TI 2021-024/II
Tinbergen Institute Discussion Paper

Balanced Externalities and the Proportional Allocation of Nonseparable Contributions

Rene van den Brink[1]
Youngsub Chun[2]
Yukihiko Funaki[3]
Zhengxing Zou[4]

[1] Department of Economics, and Tinbergen Institute, Vrije Universiteit Amsterdam
[2] Department of Economics, Seoul National University, Seoul, Korea
[3] Department of Economics, School of Political Science and Economics, Waseda University, Tokyo, Japan
[4] School of Economics and Management, Beijing Jiaotong University, Beijing, China

Balanced Externalities and the Proportional Allocation of Nonseparable Contributions

René van den Brink[1] Youngsub Chun[2] Yukihiko Funaki[3] Zhengxing Zou[4]

March 12, 2021

[1]Department of Economics, and Tinbergen Institute, Vrije Universiteit Amsterdam, Amsterdam, The Netherlands. E-mail: jrbrink@feweb.vu.nl

[2]Department of Economics, Seoul National University, Seoul 08826, Korea. E-mail: ychun@snu.ac.kr

[3]Department of Economics, School of Political Science and Economics, Waseda University, Tokyo, 169-8050 Japan. E-mail: funaki@waseda.jp

[4]School of Economics and Management, Beijing Jiaotong University, Beijing 100044, China. Department of Economics, Vrije Universiteit Amsterdam, Amsterdam, The Netherlands. E-mail: zhengxingzou@126.com

Abstract

In this paper, we study the implications of extending the balanced cost reduction property from queueing problems to general games. As a direct translation of the balanced cost reduction property, the axiom of *balanced externalities* for solutions of games, requires that the payoff of any player is equal to the total externality she inflicts on the other players with her presence. We show that this axiom and efficiency together characterize the Shapley value for 2-additive games. However, extending this axiom in a straightfoward way to general games is incompatible with efficiency. Keeping as close as possible to the idea behind balanced externalities, we weaken this axiom by requiring that every player's payoff is the same *fraction* of its total externality inflicted on the other players. This weakening, which we call *weak balanced externalities*, turns out to be compatible with efficiency. More specifically, the unique efficient solution that satisfies this weaker property is the *proportional allocation of nonseparable contribution* (PANSC) value, which allocates the total worth proportional to the *separable costs* of the players. We also provide characterizations of the PANSC value using a reduced game consistency axiom.

Keywords: Cooperative game, balanced externalities, proportional allocation of nonseparable contributions, consistency
JEL classification number: C71

1 Introduction

A situation in which a finite set of players can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a game). A game consists of a player set, and for every subset of the player set, called a *coalition*, a real number which is the *worth* that the coalition of players can earn when they agree to cooperate. A main question is how to allocate the worth of the grand coalition consisting of all players over the individual players. Various solutions to this problem appear in the literature, one of the most famous being the Shapley value (Shapley, 1953).

Games are applied in many profit and cost allocation problems in economics and operations research. An example is the queueing problem. A *queueing problem* describes a situation where jobs need to be served on a machine one at a time. A queue is efficient if jobs are served in a non-increasing order of their urgency indices. But then the question is how jobs that are served later should be compensated for waiting in the queue. One of the most popular solutions for such queueing problems is the *minimal transfer rule* (Maniquet, 2003) which is obtained by applying the Shapley value to an associated game. Queueing games are so-called *2-additive games*, or shortly *2-games*, meaning that the worth is fully generated by coalitions of size two. For a nonnegative 2-game, it is known that the Shapley value, and thus the minimal transfer rule, coincides with several other solutions such as the *nucleolus* (Schmeidler, 1969) or *τ-value* (Tijs, 1987) of the associated queueing game (van den Nouweland et al., 1996).

Solutions for games are usually supported by axiomatizations. In van den Brink and Chun (2012), the minimal transfer rule for queueing problems is axiomatized by *efficiency*, *Pareto indifference*, and *balanced cost reduction*. Whereas efficiency and Pareto indifference are very common axioms, balanced cost reduction requires that the payoff of any player is equal to the total externality she inflicts on the other players with her presence, i.e. a player's payoff equals the sum of all changes in the payoffs of all other players if that player leaves the queueing problem.

In this paper, we study the implications of extending this balanced cost reduction property to general games. First, considering the class of *2-games*, we show that the Shapley value (and thus pre-nucleolus, τ-value) is the unique efficient solution that satisfies *balanced externalities* being a direct translation of balanced cost reduction, requiring that the payoff of any player is equal to the total externalities she inflicts on the other players. Second, we extend this axiom to k-games being games where every worth is generated by coalitions of size k, and obtain a characterization of the Shapley value for k-games. Third, it turns out that this axiom is incompatible with efficiency for general games.

Keeping as close as possible to the idea of having an efficient solution which allocates the worth of the grand coalition in a way that balances a player's payoff with the externalities she inflicts on the other players, we weaken balanced externalities by requiring that every player's payoff is the same *fraction* of her total externality inflicted on the other players. This brings in one extra parameter (the fraction of total externality that is attributed to the players), which makes this weak balanced externalities axiom compatible with efficiency. We show that the unique efficient solution that satisfies this weak balanced externalities axiom is the *proportional allocation of nonseparable contribution* (PANSC) value, which allocates the payoffs proportional to the *separable costs* (Moulin, 1985) of the players. It is interesting to note that this value is closely related to the *Separable Costs Remaining Benefits* (SCRB) method (Young et al., 1982) and *Alternative Cost Avoided* (ACA) method (Straffin and Heaney, 1981; Otten, 1993) in cost allocation problems. The SCRB method is commonly used in practice, for example in allocating the costs of multi-purpose water development projects (Straffin and Heaney, 1981; Young et al., 1982).

We also consider the dual solution of the PANSC value, being the proportional division (PD) value (Zou et al., 2021), which allocates the worth of the grand coalition proportional to the stand-alone worths of the players, and extend weak balanced externalities and the axiomatization mentioned above using mollifier games (i.e. affine combinations of a game and its dual game, see Charnes et al. (1978)). A comparison between the PANSC value and PD value in terms of optimizing satisfaction criteria and associated consistency is given in Li et al. (2020). Finally, we discuss a reduced game consistency property of the PANSC value, which, by duality, follows from the reduced game consistency property of the PD value.

The paper is structured as follows. After presenting preliminaries in Section 2, in Section 3 we extend the axiomatization of the Shapley value for queueing problems and provide an axiomatization by efficiency and balanced externalities for the classes of 2-games, and more general k-games. In Section 4, we extend this axiomatization to general games and show incompatibility of efficiency and balanced externalities. We weaken balanced externalities to get compatibility, and use this weaker axiom to characterize the PANSC value. In Section 5, we consider the dual solution of the PANSC value, i.e. the PD value. In Section 6, we consider a reduced game consistency property of the PANSC value. In Section 7, we introduce cost allocation problems and compare the PANSC value with cost allocation methods from the literature. Finally, Section 8 concludes.

2 Preliminaries

A *cooperative game with transferable utility*, shortly a *game*, is a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of players, and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the *worth* of coalition S. This is what the members of coalition S can obtain by agreeing to cooperate. We denote the class of all games with player set N by $\mathcal{G}^N$, and the class of all games by $\mathcal{G}$.

A game (N, v) is *additive* if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. For every $T \subseteq N$, $T \neq \emptyset$, the *unanimity* game $(N, u_T) \in \mathcal{G}$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that for every game (N, v), there exist unique weights $\Delta_v(T) \in \mathbb{R}$, $\emptyset \neq T \subseteq N$, such that $v = \sum_{T \subseteq N} \Delta_v(T) u_T$. The weights $\Delta_v(T)$, $\emptyset \neq T \subseteq N$, are the (Harsanyi) *dividends* (Harsanyi, 1959) of the coalitions in (N, v) and are given by $\Delta_v(T) = v(T)$ if $|T| = 1$, and $\Delta_v(T) = v(T) - \sum_{S \subset T, S \neq T} \Delta_v(S)$ if $|T| \geq 2$.

A *payoff vector* of game (N, v) is an $|N|$-dimensional real vector $x \in \mathbb{R}^N$, which represents the payoffs that players can earn by cooperation. A (point-valued) *solution* on a class of games $\mathcal{C} \subseteq \mathcal{G}$ is a function ψ which assigns a payoff vector $\psi(N, v) \in \mathbb{R}^N$ to every game $(N, v) \in \mathcal{C}$. If a solution assigns to every game a payoff vector that exactly distributes the worth of the grand coalition, then the solution is *efficient*.

- **Efficiency**. A solution ψ satisfies efficiency on $\mathcal{C} \subseteq \mathcal{G}$ if for every game $(N, v) \in \mathcal{C}$, it holds that $\sum_{i \in N} \psi_i(N, v) = v(N)$.

Efficient (point-valued) solutions are often called *values*. One of the most famous values for games is the *Shapley value* (Shapley, 1953), being the solution Sh given by

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\Delta_v(S)}{|S|} \quad \text{for all } i \in N,$$

and thus allocates the Harsanyi dividends equally over the players in the corresponding unanimity coalition.

The literature also studies several efficient egalitarian solutions, such as the *equal division* value (which allocates $v(N)$ equally over all players, axiomatized in van den Brink (2007)), the *centre of the imputation set* (CIS) value (which first assigns to every player its stand-alone worth and allocates the remainder equally over all players (Driessen and Funaki, 1991)), and the *equal allocation of nonseparable cost* (EANSC) value (which first asigns to every player its separable cost and allocates the remainder equally over all players (Moulin, 1985)). Formally, the CIS value is given by

$$CIS_i(N, v) = v(\{i\}) + \frac{1}{|N|} \left(v(N) - \sum_{j \in N} v(\{j\}) \right) \quad \text{for all } i \in N.$$

The EANSC value is given by

$$EANSC_i(N, v) = SC_i(N, v) + \frac{1}{|N|} \left(v(N) - \sum_{j \in N} SC_j(N, v) \right) \quad \text{for all } i \in N, \tag{1}$$

where

$$SC_i(N, v) = v(N) - v(N \setminus \{i\}),$$

is the *separable cost* of player $i \in N$ in game (N, v).

