Cai, Y., T.M. Lenton and T.S. Lontzek. 2016. "Risk of Multiple Climate Tipping Points Should Trigger a Rapid Reduction in CO2 Emissions", *Nature Climate Change*, 6, 520-525.

Crost, B. and C.P. Traeger. 2013. "Optimal Climate Policy: Uncertainty versus Monte Carlo", *Economics Letters*, 120, 3, 552-558.

Crost, B. and C.P. Traeger. 2014. "Optimal CO2 Mitigation under Damage Risk Valuation", *Nature Climate Change*, 4, 631-636.

Dietz, S. and N. Stern. 2015. "Endogenous Growth, Convexity of Damages and Climate Risk: How Nordhaus' Framework Supports Deep Cuts in Emissions", *Economic Journal*, 125, 583, 574-620.

Dietz, S., C. Gollier and L. Kessler. 2018. "The Climate Beta", *Journal of Environmental Economics and Management*, 87, 258-274.

Drupp, M.A., M.C. Freeman, B. Groom and F. Nesje. 2018. "Discounting Disentangled", *American Economic Journal: Economic Policy*, 10, 109-134.

Duffie, D. and L.G. Epstein. 1992. "Stochastic Differential Utility", *Econometrica*, 60, 2, 353-394.

Epstein, L.G. and S. Zin. 1989. "Substitution, Risk Aversion and the Temporal Modelling of Asset Returns: A Theoretical Framework", *Econometrica*, 57, 937-967.

Epstein, L.G., E. Farhi and T. Strzalecki. 2014. "How Much Would You Pay to Resolve Long-Run Risk?", *American Economic Review*, 104, 9, 2680-2697.

Gerlagh, R. and M. Liski. 2016. "Carbon prices for the next hundred years", *Economic Journal*, 128, 609, 728-575.

Gerlagh, R. and M. Liski. 2018. "Consistent Climate Policies", *Journal of the European Economic Association*, 16, 1, 1-44.

Gollier, C. 2002. "Discounting an uncertain future", *Journal of Public Economics*, 85, 2, 149-166.

Gollier, C. 2011. "On the Underestimation of the Precautionary Effect in Discounting", *The Geneva Risk and Insurance Review*, 36 (2), 95-11.

Gollier, C. 2012. *Pricing the Planet's Future: The Economics of Discounting in an Uncertain World*, Princeton University Press, Princeton, New Jersey.

Gollier, C. 2018. *Ethical Asset Valuation and the Good Society*, Columbia University Press, New York.

Gollier, C. and O. Mahul. 2017. "Term Structure of Discount Rates: An International Perspective", Toulouse School of Economics.

Golosov, M., J. Hassler and P. Krusell and A. Tsyvinski. 2014. "Optimal Taxes on Fossil Fuel in General Equilibrium", *Econometrica*, 82, 1, 48-88.

Hambel, C., H. Kraft and E. Schwartz. 2017. "Optimal Carbon Abatement in a Stochastic General Equilibrium Model with Climate Change, Frankfurt University.

IPCC. 2014. *Fifth Assessment Report, AR5 Synthesis Report: Climate Change 2014*, Geneva, Switzerland.

Jensen, S. and C.P. Traeger. 2014. "Optimal Climate Change Mitigation under Long-Term Growth Uncertainty: Stochastic Integrated Assessment and Analytic Findings", *European Economic Review*, 69, 104-125.

Jensen, S. and Traeger, C.P. 2019. "Pricing Climate Risk", University of Oslo.

Joos, F. et al. 2013. "Carbon Dioxide and Climate Impulse Response Functions for the Computation of Greenhouse Gas Metrics: A Multi-Model Analysis", *Atmospheric Chemistry and Physics*, 13, 2793-2825.

Kelly, D.L. and C.D. Kolstad. 1999. "Bayesian learning, pollution, and growth", *Journal of Economic Dynamics and Control*, 23, 4, 491-518.

Kelly, D.L. and Z. Tan. 2015. "Learning and climate feedbacks: optimal climate insurance and fat tails", *Journal of Environmental Economics and Management*, 72, 98-122.

Kimball, M.S. 1990. "Precautionary Saving in the Small and in the Large", *Econometrica*, 58, 53-73.

Kreps, D. and E. Porteus. 1978. "Temporal Resolution of Uncertainty and Dynamic Choice Theory", *Econometrica*, 46, 185-200.

Leland, H.E. 1968. "Savings and Uncertainty: The Precautionary Demand for Saving", *Quarterly Journal of Economics*, 82, 465-473.

Lemoine, D. 2020. "The Climate Risk Premium: How Uncertainty affects the Social Cost of Carbon", Department of Economics, Working Paper, University of Arizona.

Lemoine, D. and I. Rudik. 2017. "Managing Climate Change under Uncertainty: Recursive Integrated Assessment at an Inflection Point", *Annual Review of Resource Economics*, 9, 117-142.

Lemoine, D. and C. Traeger. 2014. "Watch your Step: Optimal Policy in a Tipping Climate", *American Economic Journal: Economic Policy*, 6, 37-166.

Lemoine, D. and C.P. Traeger. 2016a. "Economics of Tipping the Climate Dominoes", *Nature Climate Change*, 6, 514-519.

Lemoine, D. and C.P. Traeger. 2016b. "Ambigiuous Tipping Points", *Journal of Economic Behavior and Organization*, 132, 5-18.

Lontzek, T.S., Y. Cai, K.L. Judd and T.M. Lenton. 2015. "Stochastic Integrated Assessment of Climate Tipping Points Indicates the Need for Strict Climate Policy", *Nature Climate Change*, 5, 441-444.

Lucas, R.E. Jr. 1978. "Asset Prices in an Exchange Economy", *Econometrica*, 46, 1429-1455.

Martin, I.W.R. 2013. "Consumption-Based Asset Pricing with Higher Cumulants", *Review of Economic Studies*, 80, 2, 745-773.

Mattauch, L., H.D. Matthews, R. Millar, A. Rezai, S. Solomon and F. Venmans. 2020. "Steering the climate system: using inertia to lower the cost of policy: Comment", *American Economic Review*, 110, 4, 1231-1237.

Matthews, H.D., N.P. Gillett, P.A. Stott and K. Zickfeld. 2009. "The Proportionality of Global Warming to Cumulative Carbon Emissions", *Nature*, 459, 829-832.

Meehl, G.A., C.A. Senior, V. Eyring, G. Flato, J.-F. Lamarque, R.J. Stouffer, K.E. Taylor and M. Schlund. 2020. "Context for Interpreting Equilibrium Climate Sensitivity and Transient Climate Response from the CMIP6 Earth System Models", *Science Advances*, 6, 26.

Millar, R.J., Z.R. Nicholls, P. Friedlingstein and M.R. Allen. 2016. "A Modified Impulse-Response Representation of the Global Near-Surface Air Temperature and Atmospheric Concentration Response to Carbon Dioxide Emissions", *Atmospheric Chemistry and Physics*, 17, 7213-7228.

Nordhaus, W.D. 2008. *A Question of Balance: Economic Modeling of Global Warming*, Yale University Press, New Haven, CT.

Nordhaus, W.D., 2011. *Estimates of the Social Cost of Carbon: Background and Results from the RICE-2011 Model*. NBER, Cambridge, Mass.

Nordhaus, W.D. 2017. "Revisiting the Social Cost of Carbon", *Proceedings of the National Academy of Sciences*, 114, 7, 1518-1523.

Nordhaus. W.D. 1994. *Managing the Global Commons: The Economics of Climate Change*, MIT Press, Cambridge, Mass.

Nordhaus. W.D. and D. Popp. 1997. "What is the Value of Scientific Knowledge? An Application to Global Warming using the PRICE Model, *Energy Journal*, 18, 1-47.

Nordhaus, W.D. and P. Sztorc. 2013. *DICE 2013R: Introduction and User's Manual*, Yale University.

Nordhaus, W.D. and A. Moffat. 2017. "A Survey of Global Impacts of Climate Change: Replication, Survey Methods, and a Statistical Analysis", Working Paper No. 23646, NBER, Cambridge, Mass.

Pindyck, R.S. 2012. "Uncertain Outcomes and Climate Change Policy", *Journal of Environmental Economics and Management*, 63, 289-303.

Pindyck, R.S. 2017. "Coase Lecture – Taxes, Targets and the Social Cost of Carbon", *Economica*, 84, 335, 345-364.

Pindyck, R.S. and N. Wang. 2013. "The Economic and Policy Consequences of Catastrophes", *American Economic Journal: Economic Policy*, 5, 4, 306-339.

Pizer, W.A. 1999. "The Optimal Choice of Climate Change Policy in the Presence of Uncertainty", *Resource and Energy Economics*, 21, 3-4, 255-287.

Ploeg, F. Van der and A.J. de Zeeuw. 2018. "Climate Tipping and Economic Growth: Precautionary Capital and the Price of Carbon", *Journal of the European Economic Association*, 16, 5, 1577-1617.

Rich, R. and J. Tracy. 2010. "The Relationship Among Expected Inflation Disagreement and Uncertainty: Evidnece from Matched Point and Density Forecasts", *Review of Economics and Statistics*, 92, 1, 200-207.

Ricke, K.L. and K. Caldeira. 2014. "Maximum Warming Occurs About One Decade after a Carbon Dioxide Emission", *Environmental Research Letters*, 9, 12, 124002.

Roe, G.H. and M.B. Baker. 2007. "Why is Climate Sensitivity so Unpredictable?", *Science*, 318, 629-632.

Roe, G.H. and Y. Bauman. 2011. "Climate Sensitivity: Should the Climate Tail Wag the Policy Dog?", *Climatic Change*.

Rudik, I. 2020. "Optimal Climate Policy when Damages are Unknown", *American Economic Journal: Economic Policy*, forthcoming.

Sandsmark, M. and H. Vennemo. 2007. "A Portfolio Approach to Climate Investments: CAPM and Endogenous Risk", *Environmental and Resource Economics*, 37, 4, 681-695.

Stern, N. 2007. *The Economics of Climate Change: The Stern Review*, Cambridge University Press, Cambridge, U.K.

Traeger, C. 2017. "ACE – An Analytical Climate Economy (with Temperature and Uncertainty)", mimeo., University of Oslo, Stanford University.

Wachter, J.A. (2013). "Can time-varying risk of rare disasters explain aggregate market volatility?, *Journal of Finance*, 68, 987-1035.

Weitzman, M.L. 2009. "On Modelling and Interpreting the Economics of Catastrophic Climate Change", *Review of Economics and Statistics*, 91, 1-19.

Weitzman, M.L. 2012. "GHG Targets as Insurance against Catastrophic Climate Damages", *Journal of Public Economic Theory*, 14, 2, 221-244.

Zickfeld, K. et al. 2013. "Long-Term Climate Change Commitment and Reversibility: an EMIC Comparison", *Journal of Climate*, 26, 5782-5809.

Zickfeld, K. and T. Herrington 2015. "The Time Lag Between a Carbon Dioxide Emission and Maximum Warming Increases with the Size of the Emission", *Environmental Research Letters*, 10, 031001.

ONLINE APPENDIX

THE RISK-ADJUSTED CARBON PRICE

Ton van den Bremer and Frederick van der Ploeg

Appendix A. Perturbation Theory

A.1. General framework

Perturbation theory is a method for finding an approximate solution to a complicated problem by starting with the exact solution of a related, simpler problem. The problem is thus not solved exactly, but instead so-called 'small' terms are added to adjust the solution of the simpler, exactly solvable problem. Perturbation theory provides a formal framework to control how small these adjustment terms are.

In general, after substitution of the optimality conditions for the forward-looking variables, any HJB equation takes the form

(A1.1) $\quad \mathcal{F}\big[J(\mathbf{x}),\mathbf{x}\big]=0,$

where the 'operator' $\mathcal{F}$ is typically nonlinear, includes first- and second-order derivatives of the value function J with respect to the vector of states $\mathbf{x}$, which may include time, and is also a function of $\mathbf{x}$ directly. Provided a (single) small parameter ϵ, defined so that we return to the simpler, exactly solvable problem in the limit $\epsilon \to 0$, can be identified, we can solve this HJB for the value function $J(\mathbf{x})$ using perturbation theory. Following practice in perturbation theory (e.g., Van Dyke, 1975; Kevorkian and Cole, 1996; Bender and Orszag, 1999; Nayfeh, 2004), the solution for the value function then takes the form of a series in ϵ

(A1.2) $\quad J(\mathbf{x}) = J^{(0)}(\mathbf{x}) + \epsilon J^{(1)}(\mathbf{x}) + \mathcal{O}(\epsilon^2),$

where the dependence of the value function on the states $\mathbf{x}$ continues to be nonlinear. To be clear, (A1.2) is not a Taylor-series expansion. Instead, (A1.2) expresses the solution as a series of adjustments (depending on the small parameter ϵ) to the so-called zeroth-order solution $J^{(0)}(\mathbf{x})$, which corresponds to the solution of the simpler, exactly solvable problem referred to above. In the limit $\epsilon \to 0$, $J(\mathbf{x}) \to J^{(0)}(\mathbf{x})$. The zeroth-order solution

is said to be $\mathcal{O}(1)$ since $\epsilon^0 = 1$ and thus 'not small'. The subsequent terms in (A1.2) adjust the solution if $\epsilon \neq 0$, where the first-order term $\epsilon J^{(1)}(\mathbf{x})$ is the so-called 'leading-order' adjustment to the solution. Formally, in the limit of an infinite number of terms in (A1.2), the solution of the simpler problem plus all its adjustments, provided the series is convergent, become equal to the exact solution of the complicated problem $J(\mathbf{x})$. In practice, only a finite number of terms gives a reasonable approximation, and the series solution is truncated. In (A1.2), the series solution is accurate up to first order in ϵ, and the error is thus $\mathcal{O}(\epsilon^2)$.

Having expanded the value function in (A1.2), we also expand the operator $\mathcal{F}$:

(A1.3)　　$\mathcal{F} = \mathcal{F}^{(0)} + \epsilon \mathcal{F}^{(1)} + \mathcal{O}(\epsilon^2),$

where $\mathcal{F}^{(0)}$ contains all operations that leave the order unchanged and $\epsilon \mathcal{F}^{(1)}$ contains all operations that increase the order by ϵ. Combining (A1.2) and (A1.3), the general form of the HJB (A1.1) becomes

(A1.4)　　$\left(\mathcal{F}^{(0)} + \epsilon \mathcal{F}^{(1)} + \mathcal{O}(\epsilon^2) \right) \left[J^{(0)}(\mathbf{x}) + \epsilon J^{(1)}(\mathbf{x}) + \mathcal{O}(\epsilon^2), \mathbf{x} \right] = 0,$ which can be expressed as a series solution itself by expanding out the brackets

(A1.5)　　$\underbrace{\mathcal{F}^{(0)} \left[J^{(0)}(\mathbf{x}), \mathbf{x} \right]}_{O(1)} + \underbrace{\epsilon \left(\mathcal{F}^{(0)} \left[J^{(1)}(\mathbf{x}), \mathbf{x} \right] + \mathcal{F}^{(1)} \left[J^{(0)}(\mathbf{x}), \mathbf{x} \right] \right)}_{O(\epsilon)} + \mathcal{O}(\epsilon^2) = 0.$

We note that the term $\epsilon^2 \mathcal{F}^{(1)} \left[J^{(1)}(\mathbf{x}), \mathbf{x} \right]$ that arises from (A1.4) is small and of the same order as terms previously ignored in (A1.2) and (A1.3), and can therefore also be ignored in (A1.5); this term is contained in the $\mathcal{O}(\epsilon^2)$ error in (A1.5). Solving the HJB equation using perturbation theory then amounts to solving (A1.5) successively at each order. For the first two orders the resulting two equations are

(A1.6)　　$\mathcal{O}(1): \ \mathcal{F}^{(0)} \left[J^{(0)}(\mathbf{x}), \mathbf{x} \right] = 0,$

(A1.7)　　$\mathcal{O}(\epsilon): \ \mathcal{F}^{(0)} \left[J^{(1)}(\mathbf{x}), \mathbf{x} \right] + \mathcal{F}^{(1)} \left[J^{(0)}(\mathbf{x}), \mathbf{x} \right] = 0.$

We first solve (A1.6) for the zeroth-order solution and then solve (A1.7) for the first-order solution $J^{(1)}(\mathbf{x})$ using the (now known) zeroth-order solution $J^{(0)}(\mathbf{x})$ from (A1.6).

A.2. Perturbation theory applied to our model

To apply the framework introduced in section A.1, we take several steps. First, we identify the small parameter ϵ, which we find by writing the problem in non-dimensional form (section A.2.1). Second, we must choose the structure of our perturbation expansion, depending on how and where the small parameter ϵ appears in the HJB equation (section

A.2.2). Third, we perform the perturbation expansion and then solve the HJB equation at zeroth order (section A.2.3) and first order (section A.2.4), respectively.

A.2.1. Non-dimensional form and identification of the small variable

Following standard practice in the physical sciences, we begin by writing the HJB equation (14) in non-dimensional form[1]. To do so, we normalize the four states K, E, χ and λ by their initial values (at $t = 0$):[2]

(A2.1) $\qquad \hat{K} \equiv \dfrac{K}{K_0}$, $\hat{E} \equiv \dfrac{E}{E_0}$, $\hat{\chi} \equiv \dfrac{\chi}{\chi_0}$, $\hat{\lambda} \equiv \dfrac{\lambda}{\lambda_0}$,

so that all four hatted variables are equal to 1 at $t = 0$. We define non-dimensional time $\hat{t} \equiv g_0 t$ with $g_0 \equiv g(E = E_0, \chi = \chi_0, \lambda = \lambda_0)$ the growth rate of the economy without additional climate change (this growth rate is constant in time). We define the non-dimensional forward-looking variables as

(A2.2) $\qquad \hat{F} \equiv \dfrac{F}{F_0}$, $\hat{C} \equiv \dfrac{C}{C_0}$,

where $F_0 \equiv A(E_0)^{\frac{1}{\alpha}}\left((1-\alpha)/b\right)^{\frac{1}{\alpha}} K_0$ and $C_0 \equiv g_0 K_0$ (these are not the initial values of F and C, as initial values of the forward-looking variables are not known at this stage in the solution procedure). In accordance with the non-dimensional form (A2.1)-(A2.2), we further define $\hat{J} \equiv g_0 J / C_0^{1-\eta}$, $\hat{I} \equiv I/C_0$, $\hat{\Phi} \equiv \Phi/C_0$, $\hat{Y} \equiv Y/C_0$, $\hat{\phi} \equiv \phi/g_0$ and $\hat{i} \equiv i/g_0$, where $\phi \equiv \Phi / K$ and $i \equiv I / K$.

