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Irreversible investment under predictable growth: Why land stays vacant when housing demand is booming

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Abstract

The standard model of irreversible investment under uncertainty considers only the level of the cash flow that could be obtained through the investment. We present a general model that includes as state variables both the level and the growth rate of the cash flow, while the timing and size of the one-time investment are discretionary. As an illustration, we consider an investor with the exclusive right to develop a vacant piece of land, where the timing of the investment and the scale of the property are chosen optimally. We demonstrate that construction is optimally postponed when prospects are gloomy, but also when they are bright. Indeed, under sufficiently high growth it is, perversely, never optimal to invest. Under a cost-of-capital argument, the rational response to predictable growth combined with flexible investment conditions is to keep land vacant for extended periods, which may explain why construction in superstar cities often appears sluggish. Our proposed model can be used in all investment decisions, irrespective of sector, where the assumptions of predictable growth and a one-off, flexible but otherwise irreversible investment are met.

keywords: optimal stopping, real options, irreversible investment, real estate, urban growth

JEL codes: C41, C61, D81, R14, R31

1 Introduction

Consider a real-estate investor with the exclusive right to develop a vacant plot of land in Manhattan in the 1980s. For simplicity, assume she can either erect a six- or a twenty-storey building. Rental rates justify erecting the small but not the large building, as the latter is disproportionately expensive: the high cost of capital would render the net operating cash flow (the rental income minus the cost of capital) negative for some time. However, the city is booming and analysts expect rents to keep rising in the foreseeable future. The investor fears she may come to regret building the low-rise, which cannot easily be converted into a high-rise even as demand skyrockets. The taller property would allow

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her to capitalise on the anticipated future growth, while exposing her to an initial net operating loss due to the high capital cost. The solution to this dilemma is to consider a third option: to postpone investment. This would allow the investor to avoid both the short-term loss involved in building big and the long-term regret associated with erecting a smaller building. Counterintuitively, the higher the anticipated growth, the more attractive it is to wait—in lay terms, don't build when demand is booming.

This article formalises the above logic by presenting a general model with an investor who owns one unit of a fixed, unalterable factor of production (e.g. land) and is considering investing in a second factor (e.g. construction) to yield a variable amount of production capacity (e.g. usable floorspace). Our main finding that postponing investment is optimal under high growth relies on two critical assumptions: (i) the growth rate of the resulting cash-flow stream is to some extent predictable, while (ii) the one-off investment is flexible a priori but fixed once realised. In the case of real estate, the development of rental rates over time is assumed to be somewhat predictable, while construction decisions concerning a single plot of land are assumed to be flexible but otherwise irreversible.

The finding that investment is suboptimal when growth is high remains valid when this growth is entirely predictable (i.e. non-stochastic), differentiating the effect identified here from the standard real-options effect (e.g. Dixit and Pindyck, 1994). To develop the intuition, the first half of this article focuses on the case of deterministic growth, before allowing stochasticity in both cash-flow levels and growth rates. Although the term *option value* suggests uncertainty, we refer to the option value of the fixed factor (e.g. land) in both the deterministic and stochastic contexts. Our rationale is that, in both cases, the decision maker is faced with the option (but never the obligation) to invest.

Murray (2020) found that valuable land near fast-growing cities remains vacant for extended periods. Some authors (e.g. Glaeser et al., 2005 and Glaeser and Ward, 2009) attribute this to regulatory inefficiencies and rent-seeking behaviour. On the contrary, we find that postponing investment is the rational response in the face of high growth and flexible but irreversible investment. In such circumstances, to capitalise on the anticipation of future growth, a substantially larger investment would seem to be desirable. The resulting amplified capital costs will, however, initially yield negative operating cash flows—a cost-of-capital argument that stymies the desire to 'build big'. By waiting for the expected growth to materialise, a larger investment can instead be made at a later time, which will instantly yield positive operating cash flows. This finding should be relevant in all investment decisions, irrespective of sector, for which our critical assumptions (i) and (ii) are reasonable approximations.

Our first assumption—growth being partially or fully predictable—may be satisfied in many sectors. In financial markets, for example, the observed variety in price-dividend ratios across firms reflects the variety in predicted growth rates. In the case of urban development, it is well known that population growth rates in cities mean-revert only slowly, at a rate of $\sim$15% per annum or less for urban regions in the United States (Campbell et al., 2009, table 3; Desmet and Rappaport, 2017), with similar figures in other countries. A city that is currently growing faster than the nationwide average can therefore be expected to continue doing so for $\sim$7 years. Rental rates at a fixed location within the city tend to be positively related to the city's population (e.g. Albouy et al., 2018; Combes et al., 2019; Davis et al., 2021). Persistence in population growth thus implies—and has been found empirically (Sinai and Souleles, 2005, table 3; Desmet and Rappaport, 2017; Eichholtz et al., 2021) to lead to—persistence in rental price growth.

Our second assumption—a one-time, flexible but irreversible investment—may be less widely applicable, but is particularly apt for urban development. As real estate makes up half of the global capital stock, this sector is of prime importance. Today's construction decisions are likely to give rise to urban structures that remain in place for decades. The street map of Manhattan bears witness to the persistence of urban structures; e.g. the rectangular grid between Houston Street and 155th Street was laid out in 1811. While the assumption of complete irreversibility is technically false (buildings can be demolished), it is, given the high cost of retroactive adjustments (Glaeser and Gyourko, 2005), nevertheless a reasonable approximation.

To operationalise assumption (i), we deviate from the classic real-options literature in assuming that the cash flow generated by the investment follows a geometric Brownian motion with a drift that is itself a mean-reverting Brownian motion, also known as an Ornstein-Uhlenbeck process. This bivariate setup allows for prolonged periods of above- or below-average cash-flow growth, and implies that both the cash flow and its growth rate are relevant state variables for the investment decision. Technically, our bivariate setup leads to a two-dimensional optimal-stopping problem (e.g. Peskir and Shiryaev, 2006). The classic assumption of a geometric Brownian motion (i.e. with constant drift) can be recovered as a limiting case of our new model.

To operationalise assumption (ii), we assume that the investor is faced with a constant-returns-to-scale two-factor Stone-Geary production function, which covers the full range of feasible elasticities of substitution between both factors from zero to infinity. This article focuses on the range from zero to one, with Leontief and Cobb-Doublas as its two limiting cases, yielding zero and unit elasticities, respectively, while Stone-Geary function covers all intermediate cases. Our main finding holds for both *elastic* production functions, i.e. Cobb Douglas and Stone Geary, for which the investment size is flexible, but not for the (inelastic) Leontief case, as the fixed size of the investment means there is little gain to be had from postponing investment.

In general economic terms, our counterintuitive result suggests that high-growth assets, which are expensive to realise but derive much of their value from anticipated future growth, may be unattractive investment candidates. While the expected growth is accounted for in their price, these assets produce cash flows that are, currently, still low. The resulting low dividend yield (or rental yield in the case of real estate) initially fails to cover the capital cost associated with the large investment. The capital share for such investments is large, as the optimal response under high growth is to build big. The combination of a diminished dividend yield and a high capital share means that such investments would generate an initial operating loss, rendering investment suboptimal.

1.1 Comparison with the classic model of irreversible investment

The classic model of irreversible investment under uncertainty (e.g. Dixit and Pindyck, 1994) postulates that, even as postponement may be attractive for some time, there exists a critical cash-flow level such that investment is triggered as soon as this level is breached. As this standard model involves a single state variable (the cash flow), the trigger value can typically be found analytically (e.g. Huisman and Kort, 2015, Prop. 1). Instead, we propose a model with two state variables: the level of the cash flow and its growth rate. In this more general setting, one may entertain the (possibly naive) hypothesis that investment should be forthcoming if the resulting cash-flow stream is sufficiently large and/or rising sufficiently fast. In that case, one would intuitively expect—in analogy with the classic model—that sufficiently high cash-flow levels should encourage investment. High growth may similarly stimulate investment by raising future cash flows. Moreover, a trade-off would seem to be acceptable: the investor may accept a lower cash-flow level given a higher growth rate, and vice versa.

However, our main finding is that, under sufficiently high growth, investment should be postponed *irrespective* of the cash-flow level that could be obtained through the investment. This directly contradicts the main finding of the classic model in the sense that, in this region of the state space, a trigger value of the cash flow that would spur investment does not exist. Technically, it is infinite: no cash flow, however large, results in the investment being made. In the adjacent part of the state space with more moderate (but still positive) growth rates, the trigger value does exist; here, we find that the trigger value increases as the growth rate increases. As such, higher growth makes investment less (rather than more) likely.

