TI 2021-034/VI
Tinbergen Institute Discussion Paper

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Björn Brügemann[1]

[1] Vrije Universiteit Amsterdam and Tinbergen Institute

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Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900

Invariance of Unemployment and Vacancy Dynamics with Respect to Diminishing Returns to Labor at the Firm Level

Björn Brügemann*
Vrije Universiteit Amsterdam and Tinbergen Institute

April 27, 2021

Abstract

This paper show analytically that introducing diminishing returns to labor at the firm level into the Diamond-Mortensen-Pissarides model, followed by recalibration, does not change aggregate dynamics of unemployment and vacancies. This invariance result holds for several standard calibration strategies developed for the model with constant returns, alternative bargaining solutions for the setting with diminishing returns, and different sources of diminishing returns. Invariance makes precise in which sense the common practice of abstracting from diminishing returns is innocuous. It provides an analytical benchmark for quantitative findings obtained in models that do combine a Diamond-Mortensen-Pissarides labor market with diminishing returns at the firm level.

JEL Classification: E24, E32, J64

Keywords: Diminishing returns, Diamond-Mortensen-Pissarides model, Aggregate unemployment dynamics, Calibration, Bargaining

*b.a.brugemann@vu.nl

1 Introduction

In this paper I show analytically that introducing diminishing returns to labor at the firm level into the Diamond-Mortensen-Pissarides model, followed by recalibration, does not change the aggregate dynamics of unemployment and vacancies. This invariance result holds for several standard calibration strategies developed for the model with constant returns, alternative generalizations of Nash bargaining to a setting with diminishing returns, and different sources of diminishing returns at the firm level, including managerial span of control and monopolistic competition.

The purpose of this result is twofold. First, it makes precise in which sense the common practice of abstracting from diminishing returns to labor at the firm level is innocuous. The large literature that studies the quantitative properties of unemployment and vacancy dynamics in the Diamond-Mortensen-Pissarides commonly assumes constant returns to labor at the firm level, or equivalently that a firm can recruit only one worker in combination with free entry of firms. This literature has produced a wealth of findings, documenting both successes and failures regarding the model's ability to account for empirical patterns. Invariance implies that the findings in this literature are exactly unchanged when the model is generalized to allow for diminishing returns at the firm level, provided they have been obtained with a calibration strategy for which invariance applies, and the same strategy is applied to the generalized model. In particular, considering the thought experiment in which the model with strictly diminishing returns is the true data generating process, calibrating the model with constant returns yields the true dynamics of unemployment and vacancies, despite the misspecification of the strength of diminishing returns.

Second, invariance provides an analytical benchmark for quantitative findings obtained using models that do combine a Diamond-Mortensen-Pissarides labor market with diminishing returns at the firm level. This combination naturally arises when a Diamond-Mortensen-Pissarides labor market is integrated into classes of macroeconomic models that inherently feature diminishing returns to labor at the firm level, for instance models of firm size dynamics. For understanding the dynamics of unemployment and vacancies in such integrated models, the well-understood dynamics in the standalone Diamond-Mortensen-Pissarides model with constant returns are a natural reference point. Yet obtaining a precise understanding of any differences can be challenging because the models differ along several dimensions. Here invariance helps to identify the relevant mechanisms by ruling out the possibility that the presence of diminishing returns by itself is a source of differences in aggregate dynamics.

Section 2 establishes invariance for a basic Diamond-Mortensen-Pissarides model and a benchmark calibration strategy. The model has diminishing returns to labor in production with a constant elasticity, a fixed number of firms, exogenous separations, linear vacancy costs, and dynamics induced by shocks to aggregate productivity. Wages are determined through Stole-Zwiebel bargaining, the most commonly used generalization of Nash bargaining to the setting with diminishing returns. The calibration strategy consists of the targets and externally calibrated parameters that are most commonly used for the model with constant returns. It sets the discount rate and the separation rate externally, and targets the job finding rate. It provides targets for both the flow payoff from unemployment and the cost of hiring a worker in wage units, that is, relative to the steady-state wage. The dynamics of labor productivity are used as a target that is informative about the dynamics of exogenous aggregate productivity, with labor productivity defined as output per worker.

I establish invariance in two steps. First, I show that the benchmark calibration strategy determines the dynamics of unemployment and vacancies. It does so despite the fact that, as a strategy developed for the model with constant returns, it does not identify the strength of diminishing returns. Second, I show that the model can match the calibration targets for any strength of diminishing returns.

Are there reasons to expect a priori that a calibrated Diamond-Mortensen-Pissarides model with strictly diminishing returns may exhibit different dynamics? If so, what features of the benchmark calibration strategy ensure that such differences do not arise? Two identification issues are key for addressing these questions. First, the identification of the magnitude of the surplus, which matters for volatility. Second, the target informing on the dynamics of exogenous aggregate productivity, which matters for both propagation and volatility.

From studies of the Diamond-Mortensen-Pissarides model with constant returns it is well understood that the magnitude of what Ljungqvist and Sargent (2017) refer to as the fundamental surplus fraction is a key determinant of the amplification of movements in labor productivity. In the model with constant returns, this fraction is the difference between the output of a match and the flow payoff from unemployment, relative to match output. By construction, calibration strategies developed for this model identify the magnitude of this fraction. Moreover, the output of a match coincides both with the average product and with the bilateral marginal return that employment generates for the firm and a worker. In this sense, there is no distinction between the average surplus and the marginal surplus.

With strictly diminishing returns, the average and the marginal surplus become distinct, and the marginal surplus is what enters the fundamental surplus fraction and thereby governs amplification. Consequently, it matters which version of the surplus a calibration strategy developed for the model with constant returns can identify in the generalized model, if any. In particular, if such a strategy identifies the average surplus, then it cannot also identify the marginal surplus, as the difference between the two versions of the surplus depends on the strength of diminishing returns. Furthermore, extending the calibration strategy to identify the strength of diminishing returns would then result in a lower marginal surplus to the extent that there are indeed strictly diminishing returns, and thus stronger amplification of movements in labor productivity.

The benchmark calibration strategy identifies the marginal surplus, due to its feature of targeting the flow payoff from unemployment and the hiring cost in wage units. In equilibrium a firm hires up the point at which the bilateral marginal return of employing a worker covers the user cost of labor, consisting of the wage and a vacancy component. The vacancy component depends on the hiring cost, the discount factor, and the separation rate. The strategy determines this component in wage units as the hiring cost target is in wage units. In this way the strategy determines the bilateral marginal return in wage units, although this return does depend on the strength of diminishing returns when viewed as a function of structural parameters. Since the strategy also provides the flow payoff from unemployment in wage units, it follows that the fundamental surplus fraction is identified, without having to extend the strategy to identify the strength of diminishing returns. In the thought experiment in which the model with strictly diminishing returns is the true data generating process, this identification approach guarantees that calibrating the model with constant returns yields the correct fundamental surplus fraction. In contrast, an alternative calibration strategy that targets the flow payoff from unemployment and the hiring cost in units of labor productivity identifies the average surplus and does not yield invariance.

Strictly diminishing returns also introduce a distinction between labor productivity and exogenous aggregate productivity. While the two coincide with constant returns, with diminishing returns labor productivity becomes endogenous because it also depends on the level of employment. Consequently, the response of the labor market to a given change in exogenous aggregate productivity varies with the strength of diminishing returns. In particular, with strictly diminishing returns a positive exogenous productivity shocks induces an expansion of employment. This mutes the impact on labor productivity, which in turn mutes the response of unemployment and vacancies. Since the response of employment takes some time, propagation of movements in exogenous aggregate productivity differs as well.

The calibration strategy ensures that differences in the response to exogenous aggregate productivity do not translate into different dynamics of unemployment and vacancies by targeting the dynamics of labor productivity. Irrespective of the strength of diminishing returns, labor productivity is a proximate driving force that captures the influence of both exogenous aggregate productivity and the strength of diminishing returns. That is, both affect aggregate labor market dynamics only through labor productivity, and thus do not matter conditional on labor productivity dynamics. The constant elasticity of output with respect to labor plays a role here, because it implies that labor productivity is proportional to the marginal product. In combination with invariant amplification of labor productivity movement implied by identification of the fundamental surplus fraction, targeting labor productivity dynamics then implies invariance of unemployment and vacancy dynamics.

The flip side of targeting labor productivity is that exogenous aggregate productivity dynamics are not identified and vary with the strength of diminishing returns. In particular, in the previously mentioned thought experiment, an economist that calibrates the model with constant returns takes as an exogenous driving force the movements in labor productivity that are endogenous in the true model. This yields the correct dynamics of unemployment and vacancies, avoiding a mistake due to misspecification of the strength of diminishing returns, precisely because the calibrated dynamics of exogenous aggregate productivity are allowed to differ from the true dynamics.

Having established invariance for the benchmark calibration strategy, in Section 4 I extend the result to other common strategies developed for the model with constant returns. The most prominent is due to Hagedorn and Manovskii (2008). It differs from the benchmark strategy by dropping the target for the flow payoff from unemployment, targeting the cyclicality of wages instead. For typical numerical values of targets used for the two strategies, the Hagedorn-Manvoskii strategy is well known to generate a much smaller fundamental surplus fraction, and thus stronger amplification. I establishing invariance for this strategy by observing that the benchmark strategy also determines wage cyclicality and, conditional on the other targets and externally set parameters, provides a one-to-one mapping between wage cyclicality and the flow payoff from unemployment in wage units. This establishes an equivalence between using these two targets, implying that the invariance result carries over to the Hagedorn-Manvoskii strategy. I use the same idea to show that invariance carries over to other common departures from the benchmark strategy, for example calibrating the bargaining power parameter externally instead of targeting the cost of hiring.

In Section 5 I show that invariance carries over to three important extensions of the basic model. The first introduces endogenous separations due to match-specific shocks as in Mortensen and Pissarides (1994). The second allows for convex vacancy costs. The third extension introduces a variety of other aggregate shocks in addition to aggregate productivity shocks, including shocks to the discount factor, the exogenous separation probability, and the flow payoff from unemployment. The literature using the Diamond-Mortensen-Pissarides model with constant returns has explored many other extensions and modification. While I cannot consider all of them, the approach I use provides a recipe for verifying applicability of invariance to other extensions and modifications.

I also extend the invariance result to alternative ways of introducing diminishing returns in Section 6. First, invariance also holds with firm entry. Second, it applies for alternative generalizations of Nash bargaining to negotiations between a firm and multiple workers that eliminate the well-known incentive for over-hiring associated with Stole-Zwiebel bargaining. This includes the approach considered by Krause and Lubik (2013), and bargaining with commitment proposed by Hawkins (2015). Third, it applies if diminishing returns are induced by monopolistic competition with a constant elasticity of substitution, rather than diminishing returns in production.

In Section 7 I use invariance as an analytical benchmark to discuss quantitative findings of studies that examine aggregate dynamics in models that combine a Diamond-Mortensen-Pissarides labor market with diminishing returns at the firm level. First, I discuss Faccini and Ortigueira (2010) and Dao and Delacroix (2018), who study models with homogeneous firms. Both compare aggregate dynamics induced by aggregate productivity shocks to those in the model with constant returns of Shimer (2005). While Faccini and Ortigueira (2010) obtain very similar dynamics, Dao and Delacroix (2018) find that their model generates substantially higher volatility. I use the invariance result to reconcile these findings by comparing calibration strategies to the benchmark calibration strategy. Second, I discuss Elsby and Michaels (2013) and Hawkins (2011a), who study models with heterogeneity in firm size and firm growth. Both provide a comparison with a model that has constant returns to labor at the firm level. While the comparison in Elsby and Michaels (2013) demonstrates stronger amplification and propagation, the comparison Hawkins (2011a) indicates that differences in aggregate dynamics are minor. I use invariance as a reference point that contributes to a precise understanding of these findings and helps to narrow down potential explanations for the differences in findings between the two studies.

This paper contributes an analytical invariance result to the literature that studies the dynamics of unemployment and vacancies in models that combine a Diamond-Mortensen-Pissarides labor market with diminishing returns at the firm level. In the remainder of the introduction, I relate this invariance result to other analytical findings in this literature.

Hawkins (2011a) shows that productivity shocks are neutral in a particular specification of a Diamond-Mortensen-Pissarides model with diminishing returns in production, Stole-Zwiebel bargaining, and heterogeneity in firm size and firm growth, generalizing a neutrality result obtained by Shimer (2010) for the corresponding model with constant returns. The particular features of the specification are that households have balanced-growth preferences with an intertemporal elasticity of substitution equal to one, recruiting uses time, and the absence of capital prevents consumption smoothing. This combination of features implies that income and substitution effects of a productivity shock on recruitment activity cancel exactly, leaving vacancies and unemployment unchanged. The invariance result I obtain here differs in that I maintain the linear preference specification that is more customary in the Diamond-Mortensen-Pissarides framework, and which implies that income effects are absent. Thus, productivity shocks do induce movements in unemployment and vacancies. Moreover, I can allow for other shocks. To obtain an analytical result for a setting in which shocks are not exactly neutral, however, I cannot allow for heterogeneity in firm size and firm growth. Hawkins also provides a quantitative analysis of a version of his model with linear preferences, which I discuss in Section 7.

Cahuc and Wasmer (2001) incorporate the Stole-Zwiebel bargaining solution into the large-firm matching model of Pissarides (2000, Chapter 3.1), and demonstrate that the equivalence with the "single-worker"-firm model established by Pissarides continues to hold. Both versions of the model feature capital. The production function has constant returns to scale and a diminishing marginal product of labor, and capital can be adjusted costlessly and without delay. Pissarides establishes equivalence under the assumption that the firm engages in a Nash bargain with each worker separately, taking the wages of all other workers as given. Cahuc and Wasmer show that this equivalence continues to hold with Stole-Zwiebel bargaining. The mechanism underlying this result is that the marginal revenue product of labor is in fact constant once the immediate adjustment of capital is taken into account. This highlights that a diminishing marginal product of labor associated with the role of capital in production does not necessarily lead to diminishing returns to labor at the firm level. With constant returns, the outcome of Stole-Zwiebel bargaining coincides with the separate Nash bargains considered by Pissarides. This equivalence result differs from the invariance result in this paper, which applies when there are in fact diminishing returns to labor at the firm level, for example due to managerial span of control or imperfect competition. Versions of the model with different strengths of diminishing returns are not equivalent, and invariance of aggregate dynamics only holds for certain calibration strategies.

Hawkins (2011b) shows that in a Diamond-Mortensen-Pissarides models with diminishing returns and heterogeneity in firm size and firm growth, it is impossible to distinguish between Stole-Zwiebel bargaining and bargaining with commitment based on observing model-generated data on wages, worker flows, and job flows. This observational equivalence result is motivated by normative considerations, as the two bargaining solutions have different implications for whether policy makers can improve welfare. I build on this insight to show that the invariance result I obtain for Stole-Zwiebel bargaining carries over to bargaining with commitment. I also show that under additional assumptions on the timing of wage payments under bargaining with commitment, the dynamics of unemployment and vacancies are invariant with respect to the choice of bargaining solution under the benchmark calibration strategy. This additional invariance result is very closely related to the observational equivalence result of Hawkins. It differs from the main invariance result in this paper, because it takes the strength of diminishing returns as given and varies the bargaining solution, while the main result varies the strength of diminishing returns for a given bargaining solution.

2 Basic Invariance Result

2.1 Model

The model studied in this section is a textbook large-firm Diamond-Mortensen-Pissarides model similar to Pissarides (2000, Chapter 3.1), with productivity shocks as in Shimer (2005), extended with diminishing returns to labor in production at the firm level, and wages determined by Stole and Zwiebel (1996) bargaining. Potential sources of diminishing returns to labor in production are managerial span of control and other fixed factors of production, which could for example include components of capital.[1]

Time is discrete and indexed by $t = 0, 1, 2 \ldots$. A stochastic event $s_t \in S$ is realized at the beginning of each period $t \geq 1$. The event s_0 in period 0 is non-stochastic. A history of events up to and including the event at time t is denoted $s^t = (s_0, \ldots, s_t)$ and $\mathcal{S}$ is the set of all histories of any length. $\Pi_t\left(s^{t+1}\right)$ denotes the probability of s^{t+1} conditional on s^t when both histories coincide until time t.