In non-dimensional form, the HJB equation (14) now becomes

(A2.3)

$$
0 = \max_{\hat{C}, \hat{F}} \left[\frac{1}{1-\gamma} \frac{\hat{C}^{1-\gamma} - \hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} + \hat{J}_{\hat{t}} + \hat{J}_{\hat{K}}\phi(\hat{i})\hat{K} + \hat{J}_{\hat{E}}(\hat{\mu}\hat{F}e^{-\hat{g}\hat{t}} - \hat{\varphi}\hat{E}) \right.
$$

$$
\left. + \hat{J}_{\hat{\chi}}\hat{v}_{\chi}(\hat{\bar{\chi}} - \hat{\chi}) + \hat{J}_{\hat{\lambda}}\hat{v}_{\lambda}(\hat{\bar{\lambda}} - \lambda) + \frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2 + \frac{1}{2}\hat{J}_{\hat{E}\hat{E}}\hat{\sigma}_E^2 + \frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 \right]
$$

---

[1] Through normalizing all variables by typical values these variables may take, we remove the physical dimensions (e.g., time) and therefore the measurements units (e.g., years) of these variables in a process known as non-dimensionalization.

[2] We do not distinguish between χ, λ and $\max(\chi, 0)$, $\max(\lambda, 0)$ here for simplicity, as we show in section IV that the probabilities of χ and λ becoming zero or negative are negligibly small.

$$+ \hat{J}_{\hat{K}\hat{E}} \hat{K} \rho_{KE} \hat{\sigma}_K \hat{\sigma}_E + \hat{J}_{\hat{K}\hat{\chi}} \hat{K} \rho_{K\chi} \hat{\sigma}_K \hat{\sigma}_\chi + \hat{J}_{\hat{K}\hat{\lambda}} \hat{K} \rho_{K\lambda} \hat{\sigma}_K \hat{\sigma}_\lambda + \hat{J}_{\hat{E}\hat{\chi}} \rho_{E\chi} \hat{\sigma}_E \hat{\sigma}_\chi$$

$$+ \hat{J}_{\hat{E}\hat{\lambda}} \rho_{E\lambda} \hat{\sigma}_E \hat{\sigma}_\lambda + \hat{J}_{\hat{\chi}\hat{\lambda}} \rho_{\chi\lambda} \hat{\sigma}_\chi \hat{\sigma}_\lambda,$$

where $\quad \hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C} = \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})\hat{K}^\alpha \hat{F}^{1-\alpha} - \hat{b}\hat{F} - \hat{C}$ and $\quad \hat{\phi} = \hat{i} - (1/2)\hat{\omega}\hat{i}^2 - \hat{\delta} \quad$ with $\hat{A} \equiv A F_0^{1-\alpha} / g_0 K_0^{1-\alpha}$. The resulting non-dimensional parameters are

(A2.4)
$$\hat{\rho} \equiv \frac{\rho}{g_0}, \ \hat{b} \equiv \frac{bF_0}{g_0 K_0}, \ \hat{\omega} \equiv g_0\omega, \ \hat{\delta} \equiv \frac{\delta}{g_0}, \ \hat{g} \equiv \frac{g}{g_0}, \ \hat{\mu} \equiv \frac{\mu F_0}{g_0 E_0}, \ \hat{\varphi} \equiv \frac{\varphi}{g_0}, \ \hat{v}_\chi \equiv \frac{v_\chi}{g_0},$$

$$\hat{\bar{\chi}} \equiv \frac{\bar{\chi}}{\chi_0}, \ \hat{v}_\lambda \equiv \frac{v_\lambda}{g_0}, \ \hat{\bar{\lambda}} = \frac{\bar{\lambda}}{\lambda_0}, \ \hat{\sigma}_K \equiv \frac{\sigma_K}{\sqrt{g_0}}, \ \hat{\sigma}_E \equiv \frac{\sigma_E}{\sqrt{g_0}E_0}, \ \hat{\sigma}_\chi \equiv \frac{\sigma_\chi}{\sqrt{g_0}\chi_0}, \ \hat{\sigma}_\lambda \equiv \frac{\sigma_\lambda}{\sqrt{g_0}\lambda_0}.$$

Except for $\hat{b}$, $\hat{\mu}$, $\hat{\bar{\chi}}$ and $\hat{\bar{\lambda}}$, which respectively measure the relative cost of fossil fuel use, the relative contribution of new emissions to the total atmospheric carbon stock, and the ratios of the steady-state and initial values of the climate sensitivity and the climate damage parameters, the non-dimensional parameters in (A2.4) measure the different rates in the economy relative to the growth rate g_0. We assume all these non-dimensional parameters are $O(1)$. That is, they are not small parameters, and their effects must be fully accounted for in our solutions and cannot be approximated using perturbation theory.

Having assumed that all non-dimensional parameters in (A2.4) are $\mathcal{O}(1)$, it is not immediately obvious how we can use perturbation theory to simplify the solutions to the HJB equation (A2.3). However, one additional non-dimensional parameter arises when we define non-dimensional damages and total factor productivity[3]

(A2.5) $\hat{D}(\hat{E},\hat{\chi},\hat{\lambda}) \equiv \hat{\lambda}^{1+\theta_\lambda} \hat{\chi}^{1+\theta_{\chi T}} \hat{E}^{1+\theta_{ET}}$ and $\hat{A} \equiv \hat{A}^*(1 - \epsilon\hat{D}) = \hat{A}^*(1 - \epsilon\hat{\lambda}^{1+\theta_\lambda} \hat{\chi}^{1+\theta_{\chi T}} \hat{E}^{1+\theta_{ET}})$,

where $\hat{D} \equiv D/D_0 = D/\epsilon$, $\hat{A} \equiv A F_0^{1-\alpha} / (g_0 K_0^{1-\alpha})$ and $\hat{A}^* \equiv A^* F_0^{1-\alpha} / (g_0 K_0^{1-\alpha})$.

Assumption A: The final additional non-dimensional parameter, which we will assume to be the small parameter of our problem, is defined as

(A2.6) $\quad \epsilon \equiv D_0 \equiv \lambda_0^{1+\theta_\lambda} \chi_0^{1+\theta_{\chi T}} \left(\dfrac{E_0}{S_{PI}}\right)^{1+\theta_{ET}}$.

---

[3] The term 'normalization' or 'scaling' is perhaps more appropriate than 'writing in non-dimensional form' for the damage ratio D, which is already non-dimensional. We avoid this ambiguity by using the three terms interchangeably.

The small parameter ϵ equals the initial damage ratio D_0, which is known a priori and empirically also small (see section IV). The limit $\epsilon \to 0$ thus corresponds to the case in which climate damages are zero. In this limit, the value function does not depend on the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$, and all terms in the HJB equation (A2.3) involving derivatives with respect to $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ disappear:

$$(A2.7) \quad \max_{\hat{C},\hat{F}} \left[\frac{1}{1-\gamma} \frac{\hat{C}^{1-\gamma} - \hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} + \hat{J}_{\hat{t}} + \hat{J}_{\hat{K}}\phi(\hat{i})\hat{K} + \frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2 \right] = 0,$$

which can be solved for the value function in closed form (e.g., Pindyck & Wang, 2013) to give the solution to the simpler, exactly solvable problem we perturb here. Note, however, that a small but non-zero value of the small parameter ϵ reduces total factor productivity via the damage ratio according to (A2.5) and introduces the solution's dependence on $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$. In the original HJB equation (A2.3), ϵ will change the investment level $\hat{i}$, which directly affects the term $\hat{J}_{\hat{K}}\phi(\hat{i})\hat{K}$ and indirectly all others. Perturbation theory now allows us to re-introduce ϵ into (A2.7) in a controlled fashion. Before we do so, we emphasize that except for the small variable ϵ, all (hatted) variables in the HJB (A2.4) are $\mathcal{O}(1)$.

A.2.2. Perturbation expansion

We now seek a perturbation series solution for the value function of the following form:

$$(A2.8) \quad \hat{J}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) = \hat{J}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right) + \epsilon\hat{J}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) + \mathcal{O}(\epsilon^2),$$

where the structure of the solution is based on the underlying HJB (A2.3), as explained below. Because the zeroth-order solution will only be affected by the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ through the total factor productivity of capital (A2.5), the zeroth-order solution only depends on these three states through the damage ratio $\hat{D} = \lambda^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}$. Moreover, in the functional dependence of the zeroth-order solution $\hat{J}^{(0)}$, the $\mathcal{O}(1)$ damage ratio $\hat{D}$ is always multiplied by the small parameter ϵ (i.e. the functional dependence is on $D = \epsilon\hat{D}$). Importantly, as a result, changes in the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ have a smaller effect

on the zeroth value function than changes in the capital stock $\hat{K}$. To illustrate this, consider the expected rate of change of the zeroth-order value function through its total derivative:

$$(A2.9) \qquad \frac{1}{d\hat{t}}\mathrm{E}_{\hat{t}}\left[d\hat{J}^{(0)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})\right]=\underset{\mathcal{O}(1)}{\hat{J}_{\hat{t}}^{(0)}}+\underset{\mathcal{O}(1)}{\hat{J}_{\hat{K}}^{(0)}}\underset{\mathcal{O}(1)}{\underbrace{\frac{1}{d\hat{t}}\mathrm{E}_{\hat{t}}\left[d\hat{K}\right]}}$$

$$+\underset{\mathcal{O}(\epsilon)}{\epsilon}\underset{\mathcal{O}(1)}{\hat{J}_{\hat{D}}^{(0)}}\left(\underset{\mathcal{O}(1)}{\frac{\partial\hat{D}}{\partial\hat{E}}}\underset{\mathcal{O}(1)}{\underbrace{\frac{1}{d\hat{t}}\mathrm{E}_{\hat{t}}\left[d\hat{E}\right]}}+\underset{\mathcal{O}(1)}{\frac{\partial\hat{D}}{\partial\hat{\chi}}}\underset{\mathcal{O}(1)}{\underbrace{\frac{1}{d\hat{t}}\mathrm{E}_{\hat{t}}\left[d\hat{\chi}\right]}}+\underset{\mathcal{O}(1)}{\frac{\partial\hat{D}}{\partial\hat{\lambda}}}\underset{\mathcal{O}(1)}{\underbrace{\frac{1}{d\hat{t}}\mathrm{E}_{\hat{t}}\left[d\hat{\lambda}\right]}}\right)+...,$$

where we have left out the stochastic terms for ease of exposition. The contributions to the rate of change of the zeroth-order value function (left-hand side of (A2.9)) from changes in time $\hat{t}\equiv g_0 t$ (the first term on the right-hand side) and in the capital stock $\hat{K}$ (the second term) are $\mathcal{O}(1)$, whereas the contributions from changes in the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ (the remaining terms on the right-hand side) are $\mathcal{O}(\epsilon)$ and thus smaller than the first two terms by a factor ϵ. The functional dependence of $\hat{J}^{(0)}$ on the climatic states is said to be 'slow'. When solving the HJB equation (A2.3) using perturbation methods, this means that some of the derivatives of $\hat{J}^{(0)}$ with respect to the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be ignored because their order in ϵ is too high. For the first-order term $\hat{J}^{(1)}$ in (A2.8), we do not assume a slow dependence on any of the states a priori. Precisely which terms in (A2.3) will be included at which order is considered in detail in sections A.2.3 and A.2.4 below, where we set out to find the zeroth- and first-order solutions.

To cast the HJB equation (A2.3) into the form $\mathcal{F}\left[J(\mathbf{x}),\mathbf{x}\right]=0$ in (A1.1), we must first find solutions for the forward-looking variables. The optimality conditions of (A2.3) with respect to $\hat{C}$ and $\hat{F}$ are, respectively,

$$(A2.10)\quad \frac{\hat{C}^{-\gamma}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}}-\phi'\left(\hat{i}\right)\hat{J}_{\hat{K}}=0 \;\Rightarrow\; \hat{C}=\left(\phi'\left(\hat{i}\right)\hat{J}_{\hat{K}}\right)^{-\frac{1}{\gamma}}\left((1-\eta)\hat{J}\right)^{-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}},$$

$$(A2.11)\quad \hat{J}_{\hat{K}}\left((1-\alpha)\hat{A}\hat{K}^{\alpha}\hat{F}^{-\alpha}-\hat{b}\right)\phi'\left(\hat{i}\right)+\hat{J}_{\hat{E}}e^{-\hat{g}\hat{t}}\hat{\mu}=0 \;\Rightarrow\; \hat{F}=\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K},$$

where we define the optimal SCC in non-dimensional form as $\hat{P}\equiv F_0 P/(g_0 K_0)$, which is given by

(A2.12) $\quad \hat{P} = -\hat{\mu} \dfrac{\hat{J}_{\hat{E}}}{\phi'(\hat{i})\hat{J}_{\hat{K}}}.$

Upon substituting equations (A2.10)-(A2.11) into the HJB equation (A2.3) and recognizing that $\hat{J}$, $\hat{C}$, $\hat{F}$, $\hat{A}$, $\hat{i}$ and $\hat{P}$ are functions of the state variables collected in $\mathbf{x}$, we obtain an equation of the form $\mathcal{F}[J(\mathbf{x}),\mathbf{x}] = 0$ as in (A1.1).

By substituting our series solution for the value function (A2.8) into (A2.12), the leading-order estimate of the optimal SCC is given by:

(A2.13) $\quad \hat{P} = -\hat{\mu} \dfrac{\hat{J}_{\hat{E}}^{(0)} + \epsilon \hat{J}_{\hat{E}}^{(1)}}{\phi'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}} + \mathcal{O}(\epsilon^2),$

which is accurate up to $\mathcal{O}(\epsilon^2)$. For completeness, we note that both $\hat{J}_{\hat{E}}^{(0)}$ and $\hat{J}_{\hat{E}}^{(1)}$ are $\mathcal{O}(\epsilon)$. We therefore need to obtain both the zeroth- and the first-order solution for the value function to obtain a consistent leading-order estimate of the SCC.

A.2.3. Zeroth-order solution (see also appendix B)

Substituting the series solution for the value function (A2.8) into the HJB equation (A2.3), in which we have substituted for the forward-looking variables from (A2.10)-(A2.11), and collecting zeroth-order terms in ϵ, we obtain a nonlinear second-order ordinary differential equation (as the dependence on time has disappeared) given by (B1) in Appendix B, which we can write generally as $\mathcal{F}^{(0)}\left[J^{(0)}(\mathbf{x}),\mathbf{x}\right] = 0$ (cf. (A1.6)). We can solve $\mathcal{F}^{(0)}\left[J^{(0)}(\mathbf{x}),\mathbf{x}\right] = 0$ to give a solution of the form (see Appendix B):

(A2.14) $\quad \hat{J}^{(0)} = \psi_0(\epsilon\hat{D})K^{1-\eta},$

where the function $\psi_0(\epsilon\hat{D})$ captures the slow dependence on the climatic states through $\hat{D} = \lambda^{1+\theta_\lambda} \hat{\chi}^{1+\theta_{\chi T}} \hat{E}^{1+\theta_{ET}}$ (see explanation in beginning of section A.2.2) and is given by (B3) in Appendix B.

A.2.4. First-order solution (see also Appendix C)

We proceed to collect terms in the HJB that are first order in ϵ (cf. (A1.7)). First, we ignore those derivatives of the zeroth-order value function with respect to the climatic states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ that result in terms of $\mathcal{O}(\epsilon^2)$ and higher. Second, we perform Taylor-series

expansions in ϵ (about $\epsilon = 0$) of any nonlinear function of the value function, again ignoring those terms of order $\mathcal{O}(\epsilon^2)$ and higher. To illustrate this second step, consider the optimality condition (A2.10), which becomes at $\mathcal{O}(\epsilon)$

$$(\text{A2.15}) \qquad \epsilon \hat{C}^{(1)} = \hat{C}^{(1)} \left(-\frac{1}{\gamma} \frac{\phi''(\hat{i}^{(0)})}{\phi'(\hat{i}^{(0)})} \epsilon \hat{i}^{(1)} - \frac{1}{\gamma} \frac{\epsilon \hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} - \frac{1}{\gamma} \frac{\eta - \gamma}{1 - \eta} \frac{\epsilon \hat{J}^{(1)}}{\hat{J}^{(0)}} \right)$$

where we have used the product and chain rules of differentiation repeatedly ((A2.10) is the product of three functions), noting that we have used the expansions $\hat{C} = \hat{C}^{(0)} + \epsilon \hat{C}^{(1)} + \mathcal{O}(\epsilon^2)$, $\hat{i} = \hat{i}^{(0)} + \epsilon \hat{i}^{(1)} + \mathcal{O}(\epsilon^2)$, $\hat{J}_{\hat{K}} = \hat{J}_{\hat{K}}^{(0)} + \epsilon \hat{J}_{\hat{K}}^{(1)} + \mathcal{O}(\epsilon^2)$ and, of course, $\hat{J} = \hat{J}^{(1)} + \epsilon \hat{J}^{(1)} + \mathcal{O}(\epsilon^2)$. A single nonlinear term in the HJB equation can thus give rise to multiple terms upon expansion. Performing the first and second step explicitly and consistently is straightforward yet cumbersome, and details are given in Appendix C. Because we have chosen $\hat{D} = \lambda^{1+\theta_\lambda} \hat{\chi}^{1+\theta_{\chi T}} \hat{E}^{1+\theta_{ET}}$ to be a product of power functions, we can solve the resulting partial differential equation in closed form.

A.3. Result A

Combining the zeroth- and first-order solutions for the value function according to (A2.13), we obtain the following (dimensional) leading-order estimate of the optimal SCC (corresponding its non-dimensional equivalent (C3.19) in Appendix C). We present results in dimensional form here, so that they can be referred to directly by the reader of the main paper.