1.2 Outline

Section 2 solves the deterministic version of model, which we present in general terms using the case of real estate for the purposes of illustration. We assume that the 'excess' growth rate follows a deterministic Ornstein-Uhlenbeck process; i.e. it is mean-reverting to zero in an exponential fashion. This setup allows us to analytically characterise the optimal timing of investment for positive excess growth rates. For the Cobb-Douglas case, the decision whether to invest turns out to be independent of the cash-flow level that could have been obtained though the investment. Instead, the decision optimally hinges only on the growth rate of the cash flow; specifically, investment should be forthcoming if and only if the growth rate is sufficiently *low*. For the Stone-Geary case, we find that while both the cash flow and its growth rate are relevant for the investment decision, they tend to be positively rather than negatively correlated at the moment of investment. That is, contrary to the trade-off implied by our initial hypothesis—that a low cash-flow level can be compensated by a high growth rate, and vice versa—the two factors are complements: higher growth rates necessitate ever higher cash-flow levels to trigger investment. Only for the Leontief production function does the standard result hold that investment should be forthcoming if the resulting cash-flow stream is sufficiently high.

While illuminating the drivers behind our main result, the deterministic model has two important drawbacks. First, deviations of the growth rate from its long-term average cannot be endogenously realised. To generate an initial deviation, we would have to rely on an 'MIT shock': a sizeable and unexpected one-time shock to the system that is entirely external to the model in the sense that agents are unaware even of the *possibility* of its arrival (e.g. Boppart et al., 2018). In reality, market participants are well aware of the stochasticity of both cash flows and growth rates; this knowledge could alter their behaviour in complicated ways. Second, the deterministic model is less general and robust than a full-fledged stochastic version would be.

Section 3 addresses these concerns by postulating stochastic laws of motion for the cash-flow level and its growth rate. Our analytic valuation formula for existing assets, which we believe to be new, is not only relevant for real-estate investment decisions but is more broadly applicable in financial economics, e.g. to the valuation of (growth) stocks. Stocks for which the dividend-growth rate exceeds the time-preference rate cannot be valued by models that assume a static (i.e. non-mean-reverting) growth rate, as the resulting present value would be unbounded. Conversely, modelling the growth rate using a mean-reverting process, as in this article, implies a bounded asset value even when the growth rate temporarily outstrips the time-preference rate.

Section 3 also provides a closed-form solution for the two-dimensional option-valuation problem with a Cobb-Douglas production function. For the Stone-Geary and Leontief cases, we employ two numerical methods, which produce near-identical results that are fully in line with the theory and robust under sensitivity checks. For all three production functions, the conclusions of the deterministic model continue to hold up under stochasticity; indeed, our main finding is reinforced, as uncertainty creates an additional incentive for postponing investment. For both elastic production functions, the critical level of the growth rate beyond which investment is suppressed is reduced relative to the deterministic case, thereby making the effect more pronounced. For realistic parameter values, we find that positive growth shocks tend to suppress rather than boost new investment.

Finally, Section 4 discusses our results in the broader context of urban growth. While the investor's actions are efficient in the context of a single plot of vacant land, positive agglomeration externalities in cities suggest that even further delays may be societally optimal. We conclude by posing an empirical litmus test that can verify or falsify the model's validity.

1.3 Related literature

Classic real-options models take cash flows to be stochastic processes with constant drifts: see e.g. Titman (1985), Geltner (1989), Smith (1984), Quigg (1993), Williams (1993), Grenadier (1996), Merton (1998), Foo Sing (2001) and Peng (2016) for applications in real estate. These models have a single state variable (the cash flow), which facilitates a straightforward threshold strategy: invest when the cash flow is above a constant trigger level that can be identified analytically by standard real-options methods (e.g. Dixit and Pindyck, 1994).

The introduction of persistent growth rates, as in the present article, may go some way to explaining why superstar cities tend to have low rent-to-price ratios (Amaral et al., 2021, Hilber and Mense, 2021). Even as the rental income is unchanged, the expectation of rent growth is sufficient to boost property prices, thereby lowering rent-to-price ratios. Persistent growth may also explain why land on the outskirts of superstar cities is more valuable than would otherwise be the case (i.e. without persistent growth). While it is well known in the real-options literature that uncertainty can drive investment delays—consider the Abel-Caballero-Hartman-Oi effect (e.g. Bloom et al., 2018)—the rationale for postponing investment in our model is different in that it persists even when growth rates evolve entirely deterministically.

In finance, persistent growth in dividends has received some attention, as it could help explain high price-to-earnings ratios of growth firms. This is because market expectations of continued growth boost equity prices (e.g. Chan et al., 2003; Chen, 2017). That housing can similarly be viewed as an asset that generates 'dividends' (i.e. rent) is also well known (e.g. Sinai and Souleles, 2005; Fairchild et al., 2015). To the best of our knowledge, however, the problem of determining the net present value of a cash-flow stream involving persistent growth has not been solved. Our closed-form solution may thus be applicable to questions such as the valuation of (growth) stocks.

Several authors have assumed the scale of the investment to be flexible a priori but fixed a posteriori (e.g. Dixit, 1993, Bertola and Caballero, 1994, Abel and Eberly, 1996, Balter et al., 2022). In the context of cities, Arnott and Lewis (1979) and Capozza and Helsley (1989) apply a model with growth-rate differentials between cities to show that land close to fast-growing cities commands a higher option value and will be developed more densely. Our model is reminiscent of that of Capozza and Li (1994), who allow for stochastic growth around a deterministic trend that differs between cities. In fact, these authors suggest modelling persistent growth rates, as we do, and conjecture some of our findings, but not the positive relation between the cash-flow level and its growth rate at the moment of investment.

From a technical perspective, our work is related to the growing body of literature on optimal stopping in multidimensional models, e.g. Rogers (2002), Andersen and Broadie (2004), Bally and Printems (2005) and Compernolle et al. (2021). It also ties in directly with Strulovici and Szydlowski's (2015, p. 1042) call for "a better understanding of the properties of optimal policies and value functions with a multidimensional state space" as well as for the construction of explicit solutions. To solve our most general model, we apply two numerical methods that yield identical results. First, we use Poisson optional stopping times (POST; Lange et al., 2020), a robust method for constructing solutions that finds the value function as an increasing sequence of lower bounds; this property persists after discretisation when using standard finite-difference stencils. The theoretical properties of the algorithm (monotone and geometric convergence) imply that the discretised problem can be solved reliably. Second, we follow Compernolle et al. (2021) in directly discretising the partial differential equation, and impose relevant constraints using the theory of linear complementarity problems (LCPs, Cottle et al., 2009). The LCP can then be solved using standard Newton-type methods as in e.g. Bazaraa et al. (2013).

2 Deterministic model yielding the main result

This section presents the simplest version of the model that yields our main result, namely that investment is suboptimal when growth is high. This model features deterministic state dynamics and can be solved almost entirely in closed form. The main result follows from simple algebra, which for transparency is contained in the main text. Our aim here is to highlight the key drivers of the main result in general economic terms and in easily verifiable form. The model is not intended to be exclusively applicable to urban development; we do, however, rely on a real-estate application for the purposes of illustration.

2.1 Model

Overall setting. There are two factors of production. The first factor has been previously committed and is unalterable (e.g. land). The decision maker owns one unit of this first factor, which can be combined with a one-off capital investment in the second factor (e.g. construction) to yield production capacity (e.g. a building with usable floorspace). Both the timing of this investment and its scale (e.g.

square footage) are chosen optimally. Once realised, the amount of production capacity is forever fixed (e.g. buildings are immutable and everlasting); neither factor depreciates. Each unit of production capacity produces one unit of a commodity per unit of time, which is sold on a perfectly competitive market; this generates a cash flow that can be viewed as dividend (e.g. rent for real estate). In sum, we assume that one factor is fixed, while the other has a 'putty-clay' capital structure, meaning it is flexible at the time of investment but immutable thereafter.