3 Balanced externalities and 2-games

A game (N, v) is a *2-additive game*, or shortly a *2-game*, if $v(S) = 0$ for all $S \subseteq N$ with $|S| \leq 1$, and $v(S) = \sum_{\substack{T \subseteq S \\ |T|=2}} v(T)$ otherwise. Equivalently, a game (N, v) is a 2-game if and only if only coalitions of size two can have a nonzero dividend, i.e. $\Delta_v(S) \neq 0$ implies that $|S| = 2$. Therefore, in 2-games, all the worth is generated by coalitions of size two. It is known that for nonnegative 2-games (i.e. 2-games in which all worths are nonnegative), the Shapley value coincides with several other values such as the nucleolus and the τ-value (van den Nouweland et al., 1996). For 2-games on N with $|N| \geq 2$, this value is given by $Sh_i(N, v) = \frac{1}{2}(v(N) - v(N \setminus \{i\}))$ for all $i \in N$.

A 2-game can be generalized to a *k-additive game*, or shortly a *k-game*. A game (N, v) is a k-game, if $v(S) = 0$ for all $S \subseteq N$ with $|S| < k$, and $v(S) = \sum_{\substack{T \subseteq S \\ |T|=k}} v(T)$ otherwise. Equivalently, a game (N, v) is a k-game if and only if only coalitions of size k can have a nonzero dividend, i.e. $\Delta_v(S) \neq 0$ implies that $|S| = k$. It is known that for k-games on N with $|N| \geq k$, the Shapley value is given by $Sh_i(N, v) = \frac{1}{k}(v(N) - v(N \setminus \{i\}))$ for all $i \in N$ (van den Nouweland et al., 1996). Also for nonnegative k-games, the Shapley value coincides with the τ-value, but for $k > 2$, the payoff vector assigned to a game by the nucleolus need not coincide with the payoff vector assigned by the Shapley value. In this paper, we do not restrict the sign of the worths of coalitions in 2-games, as well as k-games.

As mentioned in the introduction, queueing games form a proper subset of the class of 2-games. One of the most famous solutions for queueing problems is the minimal transfer rule that is obtained as the Shapley value of an associated queueing game (Maniquet, 2003). In van den Brink and Chun (2012), the minimal tranfer rule is characterized as the unique solution for queueing problems that satisfies efficiency, Pareto indifference, and balanced cost reduction. The question that we address in this paper is which solutions for games satisfy, or are characterized by, (an extension of) these axioms for general games. We first consider the class of 2-games. Throughout the sequel, we denote by $(N \setminus \{h\}, v^{-h})$ the restricted game on $N \setminus \{h\}$, given by $v^{-h}(S) = v(S)$ for all $S \subseteq N \setminus \{h\}$.

A direct translation of balanced cost reduction for 2-games gives the following property.

- **Balanced externalities**. A solution ψ on the class of 2-games satisfies balanced externalities if

$$\psi_h(N,v) = \sum_{i\in N\setminus\{h\}} \left(\psi_i(N,v) - \psi_i(N\setminus\{h\}, v^{-h})\right),$$

for every 2-game (N,v) with $|N| \geq 2$ and $h \in N$.

Notice that this axiom is well-defined since $(N\setminus\{h\}, v^{-h})$ is a 2-game if (N,v) is a 2-game. Together with efficiency, this axiom characterizes the Shapley value on the class of 2-games.[1]

Theorem 1 *For 2-games, a solution satisfies efficiency and balanced externalities if and only if it is the Shapley value.*

Proof. It is well-known that the Shapley value is efficient. To show that the Shapley value satisfies balanced externalities, let (N,v) be a 2-game such that $|N| \geq 2$, and $h \in N$. Then

$$\begin{aligned}
\sum_{i\in N\setminus\{h\}} \left(Sh_i(N,v) - Sh_i(N\setminus\{h\}, v^{-h})\right) &= \sum_{i\in N\setminus\{h\}} \left(\sum_{\substack{S\subseteq N, |S|=2\\ i\in S}} \frac{\Delta_v(S)}{2} - \sum_{\substack{S\subseteq N, |S|=2\\ i\in S, h\notin S}} \frac{\Delta_v(S)}{2} \right) \\
&= \sum_{i\in N\setminus\{h\}} \sum_{\substack{S\subseteq N, |S|=2\\ i,h\in S}} \frac{\Delta_v(S)}{2} \\
&= \sum_{\substack{S\subseteq N, |S|=2\\ h\in S}} \frac{\Delta_v(S)}{2} \\
&= Sh_h(N,v),
\end{aligned}$$

where the first equality follows since $\Delta_v(S) = \Delta_{v^{-h}}(S)$ for all $S \subseteq N\setminus\{h\}$, and the last equality follows since only coalitions of size 2 have a nonzero dividend. This shows that the Shapley value satisfies balanced externalities.

We show the 'only if' part by induction on $|N|$. If $|N| = 1$, then $\psi_i(N,v) = v(\{i\}) = 0 = Sh_i(N,v)$ by efficiency. If $|N| = 2$ such that $N = \{i,j\}$, then balanced externalities implies that $\psi_i(N,v) = \psi_j(N,v) - \psi_j(\{j\}, v^{-i}) = \psi_j(N,v)$. With efficiency and the case $|N| = 1$ above, it then follows that $\psi_i(N,v) = \psi_j(N,v) = \frac{v(N)}{2}$.

We will establish the claim for an arbitrary number of players by an induction argument. As induction hypothesis, suppose that uniqueness holds for all $N' \subset \mathbb{N}$ such that $2 \leq |N'| \leq |N| - 1$. Consider 2-game (N,v). For any $h \in N$, balanced externalities yields

$$\psi_h(N,v) = \sum_{i\in N\setminus\{h\}} \left(\psi_i(N,v) - \psi_i(N\setminus\{h\}, v^{-h})\right). \tag{2}$$

[1] We can even prove the uniqueness with the weaker axiom of 2-efficiency, requiring efficiency only for games with at most two players.

By the induction hypothesis, the $\psi_i(N\backslash\{h\}, v^{-h}),\ i,h \in N,\ i \neq h$, are uniquely determined. Since $|N| \geq 3$, (2) and efficiency yield a system of $(|N|-1)+1 = |N|$ linearly independent equations in the $|N|$ unkowns $\psi_h(N,v),\ h \in N$, which thus are uniquely determined. □

As a corollary, we have that, on the class of 2-games, the pre-nucleolus and the τ-value are also characterized by efficiency and balanced externalities.

We can generalize this result straightforward to the class of k-games by introducing a generalization of balanced externalities for k-games, $k \geq 2$.

- **k-balanced externalities**. Let $k \geq 2$. A solution ψ on the class of k-games satisfies k-balanced externalities if

$$\psi_h(N,v) = \frac{1}{k-1} \sum_{i \in N\backslash\{h\}} \left(\psi_i(N,v) - \psi_i(N \setminus \{h\}, v^{-h})\right),$$

  for every k-game (N,v) with $|N| \geq k$ and $h \in N$.

Notice again that this axiom is well-defined since $(N \setminus \{h\}, v^{-h})$ is a k-game if (N,v) is a k-game. Together with efficiency and symmetry, this axiom characterizes the Shapley value on the class of k-games.[2]

- **Symmetry**. A solution ψ satisfies symmetry on $\mathcal{C} \subseteq \mathcal{G}$ if for every game $(N,v) \in \mathcal{C}$ and $i,j \in N$ such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i,j\}$, it holds that $\psi_i(N,v) = \psi_j(N,v)$.

Theorem 2 *For k-games, a solution satisfies efficiency, symmetry, and k-balanced externalities if and only if it is the Shapley value.*

Although the proof is almost the same as that of Theorem 1, for completeness, it is given in the Appendix. Notice that Theorem 1 is a special case of Theorem 2 by taking $k = 2$. In this case, symmetry is superfluous. However, for $k \geq 3$, symmetry cannot be taken out since, k-balanced externalities has no bite if $|N| < k$ in which case the game is a null game where the worths of all coalitions equal zero.

[2]Similar as for 2-games, it is sufficient to require k-efficiency which requires efficiency only for games with at most k players.

4 Balanced externalities and the proportional allocation of nonseparable contributions for general games

Translating the idea of balanced externalities to general games, we first consider the axiom which requires that the payoff of any player is equal to the total externality she inflicts on the other players with her presence. We call a class of games $\mathcal{C} \subseteq \mathcal{G}$ *subgame closed* if $(N \setminus \{h\}, v^{-h}) \in \mathcal{C}$ for all $(N, v) \in \mathcal{C}$ and $h \in N$.

- **Balanced externalities**. A solution ψ on a subgame closed class $\mathcal{C} \subseteq \mathcal{G}$ satisfies balanced externalities if

$$\psi_h(N, v) = \sum_{i \in N \setminus \{h\}} \big(\psi_i(N, v) - \psi_i(N \setminus \{h\}, v^{-h})\big),$$

for every game $(N, v) \in \mathcal{C}$ with $|N| \geq 2$, and $h \in N$.

It turns out that this axiom is incompatible with efficiency.

Proposition 3 *There is no solution on $\mathcal{G}$ that satisfies efficiency and balanced externalities.*

Proof. Consider a three-player game $(N, v) \in \mathcal{G}$ with $N = \{1, 2, 3\}$. Let ψ be a solution satisfying efficiency and balanced externalities. Then, we have the following six equations from balanced externalities (the first equality in each line) and efficiency (the second equality in each line):

$$\begin{aligned}
-\psi_1(N, v) + \psi_2(N, v) + \psi_3(N, v) &= \psi_2(N \setminus \{1\}, v^{-1}) + \psi_3(N \setminus \{1\}, v^{-1}) = v(\{2, 3\}),\\
\psi_1(N, v) - \psi_2(N, v) + \psi_3(N, v) &= \psi_1(N \setminus \{2\}, v^{-2}) + \psi_3(N \setminus \{2\}, v^{-2}) = v(\{1, 3\}),\\
\psi_1(N, v) + \psi_2(N, v) - \psi_3(N, v) &= \psi_1(N \setminus \{3\}, v^{-3}) + \psi_2(N \setminus \{3\}, v^{-3}) = v(\{1, 2\}).
\end{aligned}$$

Further, efficiency implies that

$$\psi_1(N, v) + \psi_2(N, v) + \psi_3(N, v) = v(\{1, 2, 3\}).$$

These four equations can be simplified to

$$\begin{aligned}
\psi_1(N, v) &= \frac{1}{2}(v(\{1, 2\}) + v(\{1, 3\})),\\
\psi_2(N, v) &= \frac{1}{2}(v(\{1, 2\}) + v(\{2, 3\})),\\
\psi_3(N, v) &= \frac{1}{2}(v(\{1, 3\}) + v(\{2, 3\})), \text{ and}\\
\psi_3(N, v) &= \frac{1}{2}(v(\{1, 2, 3\}) - v(\{1, 2\})).
\end{aligned}$$

Looking at the last two equations, this system of equations clearly has only a solution if $v(\{1,3\}) + v(\{2,3\}) = v(\{1,2,3\}) - v(\{1,2\})$, or $v(\{1,2,3\}) = v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})$, which implies that (N, v) is a 2-game. □

Next, we explore whether we can characterize (subclasses of) the class of 2-games as those classes of games where the Shapley value is characterized by efficiency and balanced externalities.