Result A: The optimal risk-adjusted SCC is:

$$(\text{A3.1}) \qquad P = \frac{\mu \, \Theta(E, \chi, \lambda) Y \big|_{P=0}}{r^*} \left(1 - \frac{\Omega}{E^{\theta_{ET}} \chi^{1+\theta_{ET}} \lambda^{1+\theta_\lambda} K^{1-\eta}} \right) + \mathcal{O}(\epsilon^2),$$

where $\Theta \equiv D_E / (1 - D)$ and $r^* \equiv r^{(0)} - g^{(0)} = \rho + (\gamma - 1)(g^{(0)} - \eta \sigma_K^2 / 2)$. Further,

$$(\text{A3.2}) \quad \Omega = E_t \left[\int_t^\infty \Gamma e^{-r_\Omega(s-t)} ds \right] \text{ with } r_\Omega \equiv r^* - (\eta - 1)\left(\phi(i^{(0)}) - \eta \sigma_K^2 / 2 \right) + \varphi,$$

where $\phi \equiv \Phi/K = i - \omega i^2 / 2 - \delta$, $i = I / K$. The term Γ is given dimensionally by

A8

$$(A3.3)$$

$$\Gamma \equiv \Big((1+\theta_{ET})\varphi X\Lambda - v_\chi(\bar{\chi}-\chi)X_\chi\Lambda - v_\lambda(\bar{\lambda}-\lambda)X\Lambda_\lambda$$

$$-\frac{1}{2}\sigma_\chi^2 X_{\chi\chi}\Lambda - \frac{1}{2}X\Lambda_{\lambda\lambda}\sigma_\lambda^2 - (1-\eta)X_\chi\Lambda\rho_{K\chi}\sigma_K\sigma_\chi$$

$$-(1-\eta)X\Lambda_\lambda\rho_{K\lambda}\sigma_K\sigma_\lambda - X_\chi\Lambda_\lambda\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda\Big)K^{1-\eta}E^{\theta_{ET}}$$

$$-\theta_{ET}\mu A^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}}X\Lambda K^{2-\eta}E^{\theta_{ET}-1}e^{-g^{(0)}t} - \frac{1}{2}\theta_{ET}(\theta_{ET}-1)\sigma_E^2 X\Lambda K^{1-\eta}E^{\theta_{ET}-2}$$

$$-\Big((1-\eta)\theta_{ET}X\Lambda\rho_{KE}\sigma_K\sigma_E + X_\chi\Lambda\rho_{E\chi}\sigma_E\sigma_\chi + X\Lambda_\lambda\rho_{E\lambda}\sigma_E\sigma_\lambda\Big)K^{1-\eta}E^{\theta_{ET}-1},$$

where $X \equiv \chi^{1+\theta_{ET}}$ and $\Lambda \equiv \lambda^{1+\theta_\lambda}$. □

The term in (A3.1) in front of the brackets is the net present value of marginal damages if only economic growth or asset return uncertainty is considered, and the atmospheric carbon stock does not decay; the second term in the large brackets is the mark-up for carbon stock, climate sensitivity and damage ratio uncertainties and carbon stock decay. The integral to evaluate Ω is discounted with a rate r_Ω that differs from r^* in that it corrects for net growth in the capital stock (including a term depending on risk aversion and the volatility of the capital stock) and the rate of decay of atmospheric carbon.

The optimal SCC given in (A3.1) is proportional to world GDP, which is given to leading order by its value when there is no climate policy ($P=0$) and depends on the stock of atmospheric carbon and the climate sensitivity and damage ratio parameters through the function $\Theta(E,\chi,\lambda)$. It depends on preferences (ρ, γ and η), geophysical parameters (μ, φ and v_χ), and the properties of the stochastic processes driving GDP, the carbon stock, climate sensitivity and damages. The optimal SCC depends on the growth-corrected return on capital r^*, which is given to leading order by its value when there is no climate policy ($P=0$). The expected return on investment $r^{(0)}$ is the risk-free rate, $r_{rf}^{(0)} = \rho + \gamma g^{(0)} -(1+\gamma)\eta\sigma_K^2/2$, plus the risk premium $\eta\sigma_K^2$. [4]

Result A indicates that the absolute error in our expression for the optimal SCC is $\mathcal{O}(\epsilon^2)$ and that the error as fraction of the SCC (which is $\mathcal{O}(\epsilon)$ itself) is thus $\mathcal{O}(\epsilon)$. Consistently, we can ignore the slow dependence of the discount rate on the atmospheric carbon stock (via the marginal productivity of capital) when evaluating the discounting integral in Result

---

[4] The investment and growth rates of GDP are given to leading order by their values without climate policy (cf. (C7)). Implicitly, we get from the Euler and capital accumulation equations $i^{(0)} = Y_K|_{P=0} - q^{(0)}\big(\rho + (\gamma-1)(\phi(i^{(0)}) - \eta\sigma_K^2/2)\big)$ with $Y_K|_{P=0} = \alpha A(E,\chi,\lambda)^{1/\alpha}\big((1-\alpha)/b\big)^{(1-\alpha)/\alpha}$ and $g^{(0)} = i^{(0)} - \omega(i^{(0)})^2/2 - \delta \equiv \phi(i^{(0)})$. Tobin's q is $q(i) = 1/\phi'(i)$.

A. As $\epsilon \rightarrow 0,$ the SCC in Result A becomes exact. Generally, a closed-form solution to the 5-dimensional integral (over time and to evaluate the 4-dimensional expectations operator over the stochastic states in Ω) is unavailable, so Result A must be evaluated numerically.[5]

A.4. Results 1 and 2 (see also Appendix D)

To simplify Result A, we make three additional assumptions.

Assumption I: The future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through its dependence on the stochastic capital stock.

Assumption II: We include only the leading-order effects of uncertainty by performing an additional perturbation expansion.[6]

Assumption III: We set the initial and steady-state values of the damage ratio parameter λ_0 and $\bar{\lambda}$ to be equal, so deterministic damages are not subject to a delay. (We do not make the same assumption for the climate sensitivity parameter χ).

Owing to *Assumptions I-III*, we can derive closed-form solutions for the optimal risk-adjusted SCC by evaluating the 4-dimensional integral in Result A explicitly with all details in Appendix D. In doing so, we derive Results 1 and 2. We show in Appendix F that Results 1 and 2 only have minimal quantitative errors compared to Result A and that *Assumptions I, II and III* are therefore justified ex post.

A.4.1. Result 1

Result 1 gives the simpler case under two additional assumptions.

Assumption IV: Proportional reduced-form damages ($\theta_{ET} = 0$).

Assumption V: An initial climate sensitivity parameter that is equal to its steady-state value ($\chi_0 = \bar{\chi}$).

So-called reduced-form damages (or simply 'damages' below) are obtained when the temperature-carbon stock relationship $T(E)$ is substituted into the damage-temperature relationship $D(T)$, and damages become a direct function of the carbon stock: $D(E)$. Under *Assumption IV*, damages are proportional to the atmospheric carbon stock (i.e.

---

[5] This requires five-dimensional numerical integration over the probability space corresponding to the four states and with respect to time. If the processes are independent, the integrals over the probability space of states can be evaluated independently.

[6] For completeness, we note that in the limit of small uncertainty in which Results 1 and 2 are valid, the atoms of probability associated with all non-negativity constraints disappear.

$D \propto E$), and marginal damages are constant (i.e. $D_E \neq f(E)$) and thus unaffected by future emissions. Under *Assumption V*, the deterministic climate sensitivity parameter does not vary with time (i.e. $\mu_\chi = \chi_0 = \bar{\chi}$). We emphasize that expected climate sensitivity $(\mathrm{E}_t[T_2] = \mathrm{E}_t[\chi(s)^{1+\theta_\chi}])$) does increase the further we are looking into the future $(s > t)$ due to increased uncertainty on longer horizons and the convex dependence of climate sensitivity T_2 on the climate sensitivity parameter χ. Under *Assumption V* only deterministic delays in the climate sensitivity are thus ignored motivated by simplicity alone. Result 1 and its 'risk adjustments' are given in dimensional form in (17) in the paper.

A.4.2. Result 2

Result 2 allows for convex damages ($\theta_{ET} \neq 0$) and an initial climate sensitivity parameter that differs from its steady-state value ($\chi_0 \neq \bar{\chi}$) and thus relaxes *Assumptions IV and V*. In addition to 'risk adjustments', the SCC in Result 2 given by (20) in the paper includes additional so-called 'correction factors', which can be evaluated as simple, one-dimensional integrals. We distinguish two types of so-called 'correction factors', denoted by the symbol Υ with subscripts again denoting the state variable(s) from which the risk originates: for $\theta_{ET} \neq 0$ and for $\chi_0 \neq \bar{\chi}$.

Result 1 and 2 are different in three ways. First, the correction factors in Result 2, $\Upsilon_{\theta_{ET} \neq 0}$ and $\Upsilon_{\chi_0 \neq \bar{\chi}}$, provide deterministic corrections for $\theta_{ET} \neq 0$ and $\chi_0 \neq \bar{\chi}$, respectively. Second, in Result 2, the adjustments for uncertainty in the carbon stock, climate sensitivity, the damage ratio and the interaction between the two are now multiplied by their respective correction factors (Υ). Third, the effective discount rate r^* in Result 1 is replaced by $r^* \equiv r^* + (1 + \theta_{ET})\varphi$ (and $r^{**} \equiv r^* + (\eta - 1)\sigma_K^2 - \varphi$) in Result 2. The risk adjustments in Result 2 are given by:

$$(A4.1) \qquad \Delta_{EE} = -\frac{1}{2}\theta_{ET}(1-\theta_{ET})\left(\frac{\sigma_E}{E}\right)^2 \frac{1}{r^* - 2\varphi}\Upsilon_{EE},$$

$$(A4.2) \quad \Delta_{\chi\chi} = \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{(\sigma_\chi/\chi_0)^2}{r^* + 2v_\chi}\Upsilon_{\chi\chi}, \quad \Delta_{\lambda\lambda} = \frac{1}{2}\theta_\lambda(1+\theta_\lambda)\frac{(\sigma_\lambda/\lambda_0)^2}{r^* + 2v_\lambda}\Upsilon_{\lambda\lambda},$$

$$(A4.3) \qquad \Delta_{\chi\times\lambda} = \frac{1}{4}\theta_{\chi T}(1+\theta_{\chi T})\theta_\lambda(1+\theta_\lambda)\frac{\sigma_\lambda^2 \sigma_\chi^2}{2v_\chi 2v_\lambda}\Upsilon_{\chi\times\lambda}.$$

The adjustments for correlated climate and economic risk are

$$(A4.4)\ \Delta_{CK} = -(\eta-1)\sigma_K \left(\theta_{ET} \frac{\rho_{KE}\dfrac{\sigma_E}{E}}{(r^\star-\varphi)}\Upsilon_{KE} + (1+\theta_{\chi T})\frac{\rho_{K\chi}\dfrac{\sigma_\chi}{\chi_0}}{r^\star+v_\chi}\Upsilon_{K\chi} + (1+\theta_\lambda)\frac{\rho_{K\lambda}\dfrac{\sigma_\lambda}{\lambda_0}}{r^\star+v_\lambda}\Upsilon_{K\lambda} \right).$$

The adjustment for correlated climate sensitivity and damage ratio risk is

$$\Delta_{CC} = \theta_{ET}(1+\theta_{\chi T})\frac{\rho_{E\chi}\sigma_E\sigma_\chi/\chi_0}{(r^\star+v_\chi)E}\frac{r^\star}{r^\star-\varphi}\Upsilon_{E\chi}$$

(A4.5)

$$+(1+\theta_\lambda)\left(\theta_{ET}\frac{\rho_{E\lambda}\sigma_E\sigma_\lambda/\lambda_0}{(r^\star+v_\lambda)E}\frac{r^\star}{r^\star-\varphi}\Upsilon_{E\lambda} + (1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/(\chi_0\lambda_0)}{r^\star+v_\chi+v_\lambda}\Upsilon_{\chi\lambda} \right).$$

The correction factors (Υ) in (A.4.1)-(A4.5) multiply a risk adjustment (Δ) and must be linearly combined with unity, so that, for example, $\Upsilon_{\chi\chi} \equiv 1 + \Upsilon_{\chi\chi,\theta_{ET}\neq 0} + \Upsilon_{\chi\chi,\chi_0\neq\bar{\chi}}$. These combined correction factors are equal to unity when $\theta_{ET}\neq 0$ and $\chi_0 \neq \bar{\chi}$ (e.g., $\Upsilon_{\chi\chi}\equiv 1$). We give the correction factors in terms of dimensional quantities given in (D3.4)-(D3.5) in Appendix D.

□

Convexity of damages ($\theta_{ET}>0$) (*Assumption IV*) causes Result 2 to be different from Result 1 in four ways. First, it changes the normalized marginal damage ratio $\Theta(E)$. From (6), we obtain $\Theta_E(E) = (1/S_{PI})^2 (1+\theta_{ET})\theta_{ET}(E/S_{PI})^{\theta_{ET}-1}\chi^{1+\theta_{\chi T}}\lambda^{1+\theta_\lambda}$ to leading order in our small parameter. With convex damages ($\theta_{ET}>0$), the normalized marginal damage ratio thus rises with the stock of atmospheric carbon. The time path for the carbon price is then steeper than that of world GDP. Its effect on the deterministic SCC is captured through the correction factor $\Upsilon_{\theta_{ET}\neq 0} > 0$, reflecting the more harmful effect of future emissions (when the stock is higher). Second, convex damages boost the effective discount rate $r^\star = \rho + (\gamma-1)g^{(0)} + (1+\theta_{ET})\varphi$, because the marginal damage of a unit of CO_2 decays more quickly than the unit itself, depressing the SCC. Combining the first and second effects, the net effect on the SCC is positive for small decay rates of atmospheric carbon. Third, a new adjustment (A4.1) needs to be made for carbon stock uncertainty. For damages that are not too convex ($0<\theta_{ET}<1$), this adjustment is negative, reflecting concave

marginal damages $D_E \propto (1+\theta_{ET})E^{\theta_{ET}}$ with $D_{EEE} \propto \theta_{ET}(\theta_{ET}-1)(1+\theta_{ET})E^{\theta_{ET}-2} < 0,$ which holds for $0 < \theta_{ET} < 1$ (see section IV). We emphasize that with proportional damages $(\theta_{ET} = 0)$, the adjustment to the SCC for carbon stock uncertainty is zero in Result 1. Fourth, the adjustments for the other two climatic uncertainties in (A4.2) are now multiplied by correction factors that are greater than unity, reflecting rising marginal damages due to future emissions, as in the deterministic case. The same applies to the terms adjusting for correlations in (A4.4)-(A4.5), with new correlation terms with the carbon stock arising there. Finally, Result 2 allows for a higher-order term (A4.3), which is may be non-negligibly small if $\theta_{\chi T}$ is large enough (see also Appendix D).

The effect of the initial climate sensitivity parameter differing from its steady-state value $(\chi_0 \neq \bar{\chi})$ (*Assumption V*) is captured as follows. The normalized marginal ratio Θ is evaluated at the initial (low) temperature. The term multiplying $\Upsilon_{\chi_0 \neq \bar{\chi}} > 0$ is positive and captures this delayed deterministic temperature rise. Similarly, all the adjustments are corrected by their respective correction factors to take this delayed deterministic temperature increase into account.

A.5. Comparison with other types of perturbation theory in economics

The type of perturbation theory we apply is different (but not fundamentally so) from the types of perturbation theory that are commonly applied in (macro-)economics and finance (e.g., Judd, 1996, 1998). Typically, in this literature, the value function is expanded in powers of the states themselves (e.g., $J(K) = \sum_{n=1}^{N} c_n (K - K_0)^n$), sometimes preceded by a transformation of variables using a logarithm or power function). Examples are Judd and Guu (1997), Schmitt-Grohé and Uribe (2004) and van Binsbergen et al. (2012). Our approach is different, as we retain the nonlinear dependence on the states without approximation at every order in ϵ, which is made possible because of our use of power functions. Sometimes in the literature, the relative standard deviation of the stochastic process is the small parameter of the perturbation expansion (e.g., Judd and Guu, 2001).[7] Both methods (expansion in the states and expansion in the small relative standard deviation) can be combined (e.g., Boragan Aruoba, Fernandez-Villaverde and Rubio-Ramirez, 2006). Although we choose a different small parameter, our approach is similar to Judd and Guu (2001) with one fundamental difference: we also make use of the concept of slow functional dependence and slow derivatives (from slow-fast dynamics in the

---

[7] We also use this type of perturbation expansion to take only leading-order climatic uncertainty into account (i.e. *Assumption II*) when we obtain Results 1 and 2 from *Result A*.

differential equations literature), which we have not seen applied in economics although it has been used in financial mathematics (e.g., Fouque, Papanicolaou and Sircar (2000)).

References

Bender, C.M. and S.A. Orszag (1999). *Advanced Mathematical Methods for Scientists and Engineers*, pp. 544-568, Springer.

Binsbergen, J.H., J. Fernández-Villaverde and R. van Koijen (2012). The term structure of interest rates in a DSGE model with recursive preferences, *Journal of Monetary Economics*, 59, 634-648.

Borogan Aruoba, S., J. Fernandez-Villaverde and J.F. Rubio-Ramirez (2006). Comparing solution methods for dynamic equilibrium economies, *Journal of Economic Dynamics and Control*, 30, 12, 2477-2508.

Fouque, J.P., G. Papanicolaou and K. R. Sircar (2000). *Derivatives in Financial Markets with Stochastic Volatility,* Cambridge University Press.

Judd, K.L. 1996. Approximation, perturbation and projection methods in economic analysis, in H.M. Amman, D.A. Kendrick and J. Rust (eds.), *Handbook of Computational Economics, Volume 1*, North-Holland, Amsterdam.

Judd, K.L. 1998. *Numerical Methods in Economics*, MIT Press, Cambridge, Mass.

Judd, K.L. and S.-M. Guu (1977). Asymptotic methods for aggregate growth models, *Journal of Economic Dynamics and Control*, 21, 6, 1025-1042.

Judd, K.L. and S.-M. Guu (2001). Asymptotic methods for asset market equilibrium analysis, *Journal of Economic Theory*, 18, 127-157.

Kevorkian, J. and J.D. Cole (1996). *Multiple Scale and Singular Perturbation Methods*, Springer.

Nayfeh, A.H. (2004). *Perturbation Methods*, Wiley.

Schmitt-Grohé, S. and M. Uribe (2004). Solving dynamic general equilibrium models using a second-order approximation to the policy function, *Journal of Economic Dynamics and Control*, 28, 4, 755-775.

Van Dyke, M. (1975). *Perturbation Methods in Fluid Mechanics, Parabolic Press, Stanford*.

Appendix B: Derivation of Zeroth-Order Solution (For Online Publication)

After substituting the series solution for the value function (A2.8), the Hamilton-Jacobi-Bellman equation (A2.3) can be written at $\mathcal{O}(1)$ as

(B1)
$$\frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}\frac{\eta-\gamma}{1-\eta}} - \hat{\rho}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}}}{(1-\gamma)\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}-1}} + \hat{J}_{\hat{t}}^{(0)} + \hat{J}_{\hat{K}}^{(0)}\hat{\phi}(\hat{i}^{(0)})\hat{K}$$
$$+\frac{1}{2}\hat{J}_{\hat{K}\hat{K}}^{(0)}\hat{K}^2\hat{\sigma}_K^2 = 0,$$

where we have substituted for the forward-looking variables $\hat{C}$ and $\hat{F}$ at $\mathcal{O}(1)$ from (A2.10) and (A2.11) and we have used

(B2) $\underbrace{\frac{1}{d\hat{t}}\mathrm{E}_{\hat{t}}\left[d\hat{K}\right]}_{\mathcal{O}(1)} = \hat{\phi}(\hat{i}^{(0)})\hat{K}$.

