TI 2021-045/VI
Tinbergen Institute Discussion Paper

On current and future carbon prices in a risky world

Stan Olijslagers[1]
Rick van der Ploeg[1,2,3]
Sweder van Wijnbergen[1,3]

[1] ASE, University of Amsterdam, 1018 WB Amsterdam, The Netherlands
[2] Department of Economics, University of Oxford, Oxford OX1 3UQ, United Kingdom
[3] Centre for Economic Policy, Research, London EC1V 0DX, United Kingdom

ON CURRENT AND FUTURE CARBON PRICES IN A RISKY WORLD

Stan W.J. Olijslagers[a], Frederick van der Ploeg[a,b,c], and Sweder van Wijnbergen[a,c]

Abstract

We analyse optimal abatement and carbon pricing strategies under a variety of economic, temperature and damage risks. Economic growth, convex damages and temperature-dependent risks of climatic tipping points lead to higher growth rates, but gradual resolution of uncertainty lowers them. For temperature-dependent economic damage tipping points, carbon prices are higher, but when the tipping point occurs, the price jumps downward. With only a temperature cap the carbon price rises at the risk-adjusted interest rate. Adding damages leads to a higher carbon price that grows more slowly. But as temperature and cumulative emissions get closer to their caps, the carbon price is ramped up ever more. Policy makers should commit to a rising path of carbon prices.

JEL codes: H23, Q44, Q51, Q54

Keywords: CO_2 prices, growth uncertainty, tipping points, damages, gradual resolution of damage uncertainty, temperature caps

May 2021

[a] ASE, University of Amsterdam, 1018 WB Amsterdam, The Netherlands

[b] Department of Economics, University of Oxford, Oxford OX1 3UQ, United Kingdom

[c] Centre for Economic Policy, Research, London EC1V 0DX, United Kingdom

1. Introduction

Rising temperatures and the threat to our planet and the economy constitute the biggest market failure we know of (Stern, 2007). One solution is to price carbon as this reduces demand for carbon-intensive goods, encourages green innovation and carbon capture and sequestration, and locks up fossil fuel in the crust of the earth. This Pigouvian solution charges emissions at a price that implements the optimal policy, namely the price that internalizes the global warming externality (Pigou, 1920). There is a burgeoning debate on how high that tax should be, but with implications for future values of that tax often delivered as a by-product only. Yet *future* tax rates are a crucial determinant of the investment decisions that need to be taken *today* to implement an efficient and timely transition to a climate-neutral future. In this paper our emphasis is therefore not only on the level but more on the shape of the time path of the optimal carbon price under a wide range of economic, climatic and damage uncertainties.

Time paths matter as much as the initial price level since much of the adjustment and mitigation efforts will have to take place through new investments and these depend on the trade-off between current costs and *future* prices. In addition, as investments in both carbon-intensive and green technologies are to a large extent irreversible, there are strong option arguments for announcing a growing path of carbon prices and for policy makers to stick to it so as to lower perceived volatility and thereby reducing the incentive to delay investment (Dixit and Pindyck, 1994).

This price can be implemented as a carbon tax with the revenue rebated in lump-sum manner to the private sector. An alternative and increasingly popular method is to set up a competitive market for emission permits. Instead of the Pigouvian approach, one could also adopt a Coasian approach where property rights to emit or the right to a clean planet are allocated (Coase, 1960), with subsequent trade allowed. If there are other market failures, they should be dealt with using separate instruments. For example, learning by doing externalities in the production of green energy require early and direct subsidies of green energy. If this subsidy is lumped together with carbon prices, as is sometimes done in the literature (e.g. Daniel et al., 2019), this leads to an unwarranted early spike in carbon prices which may actually discourage investment in clean technology by rising input costs while not raising future prices commensurately.

In climate economics the Pigouvian price is referred to as the social cost of carbon or the SCC. This is defined as the expected present discounted value of all present and future damages caused by emitting one additional ton of carbon today. Strictly speaking, the SCC is a more general concept than a Pigouvian tax as it can be evaluated along other paths than the optimal path. For example, the SCC evaluated along a business-as-usual path where global warming externalities are not internalized, will be higher than along the optimal path if damages are convex enough (e.g. Olijslagers, 2020). Policy makers must evaluate the SCC under big uncertainties regarding the wealth of future generations and future global warming damages resulting from emissions today. This involves difficult trade-offs between consumption today and (the risks of) damages from global warming to consumption in the distant future.

We thus focus on the main drivers for the growth, or decline, of the optimal carbon price. Our benchmark is the case where damages to aggregate production are linear in temperature. Given that recent insights in atmospheric science suggest that temperature is linear in cumulative emissions (Matthews et al., 2009, Allen et al., 2009; van der Ploeg, 2018; Dietz and Venmans, 2018), the function relating the percentage loss in aggregate production to cumulative emissions is then also linear.[1] We then see that the optimal carbon price grows at the same rate of growth of the economy. The reason for this is that damages are proportional to aggregate production. We then consider step by step four generalizations of our benchmark and how they impact the qualitative pattern of the time path of optimal carbon prices.

First, we show that if damages are a convex function of temperature as has been argued by Weitzman (2009) and Dietz and Stern (2015), the optimal carbon price will start at a higher level and will also grow faster than the economy.

Second, we confirm an earlier result by Daniel et al. (2019) that if there is gradual resolution of uncertainty in the damage ratio, there is a component of the optimal carbon price which falls over time.[2] But we also show that when there is sufficient growth of the economy, this component is outweighed by the growing component of the carbon price resulting from growing damages. The key insight is thus that gradual resolution of uncertainty slows down the rate of growth of the optimal carbon price but under plausible assumptions does not reverse it. Gerlagh and Liski (2016) also find that learning and resolution of uncertainty slows the rise in the optimal carbon price.[3]

Third, we show that climatic and economic tipping points whose arrival rates increase in temperature boosts the carbon price. Once a climate tipping point occurs, it will suddenly increase

---

[1] This is related to Golosov et al. (2014), who take a different perspective. Their damage function is a convex function of temperature and their temperature relationship is a concave function of the stock of atmospheric carbon. They then notice that their exponential damage function is roughly a linear function of the stock of atmospheric carbon.

[2] Daniel et al. (2019) employ the workhorse recursive dynamic asset pricing model consisting of a discrete-time decision tree with a finite horizon extended to allow for Epstein-Zin preferences (Epstein and Zin, 1989, 1991) and generate optimal carbon dioxide (CO_2) price paths based on probabilistic assumptions about climate damages. They argue that it is optimal to have a high price today that is expected to decline over time as the "insurance" value of mitigation declines and technological change makes emission cuts cheaper.

[3] For learning and optimal climate policy, see also Kelly and Kohlstad (1999) and Kelly and Tan (2015).

the sensitivity of temperature to cumulative emissions which in turn should prompt policy makers to boost carbon prices and abatement significantly right now. Immediately after the tip has occurred, climate policy is ramped up resulting in an instantaneous further upward jump in the carbon price and abatement. A temperature-dependent risk of an economic tipping point that abruptly leads to a percentage destruction of production also leads to a higher path of carbon prices and abatement *ex ante*, but immediately after the tip the carbon price and abatement jump down. Different types of tipping points thus have radically different implications.

Fourth, although economists usually take a conventional welfare-maximizing approach, the International Governmental Panel on Climate Change (IPCC) and most countries have adopted the more pragmatic approach of agreeing that policy makers will do their utmost best to keep global mean temperature well below 2 degrees Celsius and aim for 1.5 degrees Celsius. A temperature cap which bites implies that the optimal carbon price should grow at a rate equal to the risk-adjusted interest rate (cf. Gollier, 2020).[4] Once allowance is made for the risk premium, this Hotelling path for the carbon price is typically faster than the rate of growth of the economy (even when the safe return is below the economic growth rate). Hence, the initial carbon price and abatement will be lower upfront but higher in the future. We find that taking account of the risk and uncertainty climate policy is stepped up hugely as temperature gets closer to its cap. The reason is that policy makers must prevent temperature overshooting the cap. If policy makers adopt a tighter cap, they need to boost the carbon price and abatement. We also show that if policy makers take account of a temperature cap and internalize damages from global warming to aggregate production, the optimal carbon price will grow faster than if only damages are internalized but slower than if only a temperature cap is imposed.

Overall, our results suggest that in face of a wide range of risks and uncertainties policy makers should commit to a gradually rising path of carbon prices. This has the added advantage that businesses get clear incentives to invest in long-term projects necessary to make the transition from carbon-intensive to carbon-free production

Our framework of analysis is a simple endowment economy where the endowment is subject to normal economic shocks (modelled by a geometric Brownian motion) and by macroeconomic disasters as in Barro (2006, 2009) and Barro and Jin (2011). Temperature is driven by cumulative emissions, and the fraction of damages lost due to global warming is a power function of

[4] Gollier shows in his analysis of the optimal carbon price needed to ensure that a temperature cap is not violated that this rate equals the safe rate plus the *beta* (the regression coefficient if rate of change in marginal abatement costs is regressed on rate of growth in aggregate consumption) times the aggregate risk premium.

temperature and is subject to stochastic shocks with a distribution that is skewed and has mean reversion as in van den Bremer and van der Ploeg (2021). Our short-cut approach to modelling gradual resolution of damage uncertainty is slow release of information.[5] We distinguish aversion to risk from aversion to intertemporal fluctuations, so we use recursive preferences (Epstein and Zin, 1989, 1991; Duffie and Epstein, 1992). This allows for a preference for early resolution of uncertainty when the coefficient of relative risk aversion exceeds the inverse of the elasticity of intertemporal substitution in accordance with empirical evidence. This is a precondition for our result that gradual resolution of uncertainty gives a declining component of the optimal carbon price.

Our paper is closely related to a recent interesting contribution by Lemoine (2021) who also studies the effect of damage ratio uncertainty and uncertainty about the economic growth rate in an endowment economy and offers analytical insights into the deterministic, precautionary, damage scaling and growth insurance determinants of the optimal social cost of carbon (cf. van den Bremer and van der Ploeg) and crucially gives simulations that show these components of the optimal carbon price. Our model differs in that we distinguish relative risk aversion from the inverse of the elasticity of intertemporal substitution and that that we have more general forms of uncertainty, i.e., we allow for skewness and declining volatility of the shocks to the damage ratio (cf. Daniel et al., 2019), the risk of rare macroeconomic disasters, and both economic and climatic tipping risks. We also allow for learning-by-doing effects in mitigation and thus for the consequent need for renewable energy subsidies. Furthermore, another contribution of our study is that we analyse the effects of temperature caps under uncertainty (both with and without damages to the economy) on the time paths of the optimal carbon price under uncertainty.

Our paper is also related to an extensive literature on optimal discounting under uncertainty (e.g. Gollier, 2002ab, 2008, 2011, 2012; Weitzman, 1998, 2007, 2009, 2011; Olijslagers and van Wijnbergen, 2020) and optimal climate policy under uncertainty (e.g. Crost and Traeger, 2013, 2014; Jensen and Traeger, 2014, Traeger, 2020; van den Bremer and van der Ploeg, 2021). It also relates to a growing literature under optimal climate policy in the presence of climatic and economic tipping points (Lemoine and Traeger, 2014, 2016b; van der Ploeg and de Zeeuw, 2018; Cai et al., 2016; Cai and Lontzek, 2019). Our contribution is to present a simple asset pricing model to answer many of the questions regarding uncertainty and tipping points in this literature. Our focus is, however, different in that we aim to understand the *qualitative* nature of the time path of the path

[5] A detailed analysis of the potential for learning and its implications for optimal policy is beyond the scope of this paper. But see van Wijnbergen and Willems (2015) on the implications for optimal climate policy.

of optimal carbon prices and abatement. A novel contribution of our approach is to also allow for temperature caps. Although Gollier (2020) has analysed these in a 2-period model, we study temperature caps in a continuous-time, infinite-horizon integrated assessment model of the economy and the climate. In the absence of damages from global warming to the economy, we show that the expected growth in the marginal abatement cost and the price of carbon equals the risk-free rate plus the insurance premium. Compared to Gollier (2020), we additionally consider the implementation of a temperature cap while at the same time internalizing the damages to aggregate production caused by climate change. This gives an expected growth of the carbon price that is in the between the growth rate of the economy and the risk-adjusted interest rate.

The outline of our paper is as follows. Section 2 sets up our asset-pricing model of the economy and the climate. Section 3 discusses our calibration and presents our benchmark result for optimal carbon pricing and abatement. Section 4 discusses the four generalizations of our benchmark and how they impact the level and the growth rate of the optimal carbon price. Section 5 discusses our results and offers a more general perspective on why it is important for policy makers to credibly commit to a path growing carbon prices. Section 6 concludes.

2. An integrated model for optimal climate policy evaluation under risk

To make the trade-off between sacrifices in current consumption against less consumption due to global warming in the future, we use recursive preferences which recursively defines a value function giving the expected welfare from time *t* onwards, i.e. V_t (Epstein and Zin, 1989, 1991; Duffie and Epstein, 1992). This formulation distinguishes the coefficient of relative risk aversion, denoted by *RA*, from the inverse of the elasticity of intertemporal substitution, *EIS*.[6] Policy makers prefer early (late) resolution of uncertainty if *RA* exceeds (is less than) 1/*EIS*. Econometric evidence on financial markets strongly suggests this separation and that *RA* exceeds 1/*EIS* (Vissing-Jørgensen and Attanasio, 2003; van Binsbergen et al., 2012). Hence, the risk-adjusted interest rate incorporates a so-called "timing premium" (Epstein et al., 2014). If *RA = 1/EIS* as with the power utility function, policy makers are indifferent about the timing of the resolution of uncertainty and there is no timing premium in interest rates. Mathematically, this is represented as follows. All

[6] 1/EIS can also be interpreted as a coefficient measuring aversion to intertemporal fluctuations.

agents have identical preferences and endowments, so all the agents can be replaced by one representative agent. If $RA = \gamma$ and $EIS = \eta$, preferences of this agent follow recursively from

(1) $$V_t = \max_{\{h_t\}_{t=0}^{\infty}} E_t\left[\int_0^{\infty} f(C_s, V_s)ds\right] \text{ with } f(C,V) = \frac{\beta}{1-1/\eta}\frac{C^{1-1/\eta} - \left((1-\gamma)V\right)^{\frac{1}{\zeta}}}{\left((1-\gamma)V\right)^{\frac{1}{\zeta}-1}},$$

where $\zeta \equiv (1-\gamma)/(1-1/\eta)$ and $\beta > 0$ denotes the utility discount rate or rate of time impatience. If RA = 1/EIS, i.e. $\gamma = 1/\eta$,equation (1) boils down to the expected utility approach with no preference for early (or late) resolution of uncertainty.

The endowment of the economy Y_t follows a geometric Brownian motion with drift μ and volatility σ^Y and includes additional terms to allow for disaster shocks with constant mean arrival rate λ_1. The size of the shocks is a random variable with time-invariant distribution. The endowment thus follows the stochastic process

(2) $$dY_t = \mu Y_t dt + \sigma^Y Y_t dW_t^Y - J_1 Y_t dN_{1,t}$$

where W_t^Y is a standard Wiener process, $N_{1,t}$ is a Poisson process with mean arrival rate λ_1, and J_1 is a random variable and corresponds to the share of endowment destroyed if a disaster hits the economy. We assume that $x = 1 - J_1$ has a power distribution with probability density $f(x) = \alpha_1 x^{\alpha_1 - 1}$, so $E[x^n] = \alpha_1/(n + \alpha_1)$ and $0 \leq E[J_1] = 1/(1+\alpha_1) \leq 1$. For all moments to exist, we assume that $\gamma < \alpha_1$. This process for the evolution of the economy thus incorporates both normal macroeconomic uncertainty (captured by the geometric Brownian motion) and macroeconomic disaster risks as in Barro (2006, 2009) and Barro and Jin (2011).