Proposition 4 *Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games that contains at least one game (N, v) with $|N| \geq 3$. Then the Shapley value is the unique solution that satisfies efficiency and balanced externalities on $\mathcal{C}$ if and only if $\mathcal{C}$ is a subclass of the class of 2-games.*

Proof. Let $\mathcal{C} \subseteq \mathcal{G}$ be a subgame closed class of games that contains at least one game (N, v) with $|N| \geq 3$. For the class of 2-games, the 'if' part follows from Theorem 1. For any subgame closed subclass of the class of 2-games, the proof goes in a similar way.

Next, we prove the 'only if' part by induction on $|N|$. Suppose that the Shapley value is the unique solution that satisfies efficiency and balanced externalities on $\mathcal{C}$.

Initialization. For $|N| = 3$, it follows from the proof of Proposition 3 that $\mathcal{C}$ should be a subclass of 2-games.

Induction hypothesis. Suppose that the Shapley value being characterized by efficiency and balanced externalities on every class $\mathcal{C} \subseteq \mathcal{G}$ with $|N| \leq d$ $(d \geq 3)$ for every $(N, v) \in \mathcal{C}$, implies that $\mathcal{C}$ is a class of 2-games.

Induction step. Consider any game (N, v) such that $|N| = d + 1$. We already know that $v(S) = v^{-h}(S) = \sum_{\substack{T \subseteq S \\ |T|=2}} v(T)$ for all $S \subseteq N \setminus \{h\}$ and $h \in N$, since $(N \setminus \{h\}, v^{-h})$ is a 2-game by the induction hypothesis. Let ψ be a solution satisfying efficiency and balanced externalities. Then,

$$\begin{aligned} \psi_h(N, v) &= \sum_{i \in N \setminus \{h\}} \psi_i(N, v) - \sum_{i \in N \setminus \{h\}} \psi_i(N \setminus \{h\}, v^{-h}) \\ &= \sum_{i \in N \setminus \{h\}} \psi_i(N, v) - v^{-h}(N \setminus \{h\}), \end{aligned}$$

where the first equality follows from balanced externalities and the second equality follows from efficiency.

Summing this equality over all $h \in N$ yields

$$\sum_{h \in N} \psi_h(N, v) = (|N| - 1) \sum_{h \in N} \psi_h(N, v) - \sum_{h \in N} v^{-h}(N \setminus \{h\}),$$

so $(|N|-2)\sum_{h\in N}\psi_h(N,v)=\sum_{h\in N}v^{-h}(N\setminus\{h\})$. Efficiency of ψ then implies that

$$(|N|-2)v(N)=\sum_{h\in N}v^{-h}(N\backslash\{h\})=\sum_{h\in N}\sum_{\substack{S\subseteq N\backslash\{h\}\\|S|=2}}v(S)=(|N|-2)\sum_{\substack{S\subseteq N\\|S|=2}}v(S),$$

where the second equality follows from the induction hypothesis, and the third follows since in $\sum_{S\subseteq N\backslash\{h\},|S|=2}v(S)$, the two-player coalition worth $v(S)$, $S=\{i,j\}$, appears once for each $h\in N\backslash\{i,j\}$. Therefore, we obtain that $v(N)=\sum_{S\subseteq N,|S|=2}v(S)$, which implies that (N,v) must be a 2-game. □

Notice that for $|N|=2$, every game with $v(\{i\})=v(\{j\})=0$ is a 2-game.

Keeping as close as possible to the idea of having an efficient solution which allocates the payoffs of the players to somehow 'balance' the externalities inflicted on the other players, we weaken balanced externalities by requiring that every player's payoff is the same *fraction* of her total externality inflicted on the other players.

- **Weak balanced externalities**. A solution ψ on a subgame closed class $\mathcal{C}\subseteq\mathcal{G}$ satisfies weak balanced externalities if for every $(N,v)\in\mathcal{C}$ with $|N|\geq 2$, there exists $\alpha\in\mathbb{R}$ such that for every $h\in N$, if $\sum_{i\in N\backslash\{h\}}\big(\psi_i(N,v)-\psi_i(N\setminus\{h\},v^{-h})\big)\neq 0$, then

$$\psi_h(N,v)=\alpha\sum_{i\in N\backslash\{h\}}\big(\psi_i(N,v)-\psi_i(N\setminus\{h\},v^{-h})\big)\,. \tag{3}$$

Notice that we require the balanced externalities condition to hold only in the case that $\sum_{i\in N\backslash\{h\}}\big(\psi_i(N,v)-\psi_i(N\setminus\{h\},v^{-h})\big)\neq 0$ since otherwise the payoff of h must be zero and is not dependent on α anymore. In the extreme case where equality would hold for all players, then all payoffs would be zero, which would be incompatible with efficiency if $v(N)>0$.[3]

This weakening of balanced externalities brings in one extra parameter (the fraction of total externality that is attributed to the players), which makes this weak balanced externalities compatible with efficiency. It turns out that the unique efficient solution that satisfies this axiom is the *proportional allocation of nonseparable contribution* (PANSC) value, which allocates the worth of the grand coalition proportional to the separable costs of the players. This value coincides with the *proportional repartition of the non-marginal costs* value (Lemaire, 1984) in cost allocation problems.

Let $\mathcal{G}_{sc+}=\{(N,v)\in\mathcal{G}\mid SC_i(N,v)>0\text{ for all }i\in N\}$ be the class of games with player set N where all players have positive separable cost. Let $\mathcal{G}^2_{sc+}=\{(N,v)\in\mathcal{G}_{sc+}\mid |N|=2\}$ and $\mathcal{G}^{\geq 2}_{sc+}=\{(N,v)\mid (N,v)\in\mathcal{G}_{sc+},|N|\geq 2\}$.

[3]This would occur if $v(N)=v(N\backslash\{h\})>0$ for all $h\in N$.

Definition 1 *The* **proportional allocation of nonseparable contribution** *(PANSC) value on $\mathcal{G}^{\geq 2}_{sc+}$ is the function PANSC which assigns to every game $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$, the payoff vector*

$$PANSC_i(N,v) = \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} v(N) \text{ for all } i \in N.$$

We restrict ourselves to the class $\mathcal{G}^{\geq 2}_{sc+}$ in order to avoid dividing by a zero denominator. We remark that the class $\mathcal{G}^{\geq 2}_{sc+}$ contains the *almost diminishing marginal contributions* games (Leng et al., 2020) with positive stand-alone worths.

Notice that equivalently the PANSC value first assigns to every player its separable cost and allocates the remainder (the total nonseparable cost) proportional to the separable costs. Thus, the difference from the EANSC value (see (1)) is that, after each player getting its separable cost, the PANSC allocates the total nonseperable cost proportional to the separable costs, while that EANSC allocates it equally over all players. (We provide a further comparison between the PANSC and EANSC value in Section 7.)

Theorem 5 *A solution on $\mathcal{G}^{\geq 2}_{sc+}$ satisfies efficiency and weak balanced externalities if and only if it is the PANSC value.*

Proof. It is obvious that the PANSC value is efficient. To show that the PANSC value satisfies weak balanced externalities, take any $h \in N$. If $\sum_{j\in N} SC_j(N,v) \neq v(N)$, taking $\alpha = \frac{v(N)}{\sum_{j\in N} SC_j(N,v) - v(N)}$ yields

$$\begin{aligned}
& \alpha \sum_{i \in N\setminus\{h\}} \left(PANSC_i(N,v) - PANSC_i(N\setminus\{h\}, v^{-h})\right) \\
= \; & \alpha \left(v(N) - PANSC_h(N,v) - v(N\setminus\{h\})\right) \\
= \; & \alpha \left(SC_h(N,v) - PANSC_h(N,v)\right) \\
= \; & \alpha \left(SC_h(N,v) - \frac{SC_h(N,v)}{\sum_{j\in N} SC_j(N,v)} v(N)\right) \\
= \; & \alpha \cdot SC_h(N,v) \cdot \left(1 - \frac{v(N)}{\sum_{j\in N} SC_j(N,v)}\right) \\
= \; & \alpha \cdot SC_h(N,v) \cdot \left(\frac{\sum_{j\in N} SC_j(N,v) - v(N)}{\sum_{j\in N} SC_j(N,v)}\right) \\
= \; & \left(\frac{v(N)}{\sum_{j\in N} SC_j(N,v) - v(N)}\right) \cdot SC_h(N,v) \cdot \left(\frac{\sum_{j\in N} SC_j(N,v) - v(N)}{\sum_{j\in N} SC_j(N,v)}\right) \\
= \; & \frac{SC_h(N,v)}{\sum_{j\in N} SC_j(N,v)} \cdot v(N) \\
= \; & PANSC_h(N,v),
\end{aligned}$$

where in the first equality we twice use that the PANSC value is efficient (once on game (N,v) and once on game $(N\setminus\{h\},v^{-h})$). This shows that the PANSC value satisfies weak balanced externalities if $\sum_{j\in N} SC_j(N,v)\neq v(N)$.

If $\sum_{j\in N} SC_j(N,v)=v(N)$, then $PANSC_i(N,v)=SC_i(N,v)$ for all $i\in N$, and thus we have by efficiency of the PANSC value that $\sum_{i\in N\setminus\{h\}}(PANSC_i(N,v)-PANSC_i(N\setminus\{h\},v^{-h}))=v(N)-PANSC_h(N,v)-v(N\setminus\{h\})=SC_h(N,v)-SC_h(N,v)=0$, and weak balanced externalities does not have any bite.