In (B1)-(B2), $\hat{i}^{(0)}$ is the (constant) optimally chosen investment rate. We note that there is no variation with time $\hat{t}$ in equation (B1), so $\hat{J}_{\hat{t}}^{(0)} = 0$, and (B1) is a second-order ordinary differential equation in $\hat{K}$. Equation (B1) has a power-law solution of the form $J^{(0)} = \psi_0\hat{K}^{1-\eta}$, and following some algebraic manipulation we obtain

(B3) $\hat{J}^{(0)} = \psi_0\hat{K}^{1-\eta}$ with $\psi_0 = \frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-(1-\eta)}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right)^{-\gamma\frac{1-\eta}{1-\gamma}}$,

where $\psi_0 = \psi_0(\epsilon\hat{D})$ is a slow function of $\hat{D}$ through $\hat{i}^{(0)} = \hat{i}^{(0)}(\epsilon\hat{D})$. From the first-order optimality condition for $\hat{C}$, i.e. (A2.10), at $\mathcal{O}(1)$, we obtain

(B4) $\hat{C}^{(0)} = \hat{c}^{(0)}\hat{K}$ with $\hat{c}^{(0)} = \frac{1}{\hat{\phi}'(\hat{i}^{(0)})}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right)$,

where $q(\hat{i}) = 1/\phi'(\hat{i})$ denotes Tobin's q, the price of capital in consumption terms.[8] We can thus write the value function (B3) as

---

[8] The value of the capital stock is $\hat{q}\hat{K}$, or dimensionally qK, where $\hat{q} = 1/\hat{\phi}'(\hat{i}) = 1/\phi'(i)$ is already a fraction and is left unchanged by the normalization (cf. $\hat{q} = q$ or $\omega i = \hat{\omega}\hat{i}$).

(B5) $\quad J^{(0)} = \dfrac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}}\hat{K}^{1-\eta}.$

Substituting in for $\hat{F}$ from the first-order optimality condition (A2.11), we obtain from $\hat{I} = \hat{Y} - \hat{C} - \hat{b}\hat{F}$:

(B6) $\quad \hat{i}^{(0)} = \hat{r}_{\mathrm{mpk}}^{(0)} + \hat{\delta} - \hat{c}^{(0)} = \hat{r}_{\mathrm{mpk}}^{(0)} + \hat{\delta} - \hat{q}^{(0)}\left(\hat{\rho} - (1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)}) - \dfrac{1}{2}\eta\hat{\sigma}_K^2\right)\right),$

where $\hat{r}_{\mathrm{mpk}}^{(0)}(\epsilon\hat{D}) \equiv \hat{Y}_{\hat{K}}^{(0)} - \hat{\delta} = \alpha\hat{A}\left(\epsilon\hat{D}\right)^{\frac{1}{\alpha}}\left((1-\alpha)/\hat{b}\right)^{\frac{1-\alpha}{\alpha}} - \hat{\delta}$ denotes the marginal productivity

of capital net of depreciation[9] at zeroth order, which is a slow function of $\hat{D}$ through its dependence on the total factor productivity. Equation (B6) implicitly defines the optimally chosen investment rate $\hat{i}^{(0)}$. From (B4), the leading-order endogenous growth rate of capital and hence of consumption is

(B7) $\quad \hat{g}^{(0)} = \underbrace{\dfrac{1}{\hat{K}}\dfrac{1}{\mathrm{d}t}\mathrm{E}_i\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(1)} = \hat{\phi}(\hat{i}^{(0)})$ and hence $\hat{g}^{(0)} = \hat{\phi}\left(\hat{i}^{(0)}\right) = 1.$

In equilibrium, the marginal propensity to consume $\hat{c}^{(0)}/\hat{q}^{(0)}$ equals the expected return on investment $\hat{r}^{(0)}$ minus the growth rate of the economy $\hat{g}^{(0)}$. The expected return on investment $\hat{r}^{(0)}$, in turn, equals the sum of the risk-free rate $\hat{r}_{\mathrm{rf}}^{(0)}$ and the risk premium $\Delta\hat{r}^{(0)}$. Hence, $\hat{c}^{(0)}/\hat{q}^{(0)} = \hat{r}^{(0)} - \hat{g}^{(0)} = \hat{r}_{\mathrm{rf}}^{(0)} + \Delta\hat{r}^{(0)} - \hat{g}^{(0)}$, and with a risk premium of $\Delta\hat{r}^{(0)} = \eta\hat{\sigma}_K^2$ in the absence of any climate risk at zeroth-order, the risk-free rate is:

(B8) $\quad \hat{r}_{\mathrm{rf}}^{(0)} = \hat{\rho} + \gamma\hat{g}^{(0)} - (1+\gamma)\eta\hat{\sigma}_K^2/2 .$

Although $\hat{J}_{\hat{E}}^{(0)}$ can be computed from (B5), a consistent leading-order estimate of the optimal SCC also requires $\hat{J}_{\hat{E}}^{(1)}$ and thus the next order in the perturbation expansion.

---

[9] Dimensionally, we have $r_{\mathrm{mpk}}^{(0)} = \hat{r}_{\mathrm{mpk}}^{(0)}g_0.$

Appendix C: Derivation of First-Order Solution (For Online Publication)

To derive the first-order solution, we follow the following steps. We first find the evolution equations for $\hat{K}$ and $\hat{E}$ (section C1). We then solve the multi-variate Ornstein-Uhlenbeck process that describes all our states (section C2). In section C3, we will substitute all these results into the HJB equation and retain only terms at $\mathcal{O}(\epsilon)$. It will become clear there that we only need to derive the evolution equation for $\hat{K}$ up to $\mathcal{O}(\epsilon)$ and for $\hat{E}$ at $\mathcal{O}(1)$ in section C1. The terms associated with uncertainty of the climatic variables and their covariances only need to be derived at $\mathcal{O}(1)$ in section C2.

C1. Expected evolution equations for $\hat{K}$ and $\hat{E}$

We consider the expected evolution equations of the states $\hat{K}$ and $\hat{E}$ at $\mathcal{O}(\epsilon)$ and $\mathcal{O}(1)$, respectively. At this order, we have for the expected evolution of $\hat{K}$:

(C1.1) $\quad \underbrace{\frac{1}{d\hat{t}} \mathrm{E}_{\hat{t}}\left[d\hat{K}\right]}_{\mathcal{O}(\epsilon)} = \hat{\phi}'(\hat{i}^{(0)})\epsilon\hat{I}^{(1)} = -\hat{\phi}'(\hat{i}^{(0)})\epsilon\hat{C}^{(1)} = \frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}} \hat{K}\left(\frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\eta - \gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right),$

where the first identity makes use of $\hat{\Phi} = \epsilon\hat{I}^{(1)} - \hat{\omega}\epsilon\hat{I}^{(1)}\hat{I}^{(0)}/\hat{K} = \epsilon\hat{I}^{(1)}\hat{\phi}'(\hat{i}^{(0)})$ at $\mathcal{O}(\epsilon)$. We further note from $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C}$ that $\hat{I}^{(1)} = -\hat{C}^{(1)}$, since production net of fossil fuel costs is unaffected by the SCC in our formulation:

(C1.2)
$$\frac{\partial}{\partial\hat{P}}\left[\hat{Y} - \hat{b}\hat{F}\right]\Bigg|_{\hat{P}=0} =$$
$$\frac{\partial}{\partial\hat{P}}\left[\hat{A}^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\hat{b} + \hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}}\hat{K} - \hat{b}\left(\frac{1-\alpha}{\hat{b} + \hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}\right]\Bigg|_{\hat{P}=0} = 0.$$

The identity in (C1.2) relies on the Cobb-Douglas nature of the production function. The third identity in (C1.1) follows from a Taylor-series expansion of $\hat{C}$, given by (A2.10), with respect to the small parameter ϵ (about $\epsilon = 0$):

$$(C1.3) \qquad \epsilon \hat{c}^{(1)} = \hat{c}^{(0)} \left(-\frac{1}{\gamma} \frac{\phi''}{\hat{\phi}'\left(\hat{i}^{(0)}\right)} \epsilon \hat{i}^{(1)} - \frac{1}{\gamma} \frac{\epsilon \hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} - \frac{1}{\gamma} \frac{\eta-\gamma}{1-\eta} \frac{\epsilon \hat{J}^{(1)}}{\hat{J}^{(0)}} \right).$$

Noting that $\hat{i}^{(1)} = -\hat{c}^{(1)}$, we can rearrange this linear equation to give

$$(C1.4) \qquad \hat{c}^{(1)} = \frac{\hat{c}^{(0)}}{1 - \frac{1}{\gamma} \frac{\hat{c}^{(0)} \phi''}{\hat{\phi}'(\hat{i}^{(0)})}} \left(-\frac{1}{\gamma} \frac{\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} - \frac{1}{\gamma} \frac{\eta-\gamma}{1-\eta} \frac{\hat{J}^{(1)}}{\hat{J}^{(0)}} \right),$$

which is used in the third identity in (C2.1).

For $\hat{E}$, we have at $\mathcal{O}(1)$:

$$(C1.5) \qquad \underbrace{\frac{1}{d\hat{t}} \mathrm{E}_{\hat{t}}\left[d\hat{E} \right]}_{\mathcal{O}(1)} = \hat{\mu} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K} e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E}.$$

C2. Solution to multi-variate Ornstein-Uhlenbeck process at $\mathcal{O}(1)$

We define $\hat{k} \equiv k \equiv \log\left(K / K_0 \right)$, so the vector of states $d\mathbf{x} = \left\{ d\hat{k}, d\hat{E}, d\hat{\chi}, d\hat{\lambda} \right\}^T$ is described by a multi-variate Ornstein-Uhlenbeck process (9), which in non-dimensional form is

$$(C2.1) \qquad d\mathbf{x} = \boldsymbol{\alpha} dt - \mathbf{v} \circ \left(\mathbf{x} - \bar{\boldsymbol{\mu}} \right) d\hat{t} + \mathbf{S} d\hat{\mathbf{W}}_{\hat{t}},$$

where we note that we have not included time $\hat{t}$ in the vector $\hat{\mathbf{x}}$ (unlike in Appendix A). The growth rate vector (10), relevant to the capital and atmospheric carbon stock processes only, is given in non-dimensional form by

$$(C2.2) \qquad \begin{aligned} \boldsymbol{\alpha} &= \left(\frac{1}{\hat{K}} \frac{1}{d\hat{t}} \mathrm{E}_{\hat{t}}\left[d\hat{K} \right] - \frac{1}{2}\hat{\sigma}_K^2, \frac{1}{d\hat{t}} \mathrm{E}_{\hat{t}}\left[d\hat{E} \right], 0, 0 \right)^T, \\ &= \left(\hat{\phi}(\hat{i}) - \frac{1}{2}\hat{\sigma}_K^2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K} e^{-\hat{g}\hat{t}}, 0, 0 \right)^T, \end{aligned}$$

A18

the mean reversion rate vector by $\mathbf{v} = \left(0, \hat{\varphi}, v_\chi, v_\lambda\right)^T$, the vector of means by $\hat{\bar{\mu}}^T = (0, 0, \hat{\bar{\chi}}, \hat{\bar{\lambda}})^T$, and the covariance matrix $\mathbf{SS}^T$ has the form

$$(C2.3) \qquad \frac{1}{d\hat{t}} \mathrm{E}_{\hat{t}}\left[d\mathbf{x}d\mathbf{x}^T\right] = \mathbf{SS}^T = \begin{pmatrix} \hat{\sigma}_K^2 & \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \\ \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \hat{\sigma}_E^2 & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda \\ \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \hat{\sigma}_\chi^2 & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \\ \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda & \hat{\sigma}_\lambda^2 \end{pmatrix}.$$

We begin by integrating the multi-variate Ornstein-Uhlenbeck process (C2.1), including only terms at zeroth order, so that the coefficients are constant, and a closed-form solution is available. It will become apparent in section C3 that $\mathcal{O}(1)$ solutions to (C2.1) are sufficient to obtain the HJB equation at $\mathcal{O}(\epsilon)$. Specifically, we have at $\mathcal{O}(1)$ that

$$\boldsymbol{\alpha}^{(0)} = \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2, \hat{\mu}\left(\frac{1-\alpha}{b}\right)^{1/\alpha}\hat{A}^{1/\alpha}, 0, 0\right)^T, \text{ where we have relied on the solution for }$$

$\hat{K}$ from the zeroth-order problem (cf. (B7)). The slow dependence of productivity $\hat{A}$ on the states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be neglected when integrating with respect to time at $\mathcal{O}(1)$. For constant coefficients, (C2.1) can be integrated to give:

$$(C2.4) \qquad \mathbf{x}(t) = \bar{\boldsymbol{\mu}} + \boldsymbol{\alpha}t + e^{\mathbf{v}\hat{t}} \circ \left(\mathbf{x}_0 - \bar{\boldsymbol{\mu}}\right) + \int_0^{\hat{t}} e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}d\mathbf{W}_{\hat{u}}.$$

The quantity $\mathbf{x}(\hat{t})$ is therefore normally distributed with covariance matrix $\boldsymbol{\Sigma}(t)$:

$$(C2.5) \qquad \boldsymbol{\Sigma}(t) = \int_0^{\hat{t}} \left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)\left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)^T d\hat{u} =$$

$$\begin{pmatrix} \hat{\sigma}_K^2\hat{t} & \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{v_\chi}\left(1-e^{-\hat{v}_\chi\hat{t}}\right) & \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{v}_\lambda}\left(1-e^{-\hat{v}_\lambda\hat{t}}\right) \\ \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\hat{\sigma}_E^2}{2\hat{\varphi}}\left(1-e^{-2\hat{\varphi}\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\hat{v}_\chi}\left(1-e^{-(\hat{\varphi}+\hat{v}_\chi)\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{v}_\lambda}\left(1-e^{-(\hat{\varphi}+\hat{v}_\lambda)\hat{t}}\right) \\ \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{v_\chi}\left(1-e^{-\hat{v}_\chi\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+v_\chi}\left(1-e^{-(\hat{\varphi}+v_\chi)\hat{t}}\right) & \frac{\hat{\sigma}_\chi^2}{2v_\chi}\left(1-e^{-2\hat{v}_\chi\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{v}_\chi+\hat{v}_\lambda}\left(1-e^{-(\hat{v}_\chi+\hat{v}_\lambda)\hat{t}}\right) \\ \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{v}_\lambda}\left(1-e^{-\hat{v}_\chi\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{v}_\lambda}\left(1-e^{-(\hat{\varphi}+\hat{v}_\lambda)\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{v}_\chi+\hat{v}_\lambda}\left(1-e^{-(\hat{v}_\chi+\hat{v}_\lambda)\hat{t}}\right) & \frac{\hat{\sigma}_\lambda^2}{2\hat{v}_\lambda}\left(1-e^{-2\hat{v}_\lambda\hat{t}}\right) \end{pmatrix}.$$

C3. The Hamilton-Jacobi-Bellman equation

Substituting for the forward-looking variables $\hat{C}$ from (A2.10) and $\hat{F}$ from (A2.11), the Hamilton-Jacobi-Bellman equation (A2.3) becomes at $\mathcal{O}(\epsilon)$:

(C3.1) $\underset{\mathcal{O}(\epsilon)}{\hat{f}^*(\hat{J})} + \epsilon\hat{J}_{\hat{i}}^{(1)} + \epsilon\hat{J}_{\hat{K}}^{(1)}\hat{K}\hat{\phi}(\hat{i}^{(0)}) + \frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma - \hat{c}^{(0)}\phi''/\hat{\phi}'(\hat{i}^{(0)})}\left(\epsilon\hat{K}\hat{J}_{\hat{K}}^{(1)} + (\eta-\gamma)\epsilon\hat{J}^{(1)}\right)$

$+\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}\right)\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{1/\alpha}\hat{A}^{1/\alpha}\hat{K}e^{-\hat{g}^{(0)}\hat{i}} - \hat{\phi}\hat{E}\right) + \left(\hat{J}_{\hat{\chi}}^{(0)} + \epsilon\hat{J}_{\hat{\chi}}^{(1)}\right)\hat{v}_\chi(\bar{\hat{\chi}} - \hat{\chi})$

$+\left(\hat{J}_{\hat{\lambda}}^{(0)} + \epsilon\hat{J}_{\hat{\lambda}}^{(1)}\right)\hat{v}_\lambda(\bar{\hat{\lambda}} - \hat{\lambda}) + \frac{1}{2}\epsilon\hat{J}_{\hat{K}\hat{K}}^{(1)}\hat{K}^2\hat{\sigma}_K^2 + \frac{1}{2}\left(\hat{J}_{\hat{E}\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}\hat{E}}^{(1)}\right)\hat{E}^2\hat{\sigma}_E^2$

$+\frac{1}{2}\left(\hat{J}_{\hat{\chi}\hat{\chi}}^{(0)} + \epsilon\hat{J}_{\hat{\chi}\hat{\chi}}^{(1)}\right)\hat{\sigma}_\chi^2 + \frac{1}{2}\left(\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(0)} + \epsilon\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(1)}\right)\hat{\sigma}_\lambda^2 + \left(\hat{J}_{\hat{K}\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{K}\hat{E}}^{(1)}\right)\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E$

$+\left(\hat{J}_{\hat{K}\hat{\chi}}^{(0)} + \epsilon\hat{J}_{\hat{K}\hat{\chi}}^{(1)}\right)\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi + \left(\hat{J}_{\hat{K}\hat{\lambda}}^{(0)} + \epsilon\hat{J}_{\hat{K}\hat{\lambda}}^{(1)}\right)\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda$

$+\left(\hat{J}_{\hat{E}\hat{\chi}}^{(0)} + \epsilon\hat{J}_{\hat{E}\hat{\chi}}^{(1)}\right)\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi + \left(\hat{J}_{\hat{E}\hat{\lambda}}^{(0)} + \epsilon\hat{J}_{\hat{E}\hat{\lambda}}^{(1)}\right)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda + \left(\hat{J}_{\hat{\chi}\hat{\lambda}}^{(0)} + \epsilon\hat{J}_{\hat{\chi}\hat{\lambda}}^{(1)}\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda = 0,$

where we have used the identity $\partial/\partial\hat{k} = \hat{K}\,\partial/\partial\hat{K}$ (chain rule), substituted the evolution equations for $\hat{K}$ at subsequent orders ((B2) and (C1.1)) and $\hat{E}$ at zeroth-order (C1.5), and defined $\hat{f}^*(J) \equiv \hat{f}\left(\hat{C}^*, \hat{J}\right)$ with $\hat{C}$ optimally chosen. From (1) and (A2.10), $\hat{f}^*(J)$ is

(C3.2) $\hat{f}^* = \frac{1}{1-\gamma}\left(\hat{\phi}'(\hat{i})\hat{J}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}} - \frac{1-\eta}{1-\gamma}\hat{\rho}\hat{J}.$

A Taylor-series expansion of $\hat{f}^*(J)$ in ϵ (about $\epsilon = 0$) gives

(C3.3)

$\underset{\mathcal{O}(\epsilon)}{\hat{f}^*} = \frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\gamma}\left(-\frac{\hat{\phi}''}{\hat{\phi}'(\hat{i}^{(0)})}\epsilon\hat{i}^{(1)} - \frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\gamma-\eta}{(1-\gamma)(1-\eta)}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right) - \frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)}$