Consumption equals

(3) $$C_t = (1 - A_t)Y_t / D_t,$$

where A_t denotes the fraction of output used for abatement and D_t is the damage ratio associated with global warming. The time path of business-as-usual emissions E_t is assumed to be exogenous. Business-as-usual emissions E_t grow at the decreasing rate $g_t^E = g_0^E e^{-\delta_E t}$, where $g_0^E > 0$ and $\delta_E > 0$ are constants. Actual emissions are $(1-u_t)E_t$ where u_t denotes the abatement rate. Without carbon capture and sequestration (CCS), the upper bound of the abatement rate equals 1 in which case all emissions are fully abated and the economy effectively only uses renewable energy.

The cost function for abatement is

(4) $$A_t = c_0 e^{-c_1 X_t} u_t^{c_2},$$

where X_t is the stock of knowledge and c_1 is the parameter that controls how fast the costs decline over time due to technological progress. The future stock of knowledge is uncertain. We assume that $c_2 > 1$, so abatement costs are a convex function of the abatement rate. We consider two different processes for the stock of knowledge. In the standard case the stock of knowledge grows linearly over time, so that

(5a) $$dX_t = 1dt + \sigma_X dW_t^X.$$

Technological progress in this case is exogenous. In the absence of abatement cost uncertainty (i.e. when $\sigma_X = 0$), the cost function is identical to the cost function in the DICE model (Nordhaus, 2017). In the alternative case we allow for learning by doing by assuming that the growth of the stock of knowledge is a function of the cumulative amount of emissions that have been abated or

(5b) $$dX_t = u_t E_t dt + \sigma_X dW_t^X.$$

Temperature is a linear function of cumulative carbon emissions and its dynamics are described by

(6) $$dT_t = \chi(1 - u_t) E_t dt,$$

where χ denotes the transient climate response to cumulative emissions or *TCRCE*. The damage ratio is a function of temperature and shocks that take some time to have their full impact and follow a skewed distribution to reflect "tail" risk. The damage ratio is given by

(7) $$D_t = T_t^{1+\theta_T} \omega_t^{1+\theta_\omega} \quad \text{with} \quad d\omega_t = \upsilon(\bar{\omega} - \omega_t)dt + \sigma_t^\omega dW_t^\omega,$$

where ω_t follows a Vasicek (or Ornstein-Uhlenbeck) process with short-run volatility σ_t^ω, mean reversion υ and long-run mean $\bar{\omega}$, and W_t^ω is a standard Wiener process (cf. van den Bremer and van der Ploeg, 2021). Here θ_T controls the convexity with respect to temperature and θ_ω controls the skew of the shocks hitting the damage ratio. Linear (convex) damages in temperature correspond to θ_T =0 (or > 0). A novel feature of our analysis is that we use the specification

(8) $$\sigma_t^\omega = \max\left[(1 - t/\bar{t}^\omega)\sigma_0^\omega, 0\right],$$

so that volatility starts with σ_0^ω and falls to zero after $\bar{t}^\omega$ years. This captures gradual resolution of damage uncertainty. Volatility is constant if $\bar{t}^\omega \to \infty$. When a temperature cap is implemented, we impose the restriction $T_t \le T^{cap}$. This is in our setup equivalent to the restriction that only renewable energy must be used once temperature is at or above its cap, i.e. $u_t = 1$ if $T_t = T^{cap}$.

Finally, we allow for the possibility of an *economic* tipping point. We assume that the probability of a tipping point increases in global mean temperature. The hazard rate equals $\lambda_2 T_t$ where λ_2 indicates the rate at which the hazard rate increases with temperature. We assume that when the system tips, a share J_2 of endowment is destroyed. J_2 is a random variable which also follows a power distribution, but with parameter α_2. The main difference between a tipping point and a disaster process is that the tipping point can only tip once, while the Barro-style macroeconomic disasters are recurring. We also consider a *climatic* tipping point for which the sensitivity of temperature to cumulative emissions suddenly increases after a tip. More specifically, we assume that before the tip the transient climate response to cumulative emissions is equal to χ_0 and after the tip it changes to χ_1. The hazard rate for this tipping points equals $\lambda_3 T_t$. We show that the two different specifications have very different implications for the optimal carbon price.

2.1. Optimal climate policies and implementation in a decentralized economy

We can solve the problem of maximizing expected welfare subject to equations (1) to (8) using the method of dynamic programming (see Appendix A). The resulting social optimum gives rise to the optimal SCC and can be sustained in a decentralized market economy when, for example, the carbon tax is set to the SCC and the revenue is rebated as lump sums (see Appendix B). The numerical implementation is discussed in Appendix C.

The social cost of carbon (SCC) corresponds to the expected present discounted value of all present and future damages to the economy resulting from emitting one ton of carbon today. It equals the welfare loss of emitting one unit of carbon divided by the instantaneous marginal utility of consumption, i.e.

$$(9) \qquad SCC_t = \Omega^{i,j}(T_t, \omega_t, X_t, t) \;\; C_t^{1/\eta} Y_t^{1-1/\eta} \; .$$

(cf. equation (A3) in Appendix A). The second part of equation (9) indicates that the optimal SCC is proportional to a weighted geometric average of aggregate consumption and the endowment with the weight to consumption equal to 1/*EIS*. The first part of equation (9) indicates that the optimal SCC depends on temperature, shocks to the damage ratio and cumulative learning by doing in renewable energy. The SCC corrected for growth of the economy only depends on the first component of (9) and is given by $\Omega^{i,j}(T_t, \omega_t, X_t, t) \;\; C_0^{1/\eta} Y_0^{1-1/\eta}$.

We consider two cases for the abatement costs. In the benchmark case abatement costs decline exogenously over time. In the learning-by-doing case abatement costs are endogenous and increase in the stock of accumulated past abatements (i.e. the stock of knowledge). The social benefit of learning corresponds to all the present and future marginal benefits in terms of lower mitigation costs resulting from using one unit of mitigation more today, i.e.

(10) $$SBL_t = \Theta^{i,j}(T_t, \omega_t, X_t, t)\, C_t^{1/\eta} Y_t^{1-1/\eta}.$$

(cf. equation (A4) in Appendix A). Like the SCC, the SBL consists of two components. The second one is proportional to a weighted average of endowment and aggregate consumption and the first one depends on temperature, damage ratio shocks and cumulative learning by doing in abatement. In the benchmark case without learning by doing, the SBL is simply equal to zero.

When choosing optimal abatement policy, policy makers must recognize that abatement serves two purposes in our set-up: 1) it reduces emissions and thus global warming, which leads to less climate damages in the future and 2) due to learning by doing, abatement reduces future abatement costs. But abatement is costly. Policy makers must sacrifice current consumption to make room for abatement if they want to curb global warming and increase (expected) future consumption. Optimal abatement u_t thus follows from the condition that the marginal abatement cost (MAC) must equal the social cost of carbon (SCC) plus the social benefit of learning, i.e.

(11) $$MAC_t = SCC_t + SBL_t$$

where $MAC_t = -\frac{\partial C_t/\partial u_t}{E_t}$ (see (A) in Appendix A). The marginal abatement cost is the cost of abating one more unit of carbon emissions. It increases in the abatement rate u_t since abatement costs are a convex function of the abatement rate. The economy increases abatement until the marginal abatement costs equal the benefits of abatement. If there is no learning by doing, the only benefit of abatement is the reduction of climate change damages. In that case the marginal abatement cost is equal to the SCC, which is the expected present discounted value of all current and future damages caused by emitting one more ton of carbon today. The learning-by-doing externality gives an additional incentive to reduce emissions. The marginal abatement cost thus equals the sum of the social cost of carbon and the social benefit of learning. We denote the optimal abatement policy that solves the dynamic programming problem by u_t^*.

When the government implements a carbon tax which is set it to $\tau_t = SCC_t$ and a renewable energy subsidy which is set to $s_t = SBL_t$, and the net revenue of these policy instruments are rebated as lump sums, the social optimum can be replicated in a decentralized market economy (see Appendix B). Competitive energy producing firms will then choose the energy mix such that the amount of fossil fuel use equals $F_t = (1-u_t^*)E_t$ and the amount of renewable energy use equals $R_t = u_t^* E_t$, where E_t is the total amount of energy use in the economy (which we have previously referred to as business-as-usual emissions).

We have adapted the simple but widely used energy model of Nordhaus (2017) and extended it to allow for uncertainty and tipping points in the economy, the climate sensitivity, and damages from global warming. One drawback of this is that in our setting, taxing carbon is equivalent to subsidizing renewable energy since total energy use is not endogenously chosen by the energy producers and since fossil and green energy are perfect substitutes. Optimal policy could thus in such a framework also be replicated by setting a carbon tax equal to $\tau_t = SCC_t + MAC_t$. However, it is important to stress that this is no longer the case in more general models. When fossil fuel and renewable energy use can be optimally chosen separately, replication of the command optimum can only be done by setting $\tau_t = SCC_t$ and $s_t = SBL_t$ (e.g. Rezai and van der Ploeg, 2017a). Taxing carbon then has different implications than subsidizing green energy. In a more general setting with directed technical change, it can be shown that when green and dirty inputs are sufficiently substitutable, a temporary green energy subsidy is optimal to fight climate change by kickstarting the economy in directions of green technical progress (e.g. Acemoglu et al., 2012).[7] Although taxing carbon and subsidizing green energy are equivalent in our simple framework, we do interpret the social cost of carbon as the optimal carbon tax and the social benefit of learning as the optimal renewable energy subsidy, to stress that the two are in general not equivalent.

We assume that negative emissions are not possible (or at least not at a competitive price) and therefore impose an upper bound on the abatement rate of 1. Hence, when it would be optimal to abate more than 100% of the emissions, the optimality condition (9) cannot be satisfied anymore. In this case the marginal abatement costs are smaller than the sum of the social cost of carbon and the renewable energy subsidy.

2.2. Effects of a temperature cap on optimal abatement and carbon pricing

Optimal policy in presence of a temperature cap still satisfies the first-order condition, but the social cost of carbon now must account for the temperature cap. A temperature cap in our model is equivalent to the restriction that $u_t = 1$ when $T_t = T^{cap}$. We show that in the case of a pure temperature cap (i.e. no effect of climate change on damages to aggregate production), intertemporal optimization implies that the expected growth rate of SCC and of the marginal abatement cost must equal the risk-free interest rate plus the risk premium (for a proof, see Appendix D). In this case, we thus have that expected growth in the marginal abatement cost and

[7] Bovenberg and Smulders (1995, 1996) offer early contributions on climate policy and endogenous growth. It has also been argued that subsidizing green energy technology is not effective to fight climate change, since it leads to higher energy use in total instead of substantially less fossil fuel use (Hassler et al., 2020).

in the optimal carbon price equals

$$(12)\qquad E_t\left[\frac{dMAC_t}{MAC_t}\right] = r_t + rp_t,$$

where the risk-free interest rate is given by

$$(13)\qquad r_t = \beta + \frac{\mu_c}{\eta} - \frac{1}{2}\left(1+\frac{1}{\eta}\right)\gamma(\sigma^Y)^2 - \lambda_1\left(\frac{\alpha_1}{\alpha_1-\gamma} - 1 - \frac{\gamma-\frac{1}{\eta}}{\alpha_1+1-\gamma}\right)$$

(cf. equation (D20)) and the risk premium is given by

$$(14)\qquad rp_t = E_t\left[-\frac{d[\pi_t, MAC_t]}{\pi_t MAC_t}\right] = E_t\left[-\frac{d\left[Y_t^{-\gamma}, Y_t\right]}{Y_t^{1-\gamma}}\right] = \gamma(\sigma^Y)^2 + \lambda_1(E[(1-J_1)^{-\gamma}] + E[1-J_1] - E[(1-J_1)^{1-\gamma}] - 1) = \gamma(\sigma^Y)^2 + \lambda_1\left(\frac{\alpha_1}{\alpha_1-\gamma} + \frac{\alpha_1}{\alpha_1+1} - \frac{\alpha_1}{\alpha_1+1-\gamma} - 1\right).$$

(cf. equation (D22)). In expectation, the growth rate of marginal abatement costs is therefore equal to the risk-free rate plus the risk premium. This result echoes the result derived by Gollier (2020) for a two-period model. It follows from the assumption that temperature is a linear function of cumulative emissions. In that case, we get an equivalent of the celebrated Hotelling rule: the price path assures that temperature does not exceed the cap and achieves intertemporal efficiency. In other words, the risk-adjusted discounted marginal cost of abatement is the same for each period.

3. The benchmark results

We discuss our benchmark calibration and then present and discuss the corresponding optimal time path for respectively the carbon price, the learning-by-doing subsidy, abatement, and temperature.

3.1. Calibration

In our benchmark calibration, we choose $RA = \gamma = 7$, $EIS = \eta = 1.5$ and the rate of impatience $\beta =$ 2% per year. These are values that are typically used in the asset pricing literature with Epstein-Zin preferences (e.g. Table 1, Cai and Lontzek, 2019) based on extensive empirical evidence. The details of our calibration are reported in Table 1.

The initial endowment is set to world GDP of 80 trillion US dollars. We suppose this endowment is subject to normal shocks captured by geometric Brownian motion with a drift of 2% per year and an annual volatility of 3%. In addition, we have macroeconomic disaster shocks along the lines of Barro (2006, 2009) and Barro and Jin (2011). Here the mean size of the disaster shocks is 8.7% and the mean arrival rate of these shocks is 0.035 per year corresponding to a mean arrival time of 29 years. This calibration yields a real risk-free interest rate of 0.75% and a risk premium of 2.65% if we abstract from the adverse effects of climate change on the economy. Since in the past century climate change has arguably had no effect on interest rates, we can compare these numbers to historical averages.