To prove the 'only if' part, suppose that a solution ψ satisfies efficiency and weak balanced externalities on $\mathcal{G}^{\geq 2}_{sc+}$. Let $(N,v)\in\mathcal{G}^{\geq 2}_{sc+}$. If $\sum_{i\in N\setminus\{h\}}(\psi_i(N,v)-\psi_i(N\setminus\{h\},v^{-h}))\neq 0$, then efficiency and weak balanced externalities together imply that there is $\alpha\in\mathbb{R}$ such that for any $h\in N$,

$$\begin{aligned}\psi_h(N,v) &= \alpha\sum_{i\in N\setminus\{h\}}(\psi_i(N,v)-\psi_i(N\setminus\{h\},v^{-h}))\\ &= \alpha\left(v(N)-\psi_h(N,v)-v(N\setminus\{h\})\right),\end{aligned}$$

and thus

$$(1+\alpha)\psi_h(N,v)=\alpha\left(v(N)-v(N\setminus\{h\})\right),$$

meaning

$$\psi_h(N,v)=\frac{\alpha}{1+\alpha}\left(v(N)-v(N\setminus\{h\})\right).$$

Efficiency determines that

$$\sum_{h\in N}\psi_h(N,v)=\sum_{h\in N}\frac{\alpha}{1+\alpha}\left(v(N)-v(N\setminus\{h\})\right)=\frac{\alpha}{1+\alpha}\sum_{h\in N}SC_h(N,v)=v(N),$$

implying that $\alpha=\frac{v(N)}{\sum_{h\in N}SC_h(N,v)-v(N)}$, and thus $\psi_h(N,v)=PANSC_h(N,v)$ for all $h\in N$.

If $\sum_{i\in N\setminus\{h\}}(\psi_i(N,v)-\psi_i(N\setminus\{h\},v^{-h}))=0$, then by efficiency, we have $v(N)-\psi_h(N,v)-v(N\setminus\{h\})=0$, implying that $\psi_h(N,v)=v(N)-v(N\setminus\{h\})=SC_h(N,v)=PANSC_h(N,v)$. □

The PANSC value can be seen as a multiplicative normalization of the separable costs to allocate $v(N)$. In the literature, this proportional allocation of nonseparable cost (or contribution) is less popular than the additive normalization of the nonseparable cost (or contribution), as done by the more famous EANSC value, that is mentioned in the preliminaries. However, as we will see in Section 7, for some cases the PANSC value coincides with well-known and applied solutions in cost allocation. In this section, we promoted the PANSC value by the weak balanced externalities axiom, which is inspired by a notion of fairness in queueing problems.

5 The dual solution: proportional division

Every solution has its dual solution which, instead of focusing on what coalitions can earn, considers what happens if any coalition leaves assuming that the grand coalition is already formed. In some cases, the dual solution equals the solution itself, in which case we call this solution *self-dual*. An example of a self-dual solution is the Shapley value.

Formally, the *dual* of game $(N, v) \in \mathcal{G}$ is the game $(N, v^*) \in \mathcal{G}$ given by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. The *dual* of solution ψ is the solution ψ^* that assigns to every game the payoff vector that ψ assigns to the dual game, i.e. $\psi^*(N, v) = \psi(N, v^*)$ for all $(N, v) \in \mathcal{G}$. The dual solution of the PANSC value is the *proportional division* (PD) value, recently studied by Zou et al. (2021), which allocates $v(N)$ proportional to the stand-alone worths of the players. The *PD value* on the class of games with positive stand-alone worths is given by

$$PD_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) \quad \text{for all } (N, v) \in \mathcal{G}_{vi+} \text{ and } i \in N,$$

where $\mathcal{G}_{vi+} = \{(N, v) \in \mathcal{G} \mid v(\{i\}) > 0 \text{ for all } i \in N\}$ is the class of games on N where all stand-alone worths are positive.[4]

Proposition 6 *For every game $(N, v) \in \mathcal{G}_{vi+}$, it holds that $PANSC(N, v^*) = PD(N, v)$.*

Proof. For every $(N, v) \in \mathcal{G}_{vi+}$ and $i \in N$, since $(N, v^*) \in \mathcal{G}_{sc+}$,

$$\begin{aligned} PANSC_i(N, v^*) &= \frac{v^*(N) - v^*(N \setminus \{i\})}{\sum_{j \in N}(v^*(N) - v^*(N \setminus \{j\}))} v^*(N) \\ &= \frac{v(N) - (v(N) - v(\{i\}))}{\sum_{j \in N}(v(N) - (v(N) - v(\{j\})))} v(N) \\ &= \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) \\ &= PD_i(N, v). \end{aligned}$$

□

Notice that, under efficiency, (3) in the definition of weak balanced externalities can be written as

$$\psi_h(N, v) = \alpha \left(\sum_{i \in N \setminus \{h\}} \psi_i(N, v) - v(N \setminus \{h\}) \right)$$

for every $h \in N$ with $\sum_{i \in N \setminus \{h\}} \psi_i(N, v) - v(N \setminus \{h\}) \neq 0$. This shows that the payoff assigned to player i equals the externality it inflicts on all other players, assuming that

[4] This value is also well-defined if all stand-alone worths are negative.

without player i, the coalition of remaining players earns its worth $v(N\setminus\{h\})$ in the original game. We can also consider other possibilities to measure this externality.

As an example, consider the class of *mollifier games* in Charnes et al. (1978), that is based on affine combinations of a game and its dual. Formally, for $\beta \in \mathbb{R}$, define the game (N, v'_β) as follows:

$$v'_\beta(S) = \beta v(S) + (1-\beta)v^*(S) \text{ for all } S \subseteq N. \tag{4}$$

- **β-weak balanced externalities**. Let $\beta \in \mathbb{R}$. A solution ψ on a subgame closed class $\mathcal{C} \subseteq \mathcal{G}$ satisfies β-weak balanced externalities if for every $(N, v) \in \mathcal{C}$ with $|N| \geq 2$, there exists $\alpha \in \mathbb{R}$ such that for every $h \in N$, if $\left(\sum_{i\in N\setminus\{h\}} \psi_i(N, v) - v'_\beta(N \setminus \{h\})\right) \neq 0$, then

$$\psi_h(N, v) = \alpha \left(\sum_{i\in N\setminus\{h\}} \psi_i(N, v) - v'_\beta(N \setminus \{h\}) \right).$$

Although the worth $v'_\beta(N \setminus \{i\})$ need not be equal to $v(N \setminus \{i\})$, $i \in N$, it turns out that this axiom is compatible with efficiency, and characterizes the following solution among the efficient solutions. Defining $\mathcal{G}_{\beta+} = \{(N, v) \in \mathcal{G} \mid \beta v^*(\{j\}) + (1-\beta)v(\{j\}) > 0 \text{ for all } j \in N\}$, the proof follows a similar line as the proof of Theorem 5.

Theorem 7 *Let $\beta \in \mathbb{R}$. A solution ψ on $\mathcal{G}_{\beta+}$ satisfies efficiency and β-weak balanced externalities if and only if $\psi = \psi^\beta$ with ψ^β given by*

$$\psi^\beta_i(N, v) = \frac{\beta v^*(\{i\}) + (1-\beta)v(\{i\})}{\sum_{k\in N} (\beta v^*(\{k\}) + (1-\beta)v(\{k\}))} v(N) \quad \textit{for all } i \in N.$$

Proof. It is obvious that ψ^β is efficient. To show that this solution satisfies β-weak balanced externalities, let $(N, v), (N, v'_\beta) \in \mathcal{G}_{\beta+}$ and $\beta \in \mathbb{R}$ be such that $v'_\beta(S) = \beta v(S) + (1-\beta)v^*(S)$ for all $S \subseteq N$. If $\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\})) \neq v(N)$, taking $\alpha = \frac{v(N)}{\sum_{j\in N}(\beta v^*(\{j\})+(1-\beta)v(\{j\}))-v(N)}$ for any $h \in N$, we have

$$\begin{aligned}
&\alpha\Big(\sum_{i\in N\setminus\{h\}} \psi^\beta_i(N, v) - v'_\beta(N\setminus\{h\})\Big) \\
&= \alpha(v(N) - \psi^\beta_h(N, v) - \beta v(N\setminus\{h\}) - (1-\beta)v^*(N\setminus\{h\})) \\
&= \alpha(v(N) - \psi^\beta_h(N, v) - \beta v(N\setminus\{h\}) - (1-\beta)v(N) + (1-\beta)v(\{h\})) \\
&= \alpha(\beta(v(N) - v(N\setminus\{h\})) + (1-\beta)v(\{h\}) - \psi_h(N, v)) \\
&= \alpha(\beta v^*(\{h\}) + (1-\beta)v(\{h\}) - \psi_h(N, v)) \\
&= \alpha\Big(\beta v^*(\{h\}) + (1-\beta)v(\{h\}) - \frac{\beta v^*(\{h\}) + (1-\beta)v(\{h\})}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))} v(N)\Big)
\end{aligned}$$

$$\begin{aligned}
&= \alpha(\beta v^*(\{h\}) + (1-\beta)v(\{h\}))\Big(1 - \frac{v(N)}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))}\Big)\\
&= \alpha(\beta v^*(\{h\}) + (1-\beta)v(\{h\}))\Big(\frac{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\})) - v(N)}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))}\Big)\\
&= \frac{\beta v^*(\{h\}) + (1-\beta)v(\{h\})}{\sum_{j\in N}(\beta v^*(\{j\}) + (1-\beta)v(\{j\}))}v(N)\\
&= \psi_h^\beta(N,v),
\end{aligned}$$

where the first equality holds from efficiency of ψ^β, and the eighth equality follows from substituting α.

If $\sum_{j\in N}(\beta v^*(\{j\})+(1-\beta)v(\{j\})) = v(N)$, then $\psi_i^\beta(N,v) = \beta v^*(\{i\})+(1-\beta)v(\{i\})$, and thus $\sum_{i\in N\setminus\{h\}}\psi_i^\beta(N,v)-v'_\beta(N\setminus\{h\},v^{-h}) = v(N)-\psi_h^\beta(N,v)-(\beta v(N\setminus\{h\})+(1-\beta)v^*(N\setminus\{h\})) = v(N)-(\beta v^*(\{h\})+(1-\beta)v(\{h\}))-\beta(v(N)-v^*(\{h\}))-(1-\beta)(v(N)-v(\{h\})) = 0$, implying that β-weak balanced externalities does not have any bite. Altogether, ψ^β satisfies β-weak balanced externalities.