There is a unit mass of ex ante identical workers. Instantaneous utility of workers is linear and equals the wage for employed workers and the flow payoff from unemployment b for unemployed workers. The discount factor is $\beta \in (0, 1)$.

There is a mass F of identical firms. Each firm employs a continuum of workers, starting with a mass n_0 at time $t = 0$. Thus, initial aggregate employment is $N_0 = Fn_0$. A firm with a mass of n workers at time t in history s^t produces $y = a\left(s^t\right) n^{1-\nu}$ units of output at the end of the period. Here $a\left(s^t\right)$ is exogenous aggregate productivity and the parameter $\nu \in [0, 1)$ governs the strength of diminishing returns. The elasticity of output with respect to employment is constant. This is important for the invariance result, as it implies proportionality of labor productivity and the marginal product. The non-stochastic steady-state value of aggregate productivity is given by a^*.

The total vacancy costs incurred in history s^t by a firm with v vacancies is cv units of history-s^t output. The number of matches formed in a period is given by a constant-returns-to-scale matching function, which is specified via the function that gives the probability with which vacancies are filled. If there are U unemployed workers at the beginning of the period and the aggregate number of vacancies in the period is V, then the probability that a vacancy is filled is given by $q(\theta)$ where $\theta \equiv V/U$ denotes the vacancy-unemployment ratio. An unemployed worker finds a job with probability $f(\theta) \equiv \theta q(\theta)$. Matches separate at exogenous rate $\lambda > 0$ at the beginning of the period. Wages are negotiated every period in each firm according to the Stole-Zwiebel bargaining solution, with $\eta \in (0, 1)$ parametrizing the bargaining power of workers.

[1] In contrast to Pissarides (2000, Chapter 3.1) capital is not modeled explicitly.

2.2 Equilibrium Conditions

I directly state a set of conditions that characterizes the equilibrium sequences of the vacancy-unemployment ratio, the wage, employment, unemployment, and vacancies. For simplicity, I restrict attention to parametrizations of the model in which aggregate vacancies are always strictly positive in equilibrium. Both the equations and their derivation are standard, Appendix A provides the derivation for completeness.

The first condition is the job creation condition in history s^t

$$\frac{c}{q\left[\theta\left(s^t\right)\right]} = m\left(s^t\right) - w\left(s^t\right) + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1})\left(1-\lambda\right) \frac{c}{q\left[\theta\left(s^{t+1}\right)\right]}. \tag{1}$$

This condition equalizes the costs and benefits of hiring an additional worker. The left-hand side gives the costs, since a vacancy costs c and is filled with probability $q\left[\theta\left(s^t\right)\right]$. The firm receives a bilateral marginal return $m\left(s^t\right)$ which I discuss in detail below. It shares this return with the worker by paying the equilibrium wage $w\left(s^t\right)$. The third term on the right-hand side represents the hiring costs that are not incurred next period since the worker hired in the current period stays with probability $1-\lambda$.

The second condition is the wage equation, giving the equilibrium wage of a worker at the time of production in history s^t:

$$w\left(s^t\right) = \eta\left[m\left(s^t\right) - b\right] + b + \eta\beta \sum_{s^{t+1}|s^t} \Pi_t\left(s^{t+1}\right) f\left[\theta(s^{t+1})\right] \frac{c}{q\left[\theta(s^{t+1})\right]}. \tag{2}$$

The term $m\left(s^t\right) - b$ is the fundamental surplus in this model, applying the definition of Ljungqvist and Sargent (2017). The worker receives a fraction η of this surplus. This is on top of the flow payoff from unemployment b, and a term due to search for a new job being part of the worker's outside option. In the event of leaving the firm, the worker finds a new job next period with probability $f\left[\theta(s^{t+1})\right]$. The bargaining solution implies that the worker's surplus associated with this job is proportional to the surplus the firm obtains by hiring this worker, and optimal job creation implies that the latter equals the cost of hiring.

Equations (1) and (2) hold in the textbook Diamond-Mortensen-Pissarides model with constant returns with the bilateral marginal return $m\left(s^t\right)$ equal to exogenous aggregate productivity $a\left(s^t\right)$. The strength of diminishing returns does not appear in these equations directly, only entering the system of equilibrium conditions via the bilateral marginal return

$$m\left(s^t\right) \equiv \frac{1-\nu}{1-\eta\nu} a\left(s^t\right)\left[N\left(s^t\right)/F\right]^{-\nu}. \tag{3}$$

Here $N\left(s^t\right) \equiv Fn\left(s^t\right)$ denotes aggregate employment, so that the term in square brackets is firm-level employment. With constant returns, this expression reduces to exogenous aggregate productivity $a\left(s^t\right)$. With strictly diminishing returns, the bilateral marginal return is given by the marginal product of labor multiplied by $1/(1-\eta\nu) > 1$. This factor captures that the return from employment for a firm and the worker in question includes a negative effect on the wage of other workers: employment of the worker keeps the marginal product and thereby the wage of other workers low. This additional return causes the well-known overhiring externality associated with Stole-Zwiebel bargaining.

Substituting the wage (2) into equation (1) yields a version of the job creation condition that involves only two endogenous variables, namely the vacancy-unemployment ratio and the bilateral marginal return:

$$\frac{c}{q\left[\theta\left(s^{t}\right)\right]} = (1-\eta)\left[m\left(s^{t}\right)-b\right] + \beta \sum_{s^{t+1}|s^{t}} \Pi_{t}(s^{t+1})\left(1-\lambda-\eta f\left[\theta\left(s^{t+1}\right)\right]\right)\frac{c}{q\left[\theta\left(s^{t+1}\right)\right]}. \quad (4)$$

The dynamics of aggregate employment are governed by the difference equation

$$N\left(s^{t}\right) = (1-\lambda)N\left(s^{t-1}\right) + f\left[\theta\left(s^{t}\right)\right]\cdot\left[1-N\left(s^{t-1}\right)\right]. \quad (5)$$

For a given parametrization of the model, equilibrium levels of $\left\{m\left(s^{t}\right),\theta\left(s^{t}\right),N\left(s^{t}\right)\right\}_{s^{t}\in\mathcal{S}}$ are determined by the system (3)– (5) together with initial aggregate employment N_0. Given $\left\{m\left(s^{t}\right),\theta\left(s^{t}\right),N\left(s^{t}\right)\right\}_{s^{t}\in\mathcal{S}}$, equilibrium wages $\left\{w\left(s^{t}\right)\right\}_{s^{t}\in\mathcal{S}}$ are determined by equation (2), and equilibrium unemployment and vacancies are obtained from the equations

$$U\left(s^{t}\right) = 1-N\left(s^{t-1}\right), \quad (6)$$

$$V\left(s^{t}\right) = \theta\left(s^{t}\right)\left[1-N\left(s^{t-1}\right)\right]. \quad (7)$$

2.3 Benchmark Calibration Strategy

The benchmark calibration strategy consists of the targets and externally calibrated parameters that are most commonly used for the model with constant returns. It coincides with the calibration strategy of Shimer (2005), except that it adopts the subsequent innovation of targeting the cost of hiring a worker in wage units rather than externally setting the bargaining power parameter. This has been widely adopted following the work of Silva and Toledo (2006, 2009). Section 4.2 shows that invariance also holds for Shimer's original strategy.

For the purposes of this paper, a calibration strategy only specifies which variables are targeted and which parameters are calibrated externally, not the numerical values assigned to targets and parameters. A particular implementation of a strategy would also assign numerical values. This is not needed here as the invariance result is analytical. Nevertheless, I do mention typical numerical values used in the literature for the U.S. labor market, to indicate the type of empirical evidence used. Here I pay special attention to the targets for the flow payoff from unemployment and the cost of hiring, as here it is important for the invariance result that the empirical evidence provides values that are relative to the wage.

When establishing invariance in Section 2.4, I keep track of variables determined by the calibration strategy by marking them with an overbar. At the outset, only targets and externally calibrated parameters are determined, and I already introduce them with the overbar notation here. I use an asterisk to indicate steady-state values.

I start with externally calibrated parameters. The first is the discount factor. In line with the notational convention just introduced, the externally calibrated discount factor is denoted $\overline{\beta}$. A typical numerical value sets $\overline{\beta}$ such that the annual discount rate is 5%. The second is the separation probability $\overline{\lambda}$. The numerical value is typically chosen to match the observed separation rate. Together with the target for the job-finding probability, this guarantees that the calibrated steady state is consistent with observed flows between employment and unemployment. For example, Shimer (2005) estimates a monthly separation rate of 0.034.

The third externally calibrated object is the matching function, specified via the function $q(\cdot)$ for the probability of filling a vacancy. The externally calibrated function is denoted $\overline{q}(\cdot)$, and the job-finding probability function is then also determined via $\overline{f}(\theta) = \theta\overline{q}(\cdot)$. The typical approach is to choose a functional form, and to externally calibrate the parameters associated with this functional form. Since the invariance result does not depend on the functional form of the matching function, however, I merely specify that the function is externally calibrated. As an illustration, consider the commonly-used Cobb-Douglas functional form. The vacancy-filling probability is $q(\theta) = \mu\theta^{-\phi}$ with elasticity parameter ϕ and scale parameter μ. Estimates of the elasticity parameter are available from the literature estimating matching functions, surveyed by Petrongolo and Pissarides (2001). A typical value adopted based on these estimates is $\overline{\phi} = 0.6$. The scale parameter can be identified through the steady-state relationship $f^* = \mu \cdot (\theta^*)^{1-\overline{\phi}}$ using data for the job-finding probability and the vacancy-unemployment ratio. With the Cobb-Douglas functional form, however, the scale parameter only matters for the scale of vacancies and vacancy-unemployment ratio and not their dynamics, making it common to use a normalization to determine μ.[2]

The final externally calibrated parameter is initial aggregate employment $\overline{N_0}$. When formally establishing the invariance result, I show that the entire history-contingent sequence of aggregate labor market variables is determined by the calibration strategy. For any calibration strategy to have a chance of determining history-contingent sequences of employment, unemployment, and vacancies, it must provide an initial condition for aggregate employment. This can be dispensed with if one only wants to show that the calibration strategy determines the ergodic distribution of aggregate labor market variables.

Next, I turn to the four targets. The first is for the job-finding probability. As with the separation probability, the numerical value is typically chosen to match observed flows.

[2]See Shimer (2005, p. 38) for a discussion.

For example, Shimer (2005) estimates a monthly job-finding rate of 0.45. Together with the externally calibrated matching function, the target $\overline{f^*}$ determines the steady-state vacancy-unemployment ratio as $\overline{\theta^*} = \overline{f}^{-1}\left(\overline{f^*}\right)$. Thereby, the strategy also determines the steady-state vacancy-filling probability $\overline{q^*} = \overline{q}\left(\overline{\theta^*}\right)$.

The next two targets provide information about the opportunity cost of employment and firms' costs of hiring workers. The first is for the ratio b/w^* between the flow payoff from unemployment and the steady-state wage. I use $\overline{b^*_w}$ to denote this target, where the subscript w indicates that it refers to the flow payoff in wage units. Shimer (2005) sets $b/p^* = 0.4$ and normalizes steady-state labor productivity p^* to one, comparing the implied value of b/w^* to the replacement rate of unemployment benefits in the US. An approach commonly used in later studies is due to Hall and Milgrom (2008). In addition to the replacement rate, Hall and Milgrom also include an estimate of the value of leisure derived from evidence on the Frisch elasticity of labor supply. As the outcome of this approach for their calibrated model, they reported the value 0.71 for b/p^*. Importantly, the steady-state wage in units of labor productivity w^*/p^* is an input in the calculation that leads to this value. Furthermore, their identification approach is such that the outcome for b/p^* is homogeneous of degree one in the input w^*/p^*, hence their approach identifies b/w^* rather than b/p^*. This distinction becomes relevant when applying this approach to models with diminishing returns to labor. Due to constant returns, in Hall and Milgrom (2008) the value $b/w^* = 0.72$ is close to $b/p^* = 0.71$ because w^*/p^* is close to one. With diminishing returns w^*/p^* can be substantially below one, potentially introducing a quantitatively important difference between b/p^* and b/w^*.

The target informative about hiring costs is for the cost of hiring a worker relative to the wage in steady state $c/\left(q^*w^*\right)$. I denote this target $\overline{h^*_w}$, again using the subscript w to indicate that it is in wage units. The most commonly used value is 14% of the quarterly wage, based on Silva and Toledo (2006, p. 10).

The final target is for the dynamics of labor productivity and its steady-state value. Labor productivity is defined as output per worker $a\left(s^t\right) n\left(s^t\right)^{1-\nu} / n\left(s^t\right)$ and thus given by

$$p\left(s^t\right) = a\left(s^t\right)\left[N\left(s^t\right)/F\right]^{-\nu} \tag{8}$$

with steady-state value $p^* = a^*[N^*/F]^{-\nu}$. The target is denoted $\left\{\overline{p}\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$ with steady-state value $\overline{p^*}$. It is informative about exogenous aggregate productivity dynamics $\left\{a\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$. Labor productivity coincides with exogenous aggregate productivity under constant returns. Matching the target then simply amounts to taking $\overline{p}\left(s^t\right)$ as the exogenous driving force. This is the approach of Shimer (2005). Labor productivity is endogenous if $\nu > 0$. In this case $\left\{a\left(s^t\right)\right\}_{s\in\mathcal{S}}$ is determined by solving for equilibrium for different values, choosing the value that matches $\left\{\overline{p}\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$. This approach is used by Elsby and Michaels (2013). Either approach is implemented in practice by specifying a stochastic process for exogenous

Table 1: Summary of Benchmark Calibration Strategy

Panel A: Externally Calibrated Parameter		
Parameter	Description	Externally Set To
β	Discount factor	$\overline{\beta}$
λ	Separation probability	$\overline{\lambda}$
$q(\cdot)$	Vacancy-filling probability function	$\overline{q}(\cdot)$
N_0	Initial employment	$\overline{N_0}$
Panel B: Other Parameters		
Parameter	Description	
b	Flow payoff from unemployment	
c	Flow cost of vacancy	
η	Bargaining power of workers	
a^*	Steady-state aggregate productivity	
$\{a(s^t)\}_{s^t \in \mathcal{S}}$	Aggregate productivity dynamics	
ν	Strength of diminishing returns	
F	Number of firms	
Panel C: Targets		
Variable	Description	Target
$f(\theta^*)$	Job-finding probability	$\overline{f^*}$
b/w^*	Flow payoff unemp. rel. to wage	$\overline{b_w^*}$
$c/(q^* w^*)$	Hiring cost rel. to wage	$\overline{h_w^*}$
p^*	Steady-state labor productivity	$\overline{p^*}$
$\{p(s^t)\}_{s^t \in \mathcal{S}}$	Labor productivity dynamics	$\{\overline{p}(s^t)\}_{s^t \in \mathcal{S}}$

aggregate productivity and choosing its parameters such that certain moments of equilibrium labor productivity match their empirical counterparts. Since the invariance result does not depend on a specific stochastic process for aggregate productivity, I do not impose one and instead use the full behavior of labor productivity as the target. Steady-state labor productivity can be normalized and it is common to set $\overline{p^*} = 1$.

Table 1 summarizes the strategy. Panel A lists externally calibrated parameters, with the third column showing the notation for the externally set values. Panel B lists all parameters that are not externally calibrated. Panel C contains the targets. Here the first column shows the targeted variable and the third column shows the notation for the target.