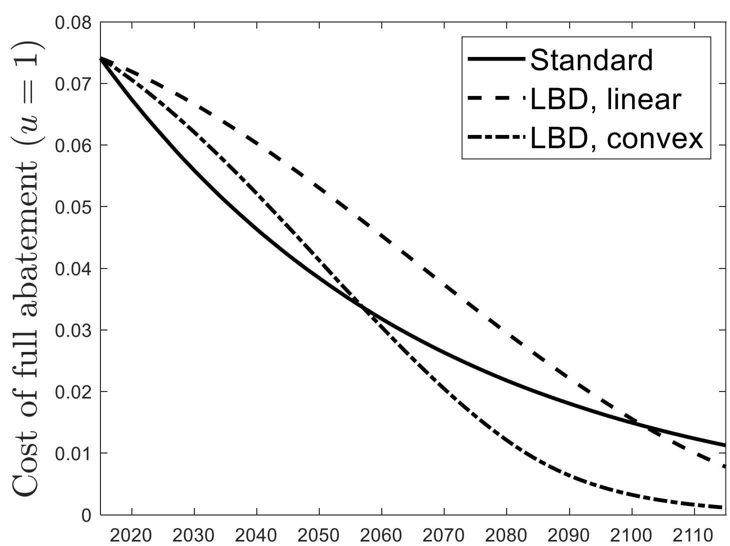
Table 1: Calibration details

Preferences	Market based: $RA = \gamma = 7$, $EIS = \eta = 1.5$, impatience = β = 2%/year
Economy	Initial endowment: Y_0 = 80 trillion US dollars *Geometric Brownian motion:* Drift in endowment = μ = 2%/year (for no growth scenario, drift = $\mu = \lambda_1 E[J_1]$ Volatility of shocks to endowment = σ^Y = 3%/√year *Macroeconomic disasters:* Mean arrival rate of disaster shocks =λ_1 = 0.035/year, Mean size of disaster shocks: $E[J_1]$ = 8.7%, Disaster shocks: shape parameter of gamma distribution α_1= 10.5
BAU emissions	Initial flow of global emissions in BAU scenario: E_0 = 10 GtC/year Initial growth of BAU emissions: g_0^E = 1.8%/year Decline of the growth rate of BAU emissions: δ_E = 2.7%/year
Abatement costs (benchmark case)	Current cost of full decarbonization: c_0 = 7.41% of initial GDP Rate of technological progress: c_1 = 1.9%/year Convexity parameter of the cost function: c_2 = 2.6 Abatement cost volatility parameter: $\sigma_X = 1$ Maximum abatement (full de-carbonization corresponds to u = 1, so $u \leq 1$)
Abatement costs (learning by doing case)	Current cost of full decarbonization: c_0 = 7.41% of initial GDP Rate of technological progress: c_1 = 0.375%/unit of knowledge Convexity parameter of the cost function: c_2 = 2.6 Abatement cost volatility parameter: $\sigma_X = 5$ Maximum abatement (full de-carbonization corresponds to u = 1, so $u \leq 1$)
Temperature	Initial temperature: T_0 = 1 °C Transient climate response to cumulative emissions before tip: $TCRCE = \chi_0$ = 1.8 °C/TtC Temperate cap: $T^{cap} = 1.5°$ or $T^{cap} = 2°$ or $T^{cap} = \infty$
Damage ratio	Linear case: convexity parameter temperature $\theta_T = 0$ Convex case: convexity parameter temperature $\theta_T = 0.56$ Skew parameter for shocks: θ_ω = 2.7 Mean reversion of shocks: υ = 0.2/year Initial and mean steady-state value of shocks: $\omega_0 = \bar{\omega} = 0.21$ Variant with constant volatility: $\sigma_0^\omega = 0.05$, $\bar{t} \to \infty$ Variant gradual resolution of uncertainty: σ_0^ω = 0.05, $\bar{t}$ = 100 years
Economic tipping point	Mean arrival rate of tipping point: $\lambda_2 = 0.01T_t$ Mean tipping damage level: $E[J_2]$ = 2.5%, Tipping damage level: shape parameter of gamma distribution α_2 = 39
Tipping point in the TCRCE	Mean arrival rate of tipping point: $\lambda_3 = 0.006T_t$ Level of TCRCE after tipping: $TCRCE = \chi_{post}$ = 2.5°C/TtC

Dimson et al. (2011) calculate that the global real risk-free rate has been on average 1% and the risk premium 4% over the period 1990-2010. We are currently in a low interest environment and in the long run it is questionable whether interest rate will return to their old average levels, which makes 0.75% a reasonable real risk-free interest rate. Our risk premium is lower than the historical average, but our main purpose is not to solve the equity premium puzzle. Furthermore, a risk premium of 2.65% is more realistic compared to most other climate-economy models in which the risk premium is often small or non-existent. These numbers are also close to Gollier (2020) who calibrates the risk-free rate at 1% and the risk premium at 2.5%. This calibration implies that in the case of a temperature cap without damages, the optimal carbon price grows in expectation at a rate equal to the risk-free rate plus the risk premium, i.e. 0.75%+2.65%=3.4%.

Parameters for business-as-usual (BAU) emissions are chosen to match the baseline emissions scenario in Nordhaus (2017) over the first century of the simulation period and afterwards BAU emissions stabilize. The parameters c_0, c_1 and c_2 of the abatement cost function in the benchmark case are taken from the DICE calibration (Nordhaus, 2017). For the learning by doing calibration, we take the same value for c_0 (cost of full abatement in initial period) and for c_2 (convexity of abatement costs in abatement rate u_t). The parameter c_1 now represents the decline in abatement costs when one additional GtC of carbon emissions is abated and is set to $c_1 = 0.375\%$ (cf. Rezai and Van der Ploeg, 2017a). With the learning by doing in renewable energy production, future abatement costs depend on cumulative past abatement efforts and thus also depend on the damage calibration. Figure 1 compares abatement costs of the benchmark case with the learning-by-doing case, both when damages from global warming are linear and when they are convex.

Figure 1: Costs of full abatement ($u_t = 1$) in the benchmark and in the learning-by-doing case for two different damage specifications (linear and convex)



We take a transient climate response to cumulative emissions (TCRCE) of 1.8 °C/TtC (cf. Matthews et al., 2009; Hambel et al., 2020). The parameters of the uncertain damage shock and of the convexity parameter θ_T are taken from van den Bremer and van der Ploeg (2021). For the variant with gradual resolution of damage uncertainty, we assume that the volatility of the damage shock is linearly declining to 0 over a period of 100 years as in equation (8).

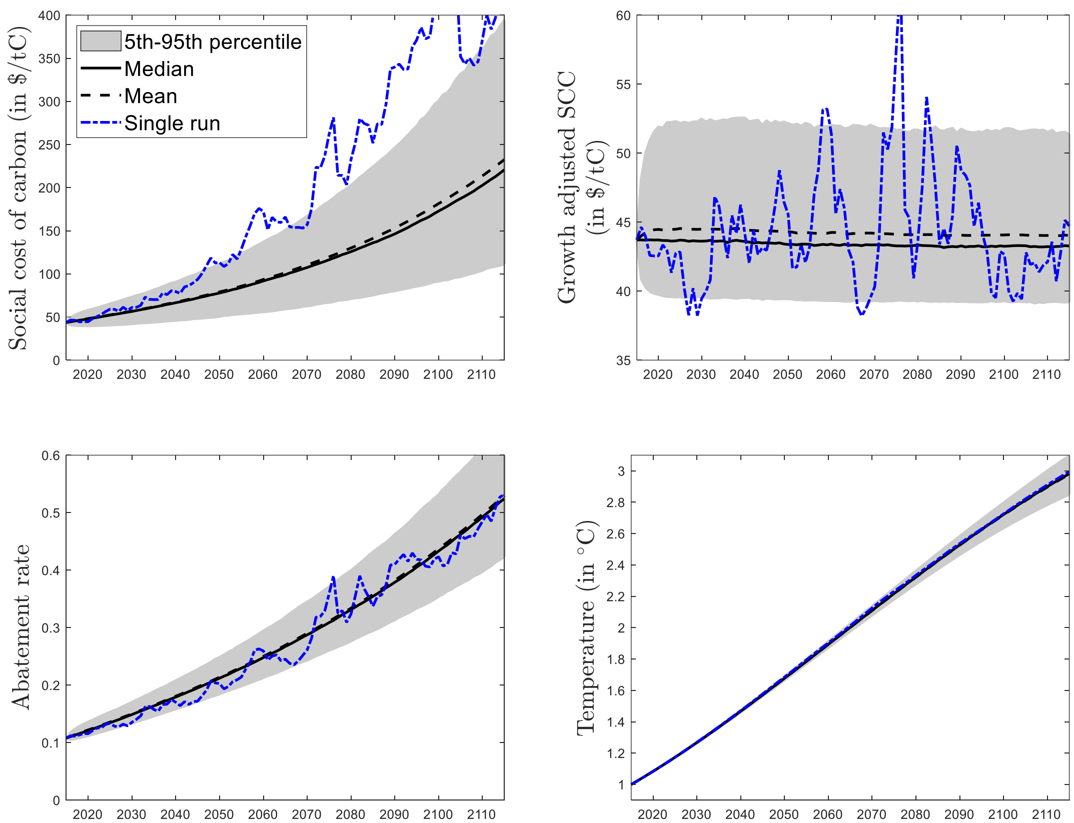
Finally, we assume that initially an economic tipping point tips on average after 100 years. When temperature increases to two (four) degrees Celsius, this becomes on average after 50 (25) years. The size of the damages caused by the tipping disaster is assumed to be on average 2.5%. For the climate tipping point, it takes initially on average 167 years for the climate system to tip. With 2 degrees Celsius warming the average time reduces to 83 years. When the system tips, the TCRCE jumps from 1.8 °C/TtC to 2.5°C/TtC. Overall, the main message of the two tipping point calibrations is that the probability of tipping in both cases is quite small, but we will show that the impact on optimal carbon prices is nevertheless considerable.

3.2. The benchmark optimal carbon prices

With this calibration, the benchmark SCC (with no learning by doing and no temperature cap) is shown in Figure 2. The SCC corresponds to the optimal carbon price. The most striking feature of the top two panels is that the ex-ante mean and median paths of the optimal carbon price start at almost $50/tC and then grow almost in tandem with the growth of the economy.

In fact, there is a modest decline in carbon price corrected for the growth of the economy as can be seen from the top right panel. The median carbon price path lies below the mean carbon price path, and the 5% and 95% bounds become wider for carbon prices that are further in the future as one should expect given that a function of GBM processes is a GBM process itself. As a result of the technological progress in abatement technology, there is a gradual rise in abatement efforts over time. Due to the rise in business-as-usual emissions, temperature rises to around 3 degrees Celsius in the next century but by rather less than in the absence of abatement efforts. The plots also indicate a sample run in blue. This suggests that for individual sample paths of the optimal carbon price there may be substantial volatility, which does not show up in the *ex-ante* time path for the mean (or median) optimal carbon price. Since we abstracted from stochastic shocks to temperature and abatement efforts are much less volatile, the temperature path itself shows hardly any volatility. When we allow for uncertain tipping points in the sensitivity of temperature to cumulative emissions, this will change (see section 4.3 below).

Figure 2: Benchmark with linear damages, no learning by doing, no gradual resolution of uncertainty, no tipping points, and no temperature cap



4. Four generalizations of the benchmark

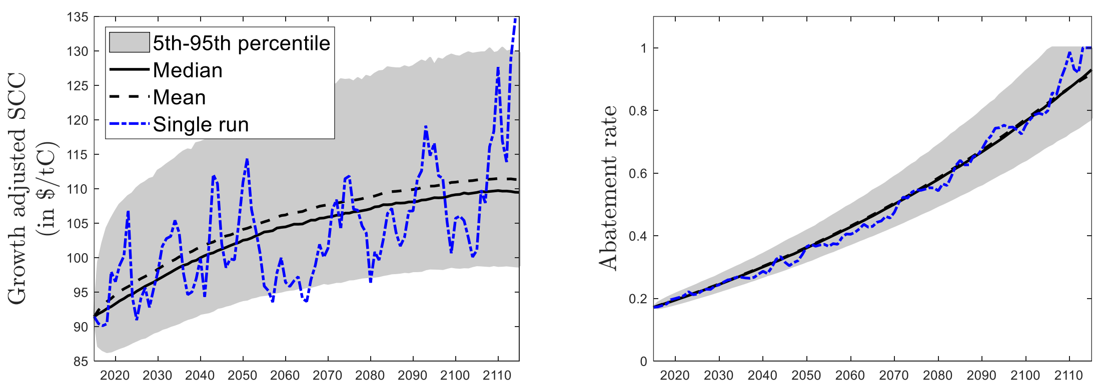
We now discuss four generalizations of the benchmark. For expositional reasons, we discuss these generalizations one by one. In practice, all these generalizations may be relevant at the same time. We discuss first the effects of convex damages, then present the effects of learning by doing and a combination of convex damages and learning by doing. After that we discuss the implications of gradual resolution of damage uncertainty and then show the differential impacts of climatic and economic tipping points. Finally, we analyse the effects on the time path of carbon prices of temperature caps.

4.1. Convex damages

Figure 3 presents the effects of convex damages captured by the proportion of output lost due to global warming being a convex rather than a linear function of temperature. Following van den Bremer and van der Ploeg (2021), we let this function be proportional to temperature to the power of 1.56. This is slightly less convex than the damage function of Nordhaus (2018) but serves to illustrate the effects of convex damages.

The most striking effect of convex damages is that the carbon price starts at a higher level, \$91/tC instead of \$44/tC, and then grows *in expectation* at a faster pace than in the benchmark. We can see this most strikingly by comparing the top right panel of Figure 1 with the left panel of Figure 2. This shows that with convex damages, the path of optimal carbon price corrected for growth of the economy rises whilst with linear damages, this path declined mildly. Hence, the abatement efforts are much stronger. The average mitigation rate now rises in a century to 92% instead of 53% in the benchmark. We thus confirm the Monte-Carlo results of Dietz and Stern (2015) in our fully stochastic framework: climate policies get intensified if damages are convex.

Figure 3: Convex damages



4.2. Learning by doing in abating emissions

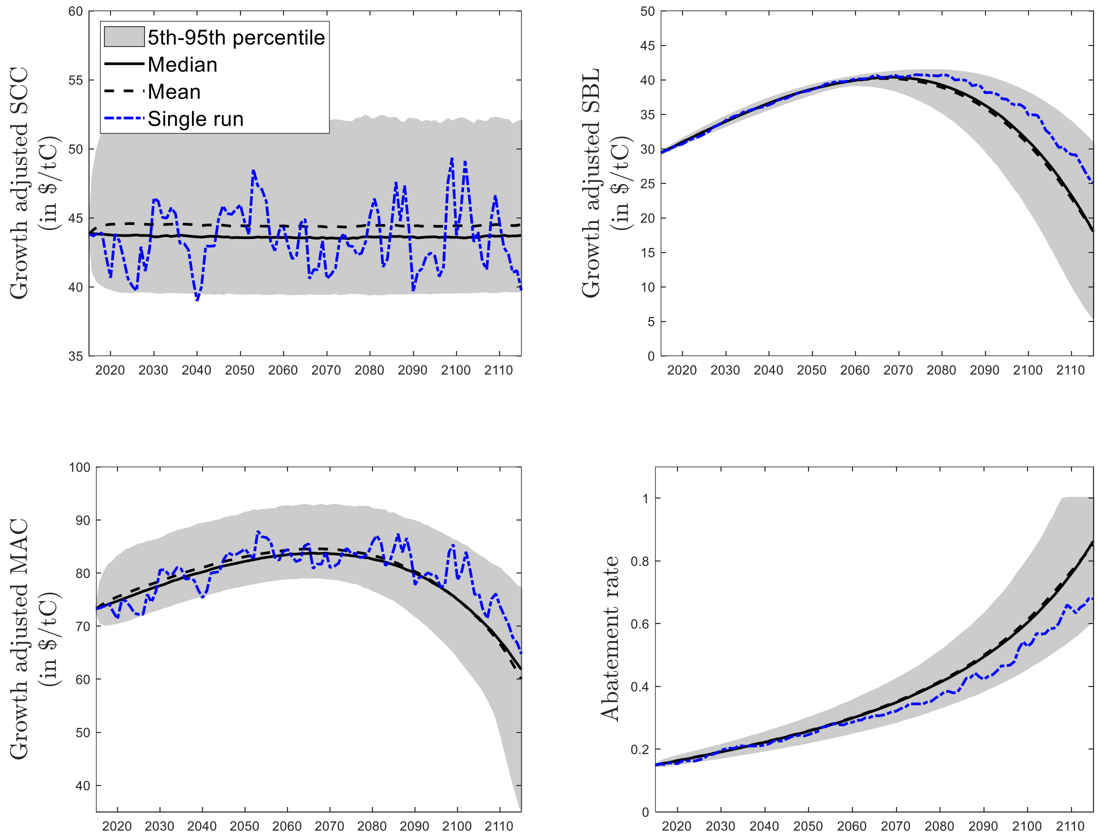
Including learning by doing into the analysis gives an additional reason for abatement. The marginal abatement cost is now equal to the social cost of carbon plus the social benefit of learning. The social cost of carbon adjusted for economic growth is almost identical to the base situation, so changing the abatement cost structure has no significant effect on optimal carbon prices. Hence, optimal carbon prices still grow in tandem with the economy (see top left panel).