To prove the 'only if' part, let ψ be a solution satisfying efficiency and β-weak balanced externalities on $\mathcal{G}_{\beta+}$. Let $(N,v)\in\mathcal{G}_{\beta+}$. If $\sum_{i\in N\setminus\{h\}}\psi_i(N,v) - v'_\beta(N\setminus\{h\}) \neq 0$, then efficiency and β-weak balanced externalities imply that there is $\alpha\in\mathbb{R}$ such that for $h\in N$,

$$\begin{aligned}
\psi_h(N,v) &= \alpha\Big(\sum_{i\in N\setminus\{h\}}\psi_i(N,v) - v'_\beta(N\setminus\{h\})\Big)\\
&= \alpha(\beta v^*(\{h\}) + (1-\beta)v(\{h\}) - \psi_h(N,v)),
\end{aligned}$$

where the second equality holds from the first four equalities shown in the existence part. Thus,

$$\psi_h(N,v) = \frac{\alpha}{1+\alpha}(\beta v^*(\{h\}) + (1-\beta)v(\{h\})).$$

Efficiency determines that

$$\sum_{h\in N}\psi_h(N,v) = \frac{\alpha}{1+\alpha}\sum_{h\in N}(\beta v^*(\{h\}) + (1-\beta)v(\{h\})) = v(N),$$

and thus

$$\frac{\alpha}{1+\alpha} = \frac{v(N)}{\sum_{h\in N}(\beta v^*(\{h\}) + (1-\beta)v(\{h\}))},$$

which yields the desired formula.

If $\sum_{i\in N\setminus\{h\}}\psi_i(N,v) - v'_\beta(N\setminus\{h\}) = 0$, then by efficiency, $v(N)-\psi_h(N,v)-v'_\beta(N\setminus\{h\}) = 0$, implying that $\psi_h(N,v) = v(N) - v'_\beta(N\setminus\{h\}) = v(N) - (\beta v(N\setminus\{h\}) + (1-$

$\beta)(v(N) - v(\{h\}))) = \beta(v(N) - v(N\backslash\{h\})) + (1-\beta)v(\{h\}) = \beta v^*(\{h\}) + (1-\beta)v(\{h\})$. By efficiency, $\sum_{h\in N}\psi_h(N,v) = \sum_{h\in N}[\beta v^*(\{h\}) + (1-\beta)v(\{h\})] = v(N) > 0$ since $(N,v) \in \mathcal{G}_{\beta+}$. Therefore, $\psi_i(N,v) = \beta v^*(\{i\}) + (1-\beta)v(\{i\}) = \frac{\beta v^*(\{i\})+(1-\beta)v(\{i\})}{v(N)}v(N) = \frac{\beta v^*(\{i\})+(1-\beta)v(\{i\})}{\sum_{k\in N}(\beta v^*(\{k\})+(1-\beta)v(\{k\}))}v(N)$, as desired. □

Notice that ψ^1 is the PANSC value, and 1-weak balanced externalities is very similar to weak balanced externalities (and, in fact, is equivalent to weak balanced externalities under efficiency, i.e. for values). As another special case, ψ^0 coincides with the PD value.

Other worths instead of $v'_\beta(N\setminus\{h\})$ could be used for coalition $N\setminus\{h\}$ in the definition of β-weak balanced externalities, but there are some restrictions. For example, assuming that there is no impact from player h leaving, and thus the remaining players earn $v(N)$ would give that $\psi_h(N,v) = \alpha\left(\sum_{i\in N\setminus\{h\}}\psi_h(N,v) - v(N)\right) = \alpha(v(N) - \psi_h(N,v) - v(N)) = -\alpha\psi_h(N,v)$, which implies that $\alpha = -1$. Obviously, this is a restatement of efficiency.

6 Consistency

In this section, considering a variable player set, we characterize the PANSC value involving a *reduced game consistency* axiom. By duality, this result is very closely related to the reduced game consistency result for the PD value in Zou et al. (2021).

The consistency principle is widely acceptable notion in cooperative game theory. It is based on the idea that the payoffs of the remaining players should not change when a specific player leaves the game with the payoff assigned to her by the solution. In the literature, various reduced games are introduced which assess the effect on the worths of coalitions of remaining players in a different way. Here, we consider the *complement reduced game* (Thomson, 2011), also known as the *dual projection reduced game* in van den Brink et al. (2016).

If player $j \in N$ leaves game (N,v) with a certain payoff, then the complement reduced game is a game on the remaining player set that assigns to every subset of $N \setminus \{j\}$ its worth together with player j in the original game minus the payoff that is assigned to player j. We might consider that player j leaves the game with a fixed payoff, but commits to cooperate with any coalition of remaining players. In return, player j is guaranteed her payoff.

Definition 2 *Given game $(N,v) \in \mathcal{G}$ with $|N| \geq 2$, player $j \in N$ and payoff vector $x \in \mathbb{R}^N$, the complement reduced game with respect to j and x is the game $(N\backslash\{j\}, v^x)$ given by*

$$v^x(S) = \begin{cases} v(S\cup\{j\}) - x_j & \text{for any } S \subseteq N\setminus\{j\}, S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Complement consistency requires that the payoffs assigned to the remaining players in $N \setminus \{j\}$, after player j leaving the game with her payoff according to solution ψ, is the same in the reduced game as in the original game.

- **Complement consistency**. A solution ψ on $\mathcal{C} \subseteq \mathcal{G}$ satisfies complement consistency if for every game $(N, v) \in \mathcal{C}$ with $|N| \geq 3$, $j \in N$, and $x = \psi(N, v)$, it holds that $(N\backslash\{j\}, v^x) \in \mathcal{C}$, and $\psi_i(N, v) = \psi_i(N\backslash\{j\}, v^x)$ for all $i \in N\backslash\{j\}$.

The following result follows straightforwardly from Proposition 2 in Zou et al. (2021). Since their domains are different, we provide the proof for ease of the reader.

Proposition 8 *The PANSC value on $\mathcal{G}^{\geq 2}_{sc+}$ satisfies complement consistency.*

Proof. Let $(N, v) \in \mathcal{G}^{\geq 2}_{sc+}$ and $j \in N$. First, we remark that the class of games $\mathcal{G}^{\geq 2}_{sc+}$ is subgame closed under the complement reduced game operator, namely, if $(N, v) \in \mathcal{G}^{\geq 2}_{sc+}$ for $|N| \geq 3$, then $(N \setminus \{j\}, v^x) \in \mathcal{G}^{\geq 2}_{sc+}$ for any $j \in N$ and $x = PANSC(N, v)$, because $SC_i(N \setminus \{j\}, v^x) = v^x(N \setminus \{j\}) - v^x(N \setminus \{i, j\}) = v(N) - PANSC_j(N, v) - (v(N \setminus \{i\}) - PANSC_j(N, v)) = v(N) - v(N \setminus \{i\}) = SC_i(N, v)$ for any $i \in N \setminus \{j\}$.[5] Next, we have, for any $i \in N \setminus \{j\}$,

$$\begin{aligned}
PANSC_i(N \setminus \{j\}, v^x) &= \frac{SC_i(N \setminus \{j\}, v^x)}{\sum_{k\in N\setminus\{j\}} SC_k(N \setminus \{j\}, v^x)} v^x(N \setminus \{j\}) \\
&= \frac{SC_i(N, v)}{\sum_{k\in N\setminus\{j\}} SC_k(N, v)} (v(N) - x_j) \\
&= \frac{SC_i(N, v)}{\sum_{k\in N\setminus\{j\}} SC_k(N, v)} \left(v(N) - \frac{SC_j(N, v)}{\sum_{k\in N} SC_k(N, v)} v(N) \right) \\
&= \frac{SC_i(N, v)}{\sum_{k\in N\setminus\{j\}} SC_k(N, v)} \left(1 - \frac{SC_j(N, v)}{\sum_{k\in N} SC_k(N, v)} \right) v(N) \\
&= \frac{SC_i(N, v)}{\sum_{k\in N\setminus\{j\}} SC_k(N, v)} \cdot \frac{\sum_{k\in N\setminus\{j\}} SC_k(N, v)}{\sum_{k\in N} SC_k(N, v)} v(N) \\
&= \frac{SC_i(N, v)}{\sum_{k\in N} SC_k(N, v)} v(N) \\
&= PANSC_i(N, v)
\end{aligned}$$

□

Complement consistency is the dual axiom of *projection consistency* used to axiomatize the PD value in Zou et al. (2021). Axiomatizations using a reduced game consistency

[5] Notice that the class of games $\mathcal{G}^{\geq 2}_{sc+}$ is subgame closed under this reduced game operator for any solution ψ.

property usually assume a specific allocation for two-player games. For the PD value, this is proportional standardness which requires that for two-player games (with positive stand-alone worths) the worth of the grand coalition is allocated proportional to the stand-alone worths of the players. The PANSC value obviously satisfies the dual of this standardness. Recall that $\mathcal{G}^2_{sc+} = \{(N,v) \in \mathcal{G}_{sc+} \mid |N| = 2\}$.

- **Dual proportional standardness**. A solution ψ on $\mathcal{G}^2_{sc+}$ satisfies dual proportional standardness if for every $(N,v) \in \mathcal{G}^2_{sc+}$,

$$\psi_i(N,v) = \frac{v(\{i,j\}) - v(\{j\})}{(v(\{i,j\}) - v(\{i\})) + (v(\{i,j\}) - v(\{j\}))} v(\{i,j\}) \quad \text{for } i \in N = \{i,j\}.$$

Theorem 9 *A solution on $\mathcal{G}^{\geq 2}_{sc+}$ satisfies complement consistency and dual proportional standardness if and only if it is the PANSC value.*

Although the proof is almost the same as that of Theorem 6 in Zou et al. (2021), for completeness, it is given in the Appendix.

Standardness axioms are quite strong since they coincide with the definition of some solution for two-player games. Instead of dual proportional standardness, we could also use the following axiom, which requires that for two-player games where the worth of the grand coalition and the ratio of the separable costs of both players is equal, their payoffs are equal.

- **Dual proportionality**. A solution ψ on $\mathcal{G}^2_{sc+}$ satisfies dual proportionality if for every two games $(N,v),(N,v') \in \mathcal{G}^2_{sc+}$ such that (i) $v(N) = v'(N)$, and (ii) there is $\alpha > 0$ such that $SC_i(N,v) = \alpha SC_i(N,v')$ for all $i \in N$, it holds that $\psi(N,v) = \psi(N,v')$.

Dual proportionality and complement consistency are not sufficient to characterize the PANSC value on $\mathcal{G}^{\geq 2}_{sc+}$. Therefore, we additionally require the inessential game property and continuity, but only for two-player games in $\mathcal{G}^2_{sc+}$.

- **Inessential game property for two-player games**. A solution ψ on $\mathcal{G}^2_{sc+}$ satisfies the inessential game property for two-player games if for every $(N,v) \in \mathcal{G}^2_{sc+}$ such that $N = \{i,j\}$ and $v(\{i\}) + v(\{j\}) = v(\{i,j\})$, it holds that $\psi_i(N,v) = v(\{i\})$ and $\psi_j(N,v) = v(\{j\})$.