Output-price dynamics. Time is continuous. Each unit of the commodity at time t is sold at a market price (e.g. the rental rate per unit of floorspace) of $Y_t > 0$. The growth rate of this market price is $\mu + X_t$, where $\mu > 0$ is the long-term growth rate, assumed for simplicity to be strictly positive[1], while the excess growth rate X_t is mean-reverting to zero at the rate $\theta > 0$. For any time $r \geq t$, these dynamics imply

$$\dot{X}_t = -\theta X_t \qquad \Leftrightarrow \qquad X_r = X_t \, \mathrm{e}^{-\theta(r-t)}, \tag{1}$$

$$\dot{Y}_t = Y_t(\mu + X_t) \qquad \Leftrightarrow \qquad Y_r = Y_t \exp\left[\mu\,(r-t) + \int_t^r X_s \,\mathrm{d}s\right], \tag{2}$$

where a dot above a variable denotes a time derivative. The model thus features two state variables, $\{X_t\}$ and $\{Y_t\}$, which influence the optimal timing and scale of the investment. For $X_0 > 0$, the exponential growth rate of Y_t exceeds μ for all $t \geq 0$, while converging to μ asymptotically. In sum, we assume that the cash flow Y_t and its growth rate $\mu + X_t$ are at least partially (and in this case, perfectly) predictable, while X_t is mean reverting. Dynamics (1)–(2) will in Section 3 be generalised to allow for random shocks. The classic model of investment can be recovered by taking the limit $\theta \to \infty$, in which case $X_t \to 0$ for all $t > 0$, such that $Y_t = Y_0 \exp(\mu t)$ is the only state variable.

Production function. Both factors of production are combined to yield production capacity by means of a constant-returns-to-scale (CRS) Stone-Geary production function.[2] One unit of the first factor combined with a capital investment K in the second factor yields the production capacity $F(K)$:

$$F(K) = (K - \phi)^{\alpha}, \qquad K \geq \phi, \tag{3}$$

where $\alpha \in [0, 1)$ and ϕ are parameters. Production function (3) resembles the standard constant-elasticity-of-substitution (CES) production function in that it allows for an elasticity of substitution between both factors anywhere in the range $[0, \infty)$ (see Appendix A for details).

In this article, we restrict our attention to the case $\phi \geq 0$, which implies that the elasticity of substitution between both factors of production lies in the range $[0, 1]$ (see also Appendix A). For $\phi \geq 0$, Table 1 displays three collectively exhaustive cases: (i) Leontief ($\alpha = 0, \phi > 0$, zero elasticity), (ii) Stone Geary ($\alpha > 0, \phi > 0$, elasticity strictly between zero and one) and (iii) Cobb Douglas ($\alpha > 0, \phi = 0$, unit elasticity). Parameter α is critical in controlling the elasticity: as indicated in Table 1, we refer to Leontief (i.e. $\alpha = 0$) as the *inelastic* case, while Stone Geary and Cobb Douglas (for

[1] The strict positivity of μ ensures that investment is attractive in the long run.

[2] A standard Stone-Geary two-factor production function reads $\mathcal{F}(\mathcal{L}, \mathcal{K}) = \mathcal{L}^{1-\alpha} (\mathcal{K} - \phi)^{\alpha}$, which exhibits increasing returns to scale. The version with constant returns to scale reads $\mathcal{F}(\mathcal{L}, \mathcal{K}) = \mathcal{L}^{1-\alpha} (\mathcal{K} - \phi\mathcal{L})^{\alpha}$. Hence $F(K) := \mathcal{F}(1, K) = (K - \phi)^{\alpha}$ where $K := \mathcal{K}/\mathcal{L}$ is the ratio between both factors of production. We consider the case where the investor owns one unit of the factor $\mathcal{L}$, thus $\mathcal{L} = 1$.

Table 1: Elasticities of substitution for production function (3)

	α	ϕ	Elasticity of substitution	Classification
Leontief	0	>0	0	inelastic
Stone Geary	$(0,1)$	>0	$(0,1)$	elastic
Cobb Douglas	$(0,1)$	0	1	elastic

Note: Elasticity of substitution for different values of the parameters α and ϕ in the production function (3).

which $\alpha > 0$) are the *elastic* cases—this distinction will be relevant for the main result. Parameter $\phi \geq 0$ can be interpreted as (i) the fixed cost of construction, incurred irrespective of the amount of floorspace created, or (ii) the present value of the agricultural use of vacant land; the latter interpretation links our approach to seminal papers in the field of urban economics (Lucas and Rossi-Hansberg, 2002 and Ahlfeldt et al., 2015).

2.2 Valuation of existing production capacity

Value of a unit of production capacity. Let $\beta > 0$ be the investor's rate of time preference; alternatively, β can be interpreted as the (flow) cost of capital. To achieve a bounded value of existing production capacity, the long-term growth rate of Y_t should be strictly exceeded by the time-preference rate:

$$\mu \;<\; \beta, \tag{4}$$

as assumed throughout. The present value of one unit of production capacity (e.g. one unit of floorspace) at time t then equals a discounted integral (over time) involving all future cash flows:

$$B_t := \int_t^\infty \mathrm{e}^{-\beta(r-t)}\, Y_r \,\mathrm{d}r = Y_t \int_t^\infty \exp\left[-(\beta-\mu)(r-t) + \int_t^r X_s \,\mathrm{d}s\right] \mathrm{d}r, \tag{5}$$

where the equality follows by equation (2). In the point $X_t = 0$, we have $B_t = Y_t/(\beta-\mu)$. Naturally, $Y_t/(\beta-\mu)$ would also be the net present value if the growth rate were assumed constant, as in e.g. Gordon and Shapiro's (1956) classic dividend-discount model. In the point $X_t = 0$, therefore, equation (5) and the 'Gordon-growth model' coincide. For positive growth (i.e. $X_t > 0$), our valuation formula (5) suggests a higher value than the Gordon-growth formula.

Price-dividend ratio and dividend yield. It is convenient to write $B_t = Y_t\, b_t$, where $b_t := B_t/Y_t$ is the price-dividend ratio (or the price-to-rent ratio for real estate), which plays an important role. Its inverse, $b_t^{-1} = Y_t/B_t$, can be viewed as the dividend yield; alternatively, it can be interpreted as the rental yield for real estate. Using equation (5), the price-dividend ratio b_t equals

$$b_t := \frac{B_t}{Y_t} = \int_t^\infty \exp\left[-(\beta-\mu)(r-t) + \int_t^r X_s \,\mathrm{d}s\right] \mathrm{d}r, \tag{6}$$

$$= \int_0^\infty \exp\left[-(\beta-\mu)s + X_t \frac{1-\mathrm{e}^{-\theta s}}{\theta}\right] \mathrm{d}s > 0, \qquad X_t \in \mathbb{R}, \tag{7}$$

where the second line follows by equation (1) for X_s. Equation (7) implies that b_t depends on time only implicitly, via X_t. Hence, we write $b_t = b(X_t)$ for the function $b(\cdot)$ defined as

$$b(X) := \int_0^\infty \exp\left[-(\beta-\mu)s + X\frac{1-\mathrm{e}^{-\theta s}}{\theta}\right]\mathrm{d}s > 0, \qquad X \in \mathbb{R}. \tag{8}$$

This function $X \mapsto b(X)$ is positive, convex and strictly increasing, while diverging as $X \to \infty$. The inverse of $b(\cdot)$ is thus well defined; hence, X_t implies $b_t = b(X_t)$ and vice versa. When convenient, therefore, b_t can be used as a state variable instead of X_t. Importantly for the main result, the dividend yield $b(X)^{-1}$ is *decreasing* in X. This is because high dividend growth reduces the ratio of current dividends over their present value (i.e. the current dividend yield).

The dynamics of b_t are derived most easily by focusing on equation (6) in terms of t. Applying the Leibniz integral rule to take the time derivative of b_t yields

$$\dot{b}_t/b_t \;=\; \beta - b_t^{-1} - \mu - X_t. \tag{9}$$

For above-average dividend growth (i.e. $X_t > 0$), it follows[3] that the price-dividend ratio b_t is decreasing over time (i.e. $\dot{b}_t < 0$); i.e. b_t is gradually reduced to its long-term average value of $(\beta-\mu)^{-1}$.

Relation to the classic model. The classic model of investment can be recovered by taking $\theta \to \infty$, in which case $X_t \to 0$ for all $t > 0$. The cash flow Y_t then grows at the constant exponential rate μ. Equation (7) shows that the price-dividend ratio is constant at $b_t = (\beta-\mu)^{-1}$ for all t. Equivalently, the classic model is recovered by taking $t \to \infty$, in which case, similarly, $b_t \to (\beta-\mu)^{-1}$. The classic model does not allow (i) the price-dividend ratio b_t to vary over time, or (ii) the growth rate μ to exceed the time-preference rate β (as the present value of the dividend flow would be unbounded). In contrast, the new model allows the price-dividend ratio b_t to vary over time, while the short-term growth rate $\mu + X_t$ may surpass the time-preference rate, as long as the long-term growth rate μ is still strictly dominated by β.