Since the strategy has been developed for the model with constant returns, by construction it does not fully identify the parameters of the model. The number of targets falls short of the number of parameters in Panel B by two, and in particular the strategy does not identify the strength of diminishing returns ν. While invariance implies that identification of ν is not needed to determine unemployment and vacancy dynamics, the analysis in the next section shows that the labor share w^*/p^* in the model varies with ν. This suggests the straightforward approach of using w^*/p^* to identify ν. This approach is problematic, however, since capital is not modeled explicitly. As demonstrated by the findings of Cahuc and Wasmer (2001) discussed in the introduction, including capital in the model does not necessarily lead to diminishing returns to labor at the firm level. Since the importance of capital in production does matter for the labor share, however, the labor share is generally not sufficient to identify the strength of diminishing returns. For alternative approaches to identify the strength of diminishing returns due to managerial span-of-control and monopolistic competition, see for example Atkeson and Kehoe (2005) and Guner et al. (2008).

2.4 Invariance

I establish invariance in two steps. First, I take as given that the model can match the targets, and show that the calibration strategy determines the dynamics of unemployment and vacancies, despite the fact that it does not identify all the parameters of the model. Second, I show that the model can indeed match the targets for any values of the strength of diminishing returns and the number of firms. Thus, the strategy does not identify these parameters, and the dynamics of unemployment and vacancies in the calibrated model are invariant with respect to values assigned to these parameters.

I start by examining quantities that are determined by the calibration of the steady state. The steady-state versions of equations (1) and (2) are

$$m^* = w^* + [1 - \beta(1-\lambda)] \frac{c}{q(\theta^*)}, \tag{9}$$

$$w^* = \eta [m^* - b] + b + \eta\beta f(\theta^*) \frac{c}{q(\theta^*)}, \tag{10}$$

where equation (9) isolates m^* on the left-hand side. Dividing both sides of equations (9) and (10) by w^* and substituting targets and externally calibrated parameters yields

$$m_w^* = 1 + \left[1 - \overline{\beta}(1-\overline{\lambda})\right] \overline{h_w^*}, \tag{11}$$

$$1 = \eta \left(m_w^* - \overline{b_w^*}\right) + \overline{b_w^*} + \eta\overline{\beta f^* h_w^*}. \tag{12}$$

Here $m_w^* \equiv m^*/w^*$ is the bilateral marginal return in wage units. Equation (11) shows that targets and externally calibrated parameters determine m_w^*, hence I mark it with an overbar from now on. The economic interpretation is based on optimal job creation. The left-hand side is the benefit a firm obtains from employing the marginal worker measured in wage units, consisting of the marginal product and the wage reduction imposed on other workers. The right-hand side is the user cost of labor in wage units, consisting of the wage and a vacancy component. The former is one since it is measured in wage units. The vacancy component is the cost of hiring a worker this period minus the hiring costs that are not incurred next period due to hiring an additional worker this period. The calibration strategy provides the cost of hiring a worker in wage units, and it also provides the separation rate and the discount factor that enter the vacancy component of user costs. Taken together, the strategy determines the user cost of labor in wage units. Optimal job creation then reveals the marginal benefit in wage units m_w^*. It does not identify how it breaks down into the marginal product and the wage reduction imposed on other workers, but it does reveal the total.

Substituting the solution for $\overline{m_w^*}$ into equation (12) and solving for η yields

$$\eta = \frac{\left(1-\overline{b_w^*}\right)/\left[1-\overline{\beta}\left(1-\overline{\lambda}-\overline{f^*}\right)\right]}{\overline{h_w^*}+\left(1-\overline{b_w^*}\right)/\left[1-\overline{\beta}\left(1-\overline{\lambda}-\overline{f^*}\right)\right]}. \tag{13}$$

Thus, the strategy also identifies the bargaining power parameter, and the corresponding value is denoted $\overline{\eta}$ from now on. Imposing the natural restrictions $\overline{b_w^*}<1$ and $\overline{h_w^*}>0$ implies $\overline{\eta}\in(0,1)$. The identification works as follows. The strategy determines both the surplus obtained by the firm and the surplus obtained by an employed worker in wage units, given by $\overline{h_w^*}$ and $(1-\overline{b_w^*})/\left[1-\overline{\beta}\left(1-\overline{\lambda}-\overline{f^*}\right)\right]$, respectively. Since the worker receives a share η of the total surplus, it follows that η is identified.

Now consider the stochastic equilibrium. Comparing equations (3) and (8) shows that the bilateral marginal return $m\left(s^t\right)$ is proportional to labor productivity

$$m\left(s^t\right) = \frac{1-\nu}{1-\eta\nu}p\left(s^t\right) \tag{14}$$

which in turn implies $m\left(s^t\right)/m^* = p\left(s^t\right)/p^*$. Using this relationship to replace $m\left(s^t\right)$ in equation (4), dividing both sides of the resulting equation by the steady-state wage, and substituting targets, externally calibrated parameters, as well as $\overline{m_w^*}$ and $\overline{\eta}$ yields

$$\begin{aligned}\overline{h_w^*}\frac{\overline{q^*}}{\overline{q}\left[\theta\left(s^t\right)\right]} &= \left(1-\overline{\eta}\right)\left[\overline{m_w^*}\frac{\overline{p}\left(s^t\right)}{\overline{p^*}}-\overline{b_w^*}\right] \\ &\quad+\overline{\beta}\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)\left(1-\overline{\lambda}-\overline{\eta}\overline{f}\left[\theta\left(s^{t+1}\right)\right]\right)\overline{h_w^*}\frac{\overline{q^*}}{\overline{q}\left[\theta\left(s^{t+1}\right)\right]}.\end{aligned} \tag{15}$$

This stochastic difference equation for the vacancy-unemployment ratio is fully determined by the calibration strategy. Thus, the strategy determines its solution $\{\overline{\theta}\left(s_t\right)\}_{s^t\in\mathcal{S}}$.[3]

[3]For simplicity, in the text I implicitly assume that the solution is unique. Invariance also holds if there is multiplicity. In this case the set of solutions is invariant.

Substituting $\overline{\lambda}$ and $\overline{f}(\cdot)$ into equation (5) yields

$$N\left(s^{t}\right)=(1-\overline{\lambda})N\left(s^{t-1}\right)+\overline{f}\left[\theta\left(s^{t}\right)\right]\cdot\left[1-N\left(s^{t-1}\right)\right]. \tag{16}$$

Via this equation the equilibrium vacancy-unemployment ratio $\{\overline{\theta}(s_t)\}_{s^t\in\mathcal{S}}$ and the initial condition $\overline{N_0}$ determine equilibrium aggregate employment $\left\{\overline{N}(s_t)\right\}_{s^t\in\mathcal{S}}$. Equilibrium unemployment and vacancies $\left\{\overline{U}(s_t),\overline{V}(s_t)\right\}_{s^t\in\mathcal{S}}$ are then determined by equations (6) and (7). This completes the first step of establishing invariance, namely to show that the calibration strategy determines the aggregate dynamics of unemployment and vacancies, taken as given that the model can match the targets.

The second step is to show that the model can match the calibration targets for any values $\nu\in[0,1)$ and $F>0$. I proceed by fixing $\nu\in[0,1)$ and $F>0$ and deriving the values that the parameters in Panel B of Table 1 must necessarily take for the model to match the targets, after which I verify that the resulting parametrization does indeed match the targets. Of these parameters, only η is identified without a need to specify ν and F, thus it remains to determine b, c, a^*, and $\{a(s^t)\}_{s^t\in\mathcal{S}}$.

Equation (14) together with the relationship $w^*=m^*/\overline{m_w^*}$ implies

$$w_\nu^*=\frac{1-\nu}{1-\overline{\eta}\nu}\frac{\overline{p^*}}{\overline{m_w^*}}, \tag{17}$$

where the subscript of w_ν^* indicates the dependence on ν. Matching the targets $\overline{b_w^*}$ and $\overline{h_w^*}$ then requires setting b and c to $b_\nu=\overline{b_w^*}w_\nu^*$ and $c_\nu=\overline{h_w^*}\,\overline{q^*}w_\nu^*$, respectively.

Equation (8) implies that exogenous aggregate productivity in history s^t must be

$$a_{\nu,F}\left(s^{t}\right)=\overline{p}\left(s^{t}\right)\left[\overline{N}\left(s^{t}\right)/F\right]^{\nu} \tag{18}$$

to match targeted labor productivity, where the subscript indicates dependence on both ν and F. The associated steady-state labor productivity is then $a_{\nu,F}^*=\overline{p^*}\left(\overline{N^*}/F\right)^{\nu}$ where $\overline{N^*}=\overline{f^*}/(\overline{f^*}+\overline{\lambda})$ is determined by the steady-state version of equation (16).

Having fully parametrized the model for given values of ν and F, I can now verify by substitution that the equilibrium under this parametrization matches all targets in Panel B of Table 1. This is somewhat tedious, and I give the details in Appendix B. This completes the argument establishing the invariance result, stated formally in the following proposition.

Proposition (Invariance)**.** *The benchmark calibration strategy determines aggregate labor market dynamics* $\{\theta(s^t),N(s^t),U(s^t),V(s^t)\}_{s^t\in\mathcal{S}}$. *The model can match the targets of the benchmark calibration strategy for any* $\nu\in[0,1)$ *and* $F>0$.

2.5 Invariance of Steady-State Elasticities

Since the calibration strategy determines the joint dynamics of unemployment, vacancies, and labor productivity, it also determines the corresponding comparative statics of the steady state. It is instructive to establish this directly, as the formulas for steady-state elasticities are frequently used in the literature to provide insights into the determinants of the relative volatility of labor market variables and labor productivity.

Let $\varepsilon_{\theta,a}$ and $\varepsilon_{p,a}$ denote the elasticities with respect to exogenous productivity of the vacancy-unemployment ratio and labor productivity, respectively. I start by deriving the relative elasticity $\varepsilon_{\theta,p;a} \equiv \varepsilon_{\theta,a}/\varepsilon_{p,a}$, which provides insight into the relative volatility of the vacancy-unemployment ratio and labor productivity induced by productivity shocks. With constant returns, labor productivity itself is exogenous and this relative elasticity coincides with $\varepsilon_{\theta,a}$. Consider the steady-state version of equation (4). As exogenous aggregate productivity does not enter directly and labor productivity is proportional to m^*, $\varepsilon_{\theta,p;a}$ is obtained by computing the elasticity of θ^* with respect to m^* based on this equation. This yields

$$\varepsilon_{\theta,p;a} = \Upsilon \frac{m^*}{m^* - b}$$

where

$$\Upsilon \equiv \frac{[1-\beta(1-\lambda)] + \eta\beta f(\theta^*)}{\phi(\theta^*)[1-\beta(1-\lambda)] + \eta\beta f(\theta^*)}$$

and $\phi(\theta) \equiv \left|\frac{q'(\theta)\theta}{q(\theta)}\right|$ is the elasticity of the vacancy-filling probability function. $\varepsilon_{\theta,p;a}$ is the product of two factors. The first is the elasticity of θ with respect to the fundamental surplus. The second is the inverse of the fundamental surplus fraction $(m^* - b)/m^*$ and gives the elasticity of the fundamental surplus with respect to labor productivity p^*.

The benchmark calibration determines both factors. Substituting targets, externally calibrated parameters, and the calibrated bargaining power $\overline{\eta}$ gives

$$\Upsilon = \frac{[1-\overline{\beta}\left(1-\overline{\lambda}\right)] + \overline{\eta}\overline{\beta f^*}}{\overline{\phi}\left(\overline{\theta^*}\right)[1-\overline{\beta}\left(1-\overline{\lambda}\right)] + \overline{\eta}\overline{\beta f^*}}$$

where the function $\phi(\cdot)$ is determined by the strategy because it is determined by the vacancy-filling-probability function $\overline{q}(\cdot)$. This factor is bounded below by unity. As discussed by Ljungqvist and Sargent (2017), a consensus about reasonable parameter values indicates that its value is close to this lower bound, limiting its contribution to $\varepsilon_{\theta,p;a}$, making the fundamental surplus fraction the key determinant of amplification of labor productivity movements. Since the calibration strategy determines both m^* and b in wage units, it also determines this fraction: dividing both numerator and denominator by w^* yields $(m^* - b)/m^* = \left(\overline{m^*_w} - \overline{b^*_w}\right)/\overline{m^*_w}$.

Using the result that the strategy determines $\varepsilon_{\theta,p;a}$, one can show that it also determines the analogous relative elasticities for unemployment and vacancies $\varepsilon_{U,p;a}$ and $\varepsilon_{V,p;a}$.

3 Discussion

The invariance result makes precise in which sense the common practice of abstracting from diminishing returns to labor at the firm level is innocuous. If an economist uses the benchmark calibration strategy for the model with constant returns, subsequently introduces diminishing returns and recalibrates the model with the same strategy, then the dynamics of unemployment and vacancies are exactly unchanged. This holds if the models are misspecified along other dimensions, so both successes and failures in accounting for empirical patterns carry over from the model with constant returns to the model with diminishing returns. Furthermore, in the thought experiment in which the model with strictly diminishing returns is the true data generating process, an economist that calibrates the model with constant returns obtains true aggregate dynamics, despite the misspecification.

The calibration strategy has two key features that ensure invariance. First, targeting of the cost of hiring and the flow payoff from unemployment in wage units enables the strategy to identify the fundamental surplus fraction. This fraction is a key determinant of the amplification of movements in labor productivity as discussed in Section 2.5. Second, it directly targets movements in labor productivity, which are a proximate driving force that induces the same movements in the vacancy-unemployment ratio irrespective of the strength of diminishing returns, provided the fundamental surplus fraction is also the same.

To understand the first feature, consider the distinction between average and marginal surplus associated with diminishing returns. In the model with constant returns, the fundamental surplus fraction in terms of structural parameters is $(a^* - b)/a^*$. Here a^* is the output of a match, and coincides with both the average product p^* and the bilateral marginal return m^*. Thus, there is no distinction between the average and the marginal surplus. With strictly diminishing returns the average product differs from the bilateral marginal return, making the two notions of the surplus distinct. The average product p^* is associated with a surplus fraction $(p^* - b)/b$. The bilateral marginal return m^* is related to the average product through $m^* = [(1-\nu)/(1-\eta\nu)]\, p^*$ and is what enters the fundamental surplus fraction. In the model with constant returns, the calibration strategy identifies the fundamental surplus fraction, but this observation by itself does not reveal if this is based on the average or the marginal surplus. This becomes important when applying the strategy to the generalized model allowing for diminishing returns. Here it cannot identify both surplus fractions, since the difference depends on the strength of diminishing returns. In particular, if the strategy were to identify the surplus fraction based on the average product, then it would be necessary to extend the strategy in order to identify the fundamental surplus fraction, which would then vary with the strength of diminishing returns.

The calibration strategy does identify the fundamental surplus fraction. It accomplishes this by providing targets that determine both the bilateral marginal return and the flow payoff from unemployment in wage units. Equation (11) shows that the hiring cost target plays a key role in determining the user cost of labor in wage units, which identifies the bilateral marginal return by exploiting the condition for optimal job creation. Dividing both numerator and denominator of $(m^* - b)/m^*$ by w^* then establishes that the fundamental surplus fraction is identified, without a need to identify ν.

Now consider the thought experiment in which the model with strictly diminishing returns is the true data generating process, and an economist calibrates the model with constant returns using the benchmark calibration strategy, correctly measuring the targets and externally calibrated parameters of this strategy. This economist obtains the true fundamental surplus fraction. The calibration strategy accomplishes this by rescaling the structural parameters b and c, rather than using the true values. Equation (17) shows that the model has a higher calibrated steady-state wage, and the strategy adjust for this by scaling up b and c proportionally. This implies that the model does not match the true values of b and the hiring cost c/q^* in units of labor productivity p^*. Since the model is misspecified, it cannot match all features of the data. A choice has to be made, and targeting b and c/q^* in wage units allows the model to replicate the true dynamics of unemployment and vacancies.