The social benefit of learning shown in the top right panel of Figure 4 has a very different shape. It grows faster than the economy in the first 50 years[8], but after that time abatement costs have been reduced substantially because of learning by doing to such an extent that even lower abatement costs do not give much additional benefit anymore. The SBL is therefore sharply declining towards

[8] Note that the panel displays the growth-adjusted SBL. Hence, an upward-sloping time path of this SBL implies that the SBL grows at a higher rate than the economy.

zero at the end of the century. Compared to the benchmark case, the optimal abatement rate is much higher initially, only to decline later on. But at the end of the century, it is still optimal to abate around 85% of emissions, which is much higher than the 53% abatement rate in the benchmark.

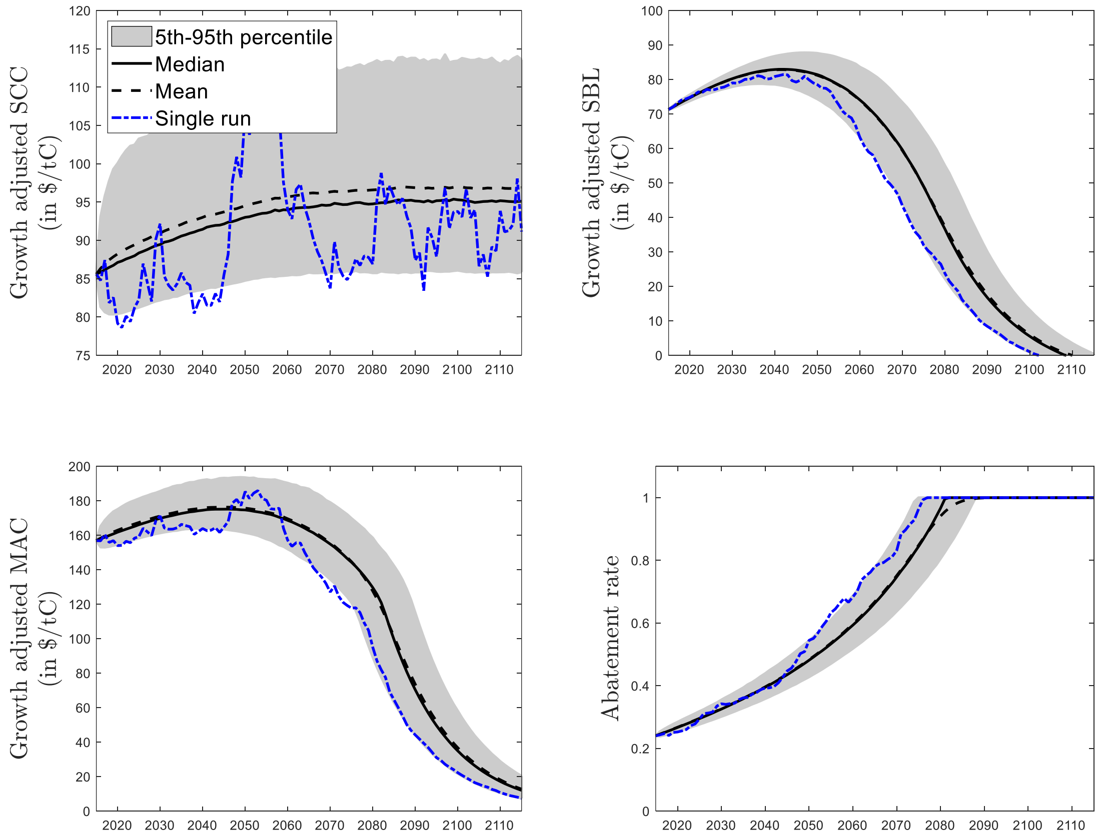
Figure 4: Learning by doing in abating emissions



4.3. Convex damages and learning by doing in abatement

Figure 5 shows that combining convex damages and learning by doing leads to an even stronger incentive for abating energy. The optimal carbon price is again very similar to the optimal carbon price without learning by doing. The social benefit of learning has a similar shape to the SBL in the linear case. However, it starts much higher and declines towards zero faster. Since damages are more severe, more abatement is optimal and lowering abatement costs by investing in knowledge is even more beneficial, which explains the higher initial level of the SBL. In this scenario it is optimal to fully decarbonize the economy around 2075, much earlier than in the previous scenarios. The main takeaway from the learning-by-doing simulations is that optimal abatement of emissions is understated if learning-by-doing externalities are not internalized.

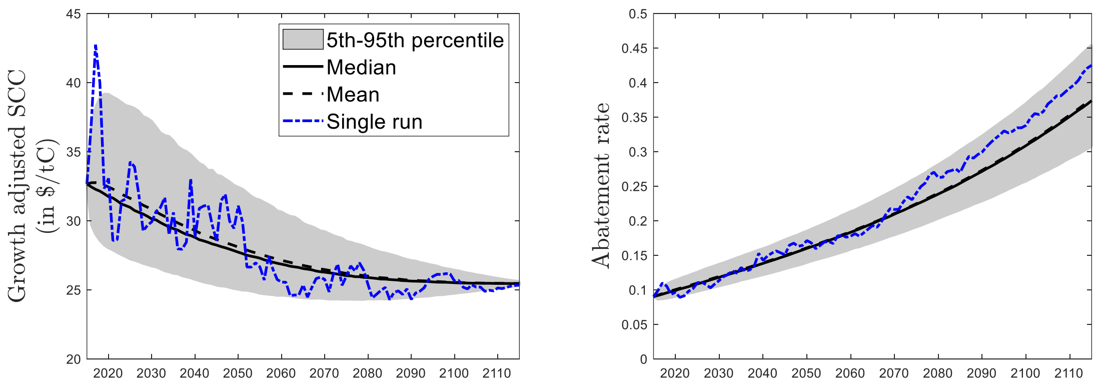
Figure 5: Convex damages and learning by doing



4.4. Gradual resolution of damage uncertainty

Our third generalization is to allow for gradual resolution of damage uncertainty. More precisely, we let the annual volatility of the damage ratio fall to zero linearly in a century. This is a shortcut to capturing slow resolution of uncertainty without delving into the intricacies of learning. The left panel of Figure 6 indicates that the *expected* optimal path of carbon prices corrected for growth of the economy now falls over time, much more strongly than the modest decline shown in the benchmark (see top right panel of Figure 1). We find that the carbon price does not only grow much more slowly than the economy, but also starts at only \$33/tC instead of \$44/tC. The fact there is a declining uncertainty about the damage ratio means that policy makers can pursue a less vigorous climate policy than in the benchmark. Declining volatility in the future already has an impact on the optimal carbon price today. This implies that the mitigation rate rises in a century to only 38% compared to 53% in the benchmark. Note that if there is no or very little growth in the economy, the optimal carbon price would decline over time as found by Daniel et al. (2019) for a 7-period model for integrated assessment of economy and the climate. The general point is that gradual resolution of damage uncertainty slows down the rate of growth of the optimal carbon price. In a more formal context of learning and resolution of uncertainty, Gerlagh and Liski (2016) show that this also tends to slow down the rise in optimal carbon prices. We should note that the impact of earlier resolution of uncertainty is reversed when ε, the IES parameter, is smaller than the inverse of the risk aversion parameter γ. But there is strong empirical support for our assumption $\varepsilon >> 1/\gamma$.

Figure 6: Gradual resolution of damage uncertainty



4.5. Climatic and economic tipping points

Our fourth generalization is to allow for climatic and economic tipping points. There is a growing literature on the effects of various stochastic tipping points on optimal climate policy (e.g. Lemoine and Traeger, 2014, 2016b; van der Ploeg and de Zeeuw, 2018; Cai and Lontzek, 2019). Most of these studies are quite challenging from a numerical point of view. Here we simply present the effects (relative to our benchmark) of two illustrative tipping points.

We first present a single climatic tipping point for which we assume that there is a risk of a regime shift in which the transient temperature response to cumulative emissions suddenly jumps up from 1.8 °C/TtC to 2.5 °C/TtC. Moreover, we assume the arrival rate to be higher at higher temperatures: the initial hazard of this tip at the initial temperature of 1 °C is 0.006, which implies an expected arrival time of 167 years, but for every increase in temperature by 1 °C we let the hazard rate rise by a further 0.006. This means that at 3 °C the hazard is 0.018 and the mean arrival time for the catastrophe is only 56 years. Although these small risks are likely to occur in the very distant future, they have consequences on optimal climate policy now already, as can be seen by comparing Figure 7 with Figure 1. We see that the mitigation rate in a century time increases from 53% to 60%. Furthermore, the mean optimal carbon price now starts somewhat higher at $48/tC than in the benchmark and then rises over time. Hence, the mitigation rate ends up higher after a century, at 60% instead of 53%. The blue lines indicate a sample path with the tipping point occurring in 2045. At that time, the carbon price jumps up substantially because of the bigger climate challenge resulting from the increased sensitivity of temperature to cumulative emissions.

Figure 7: Risk of a climatic tipping point

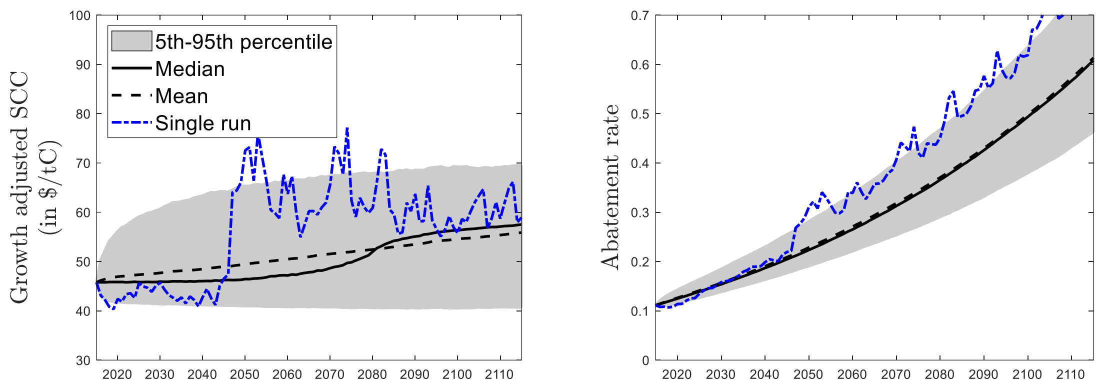
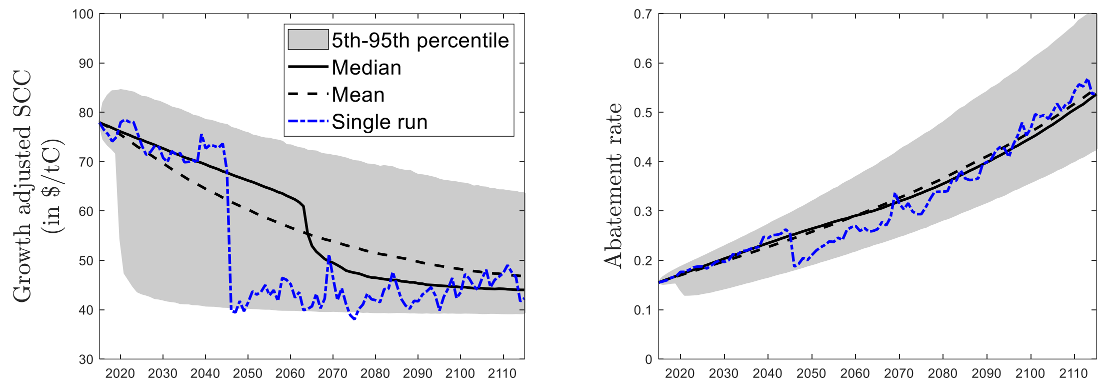


Figure 8 gives the optimal policy simulations for a different type of tipping point, namely one that leads to a higher effect of global warming on damages instead of increased temperature sensitivity. We assume that the size of the economy drops abruptly by 2.5% once this tipping point occurs. The initial hazard of this tip at initial temperature is 0.01, which implies an expected arrival time of 100 years. For each increase in temperature by 1 °C, the hazard rate is assumed to rise by 0.01. Hence, at 3 °C the hazard is 0.03 and the mean arrival time for the tipping point goes down to 33 years. This economic tipping point is thus expected to occur more rapidly than the climate tipping point. The most striking feature is that for this tipping point, the initial carbon price is much higher than in the benchmark, i.e., $78/tC instead of $44/tC, but that the mean and median paths of the optimal carbon price corrected for growth of the economy fall strongly over time. The blue line indicates a sample run where the tipping point occurs in 2045. At that time, the carbon price drops down instantaneously and, as a result, the mitigation rate drops down at that time too. The intuition behind this drop is obvious: initially, a large fraction of the carbon price is reflecting the urgency of preventing the tipping point. A higher carbon price leads to more mitigation efforts and therefore a lower probability of the tipping. But when despite these additional abatement efforts, the system tips eventually, there are no further tipping points to prevent. Moreover, after the tip has occurred the economy is smaller because of the sudden increase in damages. Furthermore, the social costs of carbon are proportional to output, which is another factor behind the drop in the SCC after the damage catastrophe occurs. The benefit of carbon reduction after the tip is the same as the benefit in the benchmark model without the tipping point for the same level of output.

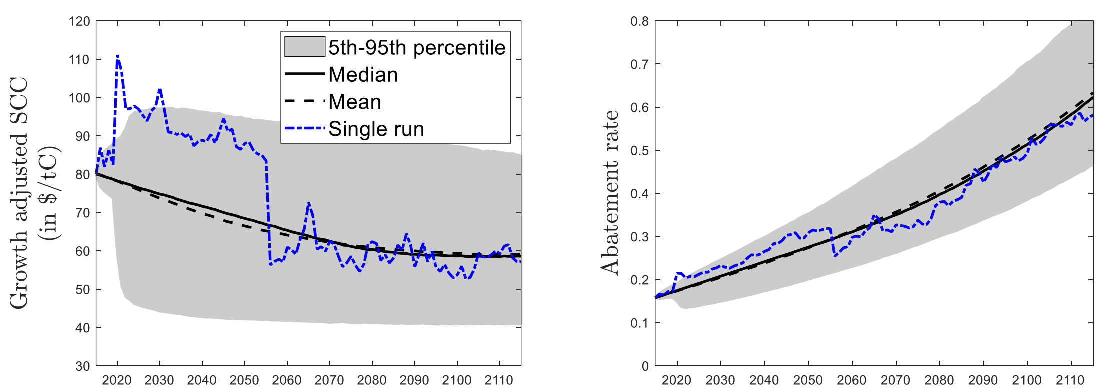
This is an important point: a tipping point in the climate system that speeds up warming or leads to a slower decay of carbon emissions has very different implications than a tipping point that directly affects the economy. In the former case abatement efforts can be higher before the tip to prevent tipping, but when the system tips eventually abatement efforts jump up even further since one unit of emissions now leads to more global warming. The expected growth adjusted carbon price is therefore growing faster than economic growth. In the latter case if an economic tipping point, abatement efforts before tipping are also higher than in the absence of a tipping point to prevent tipping, but once the damage tipping point has happened, the economy is actually smaller in the future and the carbon price jumps down since damages are still proportional to the economy.