- **Continuity for two-player games**. A solution ψ on $\mathcal{G}^2_{sc+}$ satisfies continuity for two-player games if for all sequences of games $\{(N,w_k)\}$ and game (N,v) in $\mathcal{G}^2_{sc+}$ such that $(N,w_k) \to (N,v)$, it holds that $\lim\limits_{(N,w_k)\to(N,v)} \psi(N,w_k) = \psi(N,v)$.

The above three axioms characterize the PANSC value on $\mathcal{G}^2_{sc+}$.

Lemma 10 *A solution on $\mathcal{G}^2_{sc+}$ satisfies dual proportionality, the inessential game property for two-player games, and continuity for two-player games if and only if it is the PANSC value.*

Proof. It is clear that the 'if' part is satisfied. To show the 'only if' part, suppose that ψ is a solution satisfying the three axioms. Let (N,v) be an arbitrary game in the class $\mathcal{G}^2_{sc+}$ with $N=\{i,j\}$. If $v(N)\neq 0$, let (N,v') be an additive game such that $SC_i(N,v')=\alpha SC_i(N,v)$, $SC_j(N,v')=\alpha SC_j(N,v)$ and $v'(N)=v(N)$. Clearly, $\alpha=\frac{v(N)}{SC_i(N,v)+SC_j(N,v)}\neq 0$. Dual proportionality and the inessential game property for two-player games imply that $\psi_k(N,v)=\psi_k(N,v')=\alpha SC_k(N,v)=\frac{SC_k(N,v)v(N)}{SC_i(N,v)+SC_j(N,v)}$ for all $k\in\{i,j\}$. If $v(N)=0$, continuity for two-player games implies that $\psi(N,v)=PANSC(N,v)=0$. $\square$

Logical independence of the axioms used in Lemma 10 can be shown by the following alternative solutions.

(i) The PD value satisfies all axioms except dual proportionality.

(ii) The solution $\psi_i(N,v)=\frac{SC_i(N,v)v(N)}{2\sum_{j\in N}SC_j(N,v)}+\frac{v(N)}{2|N|}$ for all $(N,v)\in\mathcal{G}^2_{sc+}$ and $i\in N$, satisfies all axioms except the inessential game property for two-player games.

(iii) The solution, for all $(N,v)\in\mathcal{G}^2_{sc+}$ and $i\in N$, given by

$$\psi_i(N,v)=\begin{cases} PANSC_i(N,v) & \text{if } v(N)\neq 0\\ CIS_i(N,v) & \text{if } v(N)=0\end{cases}$$

satisfies all axioms except continuity for two-player games.

Theorem 9 and Lemma 10 together yield the following axiomatization of the PANSC value on $\mathcal{G}^{\geq 2}_{sc+}$.

Corollary 11 *A solution on $\mathcal{G}^{\geq 2}_{sc+}$ satisfies dual proportionality, the inessential game property for two-player games, continuity for two-player games, and complement consistency if and only if it is the PANSC value.*

7 Comparison with other solutions

In this section, we discuss the relationship between the PANSC value and two other existing values, in particular the EANSC value that was mentioned in the preliminaries (see (1)), and the SCRB method which is popular for cost allocation problems.

7.1 Comparison with the EANSC value

The PANSC and EANSC values are both based on the separable costs of the players. Whereas the EANSC value assigns to every player its separable cost and allocates the remainder (the total nonseparable cost) equally over the players, the PANSC value allocates the worth of the grand coalition proportional to the separable costs, which is equivalent to first assigning to every player its separable cost and allocating the remainder (the total nonseparable cost) proportional to the separable costs. The next proposition gives two sufficient conditions for the EANSC and PANSC values giving the same payoff vector.

Proposition 12 *For every game $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$, $EANSC(N,v) = PANSC(N,v)$ if and only if $v(N) = \sum_{k\in N} SC_k(N,v)$ or $v(N\backslash\{i\}) = v(N\backslash\{j\})$ for all $i,j \in N$.*

Proof. For any $(N,v) \in \mathcal{G}^{\geq 2}_{sc+}$ and $i \in N$,

$$PANSC_i(N,v) = SC_i(N,v) + \frac{SC_i(N,v)}{\sum_{k\in N} SC_k(N,v)}[v(N) - \sum_{k\in N} SC_k(N,v)].$$

Comparing this equation with (1), we have that $EANSC(N,v) = PANSC(N,v)$ if and only if

$$[v(N) - \sum_{k\in N} SC_k(N,v)][\frac{SC_i(N,v)}{\sum_{k\in N} SC_k(N,v)} - \frac{1}{n}] = 0 \quad \text{for all } i \in N,$$

and thus $v(N) = \sum_{k\in N} SC_k(N,v)$ or $v(N\backslash\{i\}) = v(N\backslash\{j\})$ for all $i,j \in N$. □

The EANSC value satisfies the well-known standardness due to Hart and Mas-Colell (1989).

- **Standardness**. A solution ψ on $\mathcal{G}$ satisfies standardness if for every $(N,v) \in \mathcal{G}$ with $|N| = 2$, it holds that

$$\psi_i(N,v) = v(\{i\}) + \frac{1}{2}[v(\{i,j\}) - v(\{i\}) - v(\{j\})] \quad \text{for } N = \{i,j\}.$$

The EANSC value also satisfies complement consistency. Similar to Theorem 9, complement consistency and standardness together characterize the EANSC value on $\mathcal{G}$ (also on $\mathcal{G}^{\geq 2}_{sc+}$). Moulin (1985) considers this fact in cost allocation problems, but we give the exact proof for completeness. The proof is similar to that of Theorem 9, and is postponed to the Appendix.

Theorem 13 *A solution on $\mathcal{G}$ satisfies complement consistency and standardness if and only if it is the EANSC value.*

Similar as the PANSC value (respectively, the EANSC value) is the multiplicative (respectively, additive) normalization of the separable costs, the PD value (respectively, the CIS value) is the multiplicitave (respectively, additive) normalization of the stand-alone worths, as summarized in Table 1.

	Multiplicative normalization	Additive normalization
Separable cost $SC_i(N, v)$	PANSC	EANSC
Stand-alone worth $v(\{i\})$	PD	CIS

Table 1: Individual assignments and normalization

In Section 4, we considered a 'multiplicative normalization' of balanced externalities, and saw that it characterizes the PANSC value as the unique efficient solution satisfying this axiom. An alternative could be to consider an 'additive normalization'. Combining the two variations gives the following axiom (and thus weaker than both).

- α, γ**-weak balanced externalities**. A solution ψ on a subgame closed class $\mathcal{C} \subseteq \mathcal{G}$ satisfies α, γ-weak balanced externalities if for every $(N, v) \in \mathcal{C}$ with $|N| \geq 2$, there exist $\alpha, \gamma \in \mathbb{R}$ such that, for every $h \in N$,

$$\psi_h(N, v) = \alpha \sum_{i \in N \setminus \{h\}} \big(\psi_i(N, v) - \psi_i(N \setminus \{h\}, v^{-h})\big) + \gamma.$$

If ψ is efficient, then this axiom implies that

$$\psi_h(N, v) = \alpha(v(N) - \psi_h(N, v) - v(N \setminus \{h\})) + \gamma$$

and thus

$$\psi_h(N, v) = \frac{\alpha SC_h(N, v) + \gamma}{1 + \alpha}.$$

Special cases of efficient solutions are:

(i) If $\alpha = 0$, then we get the equal division value, being $ED_h = \frac{v(N)}{|N|}$ for all $h \in N$.

(ii) If $\gamma = 0$, then we get the PANSC value.

(iii) If $\alpha = 1$, then we get a modified EANSC value, being

$$MEANS_h(N, v) = \frac{1}{2} SC_h(N, v) + \frac{1}{|N|}\left(v(N) - \sum_{j \in N} \frac{SC_j(N, v)}{2}\right) \text{ for all } h \in N.$$

The MEANSC value coincides with the Shapley value for 2-games, and thus with the minimal transfer rule for queueing games, as it should by the axioms. Notice that the EANSC value does not coincide with the Shapley value for 2-games, and does not belong to this class.

7.2 Comparison with the Separable Costs Remaining Benefits (SCRB) method

In this subsection, we show that the PANSC value is closely related to the well-known *Separable Costs Remaining Benefits (SCRB) method* in cost allocation problems. Cost allocation problems have obtained much attention in the literature and have been applied to address real problems. One of the most famous empirical examples is the Tennessee Valley Authority (TVA) Act which is designed to assign the cost of TVA projects specifically among the several purposes involved (Ransmeier, 1942). The SCRB method is commonly used in practice for allocating the costs of multi-purpose water development projects (Straffin and Heaney, 1981; Young et al., 1982). It is based on a simple yet appealing idea that joint costs should be allocated in proportion to the willingness to pay of the players. For a survey of such method, we refer to Tijs and Driessen (1986).

A *cost allocation problem* is a triple (N, c, b), where $N \subset \mathbb{N}$ is a set of participants or players, $c : 2^N \to \mathbb{R}$ is a cost function with $c(\emptyset) = 0$, and $b = (b(i))_{i \in N}$ is a profile where $b(i)$ is the benefit to player i if her purposes are served. For any $S \subseteq N$, $c(S)$ is the cost of serving S which is the minimal cost of providing the service to the players in S. The objective is to allocate the total cost $c(N)$ among all players. A *cost allocation method* or *solution* is a function which assigns an allocation vector $x \in \mathbb{R}^N$ to each cost allocation problem (N, c, b).

Notice that the pair (N, c) is mathematically equivalent to a game (N, v). Because of its different interpretation, the literature often speaks about a cost game, respectively a profit game. Solutions can be defined for both types of games. In the literature, one can find solutions that are applicable for cost as well as profit games. Also the PANSC value is applicable in both contexts. In the case of cost games, we speak about $SC_i(N, c) = c(N) - c(N \setminus \{i\})$ as the separable cost of player i in cost game (N, c), and we refer to $NSC(N, c) = c(N) - \sum_{j \in N} SC_j(N, c)$ as the *nonseperable cost*.[6]

Since player i would not be willing to pay more than $\min\{b(i), c(\{i\})\}$ to participate in the joint project, $\min\{b(i), c(\{i\})\} - SC_i(N, c)$ is considered as player i's remaining benefit. The SCRB method assigns to each player her separable cost, and then allocates

[6]Since cost and profit games are mathematically equivalent, we denote the separable costs by $SC_i(N, c)$, and $SC_i(N, v)$ respectively, depending on the context.

the nonseparable cost in proportion to the remaining benefits of players. Formally, the SCRB method is given by

$$SCRB_i(N,c,b) = SC_i(N,c) + \frac{\min\{b(i), c(\{i\})\} - SC_i(N,c)}{\sum_{j\in N}(\min\{b(j), c(\{j\})\} - SC_j(N,c))} \cdot NSC(N,c).$$

A variant of the SCRB method, the Alternative Cost Avoided (ACA) method, is studied by Straffin and Heaney (1981) and Otten (1993), and is given by

$$ACA_i(N,c,b) = SC_i(N,c) + \frac{(c(\{i\}) - SC_i(N,c)) \cdot NSC(N,c)}{\sum_{j\in N}(c(\{j\}) - SC_j(N,c))}.$$

It is known that if $b(j) \geq c(\{j\})$ for all $j \in N$, then the SCRB method coincides with the ACA method.