The flip side of this observation is that an alternative strategy using targets $\overline{b_p^*}$ and $\overline{h_p^*}$ for b/p^* and $c/(q^*p^*)$, respectively, does not yield invariance. For the model with constant returns, this strategy identifies the fundamental surplus fraction in a straightforward way: since m^* and p^* coincide, the fraction is $(p^* - b)/p^*$ and its calibrated value is $1 - \overline{b_p^*}$. The economics of this identification differs from the benchmark strategy, as it does not exploit the condition for optimal job creation, and the calibrated value is independent of the target $\overline{h_p^*}$. When applying this strategy to the model with strictly diminishing returns, it still identifies the surplus fraction based on the average surplus $(p^* - b)/p^* = 1 - \overline{b_p^*}$. Consequently, it cannot identify the fundamental surplus fraction as the difference between the two fractions depends on ν. Using equation (14), for given ν the fundamental surplus fraction can be written as $(m^* - b)/m^* = 1 - \frac{1-\eta\nu}{1-\nu}\overline{b_p^*}$. This is below the surplus fraction based on the average surplus, and thus lower than the calibrated fundamental surplus fraction in the model with constant returns. Appendix C examines this strategy in more detail, and shows that it implies that b/w^* is strictly increasing in ν. As I discuss in Section 7, the lower fundamental surplus fraction explains why Dao and Delacroix (2018) find stronger amplification when they compare a model with diminishing returns to the model with constant returns in Shimer (2005), as their calibration approach results in similar values of b/p^* and $c/(q^*p^*)$ across the two models. In the familiar thought experiment, calibrating the model with constant returns using this alternative strategy results in a fundamental surplus fraction that is too high. Consequently, volatility of unemployment and vacancies is too low. By construction, this strategy succeeds in matching the true values of b^*/p^* and $c/(q^*p^*)$, but at the cost of incorrect dynamics of unemployment and vacancies.

The second key feature of the strategy is the targeting of labor productivity dynamics. Equations (4)–(7) show that the bilateral marginal return $m(s^t)$ is a proximate driving force of aggregate labor market dynamics that captures the influence of both exogenous aggregate productivity and the strength of diminishing returns. That is, both $a(s^t)$ and ν affect aggregate labor market dynamics only through $m(s^t)$, and thus do not matter conditional on the dynamics of $m(s^t)$. This property carries over to labor productivity, because the constant elasticity of output with respect to labor implies that $p(s^t)$ and $m(s^t)$ are proportional, see equation (14). Targeting this proximate driving force ensures that aggregate labor market dynamics are determined, without a need to identify ν.

The flip side of targeting labor productivity dynamics is that exogenous aggregate productivity dynamics are not identified by the strategy. Equation (18) shows that this would require extending the strategy to identify ν. In the thought experiment, the economist calibrating the model with constant returns takes as an exogenous driving force the movements in labor productivity that are endogenous in the true model, and differ from the dynamics of true exogenous aggregate productivity. Since the model with constant returns is misspecified, it cannot match the true dynamics of both $a(s^t)$ and $p(s^t)$. Choosing to match the latter guarantees that the model replicates the true dynamics of unemployment and vacancies.

For an economist calibrating the model with diminishing returns and in a position to identify ν, an alternative to targeting labor productivity dynamics is to externally calibrate exogenous aggregate productivity dynamics. Since output in the model is $y = an^{1-\nu}$, realized values of a can be inferred from data on output and employment and used to specify the stochastic sequence $\{a(s^t)\}_{s^t \in \mathcal{S}}$. In general, this does not yield aggregate dynamics that are exactly identical to those from calibrating the model with constant returns using observed labor productivity dynamics as a target. It does so only if the model with strictly diminishing returns is correctly specified, since then the alternative approach generates true labor productivity dynamics and thus the same outcome as targeting these dynamics directly. While this highlights the key role of targeting labor productivity dynamics, one can restate the invariance result in a way that is agnostic with respect to the strategy for calibrating productivity dynamics. Consider a reduced version of the benchmark strategy that drops the dynamics of labor productivity as a target. This strategy no longer uniquely identify aggregate dynamics, and instead determines a set containing all the different patterns of the joint dynamics of output and labor market variables $\{Y(s^t), \theta(s^t), N(s^t), U(s^t), V(s^t)\}_{s^t \in \mathcal{S}}$ that the model can generate by varying the $\{a(s^t)\}_{s^t \in \mathcal{S}}$. It is now this set which is invariant with respect to the strength of diminishing returns. The model with constant returns can be used to study this set, and the findings this delivers apply to the model with diminishing returns.

4 Alternative Calibration Strategies

4.1 Hagedorn-Manovksii Strategy

The benchmark strategy targets the flow payoff from unemployment in wage units b/w^*, and implementations use numerical values based on evidence for the replacement rate of unemployment insurance and households' preferences for leisure. An influential alternative is the strategy proposed by Hagedorn and Manovskii (2008), also in the setting of constant returns to labor at the firm level. Interpreting the canonical search and matching model as a linear approximation to a richer model with heterogeneity and curvature in preferences, Hagedorn and Manvoskii question the approach of targeting b/w^*. Their alternative targets the cyclicality of wages, coinciding with the benchmark strategy otherwise. Wage cyclicality is defined as the coefficient from regressing log wages on log labor productivity

$$w_{cyc} \equiv \frac{\mathrm{Cov}\left(\log\left[w\left(s^t\right)\right], \log\left[p\left(s^t\right)\right]\right)}{\mathrm{Var}\left(\log\left[p\left(s^t\right)\right]\right)}.$$

I extend the invariance result to this strategy by showing that the benchmark strategy determines wage cyclicality. In doing so, it provides a mapping from $\overline{b_w^*}$ to w_{cyc} that is determined by the targets and externally calibrated parameters that are common to both strategies, establishing an equivalence between using $\overline{b_w^*}$ and the wage cyclicality target $\overline{w_{cyc}}$.

To see that wage cyclicality is indeed determined by the benchmark strategy, apply to wage equation (2) the same steps that transform the equation for the vacancy-unemployment ratio (4) into equation (15). Specifically, use the relationship $m\left(s^t\right)/m^* = p\left(s^t\right)/p^*$ to replace $m\left(s^t\right)$, divide both sides by the steady-state wage, and substitute targets, externally calibrated parameters, as well as $\overline{m_w^*}$ and $\overline{\eta}$ to obtain

$$\frac{w\left(s^t\right)}{w^*} = \overline{\eta}\left[\overline{m_w^*}\frac{\overline{p}\left(s^t\right)}{\overline{p^*}} - \overline{b_w^*}\right] + \overline{b_w^*} + \overline{\eta}\overline{\beta}\sum_{s^{t+1}|s^t}\Pi\left(s^{t+1}\right)\overline{f}\left[\overline{\theta}(s^{t+1})\right]\overline{h_w^*}\frac{\overline{q^*}}{\overline{q}\left[\overline{\theta}(s^{t+1})\right]}$$

where the final additional step is to substitute the dynamics of the vacancy-unemployment ratio determined by the benchmark strategy $\left\{\overline{\theta}\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$. Thus, the strategy determines fluctuations of the wage around its steady-state value, which determines cyclicality since

$$\frac{\mathrm{Cov}\left(\log\left[w\left(s^t\right)\right], \log\left[p\left(s^t\right)\right]\right)}{\mathrm{Var}\left(\log\left[p\left(s^t\right)\right]\right)} = \frac{\mathrm{Cov}\left(\log\left[w\left(s^t\right)/w^*\right], \log\left[\overline{p}\left(s^t\right)\right]\right)}{\mathrm{Var}\left(\log\left[\overline{p}\left(s^t\right)\right]\right)}$$

where I substituted $\left\{\overline{p}\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$, and division by w^* leaves the covariance unaffected.

4.2 Externally Calibrating Workers' Bargaining Power

A common strategy, especially in work preceding the widespread adoption of the target $\overline{h_w^*}$, is to externally calibrate the bargaining power parameter η instead of targeting $c/(q^*w^*)$. This approach is used in Shimer (2005). In the absence of direct empirical evidence concerning η, it is common to consider the symmetric case $\eta = \frac{1}{2}$. Another common approach is to specify a Cobb-Douglas matching function and set η equal to the matching function elasticity. This is an interesting case, as it makes the equilibrium in the canonical search and matching model with constant returns socially efficient.

Under the benchmark strategy, equation (13) provides a one-to-one mapping between $\overline{\eta}$ and $\overline{h_w^*}$ conditional on other targets and externally calibrated parameters, establishing an equivalence between using the targets $\overline{\eta}$ and $\overline{h_w^*}$. For the same reason, invariance continues to hold if $\overline{\eta}$ replaces $\overline{b_w^*}$ or $\overline{f^*}$.

With strictly diminishing returns, the Hosios condition is not sufficient to imply social efficiency due to the over-hiring externality associated with Stole-Zwiebel bargaining. In Section 6.2 I show that invariance holds for alternatives to Stole-Zwiebel bargaining that do not exhibit this externality. The model with one of these alternative bargaining solution together with the modified strategy using the Hosios condition to set $\overline{\eta}$ provides a setting in which the equilibrium is socially efficient for all values of ν, and for which invariance applies.

4.3 Targeting Output Dynamics

The benchmark strategy targets labor productivity dynamics. This is an important feature, since Section 3 shows that invariance generally does not apply when exogenous aggregate productivity dynamics are calibrated externally. Yet there are alternatives for which invariance applies, in particular targeting output dynamics as in Hawkins (2011a). Let $\left\{\overline{Y}\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$ denote this target, with steady-state value $\overline{Y^*}$. Using the identity $p\left(s^t\right) = Y\left(s^t\right)/N\left(s^t\right)$ to rewrite equation (15) yields

$$\overline{h_w^*}\frac{\overline{q^*}}{\overline{q}\left[\theta\left(s^t\right)\right]} = \left(1-\overline{\eta}\right)\left[\overline{m_w^*}\frac{\overline{Y}\left(s^t\right)}{\overline{Y^*}}\frac{\overline{N^*}}{N\left(s^t\right)} - \overline{b_w^*}\right] + \overline{\beta}\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)\left(1-\overline{\lambda}-\overline{\eta}\overline{f}\left[\theta\left(s^{t+1}\right)\right]\right)\overline{h_w^*}\frac{\overline{q^*}}{\overline{q}\left[\theta\left(s^{t+1}\right)\right]}$$

where N^* is determined by the calibration strategy via $\overline{N^*} = \overline{f^*}/\left(\overline{f^*}+\overline{\lambda}\right)$. In contrast to equation (15), this is a stochastic difference equation with two rather than only one endogenous variable, involving employment $N\left(s^t\right)$ in addition to the vacancy-unemployment ratio $\theta\left(s^t\right)$. Together with the difference equation for employment in equation (16) it forms a system of two stochastic difference equations in employment and the vacancy-unemployment ratio that is determined by the calibration strategy.

5 Extensions of the Basic Model

5.1 Endogenous Separations due to Match-Specific Shocks

I extend the basic model by introducing endogenous separations due to match-specific shocks as in Mortensen and Pissarides (1994). In their model a firm can recruit only one worker. Versions of this model have been used widely for quantitative studies of aggregate labor market dynamics. I show that invariance holds when the benchmark calibration strategy is extended using common approaches for calibrating match-specific shocks.

When each firm recruits at most one worker as in Mortensen and Pissarides (1994), there is no distinction between firm-specific and match-specific shocks. For the large-firm version of the model this distinction becomes relevant. Here I consider match-specific shocks.

Idiosyncratic productivity of a match is denoted x. For new matches it is drawn from the distribution $G_{new}(x)$. Within a period, match-specific shocks occur immediately after exogenous separations. The new level is drawn from the distribution $G(x|\tilde{x})$ when the current level is $\tilde{x}$. I restrict attention to symmetric equilibria in which all firms have the same idiosyncratic productivity composition in a particular history s^t. Let $n(x|s^t)$ denote the mass of workers with idiosyncratic productivity up to x in a firm. Thus, the function $n(\cdot|s^t)$ describes employment levels at all idiosyncratic productivity levels, and $n(\infty|s^t)$ is total employment. Each firm starts with employment levels $n(\cdot|s^0) = n_0$ at time $t = 0$. Output produced by a firm with employment levels given by the function n in history s^t is

$$y = a\left(s^t\right)\left(\int xdn(x)\right)^{1-\nu}.$$

The equilibrium conditions are derived in Appendix D. In the basic model and conditional on the dynamics of the bilateral marginal return, the equilibrium sequences of $\theta\left(s^t\right)$ and $N\left(s^t\right)$ are determined by a system of two stochastic difference equations, namely job creation condition (4) and employment dynamics equation (5). Now separations are governed by a reservation rule, and the system is enlarged by equations that determines the separation threshold and track the value of a match to the firm conditional on idiosyncratic productivity.

The literature has not converged on common targets to identify the process governing idiosyncratic productivity shocks. Two types are used. The first consists of targets based on aggregate labor market dynamics. For example, Mortensen and Pissarides (1994) target the volatility of job creation, Den Haan et al. (2000) the relative volatility of employment and output, and Fujita and Ramey (2012) the volatility and autocorrelation of the separation rate. The second type consists of scale invariant moments of wage dynamics at the individual level. For example, Bils et al. (2011) target the autocorrelation of individual wage growth. For both types, the typical approach is to specify parametric functional forms for the distribution functions and use targets that exactly identify the corresponding parameters.

I extend the benchmark strategy to nest these approaches. Let $d\left(s^t\right)$ denote the aggregate separation rate in history s^t. To capture individual wage dynamics it suffices to consider an arbitrary worker. Let z^t denote the history of idiosyncratic shocks experienced by this worker up to the time of production in period t, and $\mathcal{Z}$ the set of all idiosyncratic histories of any length. The wage in aggregate history s^t and idiosyncratic history z^t is denoted $w(z^t, s^t)$. A function $l\left(\left\{\theta\left(s^t\right), N\left(s^t\right), U\left(s^t\right), V\left(s^t\right), s\left(s^t\right)\right\}_{s^t \in \mathcal{S}}, \left\{w(z^t, s^t)\right\}_{z^t \in \mathcal{Z}, s^t \in \mathcal{S}}\right)$ maps aggregate labor market dynamics and individual wage dynamics into $\mathbb{R}^o$, where l is homogeneous of degree zero in $\left\{w(z^t, s^t)\right\}_{z^t \in \mathcal{Z}, s^t \in \mathcal{S}}$ reflecting scale invariance in wages, and o is the number of parameters of the idiosyncratic productivity process. The corresponding target is $\bar{l} \in \mathbb{R}^k$. I assume that l is such that the parameters of the idiosyncratic productivity process are exactly identified, conditional on the other targets and externally calibrated parameters.

Appendix D establishes invariance with this extended calibration strategy. While the details are somewhat tedious, the mechanics underlying the result are unchanged from the basic model and the argument proceeds by showing that the enlarged system of equations determining the dynamics of the vacancy-unemployment ratio, employment, the reservation threshold, and the value of a match to the firm is determined by the calibration strategy.

With respect to the new target $\bar{l}$ the argument proceeds as follows. First, I show that both aggregate labor market and individual wage dynamics are invariant with respect to ν if the model is calibrated using the original benchmark strategy with a fixed idiosyncratic productivity process. That is, two versions of the model that differ only in the value of ν and are calibrated with the original strategy have the same aggregate and individual wage dynamics when the idiosyncratic process is identical. If a particular process matches $\bar{l}$ for one version of the model, it also matches $\bar{l}$ for the version with different ν. Since $\bar{l}$ exactly identifies the process by assumption, it follows that the calibrated processes are identical for the two version of the model, and thus aggregate labor market dynamics are also identical.

5.2 Convex Vacancy Costs

The basic model has linear vacancy costs. Yashiv (2006) finds that convex vacancy costs, and more generally convex hiring costs, improve the ability of the Diamond-Mortensen-Pissarides model to account for observed aggregate dynamics. Allowing for convex vacancy costs clarifies the mechanics of the invariance result. In the basic model, labor productivity is not only a proximate driving force of labor market dynamics. In addition, the vacancy-unemployment ratio is a jump variable with respect to labor productivity: its dynamics are fully determined by the forward-looking job creation condition (15). With convex vacancy costs, it remains true that labor productivity is a proximate driving force, while the vacancy-unemployment ratio is no longer a jump variable. Invariance continues to hold, showing that it is the former and not the latter property that is key for invariance.