Figure 8: Risk of a tipping point affecting the size of the economy



We can also combine both types of tipping points in a single simulation. Figure 9 shows a sample path in which the climate tipping point tips very early and in which the economic tipping point tips around 2055. The initial carbon price is equal to 80 $/tC. The left panel indicates that the declining effect of the economic tipping point dominates the increasing effect of the climate tipping point. However, the growth-adjusted carbon price or social cost of carbon is now much flatter compared to the left panel of Figure 8. Abatement efforts are higher when both tipping points are present; the optimal abatement rate is 63% after a century.

Figure 9: Risk of two tipping points affecting respectively the climate system and the size of the economy

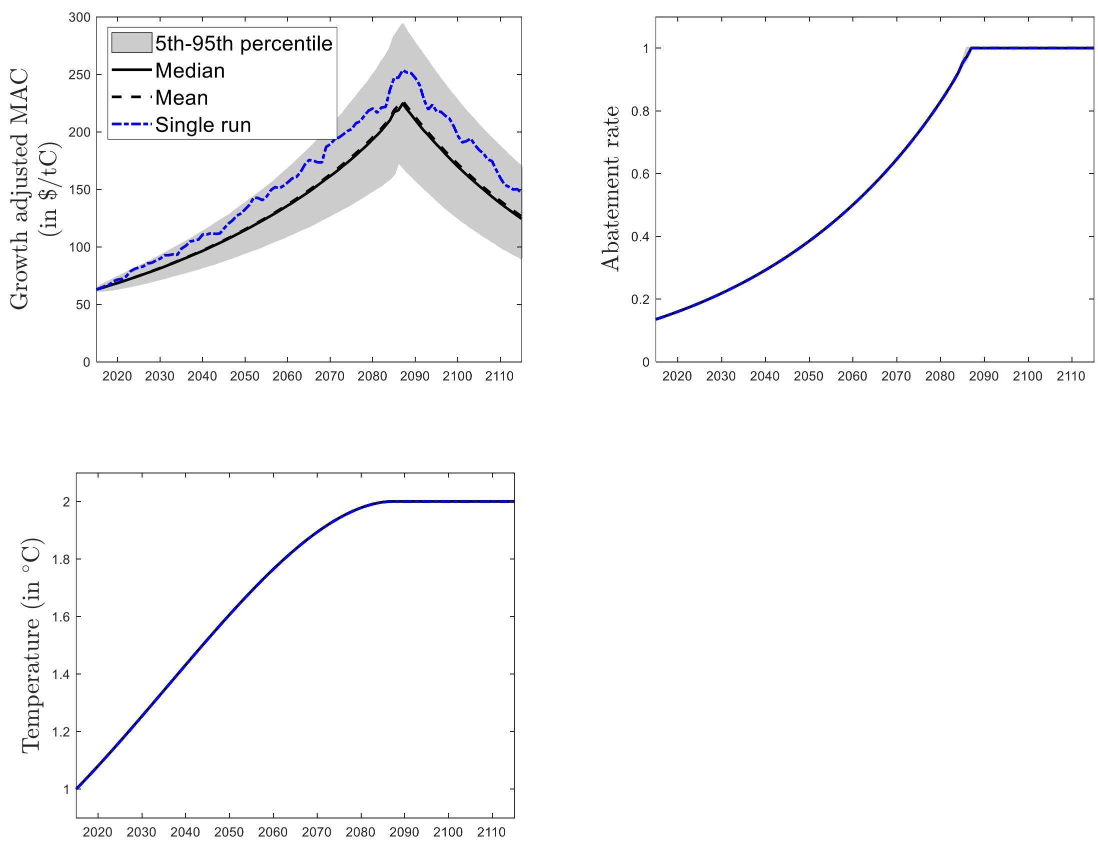


Of course, in practice, the impact of a tipping point may take a long time to materialize (e.g. van der Ploeg and de Zeeuw, 2018; Cai and Lontzek, 2019). We have abstracted from this, but protracted effects of tipping points are clearly important in terms of the resulting time path of optimal policy, which will change more gradually. It is also important to allow for cascading tipping points where the onset of one tip might increase the likelihood of another tipping point occurring, by more than implied by the temperature-dependence of the hazard rate (Lemoine and Traeger, 2016b; Cai et al., 2016). In particular, the downward jump after the damage tip occurs will be smaller in that case since there is the remaining incentive to delay future tipping points.

4.6. Temperature caps

Although most economists have adopted a welfare-maximizing approach where policymakers internalize the global warming externalities, many governments (as well as central banks and the Network of Greening the Financial System) have followed the IPCC and have decided that the best way to deal with global warming is to have a ceiling on global mean temperature.

Figure 10: Effects of temperature cap and no damages of 2 degrees Celsius



Given that temperature increases with cumulative emissions, the optimal carbon price must then grow at a rate that is equal to the risk-adjusted interest rate which is in our calibration equal to 3.4%.[9] In Figure 10 we show the optimal climate policies when a cap on global mean temperature of 2 °C is implemented and where we abstract from damages to global warming. The top left panel indicate a rapid rise in both the median carbon price and in the median carbon price even when adjusted for growth rate of the economy. The initial carbon price is about a fifth higher than in the benchmark, but the carbon price grows much faster than the growth of the economy. In fact, we numerically confirm result (12)-(14) that the expected growth rate of the carbon price and the marginal abatement cost indeed equals the risk-free interest rate plus the risk premium. This steep growth in carbon prices ensures a rapid rise in the abatement rate and quick decarbonization of the economy (top right panel). Hence, temperature is much lower in a century: 2 °C instead of 3 °C (bottom left panel). With a tighter cap of say 1.5 °C, abatement efforts must be stepped a lot more, which is induced via much higher carbon prices, and as a result the transition to the carbon-free economy occurs more rapidly (see Appendix E).

Figure 11: Optimal policy functions for the abatement rate as function of temperature

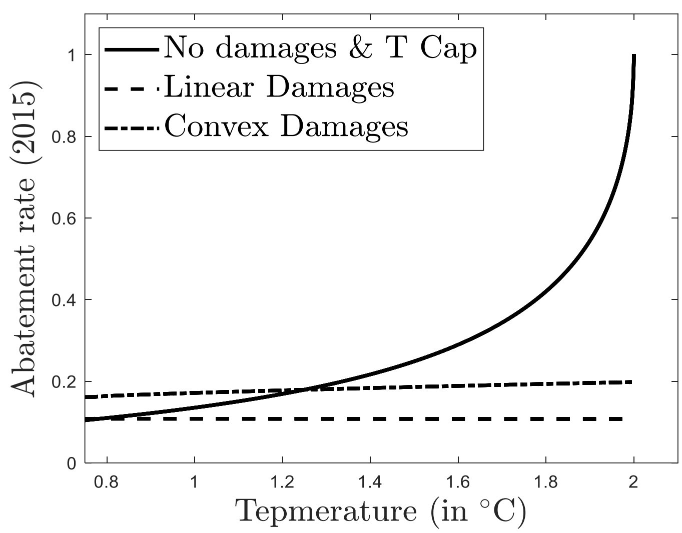


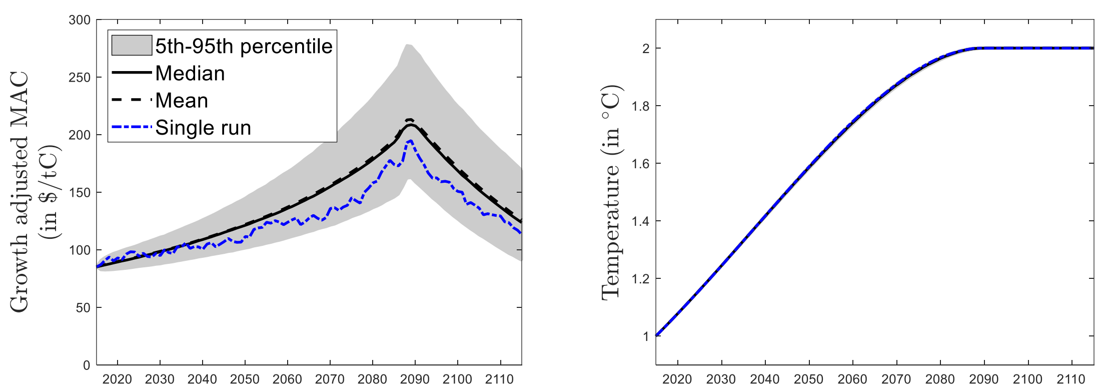
Figure 11 shows the optimal policy function for the abatement rate in state space, so as a function of temperature. The solid line shows that the abatement rate increases more and more rapidly in the direction of 100% mitigation as the temperature of cap of 2 degrees Celsius is approached. This very nonlinear feature is necessary to ensure that temperature stays below its cap. One can see that the corresponding optimal policy function for the benchmark case of linear damages (section 3) is flat. The optimal policy function for the case of convex damages (section 4.1) is, of course, much

[9] This is close to the 3.5% per year recommended by Gollier (2020) for the risk-adjusted interest rate.

higher and slopes gently upwards as the convexity of damages kicks in. Although the policy function with convex damages starts higher than the one with a temperature cap, it rapidly is overtaken as temperature increases.

Figure 12 plots the optimal climate policies under a 2 °C cap when there are also linear damages from global warming to aggregate production. We then find that the growth rate of the optimal path of carbon prices is somewhere in between the risk-adjusted rate of interest and the rate of economic growth (cf. van der Ploeg, 2018). Postponing abatement can be more cost-efficient due to discounting and technological progress in abatement technology, but that also leads to more warming and therefore more damages. The initial price, with both a temperature cap and damages, is therefore higher (90$/tC compared to 60$/tC without damages) and the growth rate lower.

Figure 12: Effects of a 2 degrees Celsius temperature cap with damages



Finally, we calculate the welfare losses relative to business as usual. The welfare loss of business as usual relative to the optimum outcome for the benchmark case of linear damages (section 3) is 0.2%. The welfare loss of enforcing a temperature cap of 2 degrees Celsius relative to the optimal outcome is also 0.2%. However, with convex damages (section 4.1), the welfare loss of business as usual relative to the optimal outcome rises to 0.8%. As a result, the welfare loss of a 2 degrees Celsius temperature cap is only 0.1%. Hence, in the benchmark case with linear damages, the damages are so small that business usual and a temperature cap give roughly the same welfare loss. In the more realistic scenario with convex damages the ambitious climate policy of enforcing a 2 degrees Celsius cap is a lot less costly than doing nothing as under business as usual. If we would also take account of tipping points (section 4.5), the difference will be even bigger. Hence, we conclude that it is better to undertake too much climate action than too little or not at all.

5. Discussion

Overall, most cases discussed in section 4 imply that the growth rate of the carbon price should be at least as high as the rate of economic growth. Combining knowledge from all simulations we conclude that as a rule of the thumb the optimal carbon price should grow at a rate that is in between the growth rate of the economy and the risk-free rate plus the risk premium. In this penultimate section, we wish to go beyond our formal modelling. We first discuss some other arguments that have been made in favour of frontloading carbon prices and then discuss the wider implications of commitment to a growing carbon price for business.

Our optimal policy simulations under a wide range of uncertainties and tipping points suggest that it is a best to have a steadily rising path of carbon prices. However, it has been argued that there may be the need for an upfront carbon spike in carbon prices followed by a decline in carbon prices if there are learning-by-doing effects in the production of renewable energies (e.g. Daniel et al., 2019). This is not strictly right, since the carbon price that is put forward is, in fact, a combination of a gradually rise in carbon prices and a temporary spike in renewable energy subsidies. To get the right economic incentives, we need to separate these two policies, since they each deal with different market failures, namely the global warming externality and the learning-by-doing externality. Hence, even though learning by doing externalities in the production of renewable energies warrant an upfront subsidy to speed up the green transition, they do not require a spike in carbon prices (cf. Bovenberg and Smulders, 1996; Goulder and Mathai, 2000; Popp, 2004; Acemoglu et al., 2012; Rezai and van der Ploeg, 2017a).

In models with fixed reserves of exhaustible fossil fuel, intertemporal arbitrage implies that a constant tax on carbon emissions, squeezes rents of the fossil fuel barons and has no effects whatsoever on the time profile of emissions. However, it has been argued that expectations of *falling* carbon taxes do postpone emissions and limit damages from global warming (Sinclair, 1992; Daniel et al., 2019). However, this result relies on some implausible features and the optimal carbon prices typically either rise all the way or rise before they fall (Ulph and Ulph, 1994).

The pattern of a rising optimal carbon price occurs in almost every integrated assessment of the economy and global warming. If on top of the normal growth uncertainty, risk of macroeconomic disasters and uncertainty about the damage ratio highlighted in our model, account is taken of climatic forms of uncertainty such as in the carbon stock and temperature dynamics (e.g. van den Bremer and van der Ploeg, 2021) or about tipping of the Greenland or Antarctic Ice Sheet or reversal of the Gulf Stream (e.g. Cai and Lontzek, 2020), the optimal response is a rising path of carbon prices. If integrated assessment models are extended to allow for long-run risk in economic growth with temperature-induced tail risks, the temperature risk premium increases with temperature (Bansal and Yaron, 2004; Bansal et al., 2016) and it is even more difficult to get a declining carbon price. Adding to all this, Olijslagers and van Wijnbergen (2019) show that ambiguity aversion (i.e. the aversion of unmeasurable or Knightian uncertainty) has a major impact on the optimal carbon price: the direct effect on the aversion-adjusted valuation of future income flows substantially exceeds the effect ambiguity aversion also has in the opposite direction because it also increases the appropriate discount rate. As one might have expected given the worst-case assumption that optimality requires one to take when faced with the multiple-priors framework (Gilboa et al., 1989). If one allows for learning after a tipping point upon which it becomes known that the climate sensitivity has increased or carbon sinks have been weakened, it has been shown that the optimal response is to have a rising path of carbon prices before and a rising but *higher* path after the tipping point (Lemoine and Traeger, 2014, Figure 4, Panel D).

Finally, we want to stress from another angle the importance of policy makers very clearly committing to the right carbon price time path early on; any uncertainty about future carbon prices in the presence of irreversible capital accumulation is an incentive to postpone investment by conferring an option value to waiting strategies (Dixit and Pindyck, 1994)

6. Conclusion

We have shown that convex damages, tipping points and temperature caps all argue in favour of a rising path of carbon prices. Only if there is gradual resolution of uncertainty will there be a declining component in the optimal carbon price, but this effect is dominated by rising components if damages and the economy are growing at empirically plausible rates. Furthermore, convex damages and especially temperature caps require that the carbon prices grow at a faster rate than the economy. Our policy recommendation is therefore that decision makers should start with a significant carbon price and at the same time commit to a steadily rising path of carbon prices. This rising path of carbon prices can, if required by learning-by-doing externalities, be supplemented with a temporary upfront spike in renewable energy subsidies. These two policies give the best guarantee for redirecting investments from carbon-intensive to green technologies.

Only by credibly committing to such a path are corporations going to make the long run and irreversible investments that are needed to transition to the carbon-free economy. Uncertainty about future prices and about the timing of a transition will cause corporations to hold back investments as carbon-intensive capital stock then acquires an option value in the likely case that capital investment is irreversible so avoiding unnecessary volatility is extremely important. A practical problem that must be dealt with is that politicians tend to procrastinate and postpone carbon pricing and prefer subsidies to higher carbon prices as they fear of losing office. This can lead to adverse Green Paradox effects, where the anticipation of a stepping up of climate policy induces owners of fossil fuel reserves to extract more quickly and accelerate emissions and global warming rather than slowing it down (Sinn, 2012; van der Ploeg and Withagen, 2015; Rezai and van der Ploeg, 2017b). Such political distortions might prevent the path of carbon prices be not high enough upfront. Credible commitment to a steadily rising path of prices is thus of paramount importance.