A cost allocation problem (N,c,b) can be transformed into a game (N,v) in two ways. One is using an *anti-game* defined by $v(S) = -c(S)$ for all $S \subseteq N$. This game assigns to every coalition the total cost to provide the service for this coalition (in nonnegative terms). The other is using a *cost saving game* (Young et al., 1982) defined by $v(S) = \sum_{k\in S} c(\{k\}) - c(S)$ for all $S \subseteq N$. This game assigns to every coalition the cost saving it can earn when cooperating and providing the service together for all players in the coalition instead of every player providing the service for itself.

It turns out that in the special cases that the individual benefits are zero, or at least equal to the individual costs, the SCRB method coincides with a variation of the PANSC value to one of the associated games.

Proposition 14 *Consider cost allocation problem* (N,c,b).

(i) *If* $b(j) = 0$ *for all* $j \in N$*, then* $SCRB(N,c,b) = -PANSC(N,v)$*, where* (N,v) *is the associated anti-game.*

(ii) *If* $b(j) \geq c(\{j\})$ *for all* $j \in N$*, then* $SCRB_j(N,c,b) = c(\{j\}) - PANSC_j(N,v)$ *for all* $j \in N$*, where* (N,v) *is the associated cost saving game.*

Proof. Consider cost allocation problem (N,c,b).

(i) Suppose that $b(j) = 0$ for all $j \in N$. Then, the SCRB solution becomes, for all $i \in N$,

$$SCRB_i(N,c,b) = SC_i(N,c) + \frac{-SC_i(N,c) \cdot NSC(N,c)}{\sum_{j\in N}(-SC_j(N,c))} = \frac{SC_i(N,c)}{\sum_{j\in N} SC_j(N,c)} \cdot c(N).$$

For the associated anti-game (N,v), since $SC_j(N,v) = -SC_j(N,c)$ for all $j \in N$, and $c(N) = -v(N)$, we have

$$SCRB_i(N,c,b) = -\frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} \cdot v(N) = -PANSC_i(N,v),$$

and thus the cost allocation determined by the SCRB method coincides with (the negative of) our PANSC value applied to the associated anti-game.

(ii) Suppose that $b(j) \geq c(\{j\})$ for all $j \in N$. In that case, for all $i \in N$,

$$SCRB_i(N,c,b) = ACA_i(N,c,b) = SC_i(N,c) + \frac{(c(\{i\}) - SC_i(N,c)) \cdot NSC(N,c)}{\sum_{j\in N}(c(\{j\}) - SC_j(N,c))}.$$

For the associated cost saving game (N,v), since $SC_j(N,v) = v(N) - v(N \setminus \{j\}) = \sum_{k\in N} c(\{k\}) - c(N) - \sum_{k\in N\setminus\{j\}} c(\{k\}) + c(N \setminus \{j\}) = c(\{j\}) - SC_j(N,c)$ for all $j \in N$, $c(N) = \sum_{k\in N} c(\{k\}) - v(N)$, and thus $NSC(N,c) = c(N) - \sum_{k\in N} SC_k(N,c) = \sum_{k\in N} c(\{k\}) - v(N) - \sum_{k\in N}(c(\{k\}) - SC_k(N,v)) = \sum_{k\in N} SC_k(N,v) - v(N) = -NSC(N,v)$, we have

$$\begin{aligned} c(\{i\}) - SCRB_i(N,c,b) &= c(\{i\}) - SC_i(N,c) - \frac{(c(\{i\}) - SC_i(N,c)) \cdot NSC(N,c)}{\sum_{j\in N}(c(\{j\}) - SC_j(N,c))} \\ &= SC_i(N,v) + \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} NSC(N,v) \\ &= \frac{SC_i(N,v)}{\sum_{j\in N} SC_j(N,v)} \cdot v(N) \\ &= PANSC_i(N,v). \end{aligned}$$

and thus the cost allocation determined by the SCRB method coincides with the individual cost of each player minus the cost share allocated by our PANSC value applied to the associated saving game. □

8 Concluding remarks

In this paper, we generalized the axiom of *balanced cost reduction* that characterizes the *minimal transfer rule* for queueing problems in van den Brink and Chun (2012), to the class of cooperative games. Although this minimal transfer rule is usually presented as the Shapley value of an associated queueing game of Maniquet (2003), since queueing games are 2-games, the minimal tranfer rule can also be obtained by applying other solutions (such as the pre-nucleolus or τ-value) to queueing games. After extending the characterization result to the class of 2-games, we show that extending this axiom in a straightfoward way to general games, is incompatible with efficiency. Therefore, we weakened the axiom by requiring that every player's payoff is the same *fraction* of its total externality inflicted on the other players. This weakening turns out to be compatible with efficiency. Moreover, these two axioms characterize a unique solution, called the PANSC value, which allocates the total worth proportional to the *separable costs* of the players. We have provided characterizations of the PANSC value. In this sense, the spirit of the balanced contributions property seems to express a feature of the PANSC value in queueing problems. Besides, we compared the PANSC value with the EANSC value, and showed that it is closely related to the well-known SCRB method in cost allocation problems, being a solution that is often applied in cost allocation problems.

Appendix

Proof of Theorem 2. It is well-known that the Shapley value satisfies efficiency and symmetry. To show that the Shapley value satisfies k-balanced externalities, consider k-game (N, v) and $h \in N$. Then

$$\begin{aligned}
\sum_{i \in N \setminus \{h\}} \left(Sh_i(N,v) - Sh_i(N \setminus \{h\}, v^{-h})\right) &= \sum_{i \in N \setminus \{h\}} \left(\sum_{\substack{S \subseteq N, |S|=k \\ i \in S}} \frac{\Delta_v(S)}{k} - \sum_{\substack{S \subseteq N, |S|=k \\ i \in S, h \notin S}} \frac{\Delta_v(S)}{k} \right) \\
&= \sum_{i \in N \setminus \{h\}} \sum_{\substack{S \subseteq N, |S|=k \\ i,h \in S}} \frac{\Delta_v(S)}{k} \\
&= (k-1) \sum_{\substack{S \subseteq N, |S|=k \\ h \in S}} \frac{\Delta_v(S)}{k} \\
&= (k-1) Sh_h(N,v),
\end{aligned}$$

where the first equality follows since $\Delta_v(S) = \Delta_{v^{-h}}(S)$ for all $S \subseteq N \setminus \{h\}$, and the third equality follows since every k-size coalition containing player h appears $k-1$ times in the summation (once for every other player $i \in N \setminus \{h\}$). This shows that the Shapley value satisfies k-balanced externalities.

We show the 'only if' part by induction on $|N|$. Let (N, v) be a k-game such that $k \geq 3$ (the case $k = 2$ is already shown by Theorem 1). If $|N| < k$, then all players are symmetric in (N, v), and thus by symmetry, $\psi_i(N, v) = \psi_j(N, v)$ for any two players $i, j \in N$. Since $v(N) = 0$, efficiency then implies that $\psi_i(N, v) = 0$ for all $i \in N$.

If $|N| = k$, then k-balanced externalities implies that $\psi_i(N, v) = \frac{1}{|N|-1} \sum_{i \in N \setminus \{h\}} (\psi_i(N, v) - \psi_i(N \setminus \{h\}, v^{-h})) = \frac{1}{|N|-1} \sum_{i \in N \setminus \{h\}} \psi_i(N, v)$ for any $i \in N$ and $h \in N \setminus \{i\}$. With efficiency it follows that $\psi_i(N, v) = \frac{1}{|N|-1}(v(N) - \psi_h(N, v))$. Summing this equality over $h \in N \setminus \{i\}$ yields $(|N|-1)\psi_i(N, v) = v(N) - \frac{1}{|N|-1} \sum_{h \in N \setminus \{i\}} \psi_h(N, v) = v(N) - \frac{1}{|N|-1}(v(N) - \psi_i(N, v))$. This is equivalent to $\left((|N|-1) - \frac{1}{|N|-1}\right) \psi_i(N, v) = \left(1 - \frac{1}{|N|-1}\right) v(N) \Leftrightarrow \frac{|N|(|N|-2)}{|N|-1} \psi_i(N, v) = \frac{|N|-2}{|N|-1} v(N)$, and thus $\psi_i(N, v) = \frac{v(N)}{|N|} = Sh_i(N, v)$.

As induction hypothesis, suppose that uniqueness holds for all $N' \subset \mathbb{N}$ such that $k \le |N'| \le |N|-1$. For any k-game (N,v) and $h \in N$, k-balanced externalities yields

$$\psi_h(N,v) = \frac{1}{k-1} \sum_{i \in N \setminus \{h\}} \left(\psi_i(N,v) - \psi_i(N \setminus \{h\}, v^{-h}) \right). \tag{5}$$

By the induction hypothesis, the $\psi_i(N \setminus \{h\}, v^{-h}),\ i,h \in N,\ i \neq h$, are uniquely determined. Since $|N| \ge 3$, (5) and efficiency yield a system of $(|N|-1)+1 = |N|$ linearly independent equations in the $|N|$ unkowns $\psi_h(N,v),\ h \in N$, which thus are uniquely determined. □

Proof of Theorem 9. It is straightforward that the PANSC value satisfies dual proportional standardness. Complement consistency follows from Proposition 8.

To prove the 'only if' part, let ψ be a solution on $\mathcal{G}^{\ge 2}_{sc+}$ which satisfies the two axioms. Let $(N,v) \in \mathcal{G}^{\ge 2}_{sc+}$, $x = PANSC(N,v)$, and $y = \psi(N,v)$. We will show that $x = y$. If $|N| = 2, x = y$ follows from dual proportional standardness. Suppose, by induction, that $PANSC_i(N',v) = \psi_i(N',v)$ holds for any game (N',v) with $|N'| < |N|$.