I replace the linear vacancy costs cv with $(1+\gamma)^{-1}cv^{1+\gamma}$ where the parameter $\gamma \geq 0$ governs convexity, with $\gamma = 0$ giving the linear specification. The marginal cost of a vacancy is now cv^{γ} rather than c, and this is the only change in the equilibrium conditions. In particular, in wage equation (2) c is replaced with $cv\left(s^{t+1}\right)^{\gamma}$. In job creation condition (4) c is replaced with $cv\left(s^{t}\right)^{\gamma}$ and $cv\left(s^{t+1}\right)^{\gamma}$ on the left- and right-hand side, respectively. Initially, I consider an extended benchmark strategy in which γ is set externally to $\overline{\gamma}$. The target $\overline{h_w^*}$ is now a target for the average cost of hiring a worker in wage units $(1+\gamma)^{-1}(v^*)^{\gamma}c/\left(q^*w^*\right)$.

When applying the calibrating strategy to the steady state, the appearances of $\overline{h_w^*}$ are replaced by $(1+\overline{\lambda})\overline{h_w^*}$ since the marginal cost of hiring exceeds the average cost by a factor $1+\overline{\lambda}$, otherwise the identification of $\overline{m_w^*}$ and $\overline{\eta}$ is unchanged. In the calibrated stochastic job creation condition (15), on the left-hand side $\overline{h_w^*}$ is replaced by $(1+\overline{\lambda})\overline{h_w^*}[V(s^t)/V^*]^{\gamma}$. The modification on the right-hand side is analogous. Using equation (7), $V(s^t)/V^*$ can be replaced by $\theta\left(s^t\right)\left[1-N\left(s^{t-1}\right)\right]/[\overline{\theta^*}(1-\overline{N}^*)]$. This job creation condition and employment dynamics equation (16) are a system of two stochastic difference equations in the two endogenous variables $\theta\left(s^t\right)$ and $N\left(s^t\right)$ that is determined by the calibration strategy. Thus, labor productivity continues to be a proximate driving force and invariance continues to hold. Yet $N\left(s^t\right)$ now appears in the job creation condition. Thus $\theta\left(s^t\right)$ is no longer a jump variable with respect to labor productivity, and the two equations must be solved simultaneously.

So far I considered an extended calibration strategy in which γ is externally calibrated. Precisely because invariance holds with this strategy, it also holds for strategies that identify γ using targets based on aggregate labor market dynamics.

5.3 Other Shocks

The literature considers fluctuations in a variety of determinants of hiring that are constant in the basic model. Shimer (2005) studies shocks to the exogenous separation rate in addition to productivity shocks. Chodorow-Reich and Karabarbounis (2016) measure the opportunity cost of employment and find that it is procyclical, pointing out that this dampens unemployment fluctuations in various models. Hall (2017) and Martellini et al. (2020) explore the hypothesis that discount factor fluctuations are an important driving force.

I start by introducing shocks to two parameters that are externally calibrated in the benchmark strategy, namely the discount factor and the separation probability. The history-invariant values β and λ are replaced by history-contingent values $\left\{\beta\left(s^t\right),\lambda\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$ with associated steady-state values (β^*,λ^*). The calibration strategy is extended by externally calibrating both types of shocks, that is, it provides targets $\left\{\overline{\beta}\left(s^t\right),\overline{\lambda}\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$ and $\left(\overline{\beta^*},\overline{\lambda^*}\right)$.

This extension does not substantially change the argument establishing invariance. The steady-state values $\overline{\beta^*}$ and $\overline{\lambda^*}$ contribute to determining $\overline{m^*_w}$ and $\overline{\eta}$ as in Section 2.3. After replacing $\left(\overline{\beta^*}, \overline{\lambda^*}\right)$ with $\left\{\overline{\beta}\left(s^t\right), \overline{\lambda}\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ in job creation condition (15) and employment condition (16), the two conditions continue to be a system of two stochastic difference equations in $\theta\left(s^t\right)$ and $N\left(s^t\right)$ that are determined by the calibration strategy. In conjucntion with initial employment $\overline{N_0}$, they can be solved recursively for $\left\{\overline{\theta}\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ and $\left\{\overline{N}\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$.

Here invariance does *not* imply that the response of the labor market to an exogenous shock to the discount factor or separation probability is invariant to ν. Consider an exogenous increase in the discount factor and contrast the cases of constant and strictly diminishing returns. The expansion of employment is larger in the former case, since with diminishing returns such an expansion reduces labor productivity, which does not happen with constant returns. This observation is consistent with invariance because the strategy targets the joint dynamics of labor productivity and the discount factor. Consider the thought experiment in which the data generating process is the model with strictly diminishing returns and discount factor shocks are the sole exogenous driving force. In the data the discount factor and labor productivity are then negatively related due to the mechanism just discussed. Calibrating the model with constant returns then generates the true labor market dynamics, as an increase in the discount factor is associated with lower labor productivity. The model with constant returns takes this co-movement as given rather than generating it endogenously.

Next, I introduce exogenous shocks to the flow payoff from unemployment and the vacancy cost. Here additional care is required in extending the calibration strategy. The benchmark strategy has the targets $\overline{b^*_w}$ and $\overline{h^*_w}$ that determine b and c relative to w^* and $w^* q^*$, respectively. The level of w^* and thus the levels of b and c are not identified. For consistency with this approach, shocks to b and c are also introduced relative to w^* and $w^*\overline{q^*}$, respectively. The extended strategy has targets for $\left\{b\left(s^t\right)/w^*\right\}_{s^t \in \mathcal{S}}$ and $\left\{c\left(s^t\right)/\left(w^* q^*\right)\right\}_{s^t \in \mathcal{S}}$ denoted $\left\{\overline{b^*_w}\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ and $\left\{\overline{h^*_w}\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$, respectively, with steady-state values $\overline{b^*_w}$ and $\overline{h^*_w}$.

Replacing the history-invariant values $\overline{b^*_w}$ and $\overline{h^*_w}$ with $\left\{\overline{b^*_w}\left(s^t\right), \overline{h^*_w}\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ in equation (15) again yields a stochastic difference equation for the vacancy-unemployment ratio that is determined by the calibration strategy. Other aggregate labor market variables are then determine exactly as before.

A type of shock for which invariance does not apply with Stole-Zwiebel bargaining are shocks to the bargaining power of workers. Equation (14) shows that such shocks break the proportionality between labor productivity and the bilateral marginal return, as the bargaining power interacts with the strength of diminishing returns in determining the strength of the overhiring externality. This is is specific to Stole-Zwiebel bargaining, and with the alternative bargaining solution discussed in Section 6.2 invariance also applies with bargaining power shocks.

6 Different Ways of Introducing Diminishing Returns

6.1 Firm Entry

The number of firms is irrelevant with homogeneous firms and constant returns to labor at the firm level. With strictly diminishing returns, it potentially becomes relevant whether entry of new firms is allowed. The basic model assumes a fixed number of firms. I now extend the invariance result to the case with entry.

At the beginning of period 0 there is a mass F_0 of active firms. The endogenous mass of active firms in history s^t is denoted $F(s^t)$. For concreteness, I assume that in every period there is a large mass of potential entrants, and that the cost of entry is given by the parameter $k > 0$. Below I show that the details of modeling the entry process do not matter for the invariance result. Active firms exit with probability $\lambda_f \in (0, 1)$ every period. I now use λ_e to denote the probability that a given worker separates exogenously. The total separation probability of a match is then $\lambda \equiv \lambda_e + \lambda_f - \lambda_e \lambda_f$. Within a period, exit and entry occur immediately before the choice of vacancies. I restrict attention to equilibria in which vacancies posted by existing firms are always positive. In such equilibria any entrant choose exactly the same size as incumbents, so the firm-size distribution remains degenerate.

Job creation condition (1), wage equation (2), and employment dynamics equation (5) continue to hold in the model with entry, where λ now refers to the total separation probability, and equation (3) for the bilateral marginal return is adapted to account for the endogenous number of firms:

$$m\left(s^t\right) = \frac{1-\nu}{1-\eta\nu} a\left(s^t\right) \left[N\left(s^t\right)/F\left(s^t\right)\right]^{-\nu}.$$

Labor productivity is

$$p\left(s^t\right) = \frac{Y(s^t)}{N(s^t)} = \frac{F\left(s^t\right) a\left(s^t\right) \left[N\left(s^t\right)/F(s^t)\right]^{1-\nu}}{N\left(s^t\right)} = a\left(s^t\right) \left[N\left(s^t\right)/F\left(s^t\right)\right]^{-\nu}. \quad (19)$$

Thus, the bilateral marginal return and labor productivity continue to be proportional. The derivation of the invariance result based on equilibrium conditions (1), (2), and (5) then goes through without change. A key feature of the calibration strategy which guarantees invariance is once again the targeting of labor productivity dynamics. Introducing entry does not change equation (15), which determines the behavior of the vacancy-unemployment ratio conditional on labor productivity dynamics. This is the reason why invariance does not depend on the details for modeling entry.

Together with invariance in the model with a fixed number of firms, this result implies that under the benchmark calibration strategy the dynamics of unemployment and vacancies are also invariant with respect to the presence of entry, as the difference between the versions of the model with and without entry becomes irrelevant in the case of constant returns.

The targeting of labor productivity dynamics is also important for this invariance with respect to the presence of entry. Introducing entry does change aggregate dynamics if the model is not recalibrated. It provides a second margin for employment to respond to an increase in exogenous aggregate productivity, in addition to an increase in the size of existing firms. Consequently, the overall response of employment is stronger, while existing firms expand less since their vacancies compete with the vacancies of entrants. The weaker employment response at existing firms implies a stronger increase in labor productivity, as the increase in exogenous aggregate productivity is not muted as much by an increase in firm size. Recalibrating the model then leads to less volatile exogenous aggregate productivity. Specifically, equation (19) implies that the dynamics of exogenous aggregate productivity needed to match labor productivity dynamics for a given value of ν are

$$a_\nu \left(s^t\right) = \overline{p}\left(s^t\right) \left[\overline{N}\left(s^t\right) / F\left(s^t\right)\right]^\nu .$$

and thus are less volatile if $F\left(s^t\right)$ is more procyclical.

6.2 Alternative Bargaining Solutions

Stole-Zwiebel bargaining is associated with an incentive to over-hire, as employment of a given worker reduces the wage of co-workers. I now show that invariance holds for two other generalizations of Nash bargaining to a setting with diminishing returns which do not exhibit this incentive, hence this incentive does not play a key role in generating invariance.

Krause and Lubik (2013) shut down the over-hiring incentive by assuming that firms behave myopically in the sense of not taking into account the dependence of wages on the level of employment. Hawkins (2015) proposes bargaining with commitment, which yields the same outcome in terms of present values as in Krause and Lubik (2013).

For simplicity of exposition, here I follow the approach of Krause and Lubik. Under this approach the timing of wage payments in ongoing employment relationships is determined, facilitating the computation of the steady-state wage when applying the calibration strategy. I examine bargaining with commitment in Appendix E. With commitment the timing of wage payments is not determined. Thus, different assumptions can be made which lead to different formulas for the average steady-state wage, slightly complicating the application of the calibration strategy. In the appendix I show that invariance holds for the different assumptions considered by Hawkins that lead to a determinate timing of wage payments.

The approach of Krause and Lubik eliminates the factor $1/(1 - \eta\nu)$ in the expression for the bilateral marginal return $m\left(s^t\right)$ in equation (3). As discussed in Section 2.2, this term causes the incentive to over-hire under Stole-Zwiebel bargaining. With this redefinition of $m(s^t)$, equilibrium conditions (1), (2), (4), and (5) continue to apply without change. Moreover, the bilateral marginal return and labor productivity remain proportional. Consequently, the argument establishing invariance goes through without change.

Invariance with respect to ν has the additional implication that for given ν, the outcome considered by Krause and Lubik and Stole-Zwiebel bargaining yield identical aggregate dynamics under the benchmark strategy: for each bargaining solution any value of ν generates the same aggregate dynamics as $\nu = 0$, and with $\nu = 0$ the two outcomes coincide because the wage under Stole-Zwiebel bargaining is then indeed independent of employment. With $\nu > 0$ the bilateral marginal return is a smaller fraction of labor productivity for the outcome considered by Krause and Lubik. Yet the calibrated bilateral marginal return in wage units is the same as under Stole-Zwiebel bargaining, because the strategy identifies this value indirectly by determining the user cost of labor in wage units, and exploiting the condition for optimal job creation. Consequently, the fundamental surplus fraction is then also the same. The difference between the bargaining solutions is reflected in the values of the parameters b and c that are needed to match the calibration targets for a given value of $\nu > 0$. The steady-state wage is now $w^*_\nu = (1-\nu)\overline{p^*}/\overline{m^*_w}$ and thus lower than the corresponding value in equation (17) for Stole-Zwiebel bargaining. The values of b and c are then scaled down along with the steady-state wage. This rescaling across the two bargaining outcomes is similar to the rescaling across different values of ν for a given bargaining solution.

In Appendix E I show that invariance with respect to the bargaining solution also holds for bargaining with commitment if the indeterminacy of the timing of wages is resolved so that in steady state wages are constant in ongoing employment relationships. This result is closely related to the finding of observational equivalence in Hawkins (2011b). Hawkins shows that for a Diamond-Mortensen-Pissarides models with diminishing returns and heterogeneity in firm size and firm growth, it is impossible to distinguish between Stole-Zwiebel bargaining and bargaining with commitment based on observing model-generated data on wages, worker flows, and job flows. In doing so, he shows that the observationally equivalent parametrizations under the two bargaining solutions are identical, except for different ratios between the parameters b and c and aggregate productivity. For the invariance result, the key question is if these parametrizations exhibit the same ratios between b and c and the steady-state wage w^*, so that they yield the same values of the targeted variables b/w^* and $c/(q^*w^*)$. This is the case if wages are constant in ongoing employment relationships.

Krause and Lubik (2013) carry out a quantitative analysis that compares aggregate dynamics under Stole-Zwiebel bargaining and their bargaining outcome. They find small differences rather than exactly identical aggregate dynamics. How is this reconciled with invariance with respect to the bargaining solution under the benchmark strategy? They examine the effect of Stole-Zwiebel bargaining for given parameters, hence they do not recalibrate when switching to their bargaining outcome. This outcome then generates a lower steady-state wage, and thus higher values of b/w^* and $c/(q^*w^*)$, as well as a lower value of θ^*. This manifests itself in small differences in aggregate dynamics.

6.3 Monopolistic Competition

The basic model has diminishing returns in production. I now extend the invariance result to a different source of diminishing returns, namely monopolistic competition.

A firm $j \in [0, F]$ with n workers at the time of production in history s^t produces $y_j = a(s^t) n$ units of its own intermediate good variety. A competitive sector produces the final good using as inputs the intermediate goods produced by all firms, with production function

$$Y = \left[\int_0^F y_j^{1-\nu} dj\right]^{\frac{1}{1-\nu}}$$

where $\nu \in [0, 1)$ and ν^{-1} is the elasticity of substitution between varieties.