2.3 Optimal level of investment

Optimal level of investment. To determine the optimal level of investment, suppose that the investor is *obliged* to invest at time t (i.e. commit a strictly positive amount of capital). We denote this 'conditionally optimal' investment by K_t, which maximises the present value of the future revenues. The present value consists in the amount of production capacity times its net present value per unit, $F(K)B_t$, minus the cost of the investment, K:

$$K_t := \arg\max_{K\geq\phi}\left[F(K)B_t - K\right] = \arg\max_{K\geq\phi}\left[(K-\phi)^\alpha Y_t\, b_t - K\right], \tag{10}$$

where we have used production function (3) and $B_t = Y_t b_t$. The first-order condition associated with the interior solution reads $\alpha(K_t-\phi)^{\alpha-1}Y_t b_t = 1$. This condition can be solved to yield the optimal

[3] This follows from $1/b_t > \beta-\mu-X_t$ for $X_t > 0$. For $X_t \geq \beta-\mu$, the inequality is trivial as the right-hand side is then weakly negative. For $0 < X_t < \beta-\mu$, the inequality follows from $b_t < 1/(\beta-\mu-X_t)$ from equation (6).

investment K_t and associated (optimal) production capacity $Q_t = F(K_t)$:

$$K_t = \phi + (\alpha Y_t b_t)^{1/(1-\alpha)} > 0, \qquad Q_t := F(K_t) = (\alpha Y_t b_t)^{\alpha/(1-\alpha)} > 0. \tag{11}$$

This interior solution remains valid as we approach the boundary case $\alpha = 0$, in which case $K_t = \phi$ and $Q_t = 1$ (as $\alpha^\alpha = 0^0 = 1$). Conveniently, therefore, equation (11) holds for the entire range $\alpha \in [0, 1)$.

Capital share. The capital share plays a key role in determining the optimal timing of investment. The capital share is the ratio of investment, K_t, over the present value of the acquired production capacity, which is the quantity times the net present value per unit, i.e. $Q_t B_t$. Using $B_t = Y_t b_t$ and equation (11) for K_t and Q_t, the capital share reads

$$\text{capital share} := \frac{K_t}{Q_t B_t} = \begin{cases} \phi \,/\, (Y_t\, b_t), & \text{Leontief:} \quad \alpha = 0, \phi > 0, \\ \alpha + \alpha\, \phi\, (\alpha\, Y_t\, b_t)^{-1/(1-\alpha)} > \alpha, & \text{Stone Geary:} \quad \alpha \in (0,1), \phi > 0, \\ \alpha, & \text{Cobb Douglas:} \quad \alpha \in (0,1), \phi = 0. \end{cases} \tag{12}$$

In the inelastic case (i.e. $\alpha = 0$), the capital share approaches zero asymptotically as $Y_t, b_t \to \infty$. This is because the investment remains fixed at ϕ, while the present value $Y_t b_t$ grows without bound. At the other end of the spectrum, it is a textbook result that the capital share for the Cobb-Douglas case is constant at $\alpha > 0$; this is due to the elasticity of substitution between both factors of production being unity. In the Stone-Geary case, the capital share exceeds α (due to the additional investment $\phi > 0$), while declining to α in the limit. In both elastic cases (i.e. $\alpha > 0$), therefore, the capital share does not vanish asymptotically but remains strictly positive at $\alpha > 0$. This distinction—i.e. whether the capital share vanishes asymptotically—is critical to the main result.

Optimal asset value. Let A_t denote the net present value (or 'asset value') at time t, still assuming that investment at time t is mandatory. This conditionally optimal asset value is the present value of the production capacity, $Q_t B_t$, minus the investment cost, K_t, i.e.

$$A_t \;:=\; Q_t B_t - K_t \;=\; (1-\alpha)\, \frac{(\alpha\, Y_t\, b_t)^{1/(1-\alpha)}}{\alpha} - \phi, \tag{13}$$

where the equality follows from $B_t = Y_t\, b_t$ and the expressions for K_t and Q_t in equation (11). For $\alpha = 0$, equation (13) simplifies to $A_t = Y_t\, b_t - \phi$ (as before, this follows from $\alpha^\alpha = 0^0 = 1$). Importantly, equation (13) yields an expression of the value of investment, A_t, in terms of $b_t = b(X_t)$ and Y_t, or, equivalently, in terms of our state variables, X_t and Y_t. While $A_t > 0$ for $\phi = 0$, A_t may be negative if $\phi > 0$; this is attributable to the mandatory nature of the investment.

2.4 Optimal timing of investment: Two necessary conditions

The option value of the committed factor (e.g. land) is obtained by optimising both the *level* and *timing* of investment. As the investment decision is irreversible, while its timing is discretionary, the optimised present value at time zero equals

$$\text{option value:} \qquad V_0 \;:=\; \sup_{t \geq 0}\, [\, \mathrm{e}^{-\beta t}\, A_t\,] \;>\; 0, \tag{14}$$

where the subscript indicates that V_0 is conditional on the starting point (X_0, Y_0), the discounting is evident from the multiplicative factor $e^{-\beta t}$ and the supremum is due to optimisation of the investment time. While the model considered here is entirely deterministic, V_0 can still be viewed as an *option value* in the sense that investment is optional and never mandatory. In the case of real estate, V_0 would be the option value of a single unit of vacant land.

The strict positivity of V_0 in equation (14) derives from the fact that, asymptotically, A_t grows without bound; i.e. a strictly positive value can be obtained by postponing investment for a sufficiently long period of time. Indeed, $A_t \propto Y_t^{1/(1-\alpha)}$ from equation (13), such that the asymptotic growth rate of A_t equals $\mu/(1-\alpha) > 0$. To ensure $V_0 < \infty$, therefore, we require

$$\text{boundedness of } V_0 : \qquad \frac{\mu}{1-\alpha} \;<\; \beta. \tag{15}$$

If optimisation (14) allows an interior solution for some $t > 0$, then the associated first-order condition reads $\dot{A}_t = \beta A_t$. If the boundary solution $t = 0$ applies, however, we may have strict inequality, i.e. $\dot{A}_t < \beta A_t$. This boundary case is relevant when $t \mapsto e^{-\beta t} A_t$ is both (i) positive at $t = 0$ and (ii) strictly decreasing for all $t > 0$, which jointly imply that immediate investment is optimal. In general, for investment to be optimal at some time $t \geq 0$, two necessary conditions must hold:

$$\text{zero-order condition:} \qquad A_t \;\geq\; 0, \tag{16}$$

$$\text{first-order condition:} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[\, e^{-\beta t} A_t \,\right] \;\leq\; 0 \qquad \Leftrightarrow \qquad \dot{A}_t \;\leq\; \beta A_t. \tag{17}$$

The zero-order condition requires the asset value at the time of investment to be weakly positive; this is a necessary condition as the value zero can be obtained by foregoing investment altogether. The first-order condition (17) posits that $t \mapsto e^{-\beta t} A_t$ should be weakly decreasing at the time of investment; if this quantity were strictly increasing, a better discounted value could be obtained by postponing investment. Due to the weak inequality, first-order condition (17) allows for both interior and boundary solutions.

2.5 Why investment is suboptimal under high growth

Here we show that, in the case of an elastic production function (i.e. $\alpha > 0$), the first-order condition (17) rules out investment when growth is high. To explain this surprising result, we demonstrate that the first-order condition (17) can be fruitfully interpreted in terms of the capital share (12).

First, we compute $\dot{A}_t$ by differentiating equation (13) and using the chain rule to account for the dependence on Y_t and b_t, yielding

$$\dot{A}_t = \frac{\mathrm{d}A_t}{\mathrm{d}Y_t}\dot{Y}_t + \frac{\mathrm{d}A_t}{\mathrm{d}b_t}\dot{b}_t = \frac{A_t + \phi}{1-\alpha}\left(\frac{\dot{Y}_t}{Y_t} + \frac{\dot{b}_t}{b_t}\right) = \frac{A_t + \phi}{1-\alpha}(\beta - b_t^{-1}), \tag{18}$$

where we have used equations (2) and (9) for $\dot{Y}_t$ and $\dot{b}_t$, respectively. Substituting this expression for

$\dot{A}_t$ into the first-order condition $\dot{A}_t \leq \beta A_t$, we obtain

$$\frac{A_t + \phi}{1-\alpha}(\beta - b_t^{-1}) \;\leq\; \beta A_t \qquad \Leftrightarrow \qquad \underbrace{\beta\, K_t}_{\substack{\text{flow cost of}\\ \text{capital investment}}} \quad \leq \quad \underbrace{Q_t\, Y_t}_{\substack{\text{cash flow}\\ \text{after investment}}} \quad , \tag{19}$$

where the reformulation on the right uses $(A_t + \phi)/(1-\alpha) = Q_t B_t$, $Q_t B_t - A_t = K_t$ and $B_t/b_t = Y_t$. The reformulation posits that the flow cost of investment, βK_t, should be covered by the cash flow generated immediately after investment, $Q_t Y_t$. This condition prohibits the investor from incurring a negative cash flow after investment, where the term βK_t accounts for the time value (or 'amortisation') of the investment K_t.