Acknowledgments

We thank Patrick Bolton, Christoph Hambel, Svenn Jensen, Holger Kraft, Armon Rezai and Christian Traeger for helpful comments on an earlier version of this paper. Olijslagers and van Wijnbergen have nothing to disclose except the pension administrator and wealth manager MN for financial support. Van der Ploeg has nothing to disclose.

References

Acemoglu, D., P. Aghion, L. Bursztyn, and D. Hemous (2012). The environment and directed technical change, *American Economic Review*, **102**, 131-162.

Bansal, R., and A. Yaron (2004). Risk for the long run: a potential resolution of asset pricing puzzles, *Journal of Finance,* 59, 4, 1481-1509.

Bansal, R., M. Ochoa, and D. Kiku (2016). Price of long-run temperature shifts in capital markets, Duke University.

Barro, R.J. (2006). Rare disasters and asset markets in the twentieth century, *Quarterly Journal of Economics*, 121, 3, 823-866.

Barro, R.J. (2009). Rare disasters, asset prices, and welfare costs, *American Economic Review*, 99, 1, 243-264.

Barro, R.J., and T. Jin (2011). On the size distribution of macroeconomic disasters, *Econometrica*, **79**, 5, 1567-1589.

Binsbergen, J.H. van, J. Fernandez-Villaverde, R.S.J. Koyen, J. Rubio-Ramirez (2012). The term structure of interest rates in a DSGE model with recursive preferences, *Journal of Monetary Economics*, **59**, 634-648.

Bovenberg, A.L., and J. Smulders (1995). Environmental quality and pollution-augmenting technical change in a two-sector endogenous growth model, *Journal of Public Economics*, 57, 3, 369-391.

Bovenberg, A.L., and J. Smulders (1996). Transitional impacts of environmental policy in an endogenous growth model, *International Economic Review,* 37, 4, 861-893.

Bremer, T.S., and F. van der Ploeg (2021). The risk-adjusted carbon price, *American Economic Review*, forthcoming.

Cai, Y., T. Lenton, and T. Lontzek (2016). Risk of multiple climate tipping points should trigger a rapid reduction in CO2 emissions, *Nature Climate Change*, **6**, 520-525.

Cai, Y., T.S. Lontzek (2019). The social cost of carbon with economic and climate risks, *Journal of Political Economy*, **127**, 6, 2684-2734.

Crost, B., and C.P. Traeger (2013). Optimal climate policy: uncertainty versus Monte Carlo", *Economics Letters*, **120**, 3, 552-558

Crost, B., and C.P. Traeger (2014). Optimal CO2 mitigation under damage risk valuation, *Nature Climate Change*, **4**, 631-636.

Daniel, K.D, R.B. Litterman, and G. Wagner (2019). Declining CO_2 price paths, *Proc. Natl. Acad. Sci. U.S.A.* **116**, 20866-20891.

Dietz, S., and N.H. Stern (2015). Endogenous growth, convexity of damages and climate risk: how Nordhaus' framework supports deep cuts in emissions, *Economic Journal*, **125**, 574-620.

Dimson, E., P. Marsh, and M. Staunton (2011). Equity premia around the world, London Business School. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1940165

Dixit, R.K. and R.S. Pindyck (1994). *Investment under Uncertainty*, Princeton University Press.

Duffie, D., L.G. Epstein (1992). Stochastic differential utility, *Econometrica*, **60**, 2, 353-394.

Epstein, L.G., S.E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework, *Econometrica*, **57**, 937-969.

Epstein, L.G. S.E. Zin (1991). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: an empirical framework, *Journal of Political Economy*, **99**, 263-286.

Epstein, L.G., E. Farhi, T. Strzalecki (2014). How much would you pay to resolve long-run risk, *American Economic Review*, 104, 9, 2680-2697.

Gerlagh, R., and M. Liski (2016). Carbon prices for the next hundred years, *Economic Journal*, 128, 609, 728-757.

Gilboa, I., and D. Schmeidler (1989). Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, **18**, 141-153

Gollier, C. (2002a). Discounting an uncertain future, *Journal of Public Economics*, **85**, 2, 149-166b

Gollier, C. (2002a). Time horizon and the discount rate, *Journal of Economic Theory*, **107**, 463-473.

Gollier, C. (2008). Discounting with fat-tailed economic growth, *Journal of Risk and Uncertainty*, **37**, 171-186.

Gollier, C. (2011). On the underestimation of the precautionary effect in discounting, *Geneva Risk and Insurance Review*, **36**, 2, 95-11.

Gollier, C. 2012. *Pricing the Planet's Future: The Economics of Discounting in an Uncertain World*, Princeton University Press, Princeton, New Jersey.

Gollier, C. (2017). *Ethical Asset Valuation and the Good Society*, Columbia University Press.

Gollier, C. (2020). The cost-efficiency carbon pricing puzzle, Toulouse University.

Hambel, C., H. Kraft, and F. van der Ploeg (2020). Asset diversification versus climate policy, Frankfurt University.

Jensen, S., and C.P. Traeger (2014). Optimal climate change mitigation under long-term growth uncertainty: stochastic integrated assessment and analytic findings, *European Economic Review*, **69**, 104-125.

Jensen, S. and C.P. Traeger (2020). Pricing climate risk, University of Oslo.

Kelly, D.L., and C.D. Kolstad (1999). Bayesian learning, pollution, and growth, *Journal of Economic Dynamics and Control*, **23**, 4, 491-518.

Kelly, D.L., and Z. Tan (2015). Learning and climate feedbacks: optimal climate insurance and fat tails, *Journal of Environmental Economics and Management*, **72**, 98-122.

Goulder, L., and K. Mathai (2000). Optimal CO2 abatement in the presence of induced technical change, *Journal of Environmental Economics and Management*, **39**, 1-38.

Lemoine, D. (2021). The climate risk premium: how uncertainty affects the social cost of carbon, *Journal of the Association of Environmental and Resource Economists*, 8, 1,

Lemoine, D., and C. Traeger (2014). Watch your step: optimal policy in a tipping climate, *American Economic Journal: Economic Policy*, **6**, 1, 137-166.

Lemoine, D.M., and C.P. Traeger (2016a). Ambiguous tipping points, *Journal of Economic Behavior and Organization*, **132**, B, 5-18.

Lemoine, D.M., and C.P. Traeger (2016b). Economics of tipping the climate dominoes, *Nature Climate Change*, **6**, 514-519.Matthews, H.D., N.P. Gillett, P.A. Stott, K. Zickfeld (2009). The proportionality of global warming to cumulative carbon emissions, *Nature* **459**, 829-832.

Nordhaus. W. (2017). Revisiting the Social Cost of Carbon, *Proceedings of the National Academy of Sciences*, **114**, 7, 1518-1523.

Nordhaus, W. (2018). Evolution of modeling of the economics of global warming: changes in the DICE model, 1992-2017, *Climatic Change*, **148**, 623-640.

Olijslagers, S. (2020). Solution methods for DSGE models in continuous time: application to a climate-economy model, University of Amsterdam.

Olijslagers, S. and S. van Wijnbergen (2019). Discounting the future: on climate change, ambiguity aversion and Epstein-Zin preferences, D.P. 2019-030/VI, Tinbergen Institute.

Pigou, A. (1920). *The Economics of Welfare*, MacMillan.

Ploeg, F. van der (2018). The safe carbon budget, *Climatic Change*, **147**, 470-59.

Ploeg, F. van der, and C. Withagen (2015). Global warming and the Green Paradox: a review of adverse effects of climate policy, *Review of Environmental Economics and Policy*, **9**, 2, 285-303.

Ploeg, F. van der, and A.J. de Zeeuw (2018). Climate tipping and economic growth: precautionary capital and the social cost of carbon, *Journal of the European Economic Association*, 16, 5, 1577-1617.

Popp, D. (2004). ENTICE: endogenous technological change in the DICE model of global warming, *Journal of Environmental Economics and Management*, **48**, 742-768.

Rezai, A., and F. van der Ploeg (2017a). Abandoning fossil fuel: how fast and how much, *Manchester School*, **85**, S2, e16-e44.

Rezai, A., and F. van der Ploeg (2017b). Second-best renewable subsidies to decarbonize the economy: commitment and the Green Paradox, *Environmental and Resource Economics*, **66**, 409-434 (2017).

Sinclair, P.J.N. (1992). High does nothing and rising is worse: carbon taxes should keep declining to cut harmful emissions, *Manchester School of Economics*, **60**, 1, 41-52.

Sinn, H.W. (2012). *The Green Paradox: A Supply-Side Approach to Global Warming*, The MIT Press.

Stern, N.H. (2007). *The Economics of Climate Change – The Stern Review*, Cambridge University Press.

Traeger, C.P. (2020). ACE – Analytic Climate Economy (with temperature and uncertainty), University of Oslo.

Ulph, A. and D. Ulph (1994). The optimal time path of a carbon tax, *Oxford Economic Papers*, **46**, 857-868.

Vissing-Jørgensen, A., and O.P. Attanasio (2003). Stock-market participation, intertemporal substitution, and risk aversion, *American Economic Review*, **93**, 2, 383-391.

Weitzman, M.L. (2009). Additive damages, fat-tailed dynamics and uncertain discounting, *Economics: The Open-Access, Open-Assessment E-Journal*, 3, 2009-39.

Weitzman, M.L. (1998). Why the far-distant discount rate should be discounted at its lowest possible rate, *Journal of Environmental Economics and Management*, **36**, 201-208.

Weitzman, M.L. (2007). Subjective expectations and asset-return puzzles, *American Economic Review*, **97**, 1102-1130.

Weitzman, M.L. (2009). On modeling and interpreting the economics of climate change, *Review of Economics and Statistics*, **91**, 1-19.

Weitzman, M.L. (2011). Fat-tailed uncertainty in the economics of catastrophic climate change, *Review of Environmental Economics and Policy*, **5**, 2, 275-292

Wijnbergen, S. van and T. Willems (2015). Optimal learning on climate change: why climate skeptics should reduce emissions, *Journal of Environmental Economics and Management*, **70**, 17-33.

Appendix A: Solving for optimal climate policy

Since we include two tipping points each of which can only tip once, we must solve four sub-problems. Define by $V_t^{1,1}$ the value function for the problem where both tipping points have already taken place. $V_t^{1,0}$ is the value function for the problem where the economic (or more precisely the endowment) tipping point has tipped but the climate tipping point has not tipped yet. $V_t^{0,1}$ is defined similarly. Lastly, $V_t^{0,0}$ is the value function before any of the two tipping points have taken place. Each of the four sub-problems satisfies its own Hamilton-Jacobi-Bellman (HJB) equation. The HJB-equation for $V_t^{i,j}$, $i \in \{0,1\}, j \in \{0,1\}$ equals:

$$0 = \max_{u_t} \left\{ \begin{array}{c} f\left(C_t, V_t^{i,j}\right) + Z_Y^{i,j} \mu Y_t + \frac{1}{2} Z_{YY}^{i,j} (\sigma^Y Y_t)^2 + Z_t^{i,j} + Z_T^{i,j} \chi_j (1-u_t) E_t + Z_\omega^{i,j} \upsilon(\overline{\omega} - \omega_t) \\ + Z_X^{i,j} \mu_X \\ + \frac{1}{2} Z_{\omega\omega}^{i,j} \max\left[\left(\left(1 - \frac{t}{\overline{t}^\omega}\right) \sigma_0^\omega \right)^2, 0 \right] + \frac{1}{2} Z_X^{i,j} \sigma_X^2 \\ + \lambda_1 E\left[Z^{i,j}\left((1-J_1)Y_t, T_t, \omega_t, X_t, t\right) - Z^{i,j}(Y_t, T_t, \omega_t, X_t, t) \right] \\ + \mathbb{1}_{i=0} \lambda_2 T_t E\left[Z^{i+1,j}\left((1-J_2)Y_t, T_t, \omega_t, X_t, t\right) - Z^{i,j}(Y_t, T_t, \omega_t, X_t, t) \right] \\ + \mathbb{1}_{j=0} \lambda_3 T_t E\left[Z^{i,j+1}(Y_t, T_t, \omega_t, X_t, t) - Z^{i,j}(Y_t, T_t, \omega_t, X_t, t) \right] \end{array} \right\}$$

(A1)

subject to $u_t = 1$ if $T_t = T^{cap}$, where the value function $V_t^{i,j} = Z^{i,j}(Y_t, T_t, \omega_t, X_t, t)$ depends on the three state variables and time and its partial derivatives are denoted by subscripts. The term μ_X is equal to 1 in the benchmark case and equal to $u_t E_t$ if there is learning by doing in abatement.

We conjecture and have verified that for each *i* and *j* the value function is of the form $V_t^{i,j} = g_t^{i,j} {Y_t}^{1-\gamma}/(1-\gamma)$ with $g_t^{i,j} = h^{i,j}(T_t, \omega_t, X_t, t)$ and rewrite equation (A1) accordingly as,

$$0 = \min_{u_t} \left\{ \begin{array}{c} \beta\zeta \left(\left(g_t^{i,j}\right)^{-\frac{1}{\zeta}} \left(\frac{C_t}{Y_t}\right)^{1-\frac{1}{\eta}} - 1 \right) g_t^{i,j} + (1-\gamma)\left(\mu - \frac{1}{2}\gamma(\sigma^Y)^2 + \lambda_1 \frac{E\left[(1-J_1)^{1-\gamma}\right]-1}{1-\gamma} \right) g_t^{i,j} \\ + h_t^{i,j} + h_T^{i,j} \chi_j (1-u_t) E_t + h_\omega^{i,j} \upsilon(\overline{\omega} - \omega_t) + h_X^{i,j} \mu_X \\ + \frac{1}{2} h_{\omega\omega}^{i,j} \max\left[\left(\left(1 - \frac{t}{\overline{t}^\omega}\right) \sigma_0^\omega \right)^2, 0 \right] + \frac{1}{2} h_{XX}^{i,j} \sigma_X^2 \\ + \mathbb{1}_{i=0} \lambda_2 T_t \left(E[(1-J_2)^{1-\gamma}] g_t^{i+1,j} - g_t^{i,j} \right) \\ + \mathbb{1}_{j=0} \lambda_2 T_t \left(g_t^{i,j+1} - g_t^{i,j} \right) \end{array} \right\}$$

(A2)

subject to $u_t = 1$ if $T_t = T^{cap}$. We define the social cost of carbon as the welfare loss of emitting one unit of carbon divided by the instantaneous marginal utility of consumption, i.e.

(A3) $$SCC_t = -\chi \frac{\partial Z_t/\partial T_t}{f_C(C_t, V_t)} = \Omega^{i,j}(T_t, \omega_t, X_t, t) C_t^{1/\eta} Y_t^{1-1/\eta},$$

where $\Omega^{i,j}(T_t, \omega_t, X_t, t) \equiv \left(\frac{\chi}{(1-\gamma)\beta} \frac{h_T^{i,j}}{(g_t^{i,j})^{1-1/\zeta}} \right)$. The first part of equation (A3) indicates that the optimal SCC depends on the shape of the reduced-form value function. The second part indicates that it is proportional to the size of the economy.

The SBL corresponds to all the present and future marginal benefits in terms of lower mitigation costs resulting from using one unit of mitigation more today, i.e.