$\frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma}\left(-\frac{\hat{\phi}''}{\hat{\phi}'(\hat{i}^{(0)})}\frac{\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}}\left(\epsilon\hat{K}\hat{J}_{\hat{K}}^{(1)} + (\eta-\gamma)\epsilon\hat{J}^{(1)}\right) - \epsilon\hat{K}\hat{J}_{\hat{K}}^{(1)} + \frac{\gamma-\eta}{(1-\gamma)}\epsilon\hat{J}^{(1)}\right) - \frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)},$

where we have substituted for $\hat{i}^{(1)} = -\hat{c}^{(1)}$ from (C2.4) and used the identity:

$$(C3.4) \qquad \frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\hat{K}\hat{J}_{\hat{K}}^{(0)}} = \hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}.$$

Substituting from (C3.2), two of the terms in (C3.1) simplify to

$$(C3.5) \qquad \underbrace{\hat{f}^*}_{\mathcal{O}(\epsilon)} + \hat{J}_{\hat{K}}^{(0)}\frac{1}{\mathrm{d}\hat{t}}\underbrace{\mathrm{E}_{\hat{t}}\left[\mathrm{d}\hat{K}\right]}_{\mathcal{O}(\epsilon)} = -\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}(\eta-\gamma)+(1-\eta)\hat{\rho}\right]\epsilon\hat{J}^{(1)}.$$

Using (C3.5), (C3.1) can be written as an equation with (derivatives of) the unknown first-order value function on the left-hand side and (derivatives of) the known zeroth-order value function on the right-hand side (cf. (A1.7)):

$$-\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}\left(\eta-\gamma\right)+\left(1-\eta\right)\hat{\rho}\right]\epsilon\hat{J}^{(1)}+\epsilon\hat{J}_{\hat{t}}^{(1)}+\epsilon\hat{J}_{\hat{K}}^{(1)}\hat{K}\hat{\phi}(\hat{i}^{(0)})$$

$$\epsilon\hat{J}_{\hat{E}}^{(1)}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}}-\varphi\hat{E}\right)+\epsilon\hat{J}_{\hat{\chi}}^{(1)}\hat{v}_{\chi}\left(\bar{\hat{\chi}}-\chi\right)+\epsilon\hat{J}_{\hat{\lambda}}^{(1)}\hat{v}_{\lambda}\left(\bar{\hat{\lambda}}-\hat{\lambda}\right)$$

$$(C3.6) \qquad +\frac{1}{2}\epsilon\hat{J}_{\hat{K}\hat{K}}^{(1)}\hat{K}^2\hat{\sigma}_K^2+\frac{1}{2}\epsilon\hat{J}_{\hat{E}\hat{E}}^{(1)}\hat{E}^2\hat{\sigma}_E^2+\frac{1}{2}\epsilon\hat{J}_{\hat{\chi}\hat{\chi}}^{(1)}\hat{\sigma}_\chi^2+\frac{1}{2}\epsilon\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(1)}\hat{\sigma}_\lambda^2$$

$$\epsilon\hat{J}_{\hat{K}\hat{E}}^{(1)}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E+\epsilon\hat{J}_{\hat{K}\hat{\chi}}^{(1)}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi+\epsilon\hat{J}_{\hat{K}\hat{\lambda}}^{(1)}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda$$

$$+\epsilon\hat{J}_{\hat{E}\hat{\chi}}^{(1)}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi+\epsilon\hat{J}_{\hat{E}\hat{\lambda}}^{(1)}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda+\epsilon\hat{J}_{\hat{\chi}\hat{\lambda}}^{(1)}\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda = -\hat{G}\left(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}\right),$$

where we will refer to the right-hand side as (minus) the 'forcing'. The forcing is defined as

$$\hat{G}(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}) \equiv \hat{J}_{\hat{E}}^{(0)}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}}-\varphi\hat{E}\right)+\hat{J}_{\hat{\chi}}^{(0)}\hat{v}_{\chi}(\bar{\hat{\chi}}-\hat{\chi})+$$

$$(C3.7) \quad \hat{J}_{\hat{\lambda}}^{(0)}\hat{v}_{\lambda}(\bar{\hat{\lambda}}-\hat{\lambda})+\frac{1}{2}\hat{J}_{\hat{E}\hat{E}}^{(0)}\hat{\sigma}_E^2+\frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}^{(0)}\hat{\sigma}_\chi^2+\frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(0)}\hat{\sigma}_\lambda^2+\hat{J}_{\hat{K}\hat{E}}^{(0)}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E+\hat{J}_{\hat{K}\hat{\chi}}^{(0)}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi$$

$$+\hat{J}_{\hat{K}\hat{\lambda}}^{(0)}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda+\hat{J}_{\hat{E}\hat{\chi}}^{(0)}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi+\hat{J}_{\hat{E}\hat{\lambda}}^{(0)}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda+\hat{J}_{\hat{\chi}\hat{\lambda}}^{(0)}\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda.$$

To obtain derivatives of the zeroth-order value function with respect to $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$, we first differentiate with respect to the marginal productivity of capital $\hat{r}_{\mathrm{mpk}}^{(0)}$, which slowly

depends on these three variables via $\hat{D}$ (i.e. the chain rule of differentiation). From (B5), we obtain:

(C3.8)
$$\frac{\partial \hat{J}^{(0)}}{\partial \hat{r}_{\text{mpk}}^{(0)}} = \hat{J}^{(0)} \left(-(1-\eta) \frac{\hat{\phi}''(\hat{i}^{(0)})}{\hat{\phi}'(\hat{i}^{(0)})} + \gamma \frac{1-\eta}{\hat{c}^{(0)}} \right) \frac{\partial \hat{i}^{(0)}}{\partial \hat{r}_{\text{mpk}}^{(0)}}.$$

Since the investment rate is implicitly defined, we obtain from (B6) by implicit differentiation:

(C3.9)
$$\frac{\partial \hat{i}^{(0)}}{\partial \hat{r}_{\text{mpk}}^{(0)}} = \frac{1}{\gamma - \hat{c}^{(0)} \hat{\phi}''(\hat{i}^{(0)}) / \hat{\phi}'(\hat{i}^{(0)})}.$$

Combining equations (C3.8) and (C3.9), we obtain

(C3.10)
$$\frac{\partial \hat{J}^{(0)}}{\partial \hat{r}_{\text{mpk}}^{(0)}} = \hat{J}^{(0)} \frac{1-\eta}{\hat{c}^{(0)}} = \left(\hat{\phi}'(\hat{i}^{(0)}) \right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)} \right)^{-\gamma \frac{1-\eta}{1-\gamma}-1} \hat{K}^{1-\eta}.$$

Using the chain rule of differentiation, we find the individual terms that contribute to the forcing (C3.7) at $\mathcal{O}(\epsilon)$:

(C3.11)
$$\hat{J}_{\hat{E}}^{(0)} = \left(\hat{\phi}'(\hat{i}^{(0)}) \right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)} \right)^{-\gamma \frac{1-\eta}{1-\gamma}-1} \hat{K}^{1-\eta} \frac{\partial \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{E}} \text{ and}$$
$$\mathcal{O}(\epsilon)$$
$$\hat{J}_{\hat{E}\hat{E}}^{(0)} = \left(\hat{\phi}'(\hat{i}^{(0)}) \right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)} \right)^{-\gamma \frac{1-\eta}{1-\gamma}-1} \hat{K}^{1-\eta} \frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{E}^2},$$
$$\mathcal{O}(\epsilon)$$

and similarly for derivatives with respect to $\hat{\chi}$ and $\hat{\lambda}$, as well as cross-derivatives. From the zeroth-order solution $\hat{r}_{\text{mpk}}^{(0)} = \alpha \hat{A} \left(\hat{E}, \hat{\chi}, \hat{\lambda} \right)^{1/\alpha} \left((1-\alpha)/\hat{b} \right)^{(1-\alpha)/\alpha} - \hat{\delta}$, we obtain

(C3.12a)
$$\frac{\partial \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{E}} = -\epsilon \hat{A}(\hat{E}, \hat{\chi}, \hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1-\alpha}{\alpha}} \hat{A}^* \left(1 + \theta_{ET} \right) \hat{E}^{\theta_{ET}} \hat{X}(\hat{\chi}) \hat{\Lambda}(\hat{\lambda}),$$
$$\frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{E}^2} = -\epsilon \hat{A}(\hat{E}, \hat{\chi}, \hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1-\alpha}{\alpha}} \hat{A}^* \theta_{ET} \left(1 + \theta_{ET} \right) \hat{E}^{\theta_{ET}-1} \hat{X}(\hat{\chi}) \hat{\Lambda}(\hat{\lambda}),$$

A22

$$(C3.12b)$$

$$\frac{\partial \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{\chi}} = -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} X_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$

$$\frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{\chi}^2} = -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} X_{\hat{\chi}\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$

$$(C3.12c)$$

$$\frac{\partial \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{\lambda}} = -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} X(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda}),$$

$$\frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{\lambda}^2} = -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} X(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda}),$$

$$\frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{E} \partial \hat{\chi}} = -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^* \left(1+\theta_{ET}\right) \hat{E}^{\theta_{ET}} X_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$

$$(C3.12d)$$

$$\frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{E} \partial \hat{\lambda}} = -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^* \left(1+\theta_{ET}\right) \hat{E}^{\theta_{ET}} X(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),$$

$$\frac{\partial^2 \hat{r}_{\text{mpk}}^{(0)}}{\partial \hat{\chi} \partial \hat{\lambda}} = -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} X_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),$$

where have used the following short hands $\hat{X} \equiv \hat{\chi}^{1+\theta_{\chi T}}$ and $\hat{\Lambda} \equiv \hat{\lambda}^{1+\theta_{\lambda}}$, so $\hat{D} = \hat{E}^{1+\theta_{ET}} \hat{X}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})$. Equations (C3.11) and (C3.12) can be substituted into (C3.7):

$$\hat{G}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) = -\epsilon \hat{A}(\hat{E},\hat{\chi})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^* \left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1} \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}$$

$$(C3.13)$$

$$\left[\left(-(1+\theta_{ET})\hat{\varphi}\hat{X}\hat{\Lambda} + \hat{v}_{\chi}(\bar{\hat{\chi}}-\hat{\chi})\hat{X}_{\hat{\chi}}\hat{\Lambda} + \hat{v}_{\lambda}(\bar{\hat{\lambda}}-\hat{\lambda})\hat{X}\hat{\Lambda}_{\hat{\lambda}} + \frac{1}{2}\hat{\sigma}_{\chi}^2 \hat{X}_{\hat{\chi}\hat{\chi}}\hat{\Lambda} + \frac{1}{2}\hat{X}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 \right. \right.$$

$$\left. + (1-\eta)\hat{X}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + (1-\eta)\hat{X}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{X}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\right)\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}}$$

$$+ (1+\theta_{ET})\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}}\hat{X}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}} e^{-\hat{g}^{(0)}\hat{i}} + \frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\hat{\sigma}_E^2 \hat{X}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}$$

$$+ \left((1-\eta)(1+\theta_{ET})\hat{X}\hat{\Lambda}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + (1+\theta_{ET})\hat{X}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}} \right.$$

$$\left. \left. + (1+\theta_{ET})\hat{X}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\right)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}} \right].$$

Because we are ultimately interested in $\hat{J}_{\hat{E}}^{(1)}$ for the computation of the social cost of carbon, we first differentiate (C3.6) with respect to $\hat{E}$ and seek a solution for $\hat{J}_{\hat{E}}^{(1)}$ of the form $\hat{J}_{\hat{E}}^{(1)} = \psi_1 (1+\theta_{ET}) \hat{\Omega}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t})$, which gives (from (C3.6)):[10]

(C3.14) $\quad \hat{J}_{\hat{E}}^{(1)} = \psi_1 (1+\theta_{ET}) \hat{\Omega}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) \Rightarrow -\hat{r}_{\Omega} \hat{\Omega} + \frac{1}{d\hat{t}} \mathrm{E}_{\hat{t}} \left[d\hat{\Omega} \right] = -\hat{\Gamma}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}),$

where we have introduced the effective discount rate

(C3.15) $\qquad \hat{r}_{\Omega} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} + (1-\eta) \left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2} \eta \hat{\sigma}_K^2 \right) + \hat{\varphi},$

and the coefficient

(C3.16) $\qquad \psi_1 \equiv \hat{A}^* \hat{A}(\hat{E}, \hat{\chi}, \lambda)^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1-\alpha}{\alpha}} \left(\hat{c}^{(0)} \right)^{-\gamma \frac{1-\eta}{1-\gamma}-1} \left(\hat{\phi}'(\hat{i}^{(0)}) \right)^{-\frac{1-\eta}{1-\gamma}}.$

The normalized forcing is defined by[11]

(C3.17) $\quad \hat{\Gamma}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) \equiv \Big((1+\theta_{ET}) \hat{\varphi} \hat{X} \hat{\Lambda} - \hat{v}_{\chi} (\hat{\bar{\chi}} - \hat{\chi}) \hat{X}_{\hat{\chi}} \hat{\Lambda} - \hat{v}_{\lambda} (\hat{\bar{\lambda}} - \hat{\lambda}) \hat{X} \hat{\Lambda}_{\hat{\lambda}} - \frac{1}{2} \hat{\sigma}_{\chi}^2 \hat{X}_{\hat{\chi}\hat{\chi}} \hat{\Lambda}$

$-\frac{1}{2} \hat{X} \hat{\Lambda}_{\hat{\lambda}\hat{\lambda}} \hat{\sigma}_{\lambda}^2 - (1-\eta) \hat{X}_{\hat{\chi}} \hat{\Lambda} \rho_{K\chi} \hat{\sigma}_K \hat{\sigma}_{\chi} - (1-\eta) \hat{X} \hat{\Lambda}_{\hat{\lambda}} \rho_{K\lambda} \hat{\sigma}_K \hat{\sigma}_{\lambda} - \hat{X}_{\hat{\chi}} \hat{\Lambda}_{\hat{\lambda}} \rho_{\chi\lambda} \hat{\sigma}_{\chi} \hat{\sigma}_{\lambda} \Big) \hat{K}^{1-\eta} \hat{E}^{\theta_{ET}}$

$-\theta_{ET} \hat{\mu} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{X} \hat{\Lambda} \hat{K}^{2-\eta} \hat{E}^{\theta_{ET}-1} e^{-\hat{g}^{(0)} \hat{t}} - \frac{1}{2} (\theta_{ET} - 1) \theta_{ET} \hat{\sigma}_E^2 \hat{X} \hat{\Lambda} \hat{K}^{1-\eta} \hat{E}^{\theta_{ET}-2}$

$-\theta_{ET} \Big((1-\eta) \hat{X} \hat{\Lambda} \rho_{KE} \hat{\sigma}_K \hat{\sigma}_E + \hat{X}_{\hat{\chi}} \hat{\Lambda} \rho_{E\chi} \hat{\sigma}_E \hat{\sigma}_{\chi} + \hat{X} \hat{\Lambda}_{\hat{\lambda}} \rho_{E\lambda} \hat{\sigma}_E \hat{\sigma}_{\lambda} \Big) \hat{K}^{1-\eta} \hat{E}^{\theta_{ET}-1}.$

Equation (C3.14) has the closed-form solution:

(C3.18) $\qquad \hat{\Omega} = \mathrm{E}_{\hat{t}} \left[\int_{\hat{t}}^{\infty} \hat{\Gamma}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{s}) e^{-\hat{r}_{\Omega}(\hat{s}-\hat{t})} d\hat{s} \right].$

---

[10] Dimensionally, we have $\Omega = E_0^{\theta_{ET}} \chi_0^{1+\theta_{\chi T}} \lambda_0^{1+\theta_{\lambda}} K_0^{1-\eta} \hat{\Omega}$.

[11] Dimensionally, we have $\Gamma = E_0^{1+\theta_{ET}} \chi_0^{1+\theta_{\chi T}} \lambda_0^{1+\theta_{\lambda}} K_0^{1-\eta} g_0 \hat{\Gamma}$.

We can now compute the SCC according to $\hat{P} = -\hat{\mu}\left(\hat{J}^{(0)}_{\hat{E}} + \epsilon \hat{J}^{(1)}_{\hat{E}}\right)\Big/ \phi'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}$:

(C3.19) $\quad \hat{P} = \dfrac{\hat{\mu}\,\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\big|_{\hat{P}=0}}{\hat{r}^*}\left(1 - \dfrac{\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})}{\hat{E}^{\theta_{ET}}\hat{X}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})\hat{K}^{1-\eta}}\right)$ with $\hat{\Theta} \equiv \dfrac{\epsilon \hat{D}_{\hat{E}}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}{1 - \hat{D}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}$,

where we have introduced $\hat{r}^* \equiv \hat{r}^{(0)} - \hat{g}^{(0)}$. Dimensionally, equations (C3.17), (C3.18) and (C3.19) correspond to Result A.

Appendix D: Leading-Order Effects of Uncertainty (For Online Publication)

To evaluate the 4-dimensional integral in Result A in closed form and thus derive Results 1 and 2, we take three steps. First, we evaluate the expected carbon stock dynamics as a function of time in section D1. Second, in section D2, we evaluate the forcing (C3.17)-(C3.18) of the first-order problem in Appendix C. In this section, we invoke three assumptions: we ignore the uncertainty in the carbon stock arising from the uncertainty of future emissions (*Assumption I*), we take account of climatic uncertainty only to leading order (*Assumption II*), and we set $\lambda_0 = \bar{\lambda}$ (*Assumption III*). Finally, we combine the zeroth- and first-order value functions and evaluate our leading-order estimate of the SCC in section D3. This is known as Result 2. Result 2 further simplifies to Result 1 for proportional damages ($\theta_{ET} = 0$) (*Assumption IV*) and with the initial climate sensitivity parameter equal to its steady-state value ($\chi_0 = \bar{\chi}$) (*Assumption V*).

D1. Expected carbon stock dynamics

The expected value of the carbon stock is governed by the differential equation (C1.5) with solution

(D1.1) $\mathrm{E}_{\hat{t}}\left[\hat{E}(\hat{s})\right] = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s}) + \hat{\mu}^*\hat{K}(\hat{t})\left[1 - \exp(-\hat{\varphi}\Delta\hat{s})\right]/\hat{\varphi} = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s})\hat{e}(\Delta\hat{s}),$

with new short hands $\hat{\mu}^* \equiv \hat{\mu}\left((1-\alpha)/\hat{b}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}$, $\Delta\hat{s} \equiv \hat{s} - \hat{t}$ and $\hat{e}(\Delta\hat{s}) = 1 + (\hat{\mu}^*\hat{K}(\hat{t})/\hat{E}(\hat{t}))(\exp(\hat{\varphi}\Delta\hat{s})-1)/\hat{\varphi}$. Dimensionally, we define μ^* so that $\mu F^{(0)} = \mu^* K$, where μ does not have units and μ^* has units TtC\$$^{-1}$year^{-1}. We can then obtain $\mu^* = \mu\left(A(1-\alpha)/b\right)^{1/\alpha}$ or $\hat{\mu}^* = \left(K_0/g_0 E_0\right)\mu^*$.