Each firm faces downward-sloping demand for its variety. The final good is the numeraire. If a firm sets the price ϖ_j, it sells $y_j = \varpi_j^{-\frac{1}{\nu}} Y(s^t)$ units. Thus $\varpi_j = [Y(s^t)/y_j]^\nu$ and revenue satisfies $\varpi_j y_j = Y(s^t)^\nu y_j^{1-\nu}$. Substituting $y_j = a(s^t) n$, revenue is given by

$$Y\left(s^t\right)^\nu a\left(s^t\right)^{1-\nu} n^{1-\nu}.$$

Revenue in the basic model is $a(s^t) n^{1-\nu}$. Here it has the same structure: an isoelastic function of n with ν parametrizing the strength of diminishing returns and a history-contingent multiplicative term that is $Y(s^t)^\nu a(s^t)^{1-\nu}$ instead of $a(s^t)$. This isomorphism implies that equilibrium conditions (1), (2), (4), and (5) hold, with formula (3) for $m(s^t)$ replaced by

$$m\left(s^t\right) = \frac{1-\nu}{1-\eta\nu} Y\left(s^t\right)^\nu a\left(s^t\right)^{1-\nu} \left[N\left(s^t\right)/F\right]^{-\nu}.$$

In equilibrium $y_j(s^t) = a(s^t) n(s^t)$ for all $j \in [0, F]$. Thus, production of the final good is $Y(s^t) = F^{\frac{1}{1-\nu}} a(s^t) [N(s^t)/F]$. Substituting this expression into the formula for the bilateral marginal return yields

$$m\left(s^t\right) = \frac{1-\nu}{1-\eta\nu} F^{\frac{\nu}{1-\nu}} a\left(s^t\right).$$

Labor productivity is

$$p\left(s^t\right) = \frac{Y\left(s^t\right)}{N\left(s^t\right)} = F^{\frac{\nu}{1-\nu}} a\left(s^t\right).$$

Thus, proportionality of $p(s^t)$ and $m(s^t)$ holds. The argument establishing invariance under the benchmark calibration then goes through without change.

Here labor productivity is proportional to exogenous aggregate productivity and thus effectively exogenous, while it is endogenous in the basic model. This has implications for the interpretation of invariance when additional shocks are introduced as in Section 5.3. While invariance holds in the basic model when these shocks are added, this does not mean that the response of unemployment and vacancies to these shocks is invariant, since these shocks can affect labor productivity as discussed in Section 5.3. In contrast, here invariance also applies to the response of unemployment and vacancies to these shocks.

The response of unemployment to changes in the flow payoff from unemployment plays an important role when using the Diamond-Mortensen-Pissarides model for policy analysis, as this payoff includes unemployment benefits. Costain and Reiter (2008) show that the model has difficulties generating a strong response of unemployment to productivity shocks without simultaneously generating an excessively strong response of unemployment to changes in unemployment benefits. Invariance in conjunction with exogeneity of labor productivity implies that this tension carries over exactly to the model with diminishing returns induced by monopolistic competition. In the next section I use this result as an analytical benchmark for the quantitative findings of Dao and Delacroix (2018), who revisit the tension identified by Costain and Reiter in a setting with diminishing returns due to monopolistic competition.

7 Quantitative Findings in the Related Literature

7.1 Models with Homogeneous Firms

Faccini and Ortigueira (2010) study a model that is very similar to the basic model of Section 2, with diminishing returns in production and Stole-Zwiebel bargaining. Their primary aim is to show that investment-specific technology shocks are an important source of labor market fluctuations, and for this purpose they introduce quadratic adjustment costs to capital as well as linear adjustment costs to labor that interact with physical investment. To highlight the role of investment-specific technology shocks, they first examine aggregate dynamics in their calibrated model when all movements in labor productivity are induced by neutral productivity shocks, that is, shocks that equally affect the production of consumption and capital. They find them to be very similar to those in the calibrated model with constant returns of Shimer (2005). This is their finding that is of primary interest here, as it involves a comparison between models with diminishing returns and constant returns.

The invariance result contributes to a precise understanding of this result. The calibration strategy of Faccini and Ortigueira (2010) largely follows Shimer (2005), including the targeting of b/w^* and labor productivity dynamics, and using very similar numerical values. The only departure is that they do not use the target $\overline{f^*}$ for the job-finding rate, and instead target vacancy costs as a fraction of output. They note, however, that the resulting job-finding rate is very close to the observed rate, hence the results would not be different if they had explicitly used $\overline{f^*}$. Ignoring for a moment that the model also features adjustment costs, the invariance result then establishes analytically that aggregate dynamics with neutral shocks must be the same as in Shimer (2005). The only potential source of differences in aggregate dynamics is then the presence of adjustment costs, and the quantitative findings of Faccini and Ortigueira (2010) in conjunction with invariance imply that the impact of adjustment costs in their calibrated model is minor for neutral productivity shocks.

Dao and Delacroix (2018) study a model with diminishing returns due to monopolistic competition, Stole-Zwiebel bargaining, and entry. Their main finding is that their calibrated model generates a substantially stronger response of unemployment to changes in labor productivity compared to the calibrated model with constant returns in Shimer (2005), accomplishing this without generating a counterfactually strong response to changes in unemployment benefits. They conclude that diminishing returns to labor at the firm level mitigate the tension identified by Costain and Reiter (2008) that I discussed in Section 6.3.

Invariance reconciles the finding of amplification in Dao and Delacroix (2018) with the lack thereof in Faccini and Ortigueira (2010). The extensions for the calibration strategy of Shimer (2005) in Section 4.2 and for monopolistic competition in Section 6.3 imply that invariance under this strategy applies to Dao and Delacroix' model.[4] Consequently, the amplification they find must be due to departures from this strategy or the use of different numerical values. The departures from the strategy consist of not targeting b/w^*, and not externally calibrating the bargaining power η. This is replaced by externally calibrating b/p^* and c/p^*, using as a reference point numerical values for these parameters obtained in other studies, primarily studies that calibrate models with constant returns. As discussed in Section 3, having the same values of b/p^* and $c/(p^*q^*)$ in a model with constant returns and in a model with diminishing returns implies a higher value of b/w^*, a smaller fundamental surplus fraction, and thus stronger amplification for the latter model. Importantly, invariance implies that the improved empirical performance of the model of Dao and Delacroix does not arise because the presence of diminishing returns enables the model to generate aggregate dynamics that are out of reach for the model with constant returns. To see this, consider the thought experiment in which the calibrated model of Dao and Delacroix is the data generating process. Invariance implies that calibrating the model with constant returns using the strategy of Shimer (2005) then generates the same aggregate dynamics.

[4]More precisely, it applies to the version of their model with a fixed number of firms. In their model with entry, they assume that the degree of competition varies with the number of firms. Dao and Delacroix (2018, p. 15) show that this mechanism provides some weak amplification and is not driving their main finding.

7.2 Models with Heterogeneous Firms

Elsby and Michaels (2013) develop a model that integrates the Diamond-Mortensen-Pissarides framework with firm-size dynamics. It has diminishing returns to labor in production, Stole-Zwiebel bargaining, a fixed number of firms, linear vacancy costs, and aggregate as well as firm-specific productivity shocks. The main finding is that the calibrated model gives a coherent account of cross-sectional features including the distribution of employment and employment growth, as well as features of aggregate dynamics such as the amplitude of fluctuations in worker flows and the propagation of aggregate shocks.[5] The calibration strategy is an extension of the benchmark strategy, except for a new approach to identify the flow payoff from unemployment b that targets the separation rate. Separations in the model are endogenous and induced by firm-specific shocks, and the process of firm-specific shocks is identified using cross-sectional features of employment growth. Thus, the separation rate is not needed to identify a match-specific shocks process, in contrast to the basic model with exogenous separations or the model with match-specific shocks in Section 5.1. This approach for identifying b yields a fundamental surplus fraction that lies between typical values implied by the benchmark strategy and the Hagedorn-Manovksii strategy, enabling the model to account for the amplitude of fluctuations in worker flows.

Elsby and Michaels compare aggregate dynamics between their model and a model with constant returns, specifically that of Mortensen and Pissarides (1994). Their strategy for identifying firm-specific shocks cannot be applied directly to the latter, as with constant returns it has no well-defined firm-size distribution. In light of this, they take the process of firm-specific shocks identified in their model as given. This reveals a tension: the model with constant returns generates excessive separations unless the flow payoff from unemployment is reduced substantially below the level in their model, resulting in a higher fundamental surplus fraction and less amplification. Targeting the separation rate then yields a calibrated value of b/w^* that is lower with constant returns. The tension arises because the strength of diminishing returns, or equivalently the difference between average and marginal surplus, does matter for how a drop in exogenous firm-specific productivity of a given magnitude affects labor productivity at the firm level. With strictly diminishing returns, the impact is muted since a decline in firm-level employment partially offsets the direct impact. This muting effect, which reduces separations, is absent with constant returns.

The invariance result for the model of Mortensen and Pissarides (1994) with match-specific shocks in Section 5.1 provides an analytical reference point that highlights the role of firm-specific shocks together with the new strategy to calibrate b for Elsby and Michael's

[5] Elsby et al. (2019) further examine the propagation mechanism in a broader class of models that includes the model of Elsby and Michaels (2013).

findings. With match-specific shocks, the introduction of diminishing returns in conjunction with Stole-Zwiebel bargaining does not change aggregate dynamics under an extension of the benchmark calibration strategy. The tension that arises for firm-specific shocks is absent with match-specific shocks, and the version of the model with constant returns does not need a lower value of b/w^* to match the observed separation rate. This is because a drop in exogenous match-specific productivity has the same impact on labor productivity of the match irrespective of the strength of diminishing return.

Hawkins (2011a) studies a model very similar to that of Elsby and Michaels, with the main difference that the model has firm entry. The main finding is that while the model succeeds in accounting for cross-sectional features, it generates neither amplification nor propagation when compared to a model with constant returns and homogeneous firms.[6] Hawkins presents two versions of this result, corresponding to two specifications of the model. The first gives rise to the analytical finding that I discuss in the introduction. The second adopts the linear preferences that are more customary in the Diamond-Mortensen-Pissarides framework and the analysis is quantitative. Hawkins finds that the model actually generates substantially less amplification than the calibrated model of Shimer (2005) with constant returns, after adjusting for differences in the value of b/w^*. Hawkins notes that there is no reason to expect the two models to produce similar results, as there is no distinction between average and marginal product in Shimer's model. In other words, the difference in amplification cannot unambiguously be attributed to firm heterogeneity, as it could also be due to the distinction between average and marginal product that is associated with diminishing returns.

The invariance result facilitates comparing aggregate dynamics in models with diminishing returns and firm heterogeneity such as Hawkins' with the well-understood aggregate dynamics in the model with constant returns. It implies that the distinction between aggregate and marginal product that is associated with diminishing returns by itself does not lead to different aggregate dynamics, provided invariance applies to the calibration strategy. Any differences can then be unambiguously attributed to firm heterogeneity, and the analytical nature of the result provides an explanation of the mechanics that lead to this outcome. Hawkins does use a strategy consistent with invariance. It departs from the benchmark strategy by targeting output dynamics instead of labor productivity dynamics. Section 4.3 shows that invariance applies with this modification. Since invariance also applies to Shimer's strategy, either strategy can be used to make this comparison.[7]

[6] Kaas and Kircher (2015, pp. 3053–55) obtain similar results in a model with competitive search. They find no amplification, and if vacancy costs are linear they also find no propagation.

[7] While invariance with respect to diminishing returns applies to each strategy, aggregate dynamics can differ between the two strategies. To ensure that any differences can only be due to firm heterogeneity, it is therefore necessary to use the same calibration for both models.

The reference point provided by invariance also helps to narrow down possible explanations for differences in findings between the two studies. While Elsby and Michaels find propagation, Hawkins does not. Moreover, Hawkins notes that differences in amplification are not fully explained by the lower surplus associated with Elsby and Michaels' calibration strategy. As the main difference between the two models, firm entry is the likely culprit. Since entry does not matter for aggregate dynamics in a model with constant returns and homogeneous firms, the question is then whether entry interacts with the introduction of diminishing returns, the introduction of firm heterogeneity, or both. Since invariance applies both with and without entry, there is no interaction between entry and the introduction of diminishing returns by itself. By ruling out this possibility, the invariance result points towards interactions between entry and firm heterogeneity as the source of the differences.

8 Conclusion

This paper shows analytically that introducing diminishing returns to labor at the firm level into the Diamond-Mortensen-Pissarides model, followed by recalibration, does not change the aggregate dynamics of unemployment and vacancies. This invariance result holds for several standard calibration strategies developed for the model with constant returns. It applies to several extensions of the basic Diamond-Mortensen-Pissarides model including endogenous separations due to match-specific shocks and convex vacancy costs, and with other aggregate shocks in addition to productivity shocks. Finally, it holds for different ways of introducing diminishing returns to labor at the firm level, whether arising in production or being due to monopolistic competition, with or without entry of new firms, and for Stole-Zwiebel bargaining as well as with alternative bargaining solutions.

The purpose of the invariance result is twofold. First, it makes precise in which sense the common practice of abstracting from diminishing returns to labor at the firm level is innocuous. A wealth of quantitative findings that have been obtained in Diamond-Mortensen-Pissarides models with constant returns remain exactly unchanged when diminishing returns to labor at the firm level are introduced, provided that the calibration strategy is subject to invariance. Second, invariance provides an analytical benchmark for quantitative findings obtained using models that do combine a Diamond-Mortensen-Pissarides labor market with diminishing returns at the firm level, such as models with firm-size dynamics. It facilitates comparing these findings to the well-understood aggregate dynamics in the standalone Diamond-Mortensen-Pissarides model with constant returns by ruling out the possibility that the presence of diminishing returns by itself is a source of differences in aggregate dynamics.

A Derivation of Equilibrium Conditions

Stole-Zwiebel bargaining implies a wage schedule that depends on firm-level employment. Let $w(n, s^t)$ denote the wage at time t in history s^t of a worker employed in a firm with n workers at the time of production. I start by analyzing the optimization problem of a firm for a given wage schedule. Then I consider the payoffs of workers and the bargaining problem to determine $w(n, s^t)$.

Let $\tilde{J}(\tilde{n}, s^t)$ denote the value of a firm with employment level $\tilde{n}$ at time t in history s^t before it chooses vacancies. Let $J(n, s^t)$ denote the value of a firm with employment level n at the time of production. Then the two values are related as follows:

$$\tilde{J}\left(\tilde{n}, s^t\right) = \max_{n,v \geq 0} \left\{-cv + J\left(n, s^t\right)\right\} \tag{20}$$

$$\text{s.t. } n = (1-\lambda)\tilde{n} + q\left[\theta\left(s^t\right)\right] v. \tag{21}$$

If the firm chooses a mass v of vacancies, it incurs a total cost of cv and has employment n at the time of production, which consists of remaining incumbents $(1-\lambda)\tilde{n}$ and new hires. The latter are given by $q[\theta(s^t)]v$ since each vacancy results in a hire with probability $q[\theta(s^t)]$. The first-order condition for vacancies is

$$c = q\left[\theta\left(s^t\right)\right] J_n\left[n, s^t\right].$$

Dividing both sides by $q[\theta(s^t)]$ and evaluating the right-hand side at the equilibrium employment level $n(s^t)$ yields

$$\frac{c}{q\left[\theta\left(s^t\right)\right]} = J_n\left(n\left(s^t\right), s^t\right). \tag{22}$$

The envelope condition for the maximization problem in equations (20)–(21) is

$$\tilde{J}_n\left(\tilde{n}, s^t\right) = (1-\lambda) J_n\left[\hat{n}\left(\tilde{n}, s^t\right), s^t\right]$$

where $\hat{n}(\tilde{n}, s^t)$ is the policy function for the level of employment at the time of production. The value of the firm at the time of production satisfies

$$J\left(n, s^t\right) = a\left(s^t\right) n^{1-\nu} - w\left(n, s^t\right) n + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1}) \tilde{J}\left(n, s^{t+1}\right), \tag{23}$$

since the firm produces output $a(s^t)n^{1-\nu}$ and pays a total wage bill $w(n, s^t)n$, and begins the next period with n workers. Thus, the relevant continuation value is $\tilde{J}(n, s^{t+1})$, which is the value of having n workers before the vacancy decision in period $t+1$.