Further economic insight is obtained by dividing both sides of condition (19) by the present value of the production capacity, $Q_t B_t = Q_t Y_t b_t$, which yields the following equivalent, first-order condition:

$$\underbrace{\beta}_{\text{flow cost of capital}} \quad \times \quad \underbrace{\frac{K_t}{Q_t\, B_t}}_{\text{capital share}} \quad \leq \quad \underbrace{\frac{1}{b_t}}_{\text{dividend yield}} \quad . \tag{20}$$

Condition (20) postulates that, at the time of investment, the dividend yield should surpass the flow cost associated with the capital share. This condition highlights why high-growth assets may in fact be *un*attractive investment candidates: while they enjoy lofty valuations, they produce relatively meagre cash flows—and thus have low current dividend yields. Indeed, the right-hand side of condition (20) is decreasing (to zero) in X_t, as higher growth rates suppress the dividend yield. However, for the two elastic cases (i.e. $\alpha > 0$), the left-hand side remains strictly positive because the capital share (12) is bounded below by $\alpha > 0$. Given that the left-hand side remains strictly positive, while the right-hand side is decreasing to zero, first-order condition (20) *cannot* hold for sufficiently high X_t. Hence we arrive at the conclusion that, for $\alpha > 0$, investment is suboptimal for sufficiently high growth.

The economic interpretation is that high-growth assets are exceedingly expensive (as their price reflects the expected dividend growth), while the current (low) dividend yield fails to cover the flow cost associated with the capital share. Critically, the capital share does not shrink to zero in the case of an elastic production function, because under high growth the optimal response is to build big; i.e. the capital investment, K_t, is proportional to the value of the associated production capacity, $Q_t B_t$, with proportionality constant $\alpha > 0$. Under high growth, the combination of (i) a diminished dividend yield and (ii) a constant capital share means that investment would generate an initial operating loss after accounting for the large capital cost; hence, investment is ruled out by the first-order condition (17) or, equivalently, (20). By waiting for the expected dividend growth to materialise, a larger investment can instead be made at a later time, which will instantly yield positive operating cash flows.

To investigate this in more detail, we substitute the capital share (12) into condition (20) to obtain three alternative versions of the first-order condition:

$$\beta\,\phi \;\leq\; Y_t, \qquad \text{for Leontief, i.e.} \qquad \alpha = 0, \phi > 0, \tag{21}$$

$$\Big(1 + \phi\,(\alpha\, Y_t\, b_t)^{-1/(1-\alpha)}\Big)\,\alpha\,\beta \;\leq\; b_t^{-1}, \qquad \text{for Stone Geary, i.e.} \qquad \alpha \in (0,1), \phi > 0, \tag{22}$$

$$\alpha\beta \leq b_t^{-1}, \qquad \text{for Cobb Douglas, i.e.} \qquad \alpha \in (0,1), \phi = 0. \tag{23}$$

Condition (21) for the Leontief case requires the cash flow, Y_t, to exceed the flow cost of investment, $\beta\phi$. Hence, condition (21) holds for sufficiently high values of Y_t; this is consistent with our initial hypothesis that investment should be forthcoming when the cash flow Y_t is sufficiently high.

Condition (23) for the Cobb-Douglas case, at the other extreme, posits that the dividend yield, $1/b_t$, should exceed the flow cost, $\alpha\beta$, associated with the (constant) capital share, α. However, the required inequality $\alpha\beta \leq 1/b_t$ cannot hold for sufficiently high X_t, because the right-hand side is decreasing to zero, while the left-hand side is constant and strictly positive. For sufficiently high X_t, investment is suboptimal regardless of Y_t; this contradicts our hypothesis that a sufficiently high cash flow Y_t should trigger investment.

Finally, the counterintuitive conclusion that investment is suboptimal when growth is high persists in the intermediate case (22), because, as with the Cobb-Douglas production function, the capital share is bounded below by α. In fact, the Stone-Geary condition (22) is even more stringent than the Cobb-Douglas condition (23), because the capital share increases due to the additional investment $\phi > 0$, further raising the hurdle for investment. Again, condition (22) cannot hold if the dividend yield b_t^{-1} falls below $\alpha\beta$, thereby rendering investment suboptimal. This simple rule is entirely agnostic about Y_t and thus contradicts our hypothesis that a sufficiently high cash flow Y_t should trigger investment. Rather, for large Y_t, the Stone-Geary condition (22) approaches the Cobb-Douglas condition (23), where X_t is the only relevant variable; moreover, the necessary condition for investment is satisfied only for sufficiently *low* growth.

2.6 Sufficiency of the first-order condition for positive excess growth

We have seen that the first-order condition (20) is highly influential in restricting the region of the state space where investment could be optimal. Here we demonstrate that, for positive excess growth rates $X_t > 0$, this condition is not only necessary but also sufficient; hence, it fully characterises optimality.

Proposition 1 (First-order condition characterises optimality for $X_t > 0$) *Let condition* (15) *hold. For positive excess growth rates $X_t > 0$, the first-order condition* (17)*, or equivalently* (20)*, is both necessary and sufficient: i.e. investment is optimal if and only if the first-order condition holds.*

We prove this result in Appendix B in three steps. First, we demonstrate that for $X_t > 0$, the first-order condition (17) implies the zero-order condition (16). Second, we establish uniqueness: there exists at most one (unique) interior solution to the optimisation problem (14). Third, we establish the existence of at least one solution to the first-order condition (17), interior or otherwise, by showing that this condition is asymptotically satisfied. Taken together, these facts imply that, for $X_t > 0$, the function $t \mapsto \mathrm{e}^{-\beta t} A_t$ has a unique maximum, while this function is strictly increasing (decreasing) at any time strictly prior (posterior) to the unique moment $t \in \mathbb{R}$ that achieves this maximum. If this unique moment lies in the past, immediate investment is optimal.

The characterisation of the optimal investment policy for $X_t < 0$ is more involved because the interior solution to the first-order condition is no longer unique. The complications are that (i) there may be a local minimum as well as a local maximum, and (ii) the local maximum is not necessarily globally optimal, as it may be dominated by the boundary solution $t = 0$. The optimal policy is to invest immediately if the local maximum lies in the past, in which case any further delay is detrimental, while if the local maximum lies in the future, the decision maker should weigh up the value of investing now against the discounted value of waiting until the local maximum; details are available on request.

2.7 Visualising the optimal investment policy

Here we show how the optimal investment decision depends on the coordinate (X, Y) in the state space $\mathbb{R} \times \mathbb{R}_{>0}$; hence, we write $(X_t, Y_t) = (X, Y)$ (i.e. we drop the subscript t). This simplification is permitted as the state space is now partitioned into (i) a region where it is optimal to wait and (ii) a region where it is optimal to invest—i.e. the investment decision depends only on the spatial coordinate. The quantities A_t, B_t, K_t and Q_t are now viewed as depending on the spatial coordinate $(X_t, Y_t) = (X, Y)$. This can be achieved by replacing, in each expression, b_t by $b(X)$ and Y_t by Y.

For the numerical illustration below, we use a set of benchmark parameter values, i.e. $\beta = 0.06$, $\mu = 0.01$, $\theta = 0.15$, measured on an annual time scale, while $\alpha = 0.70$ for both elastic cases. These parameter values, used throughout unless stated otherwise, are roughly calibrated to be relevant for urban development.[4] We normalise ϕ to unity for the Leontief and Stone-Geary cases without loss of generality; see Davis et al. (2021) for a similar argument.[5]

Our benchmark parameter values imply that assumptions (4) and (15) are satisfied, while the first-order condition (22) for the Cobb-Douglas case holds with equality at $X = X^\dagger \approx 3.45\%$, where $X^\dagger > 0$ is the unique solution to