D2. Forcing of the first-order problem with only leading-order uncertainty

To identify only leading-order contributions of uncertainty, we expand in $\Delta\hat{\chi} \equiv \hat{\chi} - \mathrm{E}_{\hat{t}}\left[\hat{\chi}\right]$, $\Delta\hat{\lambda} \equiv \hat{\lambda} - \mathrm{E}_{\hat{t}}\left[\hat{\lambda}\right]$ and $\Delta\hat{E} \equiv \hat{E} - \mathrm{E}_{\hat{t}}\left[\hat{E}\right]$ with the corresponding covariance matrix given by (C2.5) (*Assumption II*). As in footnote 15 of the paper, we will use short-hand notation for the expected values of $\hat{\chi}$ and $\hat{\lambda}$, namely $\hat{\mu}_{\chi} \equiv \mathrm{E}_{\hat{t}}\left[\hat{\chi}\right] = \exp(-\hat{v}_{\chi}\hat{t}) + \hat{\bar{\chi}}\left(1 - \exp(-\hat{v}_{\chi}\hat{t})\right)$ and

$\hat{\mu}_\lambda \equiv \mathrm{E}_{\hat{t}}\left[\hat{\lambda}\right] = \exp(-\hat{v}_\lambda \hat{t}) + \hat{\bar{\lambda}}\left(1 - \exp(-\hat{v}_\lambda \hat{t})\right)$, and we note that $\hat{\mu}_\lambda = 1$ (*Assumption III*).[12]

We begin by considering terms that only involve capital stock uncertainty, which are evaluated without approximation. The probability density function for time $\hat{s}$, but with the expectation operator evaluated at time $\hat{t}$, is

(D2.1)
$$f_k = \frac{1}{\sqrt{2\pi\hat{\sigma}_K^2 \Delta\hat{s}}} \exp\left(-\frac{1}{2}\left(\frac{(\hat{k} - \hat{\alpha}_k \hat{s})^2}{\hat{\sigma}_K^2 \Delta\hat{s}}\right)\right),$$

where $\hat{\alpha}_k = \hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2$. Combining with the discount factor $\exp\left(-\hat{r}_\Omega \Delta\hat{s}\right)$ in (C3.18) and an additional factor accounting for the decay of the atmospheric carbon stock, we have without further approximation

(D2.2)
$$\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\exp\left(-(\hat{r}_\Omega + \theta_{ET}\hat{\varphi})\Delta\hat{s}\right) = \left(\hat{K}(\hat{t})\right)^{1-\eta}\exp(-\hat{r}^\star \Delta\hat{s}) \quad \text{and}$$
$$\mathrm{E}_{\hat{t}}\left[\hat{K}^{2-\eta}\right]\exp\left(-(\hat{r}_\Omega + \hat{g}^{(0)} + (\theta_{ET}-1)\hat{\varphi})\Delta\hat{s}\right) = \left(\hat{K}(\hat{t})\right)^{2-\eta}\exp(-\hat{r}^{\star\star}\Delta\hat{s}),$$

where $\hat{r}^\star \equiv \hat{r}^* + (1+\theta_{ET})\hat{\varphi} = \hat{r}^{(0)} - \hat{g}^{(0)} + (1+\theta_{ET})\hat{\varphi}$ and
$\hat{r}^{\star\star} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} - (1-\eta)\hat{\sigma}_K^2 + \theta_{ET}\hat{\varphi} + \hat{r}^* - (1-\eta)\hat{\sigma}_K^2 - \hat{\varphi}$. We use alternative star symbols $\star$ as superscripts to denote rates corrected for atmospheric carbon stock decay. To leading order, we have for the terms involving the carbon stock:

(D2.3)
$$\mathrm{E}_{\hat{t}}\left[\hat{E}^{\theta_{ET}}\right] = \left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\left[1 + \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\left(\frac{\hat{\Sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\right)^2\right] + \mathcal{O}(\hat{\Sigma}_E^4),$$

$$\mathrm{E}_{\hat{t}}\left[\hat{E}^{\theta_{ET}-1}\right] = \left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\left[1 + \frac{1}{2}\left(\theta_{ET}-1\right)\left(\theta_{ET}-2\right)\left(\frac{\hat{\Sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\right)^2\right] + \mathcal{O}(\hat{\Sigma}_E^4),$$

$$\mathrm{E}_{\hat{t}}\left[\hat{E}^{\theta_{ET}-2}\right] = \left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-2}\left[1 + \frac{1}{2}\left(\theta_{ET}-2\right)\left(\theta_{ET}-3\right)\left(\frac{\hat{\Sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\right)^2\right] + \mathcal{O}(\hat{\Sigma}_E^4),$$

---

[12] Consistent with the other non-dimensional variables, $\hat{\mu}_\chi \equiv \mu_\chi/\chi_0$ and $\hat{\mu}_\lambda \equiv \mu_\lambda/\lambda_0$.

where we let the subscript on Σ^2 denote the relevant elements of the covariance matrix Σ (C2.35) and we have ignored any contributions to uncertainty from new emissions through their dependence on uncertain future GDP (*Assumption I*). Making *Assumption II* more precise, we retain terms up to second order in a perturbation expansion in Σ.[13] The following terms also make a contribution to the forcing (C3.17)-(C3.18): $\hat{X}\hat{\Lambda}$, $\hat{X}_{\hat{\chi}}\hat{\Lambda}$, $(\hat{\chi}-\hat{\mu}_{\chi})\hat{X}_{\hat{\chi}}\hat{\Lambda}$, $\hat{X}\hat{\Lambda}_{\hat{\lambda}}$, $(\hat{\lambda}-\hat{\mu}_{\lambda})\hat{X}\hat{\Lambda}_{\hat{\lambda}}$, $\hat{X}_{\hat{\chi}\hat{\chi}}\hat{\Lambda}$ and $\hat{X}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}$. Keeping only those terms contributing to the leading-order effect of climatic uncertainty, we have

$$\mathrm{E}_{\hat{t}}\left[\hat{X}(\hat{\chi})\right]=\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\left[1+\frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^4),$$

(D2.4a)

$$\mathrm{E}_{\hat{t}}\left[\hat{X}_{\hat{\chi}}(\hat{\chi})\right]=\hat{\mu}_{\chi}^{\theta_{\chi T}}\left[\theta_{\chi T}+1+\frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}(\theta_{\chi T}-1)\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^4),$$

$$\mathrm{E}_{\hat{t}}\left[\left(\hat{\chi}-\hat{\mu}_{\chi}\right)\hat{X}_{\hat{\chi}}\left(\hat{\chi}\right)\right]=\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\left[(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^4),$$

(D2.4b)

$$\mathrm{E}_{\hat{t}}\left[\hat{X}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\right]=\hat{\mu}_{\chi}^{\theta_{\chi T}-1}\left[(\theta_{\chi T}+1)\theta_{\chi T}\right]+\mathcal{O}(\hat{\Sigma}_{\chi}^2),$$

$$\mathrm{E}_{\hat{t}}\left[\hat{\Lambda}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{1+\theta_{\lambda}}\left[1+\frac{1}{2}(\theta_{\lambda}+1)\theta_{\lambda}\left(\frac{\hat{\Sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^4),$$

(D2.5a)

$$\mathrm{E}_{\hat{t}}\left[\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{\theta_{\lambda}}\left[(\theta_{\lambda}+1)+\frac{1}{2}(\theta_{\lambda}+1)\theta_{\lambda}(\theta_{\lambda}-1)\left(\frac{\hat{\Sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^4),$$

$$\mathrm{E}_{\hat{t}}\left[\left(\hat{\lambda}-\hat{\mu}_{\lambda}\right)\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{1+\theta_{\lambda}}\left[(\theta_{\lambda}+1)\theta_{\lambda}\left(\frac{\hat{\Sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\right)^2\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^4),$$

(D2.5b)

$$\mathrm{E}_{\hat{t}}\left[\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda})\right]=\hat{\mu}_{\lambda}^{\theta_{\lambda}-1}\left[(\theta_{\lambda}+1)\theta_{\lambda}\right]+\mathcal{O}(\hat{\Sigma}_{\lambda}^2).$$

Using (D2.2)-(D2.5), we now consider the terms in the forcing (C3.17) consecutively and let the subscript indices correspond to the sequence of terms in (C3.17) (left to right).

---

[13] We also retain the term proportional to $\hat{\Sigma}_{\chi}^2\hat{\Sigma}_{\lambda}^2$, which is fourth order, although this is inconsistent from a perturbation theory perspective. We know from comparison to Result A, which we can evaluate exactly numerically (see Appendix F), that this term is the largest higher-order term (notably, in the case of highly convex damages) we otherwise ignore. We thus increase the accuracy of Results 1 and 2 by a few percent (see Appendix F).

To consider the covariance terms in the forcing (C3.17), we also expand in $\Delta\hat{k} \equiv \hat{k} - \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2\right)\hat{t}$ and only consider deviations from the zeroth-order mean consistent with our search for leading-order terms only. The following terms arise:

$$(D2.6) \qquad \mathrm{E}_{\hat{i}}\left[\Gamma_1\right] = (1+\theta_{ET})\hat{\varphi}\left[1 + \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(\mathrm{E}_{\hat{i}}\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi} + \frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+(1-\eta)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\mathrm{E}_{\hat{i}}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}} + (1-\eta)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}$$

$$+(1-\eta)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda} + \theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\mathrm{E}_{\hat{i}}\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi}$$

$$+\theta_{ET}\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\mathrm{E}_{\hat{i}}\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda}$$

$$\left.+\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\right]\mathrm{E}_{\hat{i}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{i}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

$$\mathrm{E}_{\hat{i}}\left[\Gamma_2\right] = \hat{\nu}_\chi(1+\theta_{\chi T})\left(\theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\theta_\lambda(1+\theta_\lambda)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$(D2.7)$$

$$(1-\eta)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} + \theta_{ET}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\mathrm{E}_{\hat{i}}\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi}$$

$$\left.+\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\right)\mathrm{E}_{\hat{i}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{i}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda}$$

$$+\hat{v}_\chi(1+\theta_{\chi T})\left(\hat{\chi}(\hat{t})-\bar{\hat{\chi}}\right)e^{-\hat{v}_\chi\Delta\hat{s}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}-1\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}+\frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}$$

$$+\frac{1}{4}\theta_{\chi T}\left(\theta_{\chi T}-1\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}$$

$$+(1-\eta)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+(1-\eta)\theta_{\chi T}\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{v}_\chi\Delta\hat{s}\right)}{\hat{v}_\chi}$$

$$+(1-\eta)(1+\theta_\lambda)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{v}_\lambda\Delta\hat{s}\right)}{\hat{v}_\lambda}+\theta_{ET}\theta_{\chi T}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\chi}$$

$$+\theta_{ET}\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\lambda}$$

$$+\theta_{\chi T}\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{v}_\chi+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{v}_\chi+\hat{v}_\lambda}\left.\right]\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

$$\mathrm{E}_{\hat{t}}\left[\Gamma_3\right]=\hat{v}_\lambda\left(1+\theta_\lambda\right)\left(\theta_\lambda\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}\right.$$

$$+\frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\theta_\lambda\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}$$

(D2.8)

$$(1-\eta)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{v}_\lambda\Delta\hat{s}\right)}{\hat{v}_\lambda}+\theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\lambda}$$

$$\left(1+\theta_{\chi T}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{v}_\chi+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{v}_\chi+\hat{v}_\lambda}\right)\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda}$$

$$+\hat{v}_\lambda(1+\theta_\lambda)\left(\hat{\lambda}(\hat{t})-\bar{\hat{\lambda}}\right)e^{-\hat{v}_\lambda\Delta\hat{s}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}+\frac{1}{2}\theta_\lambda\left(\theta_\lambda-1\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}$$

$$+\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(\theta_{\lambda}-1\right)\frac{\hat{\sigma}_{\lambda}^{2}\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{v}_{\chi}\Delta\hat{s}\right)}{2\hat{v}_{\chi}}\frac{1-\exp\left(-2\hat{v}_{\lambda}\Delta\hat{s}\right)}{2\hat{v}_{\lambda}}$$

$$+\left(1-\eta\right)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E}}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+\left(1-\eta\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}}{\hat{\mu}_{\chi}}\frac{1-\exp\left(-\hat{v}_{\chi}\Delta\hat{s}\right)}{\hat{v}_{\chi}}$$

$$+\left(1-\eta\right)\theta_{\lambda}\frac{\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\hat{v}_{\lambda}\Delta\hat{s}\right)}{\hat{v}_{\lambda}}+\theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_{\chi}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_{\chi}}$$

$$+\theta_{ET}\theta_{\lambda}\frac{\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}}{\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_{\lambda}}$$

$$+\left(1+\theta_{\chi T}\right)\theta_{\lambda}\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\chi}\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{v}_{\chi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{v}_{\chi}+\hat{v}_{\lambda}}\Bigg]\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.9)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{4}\right]=\left[-\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}}-\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\chi}^{2}\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{v}_{\lambda}\Delta\hat{s}\right)}{2\hat{v}_{\lambda}}\right]$$

$$\times\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.10)

$$\mathrm{E}_{\hat{t}}\left[\Gamma_{5}\right]=\left[-\frac{1}{2}\left(1+\theta_{\lambda}\right)\theta_{\lambda}\frac{\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\lambda}^{2}}-\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\chi}^{2}\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{v}_{\chi}\Delta\hat{s}\right)}{2\hat{v}_{\chi}}\right]$$

$$\times\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(D2.11) $\quad\mathrm{E}_{\hat{t}}\left[\Gamma_{6}\right]=-\left(1-\eta\right)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$

(D2.12) $\quad\mathrm{E}_{\hat{t}}\left[\Gamma_{7}\right]=-\left(1-\eta\right)\left(1+\theta_{\lambda}\right)\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{\theta_{\lambda}},$

(D2.13) $\quad\mathrm{E}_{\hat{t}}\left[\Gamma_{8}\right]=-\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{\theta_{\lambda}},$

(D2.14)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_9\right] = -\theta_{ET}\hat{\mu}^*\left[1 + \frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}\right.$$

$$+\frac{1}{2}\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda} + \frac{1}{4}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}$$

$$+\left(2-\eta\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{v}_\chi\Delta\hat{s}\right)}{\hat{v}_\chi} + \left(2-\eta\right)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{v}_\lambda\Delta\hat{s}\right)}{\hat{v}_\lambda}$$

$$\left.+\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{v}_\chi+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{v}_\chi+\hat{v}_\lambda}\right]\mathrm{E}_{\hat{t}}\left[\hat{K}^{2-\eta}\right]e^{-\hat{g}^{(0)}\Delta\hat{s}}\left(\mathrm{E}_{\hat{t}}\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.15)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{10}\right] = -\frac{1}{2}\theta_{ET}(\theta_{ET}-1)\hat{\sigma}_E^2\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-2}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.16)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{11}\right] = -(1-\eta)\theta_{ET}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.17)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{12}\right] = -\theta_{ET}(1+\theta_{\chi T})\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(D2.18)
$$\mathrm{E}_{\hat{t}}\left[\Gamma_{13}\right] = -\theta_{ET}(1+\theta_\lambda)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda\mathrm{E}_{\hat{t}}\left[\hat{K}^{1-\eta}\right]\left(\mathrm{E}_{\hat{t}}\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{\theta_\lambda},$$

where elements of the covariance matrix have been substituted from (C2.5).

D3. Leading-order solution (Results 1 and 2)

Combining all the leading-order terms in the forcing equation (D2.6)-(D2.18) and substituting into (C3.19), further assuming $\lambda_0 = \bar{\lambda}$, so that $\hat{\mu}_\lambda = 1$ (*Assumption III*), we obtain Result 2 after considerable manipulation (including integrating by parts).

Result 2: The optimal risk-adjusted SCC is

(D3.1)
$$\hat{P} = \frac{\epsilon\hat{\mu}\ \hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\big|_{\hat{P}=0}}{\hat{r}^\star} \times$$

$$\left(1+\theta_{ET}\hat{\mu}^*\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^{\star\star}}\Upsilon_{\theta_{ET}\neq0} + (1+\theta_{\chi T})\frac{\hat{v}_\chi}{\hat{r}^\star}\frac{\bar{\hat{\chi}}-\hat{\chi}}{\hat{\chi}}\Upsilon_{\chi_0\neq\bar{\chi}} + \Delta_{EE} + \Delta_{\chi\chi} + \Delta_{\lambda\lambda} + \Delta_{\chi\times\lambda} + \Delta_{CK} + \Delta_{CC}\right),$$

where we distinguish six so-called 'risk adjustments' denoted by the symbol Δ with subscripts denoting the state variable(s) from which the risk originates.

D3.1. Risk adjustments

The risk adjustments for atmospheric carbon stock uncertainty (Δ_{EE}), climate sensitivity uncertainty ($\Delta_{\chi\chi}$), damage ratio uncertainty ($\Delta_{\lambda\lambda}$), the interaction of climate sensitivity and damage ratio uncertainty ($\Delta_{\chi\times\lambda}$), the correlation between economic risk and all three climatic risks (Δ_{CK}), and the correlation between the three climatic risks themselves (Δ_{CC}) are respectively

(D3.2)

$$
\Delta_{EE} \equiv \frac{1}{2}\theta_{ET}(\theta_{ET}-1)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^\star - 2\hat{\varphi}}\Upsilon_{EE}, \quad
\Delta_{\chi\chi} \equiv \frac{1}{2}(1+\theta_{\chi T})\theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1}{\hat{r}^\star + 2\hat{v}_\chi}\Upsilon_{\chi\chi},
$$

$$
\Delta_{\lambda\lambda} \equiv \frac{1}{2}\theta_\lambda(1+\theta_\lambda)\frac{\hat{\sigma}_\lambda^2}{\hat{r}^\star + 2\hat{v}_\lambda}\Upsilon_{\lambda\lambda}, \quad
\Delta_{\chi\times\lambda} \equiv \frac{1}{16}\theta_{\chi T}(1+\theta_{\chi T})\theta_\lambda(1+\theta_\lambda)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{v}_\chi\hat{v}_\lambda\hat{\mu}_\chi^2}\frac{\hat{r}^\star}{\hat{r}^\star - (1+\theta_{ET})\hat{\varphi}}
$$

$$
\times\left(\frac{(1+\theta_{ET})\hat{\varphi}+2\hat{v}_\chi}{\hat{r}^\star+2\hat{v}_\chi} + \frac{(1+\theta_{ET})\hat{\varphi}+2\hat{v}_\lambda}{\hat{r}^\star+2\hat{v}_\lambda} - \frac{(1+\theta_{ET})\hat{\varphi}+2(\hat{v}_\chi+\hat{v}_\lambda)}{\hat{r}^\star+2\hat{v}_\lambda+2\hat{v}_\chi} - \frac{(1+\theta_{ET})\hat{\varphi}}{\hat{r}^\star}\right)\Upsilon_{\chi\times\lambda},
$$

$$
\Delta_{CK} \equiv -(\eta-1)\hat{\sigma}_K\left(\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_E}{\hat{E}(\hat{r}^\star-\hat{\varphi})}\Upsilon_{KE} + (1+\theta_{\chi T})\frac{\rho_{K\chi}}{\hat{r}^\star+\hat{v}_\chi}\frac{\hat{\sigma}_\chi}{\hat{\mu}_\chi}\Upsilon_{K\chi} + (1+\theta_\lambda)\frac{\rho_{K\lambda}\hat{\sigma}_\lambda}{\hat{r}^\star+\hat{v}_\lambda}\Upsilon_{K\lambda}\right),
$$

$$
\Delta_{CC} \equiv \theta_{ET}(1+\theta_{\chi T})\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}\hat{\mu}_\chi}\frac{\hat{r}^\star}{\hat{r}^\star-\hat{\varphi}}\frac{1}{\hat{r}^\star+\hat{v}_\chi}\Upsilon_{E\chi}
$$

$$
+(1+\theta_\lambda)\left((1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{r}^\star+\hat{v}_\lambda+\hat{v}_\lambda}\frac{1}{\hat{\mu}_\chi}\Upsilon_{\chi\lambda} + \theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{E}}\frac{\hat{r}^\star}{\hat{r}^\star-\hat{\varphi}}\frac{1}{\hat{r}^\star+\hat{v}_\lambda}\Upsilon_{E\lambda}\right).
$$

D3.2. Correction factors

In addition to 'risk adjustments', we distinguish two types of so-called 'correction factors', denoted by the symbol Υ with subscripts again denoting the state variable(s) from which the risk originates: for $\theta_{ET}\neq 0$ and for $\chi_0\neq\bar{\chi}$. In equation (D3.1), the correction factors $\Upsilon_{\theta_{ET}\neq 0}$ and $\Upsilon_{\chi_0\neq\bar{\chi}}$ are deterministic corrections for $\theta_{ET}\neq 0$ and $\chi_0\neq\bar{\chi}$, respectively. The remaining correction factors (Υ) in (D3.2) multiply a risk adjustment (Δ) and must be linearly combined with unity, so that, for example,

$\Upsilon_{\chi\chi} \equiv 1 + \Upsilon_{\chi\chi,\theta_{ET}\neq 0} + \Upsilon_{\chi\chi,\chi_0\neq\bar{\chi}}$. These combined correction factors are equal to unity if $\theta_{ET} \neq 0$ and $\chi_0 \neq \bar{\chi}$ (e.g., $\Upsilon_{\chi\chi} \equiv 1$). We give the correction factors in terms of dimensional quantities below (using the definitions in Appendix A.2.1), so that they can be used directly in Result 2 given dimensionally in Appendix A.4.