Differentiating equation (23) with respect to n and substituting the envelope condition yields a recursive equation for the value of the marginal worker to the firm:

$$\begin{aligned} J_n\left(n, s^t\right) = {} & a\left(s^t\right)(1-\nu)n^{-\nu} - w\left(n, s^t\right) - w_n\left(n, s^t\right) n \\ & + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1})(1-\lambda) J_n\left[\hat{n}\left(n, s^{t+1}\right), s^{t+1}\right]. \end{aligned} \tag{24}$$

The utility of an employed worker at the time of production in period t in a firm with employment n satisfies the recursive equation

$$V^e\left(n,s^t\right)=w\left(n,s^t\right)+\beta\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)\left[(1-\lambda)V^e\left[\hat{n}\left(n,s^{t+1}\right),s^{t+1}\right]+\lambda V^u\left(s^{t+1}\right)\right]$$

as the flow payoff is the wage and the worker enters unemployment with probability λ, with $V^u(s^t)$ denoting the utility of an unemployed worker at the time of production. The equation for $V^u(s^t)$ is analogous, with flow payoff b and transition to employment with probability $f\left[\theta(s^{t+1})\right]$. Subtracting the latter equation from that for $V^e\left(n,s^t\right)$ and letting $V\left(n,s^t\right)\equiv V^e\left(n,s^t\right)-V^u\left(n,s^t\right)$ denote the surplus obtained by an employed worker yields

$$\begin{aligned}V\left(n,s^t\right)=w\left(n,s^t\right)-b-\beta\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)f\left[\theta(s^{t+1})\right]V\left[n\left(s^{t+1}\right),s^{t+1}\right]\\+\beta\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)(1-\lambda)V\left[\hat{n}\left(n,s^{t+1}\right),s^{t+1}\right].\end{aligned}\tag{25}$$

The total bilateral surplus for the firm and a worker is $J_n\left(n,s^t\right)+V\left(n,s^t\right)$. Stole-Zwiebel bargaining splits this surplus with shares $1-\eta$ and η between the firm and the worker:

$$\eta J_n\left(n,s^t\right)=(1-\eta)V\left(n,s^t\right).\tag{26}$$

The outcome coincides with Nash bargaining in the special case of constant returns, where $J_n\left(n,s^t\right)$ is independent of n. More generally, it coincides with the Shapley value. Brügemann et al. (2019) provide a non-cooperative foundation through an alternating-offers bargaining game in which the strategic position of each worker in the firm is symmetric.

Substituting equations (24) and (25) into equation (26) yields

$$\begin{aligned}w\left(n,s^t\right)=\eta\left[a\left(s^t\right)(1-\nu)n^{-\nu}-w_n\left(n,s^t\right)n\right]+(1-\eta)b\\+(1-\eta)\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)f\left[\theta(s^{t+1})\right]V\left[n\left(s^{t+1}\right),s^{t+1}\right]\end{aligned}\tag{27}$$

where the second lines in equations (24) and (25) cancel as the surplus split (26) also applies in period $t+1$. Equation (27) is a differential equation for $w\left(n,s^t\right)$. Solving it and using equations (26) and (22) to replace $V\left[n\left(s^{t+1}\right),s^{t+1}\right]$ with $[\eta/(1-\eta)]c/q\left[\theta(s^{t+1})\right]$ yields

$$w\left(n,s^t\right)=\eta\frac{1-\nu}{1-\eta\nu}a\left(s^t\right)n^{-\nu}+(1-\eta)b+\eta\beta\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)f\left[\theta(s^{t+1})\right]\frac{c}{q\left[\theta(s^{t+1})\right]}.\tag{28}$$

Evaluating this equation at equilibrium employment $n\left(s^t\right)$ and using the definition of $m\left(s^t\right)$ in equation (3) yields wage equation (2). Substituting the expression for $w_n\left(n,s^t\right)$ obtained by differentiating equation (28) into equation (24), evaluating the resulting equation at equilibrium employment levels, again using the definition of $m\left(s^t\right)$, and finally using first-order condition (22) to eliminate the two appearances of J_n yields job creation condition (1). Equation (21) together with $V(s^t)=Fv\left(s^t\right)$ and the relationships (6)–(7) implies that aggregate employment dynamics are governed by equation (5).

B Verification that Calibration Matches Targets

First consider the steady state. Substituting the parametrization into equations (9)–(10) and the steady-state versions of equations (3) and (5) yields the system

$$\begin{aligned}
m^* &= w^* + \left[1 - \overline{\beta}\left(1 - \overline{\lambda}\right)\right] \frac{\overline{h^*_w}\,\overline{q^*}}{\overline{q}(\theta^*)} w^*_\nu, \\
w^* &= \overline{\eta}\left[m^* - \overline{b^*_w} w^*_\nu\right] + \overline{b^*_w} w^*_\nu + \overline{\eta}\overline{\beta}\overline{f}\left(\theta^*\right) \frac{\overline{h^*_w}\,\overline{q^*}}{\overline{q}(\theta^*)} w^*_\nu, \\
m^* &= \frac{1-\nu}{1-\eta\nu}\overline{p^*}\left[N^*/\overline{N^*}\right]^{-\nu}, \\
N^* &= \left(1 - \overline{\lambda}\right) N^* + \overline{f}(\theta^*)(1 - N^*),
\end{aligned}$$

which determines the steady-state values of θ^*, N^*, w^*, and m^* and is known to have a unique solution. Substitution verifies that the solution is given by $\overline{\theta^*}$, $\overline{N^*}$, w^*_ν, and $\overline{m^*_w} w^*_\nu$, using the fact that by definition $\overline{m^*_w}$ and $\overline{\eta}$ solve equations (11)–(12). Since the steady-state vacancy-unemployment ratio is indeed $\theta^* = \overline{\theta^*}$ for this parametrization, it follows that the target $\overline{f^*}$ is matched. Since the steady-state wage is $w^* = w^*_\nu$, it follows that $b/w^* = \overline{b^*_w}$ and $c/(q^* w^*) = \overline{h^*_w}$, hence the targets $\overline{b^*_w}$ and $\overline{h^*_w}$ are matched. Substituting $a^*_{\nu,F} = \overline{p^*}\left(\overline{N^*}/F\right)^\nu$ and $N^* = \overline{N^*}$ into the steady-state version of (8) implies that the target $\overline{p^*}$ is matched.

Now consider the stochastic equilibrium. Substituting equation (3) and the parametrization into equation (4) and the parametrization into equation (5) yields

$$\begin{aligned}
\overline{h^*_w}\frac{\overline{q^*}}{\overline{q}\left[\theta\left(s^t\right)\right]} &= (1-\overline{\eta})\left\{\overline{m^*_w}\frac{\overline{p}\left(s^t\right)}{\overline{p^*}}\left[\frac{N\left(s^t\right)}{\overline{N}\left(s^t\right)}\right]^{-\nu} - \overline{b^*_w}\right\} \\
&\quad + \overline{\beta}\sum_{s^{t+1}|s^t}\Pi_t(s^{t+1})\left(1 - \overline{\lambda} - \overline{\eta}\overline{f}\left[\theta\left(s^{t+1}\right)\right]\right)\overline{h^*_w}\frac{\overline{q^*}}{\overline{q}\left[\theta\left(s^{t+1}\right)\right]}, \\
N\left(s^t\right) &= \left(1-\overline{\lambda}\right)N\left(s^{t-1}\right) + \overline{f}\left[\theta\left(s^t\right)\right]\cdot\left[1 - N\left(s^{t-1}\right)\right].
\end{aligned}$$

where I canceled w^*_ν in the first equation using equation (17). The solution to this system is given by $\{\overline{\theta}\left(s_t\right), \overline{N}\left(s_t\right)\}_{s^t\in\mathcal{S}}$ since by definition $\{\overline{\theta}\left(s_t\right), \overline{N}\left(s_t\right)\}_{s^t\in\mathcal{S}}$ satisfies equations (15)–(16). Substituting equation (18) into equation (8) and using $N\left(s^t\right) = \overline{N}\left(s^t\right)$ then shows that the parametrization matches the target $\{\overline{p}\left(s^t\right)\}_{s\in\mathcal{S}}$.

C Alternative Strategy Targeting b/p^* and $c/(q^*p^*)$

Taking the steady-state version of equation (4), dividing both sides by p^*, using equation (14) to replace m^*/p^* with $(1-\nu)/(1-\eta\nu)$, substituting targets and externally calibrated parameters of the alternative strategy, and subtracting $\overline{h_p^*}$ from both sides yields

$$0 = (1-\eta)\left[\frac{1-\nu}{1-\eta\nu} - \overline{b_p^*}\right] - \left[1-\overline{\beta}\left(1-\overline{\lambda}-\eta\overline{f^*}\right)\right]\overline{h_p^*}.$$

In contrast to the benchmark strategy, here η is not identified. For given ν, this equation determines η_ν. I assume that the right-hand side evaluated at $\eta = 0$ is strictly positive, that is, firms can cover the user cost of labor when the bargaining power of workers is zero. Since the right-hand side is strictly negative for $\eta = 1$ and the equation is a quadratic equation in η, it follows that there is a unique solution $\eta_\nu \in (0,1)$. The right-hand side cuts zero from above at η_ν, and an increase in ν shifts the right-hand side down, hence η_ν is strictly decreasing in ν. Dividing both sides of equation (10) by p^*, replacing m^*/p^* with $(1-\nu)/(1-\eta\nu)$, and substituting targets and externally calibrated parameters as well as $\eta = \eta_\nu$ yields

$$w_{p,\nu}^* = \eta_\nu\left[\frac{1-\nu}{1-\eta_\nu\nu} - \overline{b_p^*}\right] + \overline{b_p^*} + \eta_\nu\overline{\beta f^* h_p^*}.$$

where $w_{p,\nu}^*$ is w^*/p^* as a function of ν. Since η_ν is strictly decreasing in ν, it follows that $w_{p,\nu}^*$ is strictly decreasing in ν. Since $b/w^* = \overline{b_p^*}/w_{p,\nu}^*$, it follows that b/w^* is increasing in ν.

D Invariance with Match-Specific Shocks

The value of a firm before the choice of vacancies and the value at the time of production are denoted $\tilde{J}(\tilde{n}, s^t)$ and $J(n, s^t)$, respectively. The wage paid to a worker with idiosyncratic productivity x is denoted $w(x, n, s^t)$. Let R denote the separation threshold. Then

$$\tilde{J}\left(\tilde{n}, s^t\right) = \max_{v\geq 0, R, n}\left\{-cv + J\left(n, s^t\right)\right\} \tag{29}$$

$$\text{s.t. } n(x) = (1-\lambda)\int\left[G(x|\tilde{x}) - G(R|\tilde{x})\right]d\tilde{n}(\tilde{x}) + \left[G_{new}(x) - G_{new}(R)\right]q\left[\theta(s^t)\right]v. \tag{30}$$

Equation (30) states that employment with idiosyncratic productivity up to x at the time of production consists of two groups: incumbents that have not separated exogenously with new idiosyncratic productivity between R and x, and workers the firm is newly matched with for which idiosyncratic productivity is in the same range.

Adapting equation (23), the value of a firm at the time of production is

$$J\left(n, s^t\right) = a\left(s^t\right)\left(\int x dn(x)\right)^{1-\nu} - \int w\left(x, n, s^t\right)dn(x) + \beta\sum_{s^{t+1}|s^t}\Pi_t(s^{t+1})\tilde{J}\left(n, s^{t+1}\right).$$

In this setting Stole-Zwiebel bargaining yields the wage equation

$$\begin{aligned} w\left(x, n, s^t\right) = {} & \eta\frac{1-\nu}{1-\eta\nu}a\left(s^t\right)\left(\int x' dn(x')\right)^{-\nu} x + (1-\eta)b \\ & + \eta\beta\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)f\left[\theta(s^{t+1})\right]\frac{c}{q\left[\theta(s^{t+1})\right]}. \end{aligned} \tag{31}$$

This is identical to equation (28) except for taking into account that the bilateral marginal return of a match is proportional to idiosyncratic productivity x and depends on average idiosyncratic productivity given the specification of the production technology.

I start by deriving the equilibrium conditions. Substituting the wage equation into the firm value at the time of production yields

$$\begin{aligned} J(n, s^t) = (1-\eta)\frac{1-\nu}{1-\eta\nu} a\left(s^t\right)\left(\int x' dn(x')\right)^{1-\nu} - (1-\eta) b n(\infty) \\ - \eta\beta n(\infty) \sum_{s^{t+1}|s^t} \Pi_t\left(s^{t+1}\right) f\left[\theta(s^{t+1})\right] \frac{c}{q\left[\theta(s^{t+1})\right]} + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1}) \tilde{J}\left(n, s^{t+1}\right). \end{aligned}$$

The value to the firm of the marginal worker with idiosyncratic productivity x at the time of production then satisfies

$$\begin{aligned} J_n(x, n, s^t) = (1-\eta)\frac{1-\nu}{1-\eta\nu} a\left(s^t\right)\left(\int x' dn(x')\right)^{-\nu} x - (1-\eta) b \\ - \eta\beta \sum_{s^{t+1}|s^t} \Pi_t\left(s^{t+1}\right) f\left[\theta(s^{t+1})\right] \frac{c}{q\left[\theta(s^{t+1})\right]} + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1}) \tilde{J}_n\left(x, n, s^{t+1}\right). \end{aligned} \tag{32}$$

The corresponding marginal value of a worker with idiosyncratic productivity $\tilde{x}$ at the time of the choice of vacancies is

$$\tilde{J}_n\left(\tilde{x}, \tilde{n}, s^t\right) = (1-\lambda)\int_R^\infty J_n\left[x, \hat{n}\left(\cdot|\tilde{n}, s^t\right), s^t\right] dG\left(x|\tilde{x}\right) \tag{33}$$

where $\hat{n}\left(\cdot|\tilde{n}, s^t\right)$ is the policy function for optimization problem (29)–(30). Let $n(\cdot|s^t)$ denote equilibrium firm-level employment levels. Then aggregate employment levels are $N(\cdot|s^t) \equiv Fn(\cdot|s^t)$. The average bilateral marginal return can then be written as

$$m_{avg}\left(s^t\right) = \frac{1-\nu}{1-\eta\nu} a\left(s^t\right) F^\nu \left(\int x dN(x|s^t)\right)^{1-\nu} / N\left(\infty|s^t\right). \tag{34}$$

Evaluating equation (32) along the equilibrium path and substituting equations (33) and (34) yields

$$\begin{aligned} J_n(x, s^t) = (1-\eta)\left[m_{avg}\left(s^t\right)\frac{xN\left(\infty|s^t\right)}{\int x' dN(x'|s^t)} - b\right] - \eta\beta \sum_{s^{t+1}|s^t} \Pi_t\left(s^{t+1}\right)\frac{f\left[\theta(s^{t+1})\right] c}{q\left[\theta(s^{t+1})\right]} \\ + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1})(1-\lambda)\int_{R(s^{t+1})}^\infty J_n\left(x', s^{t+1}\right) dG(x'|x). \end{aligned} \tag{35}$$

The separation threshold is determined by the condition

$$J_n\left[R\left(s^t\right), s^t\right] = 0. \tag{36}$$

The condition for optimal job creation from optimization problem (29)–(30) evaluated along the equilibrium path is

$$\frac{c}{q\left[\theta(s^t)\right]} = \int_{R(s^t)}^{\infty} J_n\left(x, s^t\right) dG_{new}(x). \tag{37}$$

Evaluating wage equation (31) along the equilibrium path, integrating over $n(x|s^t)$ and substituting equation (34) yields the average wage

$$w_{avg}\left(s^t\right) = \eta\left[m_{avg}\left(s^t\right) - b\right] + b + \eta\beta \sum_{s^{t+1}|s^t} \Pi_t\left(s^{t+1}\right) f\left[\theta(s^{t+1})\right] \frac{c}{q\left[\theta(s^{t+1})\right]}. \tag{38}$$

Evaluating equation (30) along the equilibrium path then yields

$$\begin{aligned} N\left(x|s^t\right) = (1-\lambda) \int \left\{G(x|\tilde{x}) - G\left[R\left(s^t\right)|\tilde{x}\right]\right\} dN\left(\tilde{x}|s^{t-1}\right) \\ + \left\{G_{new}(x') - G_{new}\left[R\left(s^t\right)\right]\right\} f\left[\theta\left(s^t\right)\right]\left[1 - N\left(\infty|s^{t-1}\right)\right] \end{aligned} \tag{39}$$

where I also use the identity $Fv\left(s^t\right) = \theta\left(s^t\right)\left[1 - N\left(\infty|s^{t-1}\right)\right]$. The equilibrium $\{\theta\left(s^t\right), N\left(\cdot|s^t\right), R\left(s^t\right), J_n\left(\cdot|s^t\right), w_{avg}\left(s^t\right)\}_{s^t \in \mathcal{S}}$ is characterized by equations (35)–(39) together with initial aggregate employment levels N_0.