Take any $i \in N$ and $j \in N \setminus \{i\}$, and consider (N,v) and the complement reduced games $(N \setminus \{j\}, v^x), (N \setminus \{j\}, v^y)$. We have

$$\begin{aligned}
x_i - y_i &= PANSC_i(N,v) - \psi_i(N,v) \\
&= PANSC_i(N \setminus \{j\}, v^x) - \psi_i(N \setminus \{j\}, v^y) \\
&= PANSC_i(N \setminus \{j\}, v^x) - PANSC_i(N \setminus \{j\}, v^y) \\
&= \frac{SC_i(N \setminus \{j\}, v^x)}{\sum_{k \in N \setminus \{j\}} SC_k(N \setminus \{j\}, v^x)} v^x(N \setminus \{j\}) - \frac{SC_i(N \setminus \{j\}, v^y)}{\sum_{k \in N \setminus \{j\}} SC_k(N \setminus \{j\}, v^y)} v^y(N \setminus \{j\}) \\
&= \frac{SC_i(N,v)}{\sum_{k \in N \setminus \{j\}} SC_k(N,v)} v^x(N \setminus \{j\}) - \frac{SC_i(N,v)}{\sum_{k \in N \setminus \{j\}} SC_k(N,v)} v^y(N \setminus \{j\}) \\
&= \frac{SC_i(N,v)}{\sum_{k \in N \setminus \{j\}} SC_k(N,v)} (v(N) - x_j) - \frac{SC_i(N,v)}{\sum_{k \in N \setminus \{j\}} SC_k(N,v)} (v(N) - y_j) \\
&= \frac{SC_i(N,v)}{\sum_{k \in N \setminus \{j\}} SC_k(N,v)} (-x_j + y_j),
\end{aligned} \tag{6}$$

where the second equality follows from the PANSC value and ψ satisfying complement consistency, the third equality follows from the induction hypothesis, and the fifth equality follows similar as in the proof of Proposition 8.

Summing up (6) over all $i \in N \setminus \{j\}$ yields

$$\sum_{i \in N \setminus \{j\}} (x_i - y_i) = \sum_{i \in N \setminus \{j\}} \frac{SC_i(N,v)}{\sum_{k \in N \setminus \{j\}} SC_k(N,v)} (-x_j + y_j) = -x_j + y_j,$$

and thus

$$\sum_{j\in N\setminus\{i\}}(-x_j+y_j)=x_i-y_i. \tag{7}$$

Summing up (6) over all $j\in N\setminus\{i\}$ yields

$$\begin{aligned}\sum_{j\in N\setminus\{i\}}(x_i-y_i) &= (|N|-1)(x_i-y_i)\\ &= \frac{SC_i(N,v)}{\sum_{k\in N\setminus\{j\}}SC_k(N,v)}\sum_{j\in N\setminus\{i\}}(-x_j+y_j). \end{aligned}\tag{8}$$

Together (7) and (8) imply that

$$\left(|N|-1-\frac{SC_i(N,v)}{\sum_{k\in N\setminus\{j\}}SC_k(N,v)}\right)(x_i-y_i)=0.$$

Since $SC_k(N,v)>0$ for any $k\in N$ and $i\in N\setminus\{j\}$, we have

$$|N|-1-\frac{SC_i(N,v)}{\sum_{k\in N\setminus\{j\}}SC_k(N,v)}\neq 0 \text{ for any } i\in N \text{ and } j\in N\setminus\{i\}.$$

Therefore, we have $x_i=y_i$. □

Proof of Theorem 13. It is straightforward to show that the EANSC value satisfies complement consistency and standardness. To show the 'only if' part, suppose that ψ is a solution satisfying complement consistency and standardness.

If $|N|=2$, then $\psi(N,v)=EANSC(N,v)$ follows from standardness.

Proceeding by induction, for $|N|\geq 3$, suppose that $\psi(N',v)=EANSC(N',v)$ whenever $|N'|=|N|-1$. Take any $i,j\in N$ such that $i\neq j$. Let $x=\psi(N,v)$ and $y=EANSC(N,v)$. For the two reduced games $(N\backslash\{j\},v^x)$ and $(N\backslash\{j\},v^y)$, we have

$$\begin{aligned}x_i-y_i &= \psi_i(N,v)-EANSC_i(N,v)\\ &= \psi_i(N\backslash\{j\},v^x)-EANSC_i(N\backslash\{j\},v^y)\\ &= EANSC_i(N\backslash\{j\},v^x)-EANSC_i(N\backslash\{j\},v^y),\end{aligned}\tag{9}$$

where the last equality follows from the induction hypothesis.

By definitions of the EANSC value and the complement reduced game, we have

$$\begin{aligned}&EANSC_i(N\backslash\{j\},v^x)-EANSC_i(N\backslash\{j\},v^y)\\ =\;& SC_i(N\backslash\{j\},v^x)+\frac{1}{|N|-1}[v^x(N\backslash\{j\})-\sum_{k\in N\setminus\{i\}}SC_k(N\backslash\{j\},v^x)]\end{aligned}$$

$$-SC_i(N\backslash\{j\}, v^y) - \frac{1}{|N|-1}[v^y(N\backslash\{j\}) - \sum_{k\in N\backslash\{i\}} SC_k(N\backslash\{j\}, v^y)]$$

$$= v(N) - v(N\backslash\{i\}) + \frac{1}{|N|-1}[v(N) - x_j - \sum_{k\in N\backslash\{i\}} (v(N) - v(N\backslash\{k\})]$$

$$-(v(N) - v(N\backslash\{i\})) - \frac{1}{|N|-1}[v(N) - y_j - \sum_{k\in N\backslash\{i\}} (v(N) - v(N\backslash\{k\})]$$

$$= \frac{1}{|N|-1}(y_j - x_j).$$

Together with (9), we have that, for all $i, j \in N$ with $i \neq j$,

$$x_i - y_i = \frac{1}{|N|-1}(y_j - x_j). \tag{10}$$

Summing (10) over all $i \in N\backslash\{j\}$ yields $\sum_{i\in N\backslash\{j\}}(x_i - y_i) = y_j - x_j$, which implies

$$\sum_{i\in N}(x_i - y_i) = 0. \tag{11}$$

On the other hand, (10) can be written as $(|N| - 1)(x_i - y_i) = y_j - x_j$. Summing this equality over all $j \in N\backslash\{i\}$ yields

$$(|N| - 1)^2(x_i - y_i) = \sum_{j\in N\backslash\{i\}} (y_j - x_j). \tag{12}$$

Together with (11) and (12), it holds that $|N|(|N| - 2)(x_i - y_i) = 0$. Since $|N| > 2$, $x_i - y_i = 0$ for all $i \in N$. This shows that $\psi(N, v) = EANSC(N, v)$. □

Acknowledgement

Youngsub Chun is supported by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2016S1A3A2924944) and the Housing and Commercial Bank Economic Research Fund for the Institute of Economic Research of Seoul National University. Yukihiko Funaki is supported by JSPS KAKENHI Grant Numbers JP17H02503 and JP18KK0046. Zhengxing Zou is supported by the National Natural Science Foundation of China (Grant Nos. 71771025, 71801016).

References

Charnes, A., Rousseau, J., Seiford, L., 1978. Complements, mollifiers and the propensity to disrupt. International Journal of Game Theory 7, 37–50.

Driessen, T., Funaki, Y., 1991. Coincidence of and collinearity between game theoretic solutions. OR-Spektrum 13, 15–30.

Harsanyi, J.C., 1959. A bargaining model for the cooperative n-person game, in: Kuhn, H., Tucked, A. (Eds.), Contributions to the Theory of Games. Princeton University Press. IV, pp. 325–355.

Hart, S., Mas-Colell A., 1989. Potential, value, and consistency. Econometrica, 589–614.

Lemaire, J., 1984. An application of game theory: Cost allocation. ASTIN Bulletin: The Journal of the IAA 14, 61–81.

Leng, M., Luo, C., Liang, L., 2020. Multi-player allocations in the presence of diminishing marginal contributions: Cooperative game analysis and applications in management science. Management Science. Doi: 10.1287/mnsc.2020.3709

Li, W., Xu, G., Sun, H., 2020. Maximizing the minimal satisfaction: Characterizations of two proportional values. Mathematics 8, 1129.

Maniquet, F., 2003. A characterization of the Shapley value in queueing problems. Journal of Economic Theory 109, 90–103.

Moulin, H., 1985. The separability axiom and equal sharing method. Journal of Economic Theory 36, 120–148.

Otten G., 1993. Characterizations of a game theoretical cost allocation method, Mathematical Methods of Operations Research 38, 175–185.

Ransmeier, J., 1942. The Tennessee Valley Authority: A case study in the economics of multiple purpose stream planning. Nashville.

Schmeidler, D., 1969. The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics 17, 1163–1170.

Shapley, L.S., 1953. A value for n-person games, in: Kuhn, H., Tucked, A. (Eds.), Contributions to the Theory of Games. Princeton University Press. volume 28 of II, pp. 307–317.

Straffin P., Heaney J., 1981. Game theory and the Tennessee Valley Authority. International Journal of Game Theory 10, 35–43

Thomson, W., 2011. Consistency and its converse: An introduction. Review of Economic Design 15, 257–291.

Tijs, S.H., Driessen, T., 1986. Game theory and cost allocation problems. Management Science 32, 1015-1028.

Tijs, S.H., 1987. An axiomatization of the τ-value. Mathematical Social Sciences 13, 177–181.

van den Brink, R., 2007. Null or nullifying players: The difference between the Shapley value and equal division solutions. Journal of Economic Theory 136, 767–775.

van den Brink, R., Chun, Y., 2012. Balanced consistency and balanced cost reduction for sequencing problems. Social Choice and Welfare 38, 519–529.

van den Brink, R., Chun, Y., Funaki, Y., Park, B., 2016. Consistency, population solidarity, and egalitarian solutions for games. Theory and Decision 81, 427–447.

van den Brink, R., Funaki, Y., 2009. Axiomatizations of a class of equal surplus sharing solutions for games. Theory and Decision 67, 303–340.

van den Nouweland, A, Borm, P, van Golstein Brouwers, W, Groot Bruinderink, R., 1996. A game theoretic approach to problems in telecommunication. Management Science 42, 294–303.

Young H.P., Okada N., Hashimoto T., 1982. Cost allocation in water resources development. Water Resources Research 18, 463–475.

Zou, Z., van den Brink, R., Chun, Y., Funaki, Y., 2021. Axiomatizations of the proportional division value. Social Choice and Welfare. Doi: 10.1007/s00355-020-01299-3