The correction factors for $\theta_{ET} \neq 0$ are

(D3.4)

$$\Upsilon_{\theta_{ET}\neq 0} = \frac{r^{\star\star}}{1-(1+\theta_{ET})\varphi/r^{\star}} \int_0^{\infty} \left(e^{-r^{\star\star}\Delta s} - \frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s} \right) \left(e(\Delta s) \right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}} d\Delta s,$$

$$\Upsilon_{Ki,\theta_{ET}\neq 0} = \theta_{ET}\mu^{*}\frac{K(t)}{E(t)}\frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} \left[\left(1 + \frac{(1+\theta_{ET})\varphi}{v_i} \right) \int_0^{\infty} e^{-(v_i+r^{\star}-\varphi)\Delta s} \left(e(\Delta s) \right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-I_{\chi}(i)} d\Delta s \right.$$

$$-(1+\theta_{ET})\frac{\varphi}{r^{\star}}\frac{r^{\star}+v_i}{v_i}\int_0^{\infty} e^{-(r^{\star}-\varphi)\Delta s} \left(e(\Delta s) \right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-I_{\chi}(i)} d\Delta s +$$

$$\left. \frac{2-\eta}{1-\eta}\frac{v_i+r^{\star}}{v_i}\int_0^{\infty} \left(e^{-r^{\star\star}\Delta\hat{s}} - e^{-(r^{\star\star}+v_i)\Delta s} \right) \left(e(\Delta s) \right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-I_{\chi}(i)} d\Delta s \right] \text{ for } i = \chi, \lambda,$$

$$\Upsilon_{ij,\theta_{ET}\neq 0} = \theta_{ET}\mu^{*}\frac{K}{E}\frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} \left[\left(1 + \frac{(1+\theta_{ET})\varphi}{v_i+v_j} \right) \int_0^{\infty} e^{-(r^{\star}+v_i+v_j-\varphi)\Delta s} \left(e(\Delta s) \right)^{\theta_{ET}-1} \right.$$

$$\times \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-I_{\chi}(i)-I_{\chi}(j)} d\Delta s - \frac{(1+\theta_{ET})\varphi}{v_i+v_j}\frac{r^{\star}+v_i+v_j}{r^{\star}}\int_0^{\infty} e^{-(r^{\star}-\varphi)\Delta s} \left(e(\Delta s) \right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-I_{\chi}(i)-I_{\chi}(j)} d\Delta s$$

$$\left. + \frac{r^{\star}+v_i+v_j}{v_i+v_j}\int_0^{\infty} \left(e^{-r^{\star\star}\Delta s} - e^{-(r^{\star\star}+v_i+v_j)\Delta\hat{s}} \right) \left(e(\Delta s) \right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-I_{\chi}(i)-I_{\chi}(j)} d\Delta s \right] \text{ for } i, j = \chi, \lambda,$$

$$\Upsilon_{EE,\theta_{ET}\neq 0} = -\frac{\left(r^{\star}-2\varphi \right)}{2\varphi}\frac{\mu^{*}\frac{K}{E}(\theta_{ET}-2)}{1-(1+\theta_{ET})\varphi/r^{\star}} \left[\int_0^{\infty} \left(\frac{(\theta_{ET}-1)\varphi}{r^{\star}-2\varphi} e^{-(r^{\star}-3\varphi)\Delta s} - \frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s} \right. \right.$$

$$\left. \left. -e^{-(r^{\star\star}-2\varphi)\Delta s} + e^{-r^{\star\star}\Delta s} \right) \left(e(\Delta s) \right)^{\theta_{ET}-3} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}} ds, \right.$$

$$\Upsilon_{\chi\times\lambda,\theta_{ET}\neq0} = \frac{\theta_{ET}\mu^* K(t)/E(t)}{\dfrac{(1+\theta_{ET})\varphi}{r^\star} - \dfrac{(1+\theta_{ET})\varphi+2v_\chi}{r^\star+2v_\chi} - \dfrac{(1+\theta_{ET})\varphi+2v_\lambda}{r^\star+2v_\lambda} + \dfrac{(1+\theta_{ET})\varphi+2(v_\lambda+v_\chi)}{r^\star+2(v_\lambda+v_\chi)}}$$

$$\times\int_0^\infty\left(\frac{(1+\theta_{ET})\varphi}{r^\star}e^{-(r^\star-\varphi)\Delta s} - e^{-r^{\star\star}\Delta s} - \frac{(1+\theta_{ET})\varphi+2v_\chi}{r^\star+2v_\chi}e^{-(r^\star+2v_\chi-\varphi)\Delta s}\right.$$

$$+ e^{-(r^{\star\star}+2v_\chi)\Delta s} - \frac{(1+\theta_{ET})\varphi+2v_\lambda}{r^\star+2v_\lambda}e^{-(r^\star+2v_\lambda-\varphi)\Delta s} + e^{-(r^{\star\star}+2v_\lambda)\Delta s}$$

$$\left.\frac{(1+\theta_{ET})\varphi+2(v_\chi+v_\lambda)}{r^\star+2v_\lambda+2v_\chi}e^{-(r^\star+2v_\lambda+2v_\chi-\varphi)\Delta s} - e^{-(r^{\star\star}+2v_\chi+2v_\lambda)\Delta s}\right)\times\left(e(\Delta s)\right)^{\theta_{ET}-1}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-1}d\Delta s,$$

where $I_\chi(i)=1$ for $i=\chi$ and $I_\chi(i)=0$ for $i\neq\chi$ (cf. indicator function), the function that takes into account future changes to the mean carbon stock $e(\Delta s)=1+(\mu^* K(t)/E(t))(\exp(\varphi\Delta s)-1)/\varphi$, and the time-varying mean climate sensitivity $\mu_\chi(\Delta s)=\mu_\chi(t)\exp(-v_\chi\Delta s)+\bar{\mu}_\chi(1-\exp(-v_\chi\Delta s))$.

The correction factors for $\chi_0\neq\bar{\chi}$ are:

(D3.5)

$$\Upsilon_{\chi_0\neq\bar{\chi}} = r^\star\int_0^\infty\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}}e^{-(r^\star+v_\chi)\Delta s}d\Delta s,$$

$$\Upsilon_{ij,\chi_0\neq\bar{\chi}} = \left(1+\theta_{\chi T}-I_\chi(i)-I_\chi(j)\right)v_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}\frac{r^\star+v_i+v_j}{v_i+v_j}\int_0^\infty\left(e^{-(r^\star+v_\chi)\Delta s} - \frac{r^\star}{r^\star+v_i+v_j}e^{-(r^\star+v_\chi+v_i+v_j)\Delta s}\right)$$

$$\times\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-I_\chi(i)-I_\chi(j)}d\Delta\hat{s},$$

$$\Upsilon_{Ki,\chi_0\neq\bar{\chi}} = \left(1+\theta_{\chi T}-I_\chi(i)\right)v_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}\frac{r^\star+v_i}{v_i}\int_0^\infty\left(e^{-(r^\star+v_\chi)\Delta s} - \frac{r^\star}{r^\star+v_\chi}e^{-(v_\chi+v_i)\Delta s}\right)\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-I_\chi(i)}d\Delta s,$$

$$\Upsilon_{EE,\chi_0\neq\bar{\chi}} = \left(1+\theta_{\chi T}\right)v_\chi\frac{\left(\bar{\chi}-\mu_\chi(t)\right)}{\mu_\chi(t)}\int_0^\infty\left(\frac{r^\star}{2\varphi}e^{-(r^\star+v_\chi-2\varphi)\Delta s} - \frac{r^\star-2\varphi}{2\varphi}e^{-(r^\star+v_\chi)\Delta s}\right)\left(e(\Delta s)\right)^{\theta_{ET}-2}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}}d\Delta s,$$

$$\Upsilon_{\chi \times \lambda, \chi_0 \neq \bar{\chi}} = \dfrac{\left(1 - \dfrac{(1+\theta_{ET})\varphi}{r^\star}\right)\left(\theta_{\chi T} - 1\right) v_\chi \dfrac{\bar{\chi} - \mu_\chi(t)}{\mu_\chi(t)}}{-\dfrac{(1+\theta_{ET})\varphi}{r^\star} + \dfrac{(1+\theta_{ET})\varphi + 2v_\chi}{r^\star + 2v_\chi} + \dfrac{(1+\theta_{ET})\varphi + 2v_\lambda}{r^\star + 2v_\lambda} - \dfrac{(1+\theta_{ET})\varphi + 2(v_\chi + v_\lambda)}{r^\star + 2(v_\lambda + v_\chi)}}$$

$$\int_0^\infty \left(1 - \dfrac{r^\star}{r^\star + 2v_\chi} e^{-2v_\chi \Delta s} - \dfrac{r^\star}{r^\star + 2v_\lambda} e^{-2v_\lambda \Delta s} + \dfrac{r^\star}{r^\star + 2v_\lambda + 2v_\chi} e^{-2(v_\lambda + v_\chi)\Delta s}\right)$$

$$\times e^{-(r^\star + v_\chi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}} \left(\dfrac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T} - 2} d\Delta s.$$

We do not explicitly give the correction factors for the correlation terms involving carbon stock uncertainty. Equation (D3.1) together with (D3.2)-(D3.5) gives the optimal SCC according to Result 2.

Appendix E: Calibration (For Online Publication)

E1. Asset returns, risk aversion and intertemporal substitution

We follow the calibration of Pindyck and Wang (2013), but ignore the effect of catastrophic shocks.[14,15] Using monthly asset data from the S&P 500 for the period 1947-2008, we obtain an annual return on assets (capital gains plus dividends) of $r^{(0)} = 7.2\%$/year with annual volatility of $\sigma_K = 12\%$. For a return on safe assets of 0.80%/year based on the annualized monthly return on 3-months T-bills, we obtain a risk premium of $\Delta r^{(0)} \equiv r^{(0)} - r_{\mathrm{rf}}^{(0)} = 6.4\%$/year and calibrate the coefficient of relative risk aversion as $\eta = 4.3$ (cf. $\Delta r^{(0)} = \eta \sigma_K^2$). Taking the growth rate to be equal to the historical growth rate of $g^{(0)} = 2.0\%$/year, the equation $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta \sigma_K^2 / 2$ (cf. (B9)) defines the combinations of ρ and γ that are consistent with historical asset returns. Setting the coefficient of elasticity of intertemporal substitution $\mathrm{EIS} = 2/3$, we obtain $\gamma = \mathrm{EIS}^{-1} = 1.5$ and thus a rate of time preference is $\rho = 5.8\%$/year. In section V.A we also consider an alternative calibration where EIS = 1.5 (larger than one as is assumed in asset pricing theory) and adjust $\rho = 4.8\%$/year, so that the same risky and risk-free financial returns are matched.

E2. Productivity, fossil fuel, adjustment costs and the depreciation rate

To calibrate total factor productivity, we consider the production function in the absence of climate damage that can be obtained by setting $P = 0$ (i.e. at zeroth order), namely $Y^{(0)} = A^* K$ with $A^* = A^{1/\alpha} \left((1-\alpha)/b \right)^{(1-\alpha)/\alpha}$. Pindyck and Wang (2013) use empirical estimates of the physical, human and intangible capital stocks and find $A^* = 0.113$/year, which we adopt. Based on emissions of $F_0^{(0)} = 9.1$ GtC/year in 2015, energy costs making up a share $1 - \alpha = 4.3\%$ of world GDP at PPP in 2015 of \$116 trillion/year, we estimate the

---

[14] Pindyck and Wang (2013) use Poisson shocks to capture small risks of large disasters (cf. Barro, 2016) and thus match skewness and kurtosis of asset returns. These shocks are responsible for approximately 1%-point of the risk premium.

[15] The alternative is to calibrate our AK-model to the observed volatility of consumption or output (cf. Gollier, 2012), which are generally much less volatile than capital (asset returns). Because the volatilities of capital, consumption and output are equal to the volatility of capital in an AK-model, this alternative calibration gives a much lower volatility and, consequently, a higher coefficient of relative risk aversion to match the equity premium (see also the discussion in Pindyck and Wang, 2013). Historical data for the growth rate of world GDP for 1961-2015 imply a volatility of $\sigma_K = 1.5\%$/year$^{1/2}$ and thus a much higher value of risk aversion of $\eta = 2.8 \times 10^2$ for an equity premium of 6.4%/year. Kocherlota (1996) obtains $\sigma_K = 3.6\%$/year$^{1/2}$ from US annual consumption growth during 1889-1978, which gives $\eta = 49$. We use $\sigma_K = 1.5\%$/year$^{1/2}$, but not the corresponding high values of risk aversion.

fossil fuel cost to be $b = Y_0^{(0)}(1-\alpha)/F_0^{(0)} = \$5.4 \times 10^2$/tC.[16] The gross marginal productivity of capital is thus $Y_K^{(0)}\big|_{t=0} = \alpha A^* = 0.11$/year.[17] Using Pindyck and Wang's (2013) consumption-investment ratio $c^{(0)}/i^{(0)} = 2.84$ and the identity $\alpha A^* = c^{(0)} + i^{(0)}$, we obtain initial values of $c^{(0)} = 8.0\%$/year and $i^{(0)} = 2.8\%$/year. Using $q^{(0)} = c^{(0)}/(r^{(0)} - g^{(0)}) = 1.5$ and $q^{(0)} = (1 - \omega i^{(0)})^{-1}$, we get the adjustment-cost parameter $\omega = 12.5$ year. Finally, we find the depreciation rate that is consistent with the assumed rate of economic growth: $\delta = i^{(0)} - \omega(i^{(0)})^2/2 - g^{(0)} = 0.33\%$/year.

E3. Atmospheric carbon stock and uncertainty

Here we calibrate our carbon stock model (4) to the Law Dome Ice Core 2000-year data set and historical emissions. The first column of Fig. E1 shows maximum-likelihood estimates, from which it is evident that estimates displaying a certain linear relationship between φ and μ are of comparable likelihood.[18]

These loci of maximum likelihood are shown separately in Fig. E2, with the overall maximum denoted by a red circle and corresponding values given in Table E1. The remaining columns in Fig. E1 show the predicted and observed rate of change of the atmospheric carbon stock (second column), the predicted and observed atmospheric carbon stock (third column) and the remaining variability (fourth column).[19]

---

[16] We estimate the share of energy costs from data for energy use and energy costs from BP Statistical Review of World Energy 2017. Data for emissions are obtained from the same source available online at https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html. Our estimate of energy costs as a percentage of GDP is in good agreement with data from the U.S. Energy Information Administration available online at https://www.eia.gov/totalenergy/data/annual/showtext.php?t=ptb0105.

[17] This is in line with Caselli and Feyrer (2007), who estimate annual marginal products of capital of 8.5% for rich countries and 6.9% for poor countries, and an observed annual risk premium of 5-7%. They use a depreciation rate of 6.0% to calculate the capital stock from investment, include the share of reproducible capital rather than the share of total capital, account for differences in prices between capital and consumption goods and correct for inflation.

[18] Annual data from the Law Dome firn and ice core records and the Cape Grim record are available online at ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/law/law2006.txt. This data is based on spline fits to different dataset with different spline windows across time reflecting changes in the temporal resolution of the data. The discrete nature of the fitted data is evident for the early years. Annual carbon emissions from fossil fuel consumption and cement production are available online at http://cdiac.ornl.gov/trends/emis/tre_glob_2013.html.