(A4) $$SBL_t = \frac{\partial Z_t/\partial X_t}{f_C(C_t, V_t)} = \Theta^{i,j}(T_t, \omega_t, X_t, t)\; C_t^{1/\eta} Y_t^{1-1/\eta},$$

where $\Theta^{i,j}(T_t, \omega_t, X_t, t) \equiv \left(\frac{\chi}{(1-\gamma)\beta} \frac{h_T^{i,j}}{(g_t^{i,j})^{1-1/\zeta}} \right)$.

The optimality of the abatement rate implies that u_t is chosen such that the MAC is equal to the sum of the SCC and the SBL. Abatement on the one hand leads to lower emissions and on the other hand lowers the costs for future abatement, which implies that $SCC_t + SBL_t = MAC_t$, where

(A5) $$MAC_t = -\frac{\partial C_t/\partial u_t}{E_t} = \frac{\frac{Y_t}{1+D_t}}{E_t} \frac{\partial A_t}{\partial u_t} = \frac{\frac{Y_t}{1+D_t}}{E_t} c_0 e^{-c_1 X_t} c_2 u_t{}^{c_2 - 1}.$$

The relation $SCC_t + SBL_t = MAC_t$ holds if the restriction $u \leq 1$ is not binding. If $u = 1$, then the sum of the SCC and the SBL will be larger than MAC, but it is not possible to abate more. The single control variable u_t thus tackles both externalities.

The main insight is that in more disaggregated models of energy use two separate policy instruments should be included. In that case carbon emissions should be priced at the *SCC* whilst mitigation should be subsidized at the *SBL*. We also refer to the SCC as the optimal carbon price and to the SBL as the optimal mitigation subsidy, while we note that this relation only holds as long as there is an interior solution to optimal abatement.

We also report the growth-adjusted quantities of the SCC, SBL and MAC to analyse the determinants of these variables other than economic growth. We define the growth-adjusted social cost of carbon by $SCC_t \frac{C_0^{1/\eta} Y_0^{1-1/\eta}}{C_t^{1/\eta} Y_t^{1-1/\eta}}$. This implies that the growth-adjusted social cost of carbon equals the first term of equation (A3): $\frac{\chi}{(1-\gamma)\beta} \frac{-h_T}{g_t^{1-1/\zeta}}$, but scaled with $C_0^{1/\eta} Y_0^{1-1/\eta}$ to make the initial social cost of carbon equal to the actual initial social cost of carbon. The growth adjusted *SBL* and *MAC* are defined in the same way.

Appendix B: A decentralized market economy

In the decentralized market economy, we need to consider energy producers, households, and the government separately. We assume that the households own the energy producers. We denote the consumer price for fossil fuel by p_t. Since fossil fuel and renewable energy are perfect substitutes, the consumer price for renewable energy is also equal to p_t. We let τ_t and s_t denote the specific tax on fossil fuel and the subsidy on renewable energy, respectively. Fossil fuel use is denoted by F_t and renewable energy use by R_t, so that the mitigation rate is defined by $u_t = \frac{R_t}{F_t+R_t}$. Total energy use is exogenous and equal to E_t. Profits of and lump-sum rebates to energy producers are denoted by Π_t and S_t, respectively. Profits of energy firms, the household budget constraint and the government budget constraint are given by

(B1) $\Pi_t = p_t F_t + p_t R_t - \tau_t F_t + s_t R_t - A(u_t, X_t)\frac{Y_t}{1+D_t},$

(B2) $\mathrm{C}_t = \frac{Y_t}{1+D_t} + \Pi_t - \tau_t F_t - p_t F_t - p_t R_t,$

(B3) $\mathrm{S}_t = \tau_t F_t - s_t R_t.$

Provided that it is not optimal to fully decarbonize the economy, the first-order optimality conditions for fossil fuel and renewable energy use are

(B4) $p_t = \tau_t - A_u(u_t, X_t)u_t(1-u_t)\frac{Y_t}{F_t(1+D_t)},$

(B5) $p_t = -s_t + A_u(u_t, X_t)u_t(1-u_t)\frac{Y_t}{R_t(1+D_t)}.$

Now use that $F_t = (1-u_t)E_t$ and $R_t = u_t E_t$ to obtain

(B6) $p_t = \tau_t - A_u(u_t, X_t)u_t\frac{Y_t}{E_t(1+D_t)},$

(B7) $p_t = -s_t + A_u(u_t, X_t)(1-u_t)\frac{Y_t}{E_t(1+D_t)}.$

Combining equations (B6) and (B7) gives

(B8) $\tau_t + s_t = A_u(u_t, X_t)\frac{Y_t}{E_t(1+D_t)}.$

Note that $MAC_t = A_u(u_t, X_t)\frac{Y_t}{E_t(1+D_t)}$. Imposing a carbon tax and a renewable energy subsidy implies that optimal policy is chosen such that the marginal abatement cost equals the sum of the carbon tax and the renewable energy subsidy. We can therefore replicate optimal policy of the command optimum by setting $\tau_t = SCC_t$ and $s_t = SBL_t$. In our setting, both the learning-by-doing and the climate-change externality are tackled by one policy instrument, i.e. the abatement rate u_t.

Appendix C: Numerical implementation

The HJB-equation is a set, of partial differential equations. We solve this system of partial differential equations using a finite-difference method. We can solve the model analytically when there are no climate damages. We use this as our initial guess at time $t_{max} = 500$, and from there solve the system backwards with time step $\delta_t = 1$. The three-dimensional grid is equally spaced with boundaries $\begin{bmatrix} T^{max} \\ \omega^{max} \\ X^{max} \end{bmatrix} = \begin{bmatrix} T^{cap} \\ 0.7 \\ 1000 \end{bmatrix}$ and $\begin{bmatrix} T^{min} \\ \omega^{min} \\ X^{min} \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0 \\ -25 \end{bmatrix}$. Without a temperature cap, T^{max} is set to 6 degrees Celsius. Specifically, we use an upwind semi-implicit finite-difference scheme. Optimal policy is calculated every period by solving for u_t such that the SCC is equal to the marginal abatement cost. If this requires $u_t > 1$, we set $u_t = 1$. The restriction of the temperature cap is implemented by imposing $u_t = 1$ on the boundary $T_t = T^{max}$. The restriction at the boundary also affects optimal policy at all interior grid points of temperature since it will affect the derivative of the value function with respect to temperature. A temperature cap will thus lead to a higher social cost of carbon and to a higher emissions control rate u_t. More details on the finite-difference method for a more general problem are given in Olijslagers (2021).

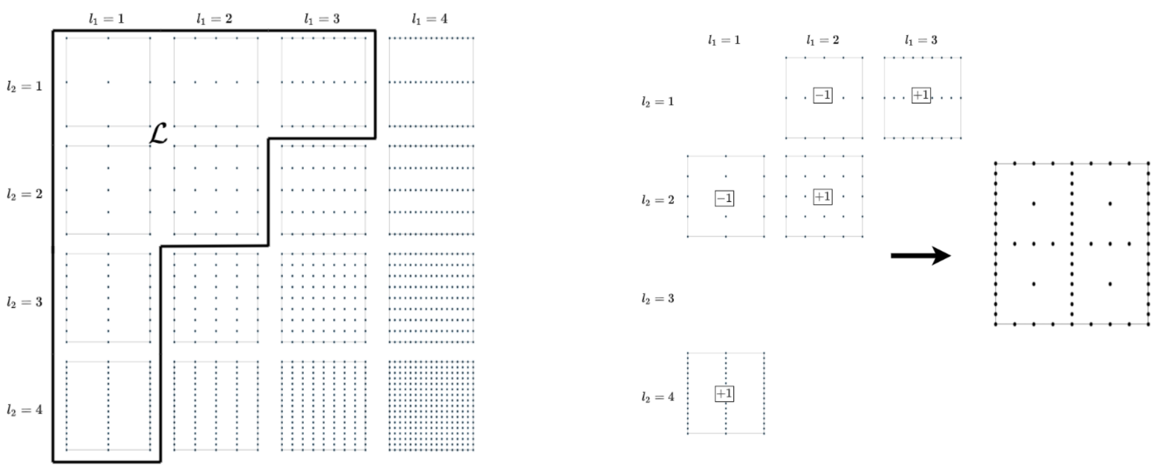
The reduce the computation time, we apply the sparse-grid combination method. The idea behind this method is to solve the problem on multiple smaller regular grids and then combine the results to obtain a solution on a sparse grid. Compared to applying a finite-difference method on an actual sparse grid directly, the combination method has several advantages. First, standard finite-difference methods on regular grids can be applied and hence this method is easier to implement. Second, all subproblems can efficiently be solved in parallel which significantly speeds up the computation.

Define the `level' of the grid for dimension i by $L_i, i \in \{T, \omega, X\}$. The number of grid points on the edge of the sparse grid in dimension i is equal to $2^{L_i} + 1$. The level therefore controls the amount of grid points and the accuracy in dimension i. When the value function is non-linear in a specific dimension it is possible to have more grid points in that dimension. This is for example useful when we solve the problem with a temperature cap, since in this case the value function becomes quite non-linear in the temperature dimension.

Let $\mathcal{L} = \left\{ l : \frac{l_T - 1}{L_T - 1} + \frac{l_\omega - 1}{L_\omega - 1} + \frac{l_X - 1}{L_X - 1} \leq 1 \right\}$ be the set of all admissible sub-grids where $l = (l_T, l_\omega, l_X)$. The weight of sub-grid l is equal to $w_l = \sum_{i_T=0}^{1} \sum_{i_\omega=0}^{1} \sum_{i_X=0}^{1} (-1)^{i_T + i_\omega + i_X} \mathbb{I}_{(l_T + i_T, l_\omega + i_\omega, l_X + i_X) \in \mathcal{L}}$. We solve for g on all subgrids that have a non-zero weight w_l. Note that all grids have different grid points. To find the approximation g_l on sub-grid l in a specific point, we use linear interpolation. We then combine the solutions on all sub-grids by summing over the product of the weight and the solutions: $g = \sum_{l \in \mathcal{L}} w_l \, g_l$.

Figure C1 shows an example of the sparse-grid combination method in two dimensions. In the example $L_1 = 3$ and $L_2 = 4$, so the sparse grid will be denser in the second dimension. First, the set $\mathcal{L}$ is constructed, which in this example consists of the following grids: $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (3,1)$. Of all grids within this set, the grids $(1,4), (2,2), (3,1)$ all have weight +1 and the grids $(1,2), (2,1)$ have weight -1. The other two grids have weight zero and therefore these do not have to be evaluated.

Figure C1: The sparse grid combination method



Appendix D: Derivation of the growth rate of marginal abatement costs with a temperature cap and no damages

If climate damages are not taken account of and a temperature cap is in place instead, it does not matter for the time at which the temperature cap is reached whether a unit of emissions is abated today or in some period in the future before that time, at least as long as the relationship between temperature and cumulative emissions is linear. Therefore, along the optimal path, a marginal increase of abatement today combined with a marginal decrease of abatement in the future should not lead to a change in welfare. The cost of a marginal increase of abatement today equals MAC_0, while the benefit of a marginal decrease of abatement in time t equals MAC_t. Optimal behaviour therefore implies that $\pi_0 MAC_0 = E_0[\pi_t MAC_t]$ where $\pi_t = \exp\left(\int_0^t f_V(C_s, V_s) ds\right) f_C(C_t, V_t)$ is the stochastic discount factor (Duffie and Epstein, 1992). We therefore must have that the product $\pi_t MAC_t$ is a martingale. Now calculate

(D1) $\frac{d\pi_t MAC_t}{\pi_t MAC_t} = \frac{d\pi_t}{\pi_t} + \frac{dMAC_t}{MAC_t} + \frac{d[\pi_t, MAC_t]}{\pi_t MAC_t}.$

Applying the martingale property and rearranging gives

(D2) $E_t\left[\frac{dMAC_t}{MAC_t}\right] = E_t\left[-\frac{d\pi_t}{\pi_t}\right] + E_t\left[-\frac{d[\pi_t, MAC_t]}{\pi_t MAC_t}\right],$

where $[\pi_t, MAC_t]$ denotes the quadratic covariation for the processes π_t and MAC_t. Note that the first term $E_t\left[-\frac{d\pi_t}{\pi_t}\right]$ is exactly equal to the real risk-free interest rate, while the second term is a risk premium related to the correlation between the stochastic discount factor and the marginal abatement costs. Equation (D2) implies that the optimal carbon price must grow at a rate equal to the sum of the real risk-free interest rate plus an interest premium to be determined (cf. Gollier, 2020). In the following we derive the risk-free rate and the risk premium in equation (D2).

Derivation of the risk-free rate (D20) and the risk premium (D22) for equation (D2)

We can work out the stochastic discount factor π_t and the marginal abatement cost function MAC_t. The model without climate damages can be written as follows. The endowment follow from

(D3) $dY_t = \mu Y_t dt + \sigma^Y Y_t dW_t^Y - J_1 Y_t dN_{1,t}.$

Consumption is equal to endowment minus abatement expenditure: $C_t = (1 - A_t)Y_t$, where the abatement cost function $A_t = c_0 e^{-c_1 X_t} u_t^{c_2}$. Define the consumption-endowment ratio $\xi_t = 1 - A_t = \nu(T_t, X_t, t)$, which depends on the two state variables and time. The two state variables X_t (abatement cost variable) and T_t (temperature) follow from

(D4)
$$\begin{aligned} dX_t &= \mu_X dt + \sigma_X dW_t^X, \\ dT_t &= \chi(1 - u_t)E_t dt. \end{aligned}$$

The temperature cap adds the restriction $u_t = 1$ if $T_t = T^{cap}$. The HJB-equation corresponding to the value function V_t for this problem is thus given by

$$0 = \max_{u_t}\left\{\begin{array}{c} f(C_t, V_t) + Z_Y \mu Y_t + \frac{1}{2} Z_{YY}(\sigma^Y Y_t)^2 + Z_t + Z_T \chi(1 - u_t)E_t \\ + Z_X \mu_X + \frac{1}{2} Z_{XX}\sigma_X^2 + \lambda_1 E\big[Z\big((1 - J_1)Y_t, T_t, X_t, t\big) - Z(Y_t, T_t, X_t, t)\big] \end{array}\right\}$$

(D5)

subject to $u_t = 1$ if $T_t = T^{cap}$, where the value function $V_t = Z(Y_t, T_t, X_t, t)$ depends on two state variables and time and its partial derivatives are denoted by subscripts. We conjecture and have verified that the value function is of the form $V_t = g_t Y_t^{1-\gamma}/(1 - \gamma)$ with $g_t = h(T_t, X_t, t)$ and rewrite equation (D5) accordingly as,

$$0 = \min_{u_t} \left\{ \begin{array}{c} \beta\zeta \left(g_t^{-\frac{1}{\zeta}} \left(\frac{C_t}{Y_t}\right)^{1-\frac{1}{\eta}} - 1 \right) g_t + (1-\gamma)\left(\mu - \frac{1}{2}\gamma(\sigma^Y)^2 + \lambda_1 \frac{E[(1-J_1)^{1-\gamma}]-1}{1-\gamma}\right) g_t \\ + h_t + h_T \chi(1-u_t)E_t + h_X \mu_X + \frac{1}{2} h_{XX}\sigma_X^2 \end{array} \right\}$$

(D6)

subject to $u_t = 1$ if $T_t = T^{cap}$.