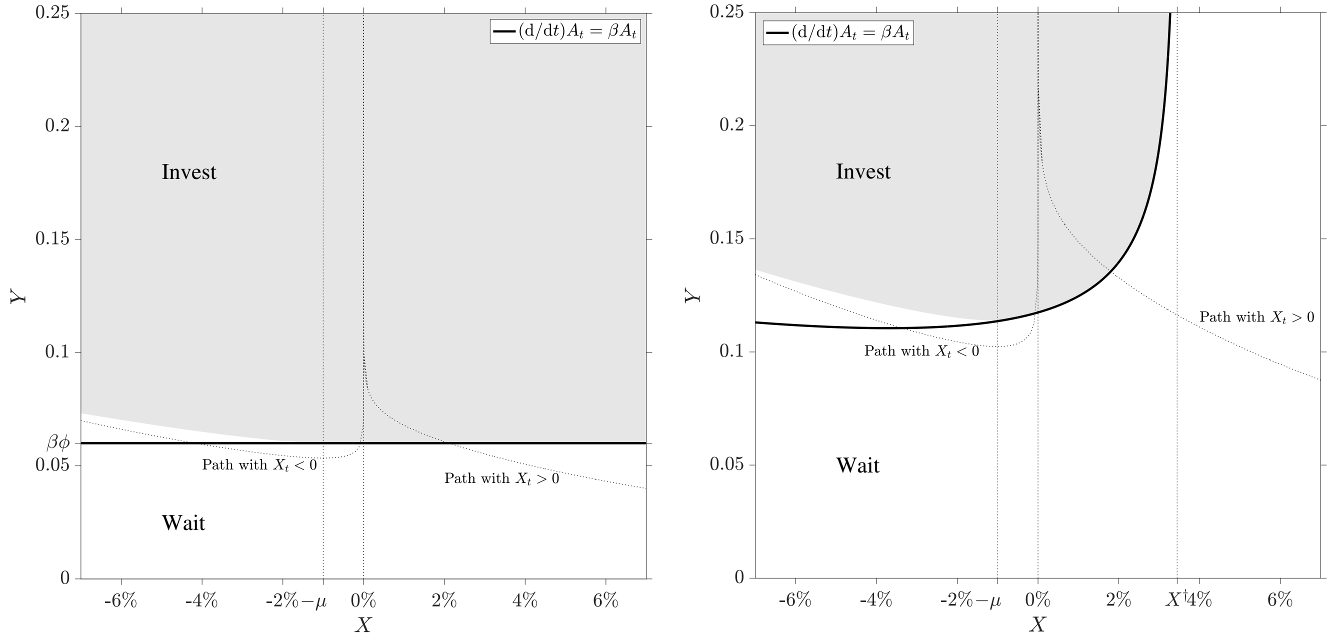
$$\alpha\,\beta \;=\; 1/b(X^\dagger). \tag{24}$$

A powerful corollary of Proposition 1 is that, in the Cobb-Douglas case, $X \leq X^\dagger$ is both necessary and sufficient for investment. Hence the optimal investment region is the half-space to the left of the vertical line $X = X^\dagger$; i.e. investment is optimal for sufficiently *low* growth.

Figure 1 shows the optimal investment policy in the state space for the Leontief (panel A) and Stone-Geary (panel B) production functions. Both panels contain the optimal investment region (shaded grey) and a solid curve that shows where the first-order condition (17) or (20) holds with equality (i.e. $\dot{A}_t = \beta A_t$ on the solid curve). Above this solid curve, the first-order condition (17) is satisfied. We see that the solid curve exactly demarcates the edge of the grey investment region for $X > 0$ in both panels, providing a visual illustration of Proposition 1; i.e. the first-order condition is both necessary and sufficient. For $X < 0$, in contrast, the panels reveal that the first-order condition $\dot{A}_t \leq \beta A_t$ is necessary but *in*sufficient, as the grey investment region is no longer demarcated by the solid curve; rather, there is a white 'gap' between the investment region and the curve. Within this gap, investment

[4] The time-preference rate $\beta = 0.06$ is motivated by the real cost of capital for real estate in Jordà et al. (2019, Table III). The long-run drift $\mu = 0.01$ is based on the average growth in land productivity reported in Davis et al. (2014, p. 732), while $\theta = 0.15$ implies an annual autocorrelation of $1 - 0.15 = 0.85$, consistent with the observed autocorrelation of population growth in US metropolitan areas (Campbell et al., 2009, table 3 and Desmet and Rappaport, 2017). Production parameter $\alpha = 0.70$ falls in the middle of the range 0.60–0.80 reported in the literature (e.g. Davis et al., 2014).

[5] Increasing the fixed cost ϕ while holding constant the excess growth rate X simply increases (by some multiplicative constant) the rental price per unit of floorspace Y for which investment becomes optimal.



Panel A: Leontief ($\alpha = 0, \phi = 1$) Panel B: Stone Geary ($\alpha = 0.7, \phi = 1$)

Figure 1: Optimal investment policy with deterministic dynamics (1)–(2).

is suboptimal even as the first-order condition is satisfied; while $t \mapsto \mathrm{e}^{-\beta t} A_t$ is *locally* weakly decreasing, waiting is nevertheless optimal because the global optimum still lies ahead.

In panel A, for the Leontief production function, investment for $X > 0$ is optimal if and only if Y exceeds the flow cost of capital, $\beta\phi$. For negative growth rates $X < 0$, the critical level of Y that triggers investment varies only weakly with X. Panel A is consistent with our hypothesis that investment should be optimal if the cash flow Y is sufficiently high. Moreover, the correlation between X and Y at the moment of investment is zero or marginally negative; i.e. a trade-off between them is deemed acceptable.

In panel B, investment is never optimal for any $X > X^\dagger = \sim 3.45\%$, irrespective of the cash-flow level Y. The grey investment region lies entirely to the left of the vertical line $X = X^\dagger$. The investment region for Stone Geary is a subset of that for Cobb Douglas, because the former case requires an additional capital investment $\phi > 0$, which increases the hurdle for investment. This hurdle is negligible for large Y, however, such that the investment region approaches the vertical line $X = X^\dagger$ in this limit, mimicking the Cobb-Douglas case. What differentiates both elastic cases from the standard Leontief case, however, is that our hypothesis that investment should be forthcoming if Y is sufficiently high does *not* hold. Indeed, panel B with $X > X^\dagger$ reveals no trigger value of Y, however large, that would spur investment. Instead, for sufficiently large Y, investment is optimal if and only if $X \leq X^\dagger$; i.e. only sufficiently *low* growth triggers investment.

Finally, both panels also contain two sample trajectories $\{(X_t, Y_t)\}_{t \in \mathbb{R}}$ through the state space, where either (i) $X_t > 0$ for all t or (ii) $X_t < 0$ for all t. For $X_t > 0$, both panels illustrate that the sample trajectory intersects the solid curve exactly once, at which point $t \mapsto \mathrm{e}^{-\beta t} A_t$ achieves its global maximum. For $X_t < 0$, in contrast, trajectories $\{(X_t, Y_t)\}_{t \in \mathbb{R}}$ through the state space are U-shaped, with a minimum on the vertical line $X = -\mu$. As illustrated in Figure 1, the trajectory may intersect the bold curve twice, corresponding to two stationary points. While downcrossings of the solid curve (which represent local minima) are ruled out as optimal investment times, upcrossings (which represent local maxima) in panel B are confined to the upward-sloping part of the solid curve, implying that higher growth rates necessitate higher cash-flow levels to trigger investment.

We conclude that, for the Stone-Geary production function, state variables X and Y tend to be positively correlated at the moment of investment, acting as complements for investment; this prediction warrants empirical testing in view of the widespread belief that (i) investment decisions depend mostly or exclusively on Y, or that (ii) as preconditions for investment, X and Y can be substituted for each other.

3 Formulation and solution of the model with stochasticity

This section presents and solves the full-fledged version of the model, i.e. with stochastic state dynamics. The deterministic model has the drawback that the system is forced to respond to an 'MIT shock' that agents are unaware could occur. In the real world, market participants are well aware of the possibility of shocks, and this could affect their behaviour. Introducing stochasticity adds an element of realism and enhances the generality and robustness of the proposed model. As we shall see, our main findings from Figure 1 remain valid in the stochastic setting. In fact, as Figure 4 illustrates, our main finding is further reinforced as uncertainty creates an *additional* incentive for postponing investment. As this section is necessarily more technically advanced, readers primarily interested in the main findings are encouraged to skip directly to the discussion of Figure 4 in Section 3.6.