[19] By setting $\varphi = 0$, we can estimate the fraction μ of emissions that stays in the atmosphere forever, whilst the remainder is instantaneously absorbed by the oceans and other carbon sinks. Calibrating to this data, we find $\mu = 0.68, 0.64, 0.56$ and 0.43 for the periods 1750-2004, 1800-2004, 1900-2004 and 1959-2004, respectively. Performing a similar analysis, Le Quéré et al. (2009) find that, between 1959 and 2008, 43% of each year's CO_2 emissions remained in the atmosphere on average.
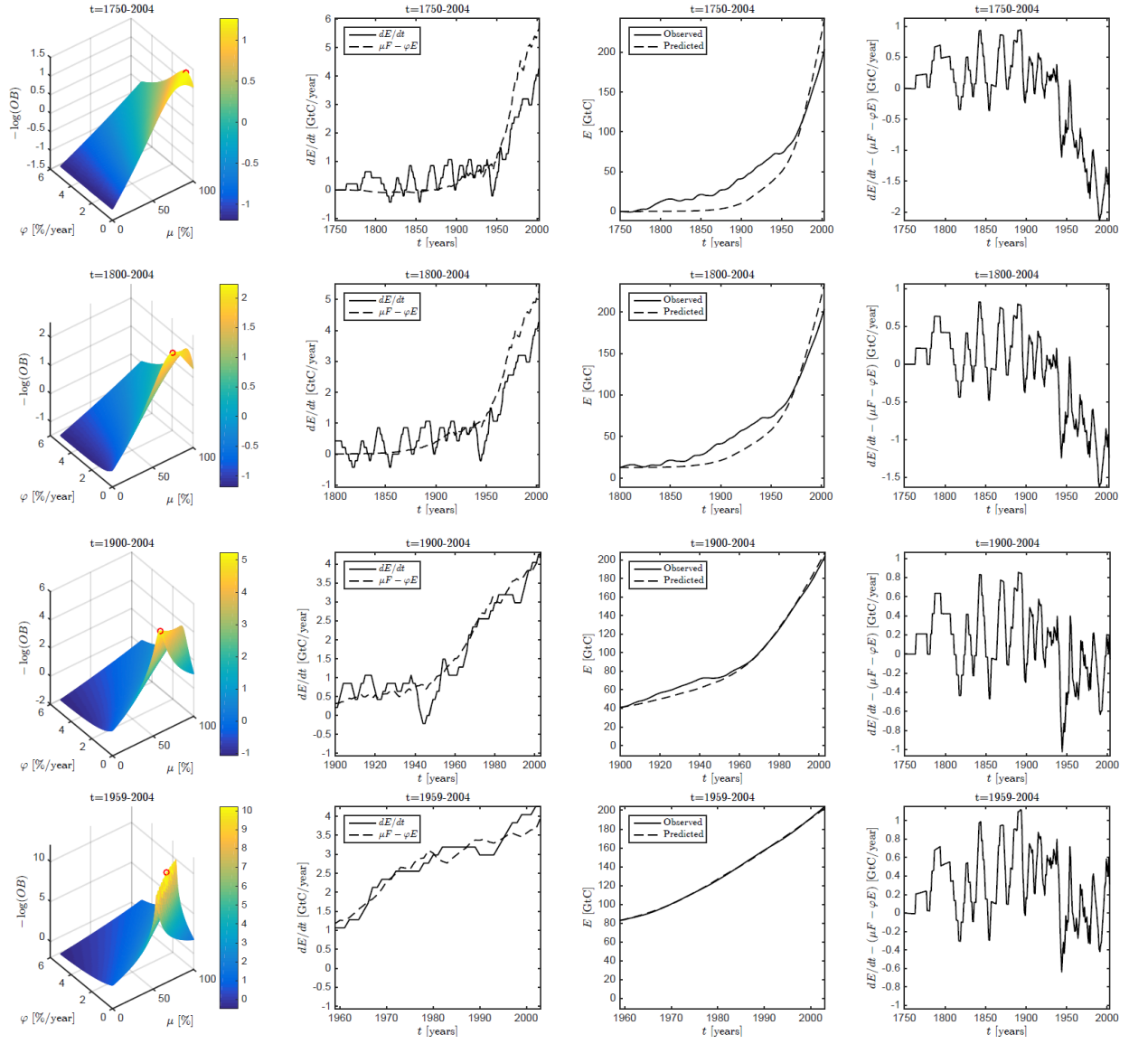
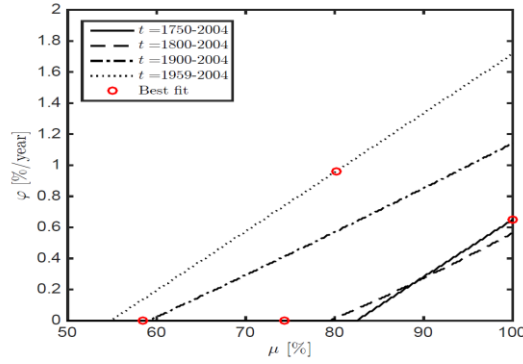FIGURE E1. HISTORICAL ATMOSPHERIC CARBON STOCK CALIBRATION

FIGURE E2. LOCI OF BEST FIT OF TMOSPHERIC STOCK CALIBRATION

Fig. E1 indicates that our model (4) captures the observed historical variations in the atmospheric carbon stock reasonably well, including for very long time periods. The final column in Table F1 shows volatility as percentage of the initial carbon stock, from which we note that the stochastic carbon stock adjustment to the optimal SCC will be tiny if estimated from historical emissions.

TABLE E1. ATMOSPHERIC CARBON STOCK CALIBRATION

Time	μ	φ [%/year]	σ_E [GtC$/$year$^{1/2}$]	σ_E/S_0 [%$/$year$^{1/2}$]	σ_E/E_0 [%$/$year$^{1/2}$]
1750-2004	1.0	0.66	0.31	0.036	0.12
1800-2004	0.75	0.00	0.26	0.029	0.10
1900-2004	0.59	0.00	0.21	0.025	0.081
1959-2004	0.79	0.91	0.23	0.027	0.089

E4. Calibration of the curvature of the temperature-carbon stock relationship

The curvature of our temperature relationship (5), $T(E, \chi) = \chi^{1+\theta_\chi}(E/S_{\mathrm{PI}})^{1+\theta_E}$, is constant: $\theta_E \equiv ET_{EE}(E, \chi)/T_E(E, \chi)$. The radiative law for global mean temperature,

$T \propto \ln(S / S_{\text{PI}}) / \ln(2) \propto \ln\left((E + S_{\text{PI}}) / S_{\text{PI}}\right) / \ln(2)$ (Arrhenius,   1896)[20]   gives $\theta_E = -E / (E + S_{PI})$. If we evaluate the temperature relationship at double (quadruple) the pre-industrial stock $E = S_{\text{PI}}$ ($E = 3S_{\text{PI}}$), we obtain $\theta_E = -0.50$ (or $\theta_E = -0.75$).[21] For $S_0 = 0.854$ TtC or $E_0 = 0.258$ TtC (Fiven $S_{\text{PI}} = 0.596$ TtC), we get $\theta_E = -0.30$. We set $\theta_E = -0.36$ for our base case calibration.

E.5. Climate sensitivity and uncertainty

If climate sensitivity parameter χ is normally distributed with mean μ_χ and standard deviation Σ_χ, the climate sensitivity $T_2 = \chi^{1+\theta_\chi}$ is described by the probability density function

$$(E1) \quad f_{T_2}\left(T_2; \mu_\chi, \Sigma_\chi, \theta_\chi\right) = \frac{1}{\sqrt{2\pi}\Sigma_\chi\left(1 + \theta_\chi\right)} T_2^{-\frac{\theta_\chi}{1+\theta_\chi}} \exp\left(-\left(T_2^{\frac{1}{1+\theta_\chi}} - \mu_\chi\right)^2 / 2\Sigma_\chi^2\right).$$

Unlike for fat-tailed distributions, which typically have algebraically decaying tails, all moments of (E1) are defined due to its exponential tail (for $\theta_\chi \geq -1$), so that Weitzman's (2009) 'dismal theorem' does not apply. Positive values of θ_χ result in a positively skewed (non-Gaussian) distribution with more probability mass at high temperatures. Leading-order central moments of climate sensitivity can be obtained from performing Taylor-series expansions of $T_2 = \chi^{1+\theta_\chi}$ about its mean μ_χ:

$$(E2a) \quad E[T_2] = \mu_\chi^{1+\theta_\chi}\left(1 + \frac{1}{2}\theta_\chi(1+\theta_\chi)(\Sigma_\chi / \mu_\chi)^2\right) + O(\Sigma_\chi^4),$$

$$(E2b) \quad \text{var}[T_2] \equiv E\left[\left(T_2 - E[T_2]\right)^2\right] = (1+\theta_\chi)^2 \mu_\chi^{2(1+\theta_\chi)}(\Sigma_\chi / \mu_\chi)^2 + O(\Sigma_\chi^4),$$

---

[20] In their table 6.2, IPCC (2001) propose a logarithmic relationship for radiative forcing as a function CO_2, also given in IPCC (1990, chapter 2, where original sources are cited), among two other non-logarithmic, but generally concave parametrizations. IPCC (1990, chapter 2, page 51) note that for "low/moderate/high concentrations, the form is well approximated by a linear/square-root/logarithmic dependence", where the limit of validity of the logarithmic calibration is said to be 1000 ppm. For other greenhouse gases alternative parametrizations are proposed: a square-root dependence for methane and a linear dependence for halocarbons.
[21] Whereas the normalized curvature of Arrhenius's (1896) logarithmic radiative law with respect to the atmospheric carbon stock S, namely $ST_{SS}(S)/T_S(S)$ is constant and equal to -1, this limit is only reached for large carbon stock in our case, in which $\theta_E \equiv ET_{EE}(E, \chi)/T_E(E, \chi)$.

(E2c) $\text{skew}[T_2] \equiv E\left[\left(T_2 - E[T_2]\right)^3\right] = 3\theta_\chi(1+\theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)}(\Sigma_\chi/\mu_\chi)^4 + O(\Sigma_\chi^6),$

(E2d) $\text{skew}^*[T_2] \equiv \text{skew}[T_2]/\left(\text{var}[T_2]\right)^{3/2} = 3\theta_\chi(\Sigma_\chi/\mu_\chi) + O(\Sigma_\chi^3).$

Our calibration of the distribution of the climate sensitivity are based on a wide range of distributions reported and used by the IPCC (2014, AR5) (see Fig. 2 in section IV.B). Combining (E1) with the expected carbon stock dynamics in our model, Fig. E3 shows the exceedance probability of temperature in our model as a function of time. The rapid broadening of the distribution with time reflects our calibration to the TCR for short time and the ECS for long time (see section IV.B).
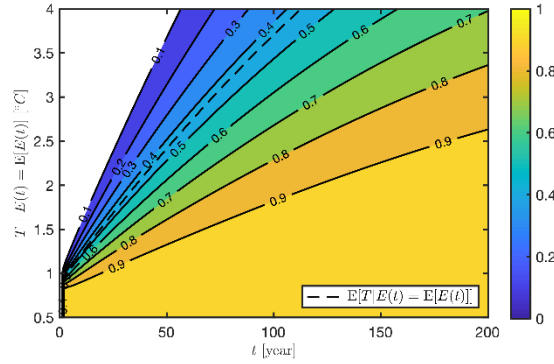

FIGURE E3. CONDITIONAL EXCEEDANCE PROBABILITY OF TEMPERATURE IN OUR MODEL

The skewness of the temperature distribution is evident from the expected temperature (dashed line) being greater than the median temperature, which is shown by the contour with an exceedance probability of 0.5.

E6. Climate damage uncertainty

In addition to the two calibrations of our model in Fig. 3, two additional calibrations have been considered in footnotes 60 and 61: a calibration based on Ackerman and Stanton (2012) that is of form $D = T^{1+\theta_{T,AS}}(C_{AS}\lambda)^{1+\theta_\lambda}$ and a calibration that is of the form $D = D_0 T^\lambda$ with $\lambda \sim N(\mu_\lambda, \Sigma_\lambda^2)$ (see footnotes 60 and 61 for details). Fig. E4 illustrates these two alternative calibrations with the continuous lines corresponding to expected damages, the shaded areas to the 90% confidence intervals, and the blue dashed line labelled AS12 to the original damage function of Ackerman and Stanton (2012). Also shown are the expected damages for the convex damage case of our model as continuous red lines and the corresponding 90% confidence bands as dashed red lines (cf. Fig. 3b).
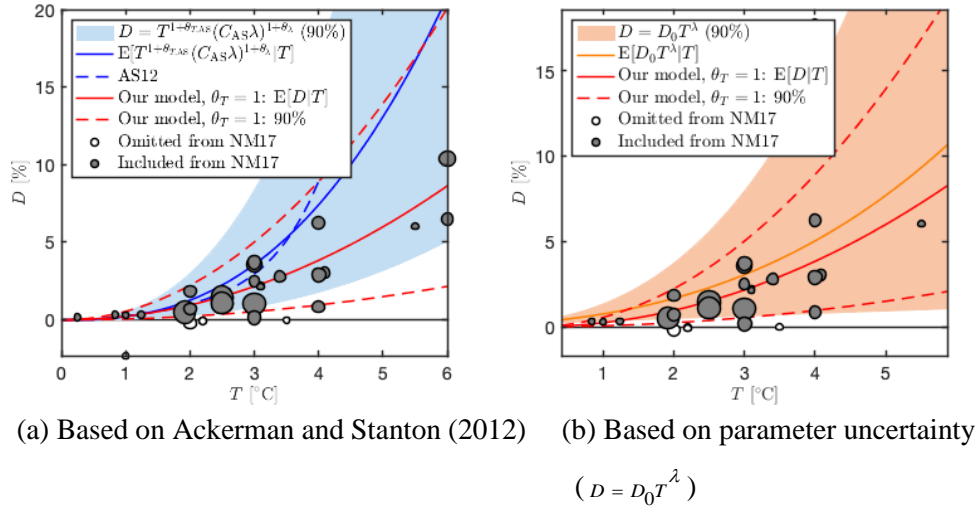
(a) Based on Ackerman and Stanton (2012)   (b) Based on parameter uncertainty

$$(D = D_0 T^\lambda)$$

FIGURE E4. ALTERNATIVE DAMAGE FUNCTION CALIBRATIONS

References

Arrhenius, S. 1896. "On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground", *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 5, 41, 237-276.

Barro, R.J. 2006. "Rare Disasters and Asset Markets in the Twentieth Century", *Quarterly Journal of Economics*, 121, 3, 823-866.

Caselli, F. and J. Feyer. 2007. "The Marginal Product of Capital", *Quarterly Journal of Economics*, 122, 2, 535-568.

IPCC. 1990. *First Assessment Report – Climate Change 1990*, *Working Group I Report*, International Panel on Climate Change, Geneva, Switzerland.

IPCC. 2001. *Third Assessment Report – Climate Change 2013*, *Working Group I Report*, International Panel on Climate Change, Geneva, Switzerland.

Kocherlota, N. 1996. "The Equity Premium Puzzle: It's Still a Puzzle", *Journal of Economic Literature*, 34, 1, 42-71.

Le Quéré, C., M.R. Raupach, J.G. Canadell, G. Marland et al. 2009. "Trends in the Sources and Sinks of Carbon Dioxide", *Nature Geoscience*, 2, 831-836.

Appendix F: Accuracy of Results 1 and 2 (For Online Publication)

Result A is evaluated numerically by discretization in time before evaluating the expectation operator numerically exactly and summing up the discounted contributions of every time step. Whereas the stochastic processes for χ and λ are autonomous, the stochastic process for K remains autonomous in Result 1, and all three have (independent) probability distributions available in closed form, the probability distribution of E at any time period in the future must combine all uncertain emissions (proportional to K) before that time. As the time integral of a Geometric Brownian motion does not have a closed-form solution, we update the probability distribution function of E every time step with the stochastic emissions and the decay in that period according to the differential equation for E and project on a fixed grid for E to enable transfer of the probability density function between time periods. Of course, the validity of Result A itself still relies on the parameter ϵ being small. Consistent with our perturbation scheme, all our optimal risk-adjusted carbon prices in Results A, 1 and 2 are evaluated along the business-as-usual path for which $P = 0$. We assess the accuracy of Results 1 and 2 for some of the calibrations examined in section V. By choosing the grid size sufficiently small and the grid sufficiently large in each case, we ensure that discretization errors for Result A are negligible.

TABLE F1. ACCURACY OF RESULT 1 OR 2 COMPARED TO RESULT A

Impatience ρ [/year]	5.8%	1.5%	0.1%	0.1%	0.1%
Economic volatility σ_K [/year$^{1/2}$]	12%	12%	1.5%	1.5%	1.5%
Damages	Proportional	Proportional	Proportional	Convex	Highly convex (AS12)
Total error in risk-adjusted SCC	-0.02%	-2.0%	0.73%	1.9%	-1.3%

Two factors determine the accuracy of using Result 1 or 2 instead of Result A. First, in Results 1 and 2 we ignore any uncertainty in the atmospheric carbon stock that arises because of the uncertain nature of future economic growth and thus of future emissions (*Assumption I*) For our base case calibration with *proportional* damages ($\theta_{ET} = 0$) (*Assumption IV*), the stochastic nature of E does not lead to a change in the SCC. Second, in Results 1 and 2 we only consider leading-order terms in the climate sensitivity uncertainty (*Assumption II*). We can confirm from Table F1 that the combined effect of these two errors is sufficiently small to be ignored for all practical purposes. As expected, it is larger for low discount rates, higher economic volatility, and *convex* damages.

Appendix G: Carbon pricing with some common calibrations

In Table G1, we evaluate the optimal risk-adjusted SCC for different calibrations in the literature. Golosov et al. (2014) use proportional damages, logarithmic utility (IIA = RRA = 1), and $\rho = 1.5\%$ per year, which gives a risk-adjusted discount rate $r^{(0)}$ of 3.5% per year. With logarithmic utility, neither the expected rate of growth nor uncertainty about the future rate of growth influences the optimal SCC. Gollier (2012) uses RRA = IIA = 2 and $\rho = 0$ and calibrates to GDP volatility, which gives a risk-adjusted discount rate $r^{(0)}$ of 4% per year and a risk-adjusted SCC of \$18.5/tCO$_2$. If the model were to be calibrated to asset return volatility, the risk-adjusted discounted rate drops to 2.5% per year and the risk-adjusted SCC rises to \$62.6/tCO$_2$. The discount rate is only substantially lowered for asset return uncertainty; asset return uncertainty depresses the discount rate and increases the risk-adjusted SCC as IIA > 1 in this calibration.

TABLE G1. ESTIMATES OF THE SCC: COMPARISON WITH OTHER CALIBRATIONS

Model	Base	Golosov et al. (2014)	Gollier (2012)		Stern (2007) +AS12
Volatility based on	asset returns	-	asset returns	GDP	GDP
Deterministic SCC (\$/tCO$_2$)	11.5	19.0	14.4	14.4	51.6
Risk-adjusted SCC (\$/tCO$_2$)	**39.8**	**24.6**	**62.6**	**18.5**	**102.9**
Economic risk mark-up	163%	0%	225%	1.1%	0.3%
Carbon stock risk mark-up	0%	0%	0%	0%	-1.1%
Climate sensitivity risk mark-up	41%	13%	57%	12%	66%
Damage ratio mark-up	43%	16%	54%	16%	21%
Total risk mark-up	247%	29%	336%	29%	90%
Discount rate $r^{(0)}$ (per year)	2.9%	3.5%	2.5%	4.0%	3.0%
Estimates in this table are for *proportional* damages ($\theta_{ET} = 0$), except for the final column, which assumes highly convex AS12 damages. The base case is for $\rho = 1.5\%$/year (*ethics-based* calibration).					

Our analytical results can also be used for stochastic carbon pricing with very convex damages, i.e., those used in Ackerman and Stanton (2012). The last column of Table 8 uses IIA = RRA = 1.45 and a very low rate of time preference of $\rho = 0.1\%$/year corresponding to a discount rate $r^{(0)}$ of 2.5% per year (for GDP-based economic volatility). These choices reflect the low discount rate and convexity of damages used by Stern (2007). This gives a very high deterministic SCC of \$52 and an even higher risk-adjusted SCC of \$103/tCO$_2$.