The derivatives of instantaneous utility $f(C_t, V_t)$ can be calculated as

(D7)
$$f_C(C_t, V_t) = \frac{\beta C_t^{-1/\eta}}{((1-\gamma)V_t)^{\frac{1}{\zeta}-1}},$$
$$f_V(C_t, V_t) = \beta\zeta \left(\left(1 - \frac{1}{\zeta}\right) C_t^{1-\frac{1}{\eta}} ((1-\gamma)V_t)^{-\frac{1}{\zeta}} - 1 \right).$$

Now substitute in $V_t = \frac{g_t {Y_t}^{1-\gamma}}{1-\gamma}$ and $\xi_t = \frac{C_t}{Y_t}$ to obtain

(D8)
$$f_C(C_t, V_t) = \beta \xi_t^{-1/\eta} g_t^{1-1/\zeta} Y_t^{-\gamma},$$
$$f_V(C_t, V_t) = \beta\zeta \left(\left(1 - \frac{1}{\zeta}\right) \xi_t^{1-\frac{1}{\eta}} g_t^{-\frac{1}{\zeta}} - 1 \right).$$

Substituting this into the stochastic discount factor gives

(D9) $$\pi_t = \exp\left(\int_0^t \beta\zeta \left(\left(1 - \frac{1}{\zeta}\right) \xi_s^{1-\frac{1}{\eta}} g_s^{-\frac{1}{\zeta}} - 1 \right) ds \right) \beta \xi_t^{-\frac{1}{\eta}} g_t^{1-\frac{1}{\zeta}} Y_t^{-\gamma}.$$

Writing π_t as a differential equation gives

(D10) $$\frac{d\pi_t}{\pi_t} = \beta\zeta \left(\left(1 - \frac{1}{\zeta}\right) \xi_t^{1-\frac{1}{\eta}} g_t^{-\frac{1}{\zeta}} - 1 \right) dt + \frac{dY_t^{-\gamma}}{Y_t^{-\gamma}} + \frac{d g_t^{1-1/\zeta}}{g_t^{1-1/\zeta}} + \frac{d\xi_t^{-1/\eta}}{\xi_t^{-1/\eta}} + \frac{d[g_t^{1-1/\zeta}, \xi_t^{-1/\eta}]}{g_t^{1-1/\zeta}\xi_t^{-1/\eta}}.$$

Applying Ito's lemma to Y_t gives

(D11) $$\frac{dY_t^{-\gamma}}{Y_t^{-\gamma}} = -\gamma\left(\mu - \frac{1}{2}(\gamma+1)(\sigma^Y)^2\right) dt - \gamma\sigma^Y dW_t^Y + ((1-J_1)^{-\gamma} - 1) dN_{1,t}.$$

Similarly, we apply Ito's lemma to g_t to get

(D12) $$\frac{dg_t}{g_t} = \left(\frac{h_t}{g_t} + \frac{h_T}{g_t}\chi(1-u_t)E_t + \frac{h_X}{g_t}\mu_X + \frac{1}{2}\frac{h_{XX}}{g_t}\sigma_X^2 \right) dt + \frac{h_X}{g_t}\sigma_X dW_t^X.$$

Define $\mu_g = \frac{h_t}{g_t} + \frac{h_T}{g_t}\chi(1-u_t)E_t + \frac{h_X}{g_t}\mu_X + \frac{1}{2}\frac{h_{XX}}{g_t}\sigma_X^2$. Then we can calculate

(D13) $$\frac{dg_t^{1-1/\zeta}}{g_t^{1-1/\zeta}} = \left(1 - \frac{1}{\zeta}\right)\left(\mu_g - \frac{1}{2}\frac{1}{\zeta}\frac{h_X^2}{g_t^2}\sigma_X^2\right) dt + \left(1 - \frac{1}{\zeta}\right)\frac{h_X}{g_t}\sigma_X dW_t^X.$$

Using a similar derivation, we calculate that

(D14) $\frac{d\xi_t^{-1/\eta}}{\xi_t^{-1/\eta}} = -1/\eta \left(\mu_\xi - \frac{1}{2}(1+\frac{1}{\eta})\frac{\nu_X^2}{\xi_t^2}\sigma_X^2\right)dt - 1/\eta\frac{\nu_X}{\xi_t}\sigma_X dW_t^X,$

where $\mu_\xi = \frac{\nu_t}{\xi_t} + \frac{\nu_T}{\xi_t}\chi(1-u_t)E_t + \frac{\nu_X}{\xi_t}\mu_X + \frac{1}{2}\frac{\nu_{XX}}{\xi_t}\sigma_X^2$. The cross terms are equal to

(D15) $\frac{d[g_t^{1-1/\zeta}, \xi_t^{-1/\eta}]}{g_t^{1-1/\zeta}\xi_t^{-1/\eta}} = -1/\eta\left(1-\frac{1}{\zeta}\right)\frac{h_X\nu_X}{g_t\xi_t}\sigma_X^2 dt.$

Putting everything together yields

(D16) $\frac{d\pi_t}{\pi_{t-}} = \Bigg\{\beta\zeta\left(\left(1-\frac{1}{\zeta}\right)\xi_t^{1-\frac{1}{\eta}}g_t^{-\frac{1}{\zeta}} - 1\right) - \gamma\left(\mu - \frac{1}{2}(\gamma+1)(\sigma^Y)^2\right) + \left(1-\frac{1}{\zeta}\right)\Big(\mu_g -$

$\frac{1}{2}\frac{1}{\zeta}\frac{h_X^2}{g_t^2}\sigma_X^2\Big) - 1/\eta\left(\mu_\xi - \frac{1}{2}(1+\frac{1}{\eta})\frac{\nu_X^2}{\xi_t^2}\sigma_X^2\right) - 1/\eta\left(1-\frac{1}{\zeta}\right)\frac{h_X\nu_X}{g_t\xi_t}\sigma_X^2\Bigg\}dt - \gamma\sigma^Y dW_t^Y +$

$\left(1-\frac{1}{\zeta}\right)\frac{h_X}{g_t}\sigma_X dW_t^X - 1/\eta\frac{\nu_X}{\xi_t}\sigma_X dW_t^X + ((1-J_1)^{-\gamma} - 1)dN_{1,t}.$

We now substitute this in the HJB-equation. The HJB-equation is equivalent to

(D17) $\mu_g = -\beta\zeta\left(g_t^{-\frac{1}{\zeta}}\xi_t^{1-\frac{1}{\eta}} - 1\right) - (1-\gamma)\left(\mu - \frac{1}{2}\gamma(\sigma^Y)^2 + \lambda_1\frac{E[(1-J_1)^{1-\gamma}]-1}{1-\gamma}\right).$

Substituting this into the stochastic discount factor gives

(D17) $\frac{d\pi_t}{\pi_{t-}} = \Big\{-\beta - \frac{\mu}{\eta} + \frac{1}{2}\left(1+\frac{1}{\eta}\right)\gamma(\sigma^Y)^2 + (\gamma - \frac{1}{\eta})\lambda_1\frac{E[(1-J_1)^{1-\gamma}]-1}{1-\gamma} - 1/\eta\Big(\mu_\xi - \frac{1}{2}(1+$

$\frac{1}{\eta})\frac{\nu_X^2}{\xi_t^2}\sigma_X^2\Big) - \frac{1}{2}\frac{1}{\zeta}\left(1-\frac{1}{\zeta}\right)\frac{h_X^2}{g_t^2}\sigma_X^2 - 1/\eta\left(1-\frac{1}{\zeta}\right)\frac{h_X\nu_X}{g_t\xi_t}\sigma_X^2\Big\}dt - \gamma\sigma^Y dW_t^Y + \left(1-\frac{1}{\zeta}\right)\frac{h_X}{g_t}\sigma_X dW_t^X -$

$1/\eta\frac{\nu_X}{\xi_t}\sigma_X dW_t^X + ((1-J_1)^{-\gamma}-1)dN_{1,t}.$

We can thus define μ_π and σ_π such that

(D18) $\frac{d\pi_t}{\pi_{t-}} = \mu_\pi dt - \gamma\sigma dW_t^Y + \sigma_\pi dW_t^X + ((1-J_1)^{-\gamma}-1)dN_{1,t}.$

We can now first calculate the risk-free rate

(D19) $r_t = E_t\left[-\frac{d\pi_t}{\pi_t}\right] = -\mu_\pi - \lambda_1(E[(1-J_1)^{-\gamma}]-1) = \beta + \frac{\mu+\mu_\xi}{\eta} - \frac{1}{2}\left(1+\frac{1}{\eta}\right)\gamma(\sigma^Y)^2 -$

$\lambda_1\left(E[(1-J_1)^{-\gamma}] - 1 + \left(\gamma - \frac{1}{\eta}\right)\frac{E[(1-J_1)^{1-\gamma}]-1}{1-\gamma}\right) - \frac{1}{2}\frac{1}{\eta}(1+\frac{1}{\eta})\frac{\nu_X^2}{\xi_t^2}\sigma_X^2 - \frac{1}{2}\frac{1}{\zeta}\left(\frac{1}{\zeta}-1\right)\frac{h_X^2}{g_t^2}\sigma_X^2 -$

$1/\eta\left(\frac{1}{\zeta}-1\right)\frac{h_X\nu_X}{g_t\xi_t}\sigma_X^2$.

The effect of abatement uncertainty σ_X on the interest rate is negligible compared to the effect of economic uncertainty σ^Y and jump risk. We can simplify the interest rate if we leave out the terms related to abatement uncertainty. Additionally, we can calculate the expectations of the jump variable J_1, since J_1 follows a power distribution. Lastly, note that the growth rate of consumption, which we call μ_c, is equal to $\mu + \mu_\xi$. The real risk-free rate is thus given by

(D20) $r_t = \beta + \frac{\mu_c}{\eta} - \frac{1}{2}\left(1 + \frac{1}{\eta}\right)\gamma(\sigma^Y)^2 - \lambda_1\left(\frac{\alpha_1}{\alpha_1 - \gamma} - 1 - \frac{\gamma - \frac{1}{\eta}}{\alpha_1 + 1 - \gamma}\right).$

Marginal abatement costs are given by:

(D21) $MAC_t = -\frac{\frac{\partial C_t}{\partial u_t}}{E_t} = \frac{Y_t}{E_t}\frac{\partial A_t}{\partial u_t} = \frac{Y_t}{E_t} c_0 e^{-c_1 X_t} c_2 {u_t}^{c_2 - 1}.$

If we again assume that abatement uncertainty has a negligible effect on the risk premium, we obtain the risk premium

(D22) $rp_t = E_t\left[-\frac{d[\pi_t, MAC_t]}{\pi_t MAC_t}\right] = E_t\left[-\frac{d\left[Y_t^{-\gamma}, Y_t\right]}{Y_t^{1-\gamma}}\right] = \gamma(\sigma^Y)^2 + \lambda_1(E[(1 - J_1)^{-\gamma}] + E[1 - J_1] - E[(1 - J_1)^{1-\gamma}] - 1) = \gamma(\sigma^Y)^2 + \lambda_1\left(\frac{\alpha_1}{\alpha_1 - \gamma} + \frac{\alpha_1}{\alpha_1 + 1} - \frac{\alpha_1}{\alpha_1 + 1 - \gamma} - 1\right).$

In expectation, the growth rate of marginal abatement costs is therefore equal to the risk-free rate plus the risk premium.

Appendix E: Simulation results with a temperature cap of 1.5 degrees Celsius

Figure E1 and E2 shows the effects on the optimal time path of the carbon price and the growth-corrected carbon price, the abatement rate and temperature for the situation when policy makers face a 1.5 cap and no damages to the economy from global warming and when they a cap with damages from global warming, respectively. In addition, Figure E1 shows the policy function for carbon prices against temperature.

Figure E1: Effects of temperature cap and no damages of 1.5 degrees Celsius

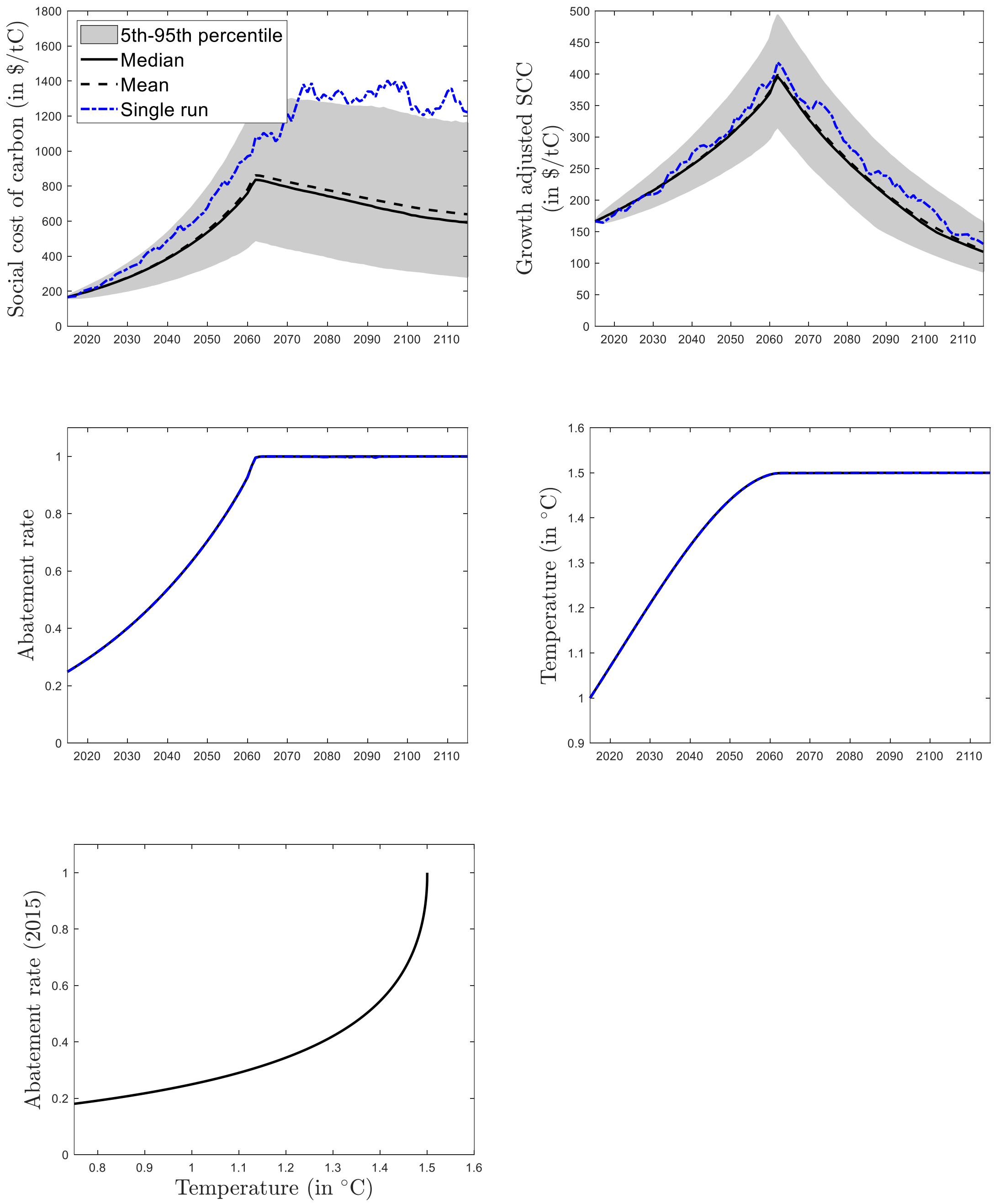


Figure E2: Effects of a 1.5 degrees Celsius temperature cap with damages