3.1 Model

The model setup is the same as in Section 2.1, except that equations (1) and (2) are generalised to allow for stochastic shocks. In analogy with equation (1), the excess growth rate $\{X_t\}$ follows a mean-reverting Brownian motion (i.e. an Ornstein-Uhlenbeck process) with mean-reversion parameter $\theta > 0$ and stochasticity driven by $\sigma_X \geq 0$. In analogy with equation (2), the cash-flow process $\{Y_t\}$ is subject to a geometric drift $\mu + X_t$ with $\mu > 0$ and geometric shocks with standard deviation $\sigma_Y \geq 0$:

$$\mathrm{d}X_t = \theta \left(- X_t \,\mathrm{d}t \;+\; \sigma_X \,\mathrm{d}W_t^X \right), \tag{25}$$

$$\mathrm{d}\log Y_t = \;(\mu + X_t)\,\mathrm{d}t \;+\; \sigma_Y \,\mathrm{d}W_t^Y. \tag{26}$$

Here $\mathrm{d}W_t^X$ and $\mathrm{d}W_t^Y$ are increments of standard Wiener processes, with $\mathrm{E}[\mathrm{d}W_t^X \mathrm{d}W_t^Y] = \rho\,\mathrm{d}t$ for a correlation parameter $\rho \in (-1, 1)$. For some results we require $\rho \in [0, 1)$, which has the advantage that upward shocks to either X_t or Y_t can be unambiguously classed as good news.[6] Lemma 1 in Appendix C demonstrates that, for $t > 0$, the distribution of $(X_t, \log Y_t)$ conditional on $(X_0, \log Y_0)$ is bivariate normal; it also gives the associated mean and covariance matrix. As is standard, the

[6] When allowing for $\rho < 0$, an upward shock to either state variable could actually reduce (rather than increase) the present value of future cash flows.

infinitesimal generator corresponding to process (25)–(26) is

$$\mathrm{L} := -\,\theta\, X\,\frac{\mathrm{d}}{\mathrm{d}X} + \frac{\sigma_X^2}{2}\theta^2\,\frac{\mathrm{d}^2}{\mathrm{d}X^2} + \left(\mu + X + \frac{\sigma_Y^2}{2}\right) Y\,\frac{\mathrm{d}}{\mathrm{d}Y} + \frac{\sigma_Y^2}{2}Y^2\frac{\mathrm{d}^2}{\mathrm{d}Y^2} + \theta\rho\sigma_X\sigma_Y Y\frac{\mathrm{d}^2}{\mathrm{d}X\mathrm{d}Y}. \tag{27}$$

Formally, $\mathrm{L}f(X_0, Y_0) := \lim_{t\downarrow 0} \mathrm{E}_0[\{f(X_t, Y_t) - f(X_0, Y_0)\}/t]$ for an appropriate test function $f : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$, such that L captures the expected change of $f(X_0, Y_0)$ during a short time interval $\mathrm{d}t$.

The stochastic laws of motion (25)–(26) may be contrasted with the classic model of investment, which takes $\log Y_t$ to be a Brownian motion with constant drift. Conveniently, as Lemma 1 in Appendix C shows, this classic model can be recovered as a limiting case of model (26) by taking $\theta \to \infty$:

$$\mathrm{d}\,\log\,Y_t \;=\; \mu\,\mathrm{d}t \,+\, \sigma\,\mathrm{d}W_t, \tag{28}$$

where $\mathrm{d}W_t$ is the increment of a standard Wiener process and where

$$\sigma^2 := \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}, \qquad \sigma_{XY} := \rho\,\sigma_X\sigma_Y. \tag{29}$$

3.2 Valuation of existing production capacity

This section computes the present value of a unit of production capacity under the stochastic dynamics (25)–(26), thus generalising the result in Section 2.2. To obtain a bounded value of existing technologies, both the new model (25)–(26) and the classic model (28) require a further parameter restriction. In both cases, $\log Y_t$ is normally distributed, where the mean and variance are asymptomatically linear in t, scaling as μt and $\sigma^2 t$, respectively. By the expectation of a log-normally distributed random variable, the discounted cash flow $\mathrm{e}^{-\beta t}\mathbb{E}[Y_t]$ is of order $\mathrm{e}^{-(\beta-\mu-\sigma^2/2)t}$ for large t. For this discounted cash flow to be integrable over time, we thus require

$$\beta_0 := \beta - \mu - \sigma^2/2 > 0. \tag{30}$$

Condition (30) generalises restriction (4), which is a special case for which $\sigma_X = \sigma_Y = \sigma = 0$.

Next, Proposition 2 computes the present value of one unit of production capacity conditional on the current state variables X_0 and Y_0. The resulting analytic valuation formula (31)—which is, to the best of our knowledge, new—could be applied to a broader set of questions, such as the valuation of (growth) stocks.

Proposition 2 (Value of unit of production capacity) *Let condition* (30) *hold. The present value of a single unit of production capacity at time* 0 *conditional on the state variables* (X_0, Y_0)*, denoted* $B(X_0, Y_0)$*, equals*

$$B(X_0, Y_0) \;:=\; \int_0^\infty \mathrm{e}^{-\beta\,t}\;\mathrm{E}_0[Y_t]\;\mathrm{d}t = \;Y_0\,b(X_0), \qquad \forall (X_0, Y_0) \in \mathbb{R} \times \mathbb{R}_{>0}, \tag{31}$$

where E_0 *is the expectation conditional on* (X_0, Y_0) *and where the function* $b(X)$ *is defined as*

$$b(X) \;:=\; \int_0^\infty \exp\left[-\beta_0\,t + \left(X - \sigma_X^2 - \sigma_{XY}\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t, \quad X \in \mathbb{R}. \tag{32}$$

The function $b(X)$ is positive, increasing and convex, while diverging to infinity as $X \to \infty$. Moreover, $B(X,Y)$ satisfies Bellman's equation

$$\beta\, B(X,Y) \;=\; \mathrm{L}\, B(X,Y) + Y. \tag{33}$$

The proof is presented in Appendix D, while Lemma 2 in Appendix E lists further properties of $b(\cdot)$ as used in the proofs of other results. The price-dividend ratio $b(X) = B(X,Y)/Y$ in equation (32) generalises equation (7), which is a special case for which $\sigma_X = \sigma_Y = 0$. Bellman's equation (33) indicates that the return on each unit of production capacity is driven by the expected change in the state variables, as measured by $\mathrm{L}B(X,Y)$, while additionally producing the dividend flow Y.

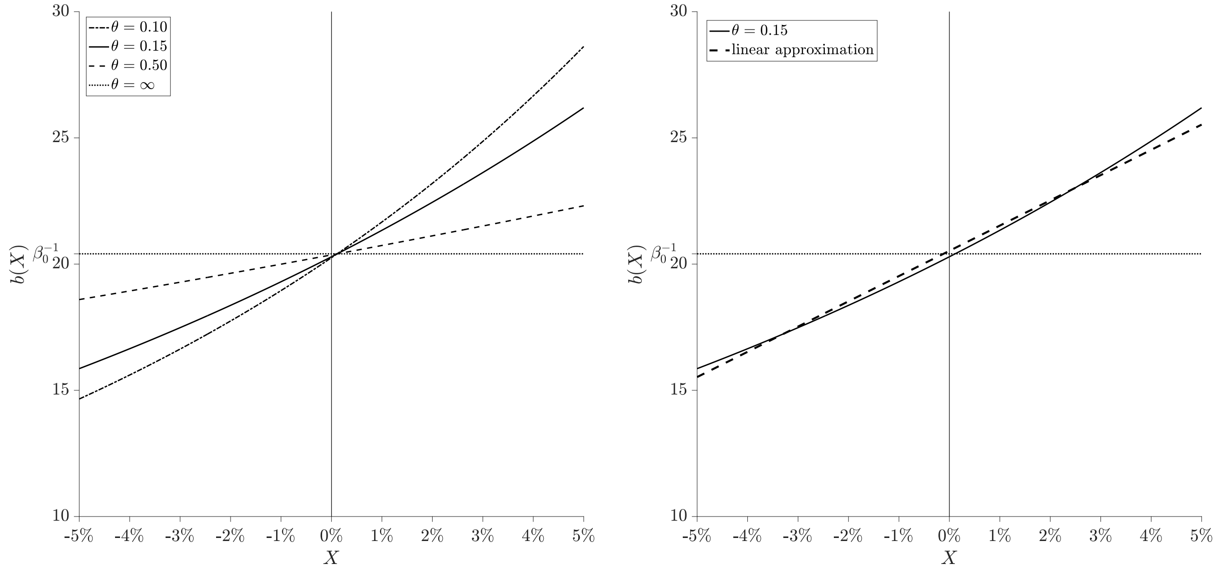
Given the mean-reverting nature of the excess growth rate X, it is perhaps unsurprising that Proposition 2 is closely related to Vasicek's (1977) zero-coupon bond-pricing formula.[7] The resulting valuation formula (31) has obvious counterparts in the finance literature, where affine diffusion models are widely used. For example, when flipping the sign of X, so that it acts as an excess discount rate rather than a growth rate, $B(X,Y)$ can be interpreted as the price of a perpetual bond for which the continuous-time coupon payments follow a geometric Brownian motion started at Y, whose increments may be correlated with those of the excess discount rate X.