Next, I turn to establishing invariance. As for the basic model, I first takes as given that the model can match the targets and show that the calibration strategy determines aggregate labor market dynamics. As explained in the text, I initially take the idiosyncratic productivity process described by G and G_{new} as fixed.

The steady-state job-finding probability is now $\theta^* q(\theta^*)\left[1 - G_{new}(R^*)\right]$, hence this is the variable for which $\overline{f^*}$ provides a target. Analogously, the vacancy-filling probability is $q(\theta^*)\left[1 - G_{new}(R^*)\right]$, hence $\overline{h_w^*}$ is a target for $c/\{q(\theta^*)\left[1 - G_{new}(R^*)\right] w_{avg}^*\}$.

In the basic model all separations are exogenous and the observed separation rate is used to externally calibrate λ. Here there are both endogenous and exogenous separations. Thus, the observed separation rate does not identify the exogenous separation probability λ, and is instead used as a target $\overline{d^*}$ for the total steady-state separation rate:

$$\lambda + (1-\lambda) \int G\left(R^*|x\right) dN^*(x)/N^*(\infty) = \overline{d^*}, \tag{40}$$

where $N^*(\cdot)$ denotes steady-state aggregate employment levels.

Taking the steady-state versions of equations (35)–(39), dividing the first four of these equations by the steady-state average wage w_{avg}^*, and substituting targets and externally calibrated parameters yields

$$\begin{aligned} J_{n,w}^*(x) = (1-\eta)\left[m_w^* \frac{xN^*\left(x_{\max}\right)}{\int x' dN^*(x')} - \overline{b_w^*}\right] - \eta\overline{\beta f^* h_w^*} \\ + \overline{\beta}(1-\lambda) \int_{R^*}^{\infty} J_{n,w}\left(x'\right) dG(x'|x) \quad \forall\, x \in [0, \infty), \end{aligned} \tag{41}$$

$$J_{n,w}^{*}\left(R^{*}\right)=0, \tag{42}$$

$$\overline{h_w^*}=\int_{R^*}^{\infty} J_{n,w}^{*}(x)dG_{new}(x)/\left[1-G_{new}(R^*)\right], \tag{43}$$

$$1=\eta\left[m_w^*-\overline{b_w^*}\right]+\overline{b_w^*}+\eta\overline{\beta f^* h_w^*}, \tag{44}$$

$$\begin{aligned} N^{*}(x)=&(1-\lambda)\int\left[G(x|\tilde{x})-G(R^*|\tilde{x})\right]dN^{*}(\tilde{x}) \\ &+\frac{G_{new}(x)-G_{new}(R^*)}{1-G_{new}(R^*)}\overline{f^*}\left[1-N^{*}(\infty)\right] \quad \forall\, x\in[0,\infty), \end{aligned} \tag{45}$$

where $J_{n,w}^{*}(x)$ and m_w^* are the steady-state values of $J_n^*\left(x,s^t\right)$ and $m_{avg}\left(s^t\right)$, respectively, divided by the steady-state average wage w_{avg}^{*}.

Everything in the system of six equations (40)–(45) is directly determined by the calibration strategy except for the six variables λ, R^*, η, m_w^*, $N^*(x)$, and $J_{n,w}^{*}(x)$. This system is the counterpart of the system (11)–(12) through which the benchmark strategy determines η and m_w^* in the basic model. Analogously, here the strategy determines the six variables listed above, and I denote the corresponding values $\overline{\lambda}$, $\overline{R^*}$, $\overline{\eta}$, $\overline{m_w^*}$, $\overline{N^*}(x)$, and $\overline{J_{n,w}^{*}}(x)$.

Returning to the stochastic equilibrium, equation (34) implies that the average bilateral return and labor productivity are proportional:

$$m_{avg}\left(s^t\right)=\frac{1-\nu}{1-\eta\nu}p\left(s^t\right).$$

Using this relationship to replace $m_{avg}\left(s^t\right)$ in equation (35), dividing equations (35)–(37) by the steady-state average wage, and substituting quantities determined by the calibration strategy into these three equations as well as into equation (39) yields the system

$$\begin{aligned} J_{n,w}(x,s^t)=&(1-\overline{\eta})\left[\overline{m_w^*}\frac{\overline{p}\left(s^t\right)}{\overline{p^*}}\frac{xN\left(\infty|s^t\right)}{\int x'dN(x'|s^t)}-\overline{b_w^*}\right]-\overline{\eta}\overline{\beta}\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)\overline{f^*h_w^*}\frac{\theta\left(s^{t+1}\right)}{\overline{\theta^*}} \\ &+\overline{\beta}\sum_{s^{t+1}|s^t}\Pi_t(s^{t+1})\left(1-\overline{\lambda}\right)\int_{R(s^{t+1})}^{\infty}J_{n,w}\left(x',s^{t+1}\right)dG(x'|x) \quad \forall\, x\in[0,\infty), \end{aligned} \tag{46}$$

$$0=J_{n,w}\left[R\left(s^t\right),s^t\right], \tag{47}$$

$$\overline{h_w^*}\frac{\overline{q^*}}{\overline{q}\left[\theta(s^t)\right]}=\int_{R(s^t)}^{\infty}J_{n,w}\left(x,s^t\right)dG_{new}(x), \tag{48}$$

$$N\left(x|s^t\right)=\left(1-\overline{\lambda}\right)\int\left\{G(x|\tilde{x})-G\left[R\left(s^t\right)|\tilde{x}\right]\right\}dN\left(\tilde{x}|s^{t-1}\right) \tag{49}$$

$$+\left\{G_{new}(x')-G_{new}\left[R\left(s^t\right)\right]\right\}\overline{f}\left[\theta\left(s^t\right)\right]\cdot\left[1-N\left(\infty|s^{t-1}\right)\right] \quad \forall\, x\in[0,\infty), \tag{50}$$

where $J_{n,w}\left(x,s^t\right)\equiv J_n\left(x,s^t\right)/w_{avg}^{*}$. This is a system of four stochastic difference equations in the four variables $J_{n,w}\left(x,s^t\right)$, $R\left(s^t\right)$, $\theta\left(s^t\right)$ and $N\left(x|s^t\right)$. It is the counterpart of system (15)–(16) through which the benchmark strategy determines $\theta\left(s^t\right)$ and $N\left(s^t\right)$ in the basic model. Here the system is enlarged, but remains determined by the calibration strategy. Unemployment and vacancies are then determined by equations (6)–(7) with $N(s^t) = N(\infty|s^t)$. The total separation rate is determined by the strategy via

$$d\left(s^t\right) = \overline{\lambda} + \left(1 - \overline{\lambda}\right) \int G\left[\overline{R}\left(s^t\right)|x\right] d\overline{N}(x|s^{t-1}) / \overline{N}(\infty|s^{t-1}).$$

Evaluating wage equation (31) along the equilibrium path, dividing by the steady-state average wage, and substituting quantities determined by the calibration strategy yields

$$\frac{w\left(x, s^t\right)}{w^*_{avg}} = \overline{\eta}\left[\overline{m^*_w}\frac{\overline{p}\left(s^t\right)}{\overline{p^*}}\frac{x\overline{N}\left(\infty|s^t\right)}{\int x' d\overline{N}(x'|s^t)} - \overline{b^*_w}\right] + \overline{b^*_w} + \overline{\eta}\overline{\beta}\sum_{s^{t+1}|s^t}\Pi_t\left(s^{t+1}\right)\overline{f^* h^*_w}\frac{\theta\left(s^{t+1}\right)}{\overline{\theta^*}}.$$

Thus, the wage $w\left(x, s^t\right)$ is determined up to scale. Consequently, the same is true for the wage $w(z^t, s^t)$ conditional on individual history z^t and aggregate history s^t.

So far I have taken the idiosyncratic productivity process as given, now I turn to its identification. Since the calibration strategy determines individual wage dynamics up to scale as well as aggregate labor dynamics for given G and G_{new}, the equation

$$l\left(\left\{\theta\left(s^t\right), N\left(s^t\right), U\left(s^t\right), V\left(s^t\right), s\left(s^t\right)\right\}_{s^t \in \mathcal{S}}, \left\{w(z^t, s^t)\right\}_{z^t \in \mathcal{Z}, s^t \in \mathcal{S}}\right) = \overline{l}$$

provides a system of equations for the parameters of the idiosyncratic productivity process that is determined by the calibration strategy. By assumption, the function l is such that this system has a unique solution. Thus, the idiosyncratic productivity process is determined by the calibration strategy, and the corresponding values are denoted $\overline{G}$ and $\overline{G}_{new}$. Since the calibration strategy determines aggregate labor market dynamics for given G and G_{new}, it then follows that the strategy determines aggregate labor market dynamics.

For the basic model I show explicitly that it can match the targets for any $\nu \in [0, 1)$ and $F > 0$. This is not possible here since I have not explicitly specified the parametrization of the idiosyncratic productivity process and the function l, and put no restrictions on $\overline{l}$. One can specify these objects in a way such that the model with constant returns cannot match the targets. If the model with constant returns can match the targets, however, then the analysis above implies that the model can match the targets for any $\nu \in [0, 1)$.

E Invariance under Bargaining with Commitment

Bargaining with commitment as proposed by Hawkins (2015) differs from Stole-Zwiebel bargaining in that when a worker and a firm meet, they sign a contract that determines the timing and structure of future wages and the continuation of employment. This ensures that, unlike under Stole-Zwiebel bargaining, wages received by a worker are not affected by the outcome of bargaining between the firm and workers that arrive in the future. As with Stole-Zwiebel bargaining, the surplus of a new employment relationship is split between the firm and the worker according to a sharing rule with shares $1-\eta$ and η, respectively.

The timing of wage payments over the course of an employment relationship is indeterminate under bargaining with commitment. For simplicity of exposition, I follow Hawkins (2011b) in deriving the equilibrium conditions using a specific timing assumption under which the firms pay an initial hiring bonus $w^h(n, s^t)$, followed for the remainder of the contract by wage payments that make workers indifferent between continued employment and unemployment. The wage paid to continuing workers after the initial bonus is then

$$w^c\left(s^t\right) = b + \beta \sum_{s^{t+1}|s^t} \Pi_t\left(s^{t+1}\right) f\left[\theta(s^{t+1})\right] w^h\left[n\left(s^{t+1}\right), s^{t+1}\right], \tag{51}$$

only compensating workers for giving up the flow payoff from unemployment and the chance to receive a bonus when meeting a new firm. The surplus obtained by a newly-hired worker is exactly the hiring bonus, hence the surplus sharing rule takes the form

$$\eta\left[J_n\left(n, s^t\right) - w^h\left(n, s^t\right)\right] = (1-\eta) w^h\left(n, s^t\right) \tag{52}$$

where $J_n\left(n, s^t\right)$ denotes the value of the marginal worker to the firm after paying the bonus, so that $J_n\left(n, s^t\right) - w^h\left(n, s^t\right)$ is the surplus the firm obtains in the bargain.

With this model of bargaining, the recruiting decision of the firm yields the condition

$$\frac{c}{q\left[\theta\left(s^t\right)\right]} + w^h\left(n, s^t\right) = J_n\left(n, s^t\right), \tag{53}$$

equalizing the sum of hiring costs and bonus to the value that the marginal worker has to the firm after the bonus is paid. Adapting equation (24), the latter satisfies

$$J_n\left(n, s^t\right) = a\left(s^t\right)(1-\nu)n^{-\nu} - w^c\left(s^t\right) + \beta \sum_{s^{t+1}|s^t} \Pi_t(s^{t+1})(1-\lambda) J_n\left[\hat{n}\left(n, s^{t+1}\right), s^{t+1}\right]. \tag{54}$$

The derivative of the wage with respect to the employment level does not appear, since here the wage in ongoing employment relationships does not depend on the employment level.

Evaluating equations (51)–(54) along the equilibrium path and combining them to eliminate the ongoing wage, the bonus and appearances of the function J_n yields equation (4) with the bilateral marginal return in equation (3) replaced by the marginal product $m\left(s^t\right) = (1-\nu)a\left(s^t\right)\left[N\left(s^t\right)/F\right]^{-\nu}$. This coincides with the corresponding equation obtained for the bargaining outcome considered by Krause and Lubik.

In steady state the mass of new hires in a period corresponds to a fraction λ of employed workers, hence the steady-state average wage is $w^* = w^{c*} + \lambda w^{h*}$. Using the steady-state versions of equations (51)–(53) to substitute for w^{c*} and w^{h*} yields

$$w^* = b + \frac{\eta}{1-\eta}\left[\lambda + \beta f\left(\theta^*\right)\right] \frac{c}{q(\theta^*)}.$$

Dividing by w^* and substituting quantities determined by the calibration strategy yields

$$1 = \overline{b_w^*} + \frac{\eta}{1-\eta}\left(\overline{\lambda} + \overline{\beta f^*}\right)\overline{h_w^*}. \tag{55}$$

Thus, as with Stole-Zwiebel bargaining, the strategy identifies η. Solving for η yields

$$\eta = \frac{\left(1-\overline{b_w^*}\right)/\left(\overline{\lambda}+\overline{\beta f^*}\right)}{\overline{h_w^*}+\left(1-\overline{b_w^*}\right)/\left(\overline{\lambda}+\overline{\beta f^*}\right)}. \tag{56}$$

Since equation (4) holds, I can solve its steady-state version for m_w^* and use the fact that η is determined by the calibration strategy:

$$m_w^* = \overline{b_w^*} + \left[1-\overline{\beta}\left(1-\overline{\lambda}-\overline{\eta}\overline{f^*}\right)\right]\overline{h_w^*}/\left(1-\overline{\eta}\right).$$

Thus, the calibration strategy also determines m_w^*. The remainder of the argument establishing invariance then goes through without change. In particular, equation (15) holds with the new expressions for $\overline{\eta}$ and $\overline{m_w^*}$.

The values of $\overline{\eta}$ and $\overline{m_w^*}$ differ from the corresponding values under Stole-Zwiebel bargaining. This does not interfere with invariance with respect to the strength of diminishing returns, which only requires that the strategy determines η and m_w^*. Yet the different values of $\overline{\eta}$ and $\overline{m_w^*}$ do imply that aggregate dynamics are not exactly identical for Stole-Zwiebel bargaining and bargaining under commitment with this timing of wage payments, even in the version of the model with constant returns. The values of $\overline{\eta}$ and $\overline{m_w^*}$ differ due to the different timing of wage payments in interaction with discounting. The hiring bonus implies that wage payments are front-loaded, hence the steady-state average wage is lower for a given present discounted value. Thus, the calibrated bargaining power must be higher than under Stole-Zwiebel bargaining to match the targets $\overline{b_w^*}$ and $\overline{h_w^*}$. This in turn implies that $\overline{m_w^*}$ must also be higher to cover the user cost of labor.

Hawkins (2015) considers two alternative assumptions concerning the timing of wage payments. Under the first assumption, firms pays a constant wage over the course of an employment relationship. Under the second, the wage of incumbents is adjusted to the level of new hires every period, and any change in the present discounted value of wages induced by this adjustment is offset with lump sum transfers. Both assumptions imply a constant wage in ongoing employment relationships in steady state, as in the bargaining outcome considered by Krause and Lubik. This reinstates equation (12). The values of η and m_w^* determined by the benchmark calibration strategy are then the same as under Stole-Zwiebel bargaining. This implies identical aggregate dynamics for bargaining with commitment and Stole-Zwiebel bargaining under the benchmark calibration strategy.

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