Returning to the case where X is interpreted as an excess growth rate, Figure 2 (panel A) plots the function $b(\cdot)$ using our benchmark parameters (see footnote 4) as well as $\sigma_X = 0.04$, $\sigma_Y = 0.02$ and $\rho = 0$, such that $\sigma \approx 0.045$ per annum from equation (29). The annual standard deviation of ~5% is roughly calibrated to be relevant for rental rates. Panel A shows that $b(\cdot)$ is steeper for lower values of θ, in which case X has greater predictive power. In the limit $\theta \to \infty$, the curve $b(\cdot)$ flattens out at the value β_0^{-1} for all X. This limit corresponds to the classic model (28) and cannot explain the empirically observed heterogeneity in price-dividend ratios.

The theoretical finding that the price-dividend ratio $b(X)$ increases with the excess growth rate X is consistent with the empirical results in Sinai and Souleles (2005, p. 765). The valuation formula (31) is therefore useful for the valuation of (growth) stocks, for which the growth rate of dividends, i.e. $\mu + X_0$, exceeds the time-preference rate β. Modelling the growth rate by a mean-reverting process is the obvious—and perhaps only—route to obtain bounded asset values in combination with growth rates that temporarily outstrip the time-preference rate. Indeed, the new model (25)–(26) allows the short-term growth rate $\mu + X_t$ to take *any* value, contingent on the long-term growth rate being dominated by β, as guaranteed by condition (30).

In their seminal work, Campbell and Shiller (1988) assume that the logarithm of the price-dividend ratio, i.e. $\log b(X)$, is linear in the dividend excess growth rate X. We consider instead a linear approximation of $b(X)$ around the point $X = \sigma = 0$, for which the associated slope and intercept are provided in closed form in Lemma 2 in Appendix E. Figure 2 (panel B) shows the resulting linear approximation of $b(X)$ with $\theta = 0.15$, which turns out to be quite accurate for a wide range of excess growth rates X.

[7] Specifically, Vasicek's formula equals the integrand in our equation (32), with some redefinitions: (a) treating our integration (dummy) variable $t \geq 0$ as the bond's maturity date; (b) flipping the sign of X such that it acts as an excess discount rate (applied on top of the time-constant discount rate β) rather than an excess growth rate; (c) setting $\mu = \sigma_Y = \sigma_{XY} = 0$ such that the randomness originates only from X; and (d) redefining σ_X as σ_X/θ. With these changes, the integrand in equation (32) is equivalent to Vasicek's bond price as in e.g. Mamon's (2004, eq. 13) review article.



Panel A: Function $b(X)$ for different θ Panel B: Linear approximation

Figure 2: Function $b(X)$ defined in equation (32).

3.3 Optimal level and timing of investment

Analogous to K_t, A_t, and V_t in the deterministic model, we define $K(X_t, Y_t) = K_t$, $A(X_t, Y_t) = A_t$, and $V(X_t, Y_t) = V_t$ to be the optimal level of investment, the value of the investor's committed factor at the time of investment, and the option value of this committed factor prior to investment, as functions of the state variables X_t and Y_t. Analogous to equations (11), (13), and (14) and $K(X, Y)$, $A(X, Y)$, and $V(X_0, Y_0)$ are:

$$K(X,Y) := \arg\max_{K \geq \phi} \left[(K-\phi)^\alpha \; B(X,Y) \; - \; K\right] = \phi + [\alpha Y b(X)]^{1/(1-\alpha)}, \tag{34}$$

$$A(X,Y) := (K(X,Y)-\phi)^\alpha \, B(X,Y) \; - \; K(X,Y) = \frac{1-\alpha}{\alpha} \left[\alpha Y b(X)\right]^{1/(1-\alpha)} - \phi, \tag{35}$$

$$V(X_0,Y_0) := \sup_{t \geq 0} \mathbb{E}_0 \left[\mathrm{e}^{-\beta t} A(X_t,Y_t)\right], \tag{36}$$

where the supremum over t in the last line is understood as a supremum over stopping times.

As $A(X_t, Y_t)$ scales with $Y_t^{1/(1-\alpha)}$, while $\log Y_t$ asymptotically resembles $\mathrm{N}(\mu t, \sigma^2 t)$, the asymptotic growth rate of $A(X_t, Y_t)$ equals $\mu/(1-\alpha) + (1/2)\sigma^2/(1-\alpha)^2$. Hence, for $V(X_0, Y_0)$ to remain bounded, we require

$$\text{boundedness of } V(X,Y): \qquad \beta_1 := \beta - \frac{\mu}{1-\alpha} - \frac{\sigma^2}{2}\frac{1}{(1-\alpha)^2} > 0. \tag{37}$$

Condition (37) generalises condition (15) for the deterministic case, which is a special case whereby $\sigma_X = \sigma_Y = 0$. Using Bellman's dynamic-programming principle[8], $V(X, Y)$ satisfies

$$\text{option value prior to investment:} \qquad \beta V(X,Y) = \mathrm{L}V(X,Y), \tag{38}$$

[8] Alternatively, equation (38) can be derived using contingent-claims analysis, in which case L is interpreted as the infinitesimal generator under the risk-neutral measure (see e.g. Dixit and Pindyck, 1994, p. 120).

subject to the boundary condition $V(X,0)=0$ for all X. Intuitively, equation (38) indicates that, prior to investment, the return on the committed factor, $\beta V(X,Y)$, is driven only by the expected change in the state variables, as measured by $\mathrm{L}V(X,Y)$. Assuming $\sigma_X,\sigma_Y>0$, the classic value-matching and smooth-pasting conditions[9] must hold for all points (X,Y) on the investment boundary:

$$V(X,Y)=A(X,Y), \qquad \frac{\mathrm{d}V(X,Y)}{\mathrm{d}X}=\frac{\mathrm{d}A(X,Y)}{\mathrm{d}X}, \qquad \frac{\mathrm{d}V(X,Y)}{\mathrm{d}Y}=\frac{\mathrm{d}A(X,Y)}{\mathrm{d}Y}. \tag{39}$$

The value at the moment of investment, $A(X,Y)$, is determined by equation (35) with equation (32) for $b(X)$. The function $V(X,Y)$ is the solution to Bellman's differential equation (38) that satisfies the value-matching and smooth-pasting conditions (39) for all points (X,Y) along the investment boundary. Unfortunately, the resulting two-dimensional free-boundary problem cannot in general be analytically solved. However, it is straightforward to formulate two necessary conditions, analogous to the zero- and first-order conditions (16)–(17), which must be satisfied at the moment of investment and are highly influential in shaping the investment region.

3.4 Two necessary conditions

For the model with stochastic dynamics (25)–(26), two necessary (but typically insufficient) conditions for investment to be optimal at the point (X,Y) are as follows:

$$\text{zero-order condition:} \qquad A(X,Y)\ \geq\ 0, \tag{40}$$

$$\text{first-order condition:} \qquad \mathrm{L}A(X,Y)\ \leq\ \beta A(X,Y). \tag{41}$$

Condition (40) is identical to the zero-order condition (16) in the deterministic case, requiring the asset value at the time of investment to be weakly positive. Condition (41) is near identical to the first-order condition (17) in the deterministic case, the only difference being that the infinitesimal generator, L, replaces the time derivative, $(\mathrm{d}/\mathrm{d}t)$.

Intuitively, the first-order condition (41) requires $A(X_0,Y_0)$ to weakly exceed $\mathrm{e}^{-\beta \mathrm{d}t}\mathbb{E}_0[A(X_{\mathrm{d}t},Y_{\mathrm{d}t})]=A(X_0,Y_0)+(\mathrm{L}-\beta)A(X_0,Y_0)\mathrm{d}t$, where $\mathrm{d}t$ denotes an infinitesimal time interval. If $(\mathrm{L}-\beta)A(X,Y)$ were strictly positive, postponing investment for a short time $\mathrm{d}t$ would be strictly preferable over investing now. Hence $(\mathrm{L}-\beta)A(X,Y)$ should be weakly negative at the time of investment. This fact is well known in optimal-stopping theory; see e.g. Øksendal (2007, remark on p. 217-18) and Øksendal and Sulem (2005, p. 30, Prop. 2.3), who restrict themselves to the case $\beta=0$. The advantage of this first-order condition is that it can be analytically derived prior to solving the option-valuation problem.

Proposition 3 (First-order condition for investment with stochasticity) *Let condition* (37) *hold and let* $\rho\in[0,1)$. *The first-order condition* (41) *can be equivalently expressed as*

$$\underbrace{\beta K(X,Y)}_{\text{flow cost of capital}} + \underbrace{f(X)\,[K(X,Y)-\phi]}_{\substack{\text{value of extra information}\\ \text{for optimal investment}}} \ \leq\ \underbrace{Y\,[K(X,Y)-\phi]^{\alpha}}_{\substack{\text{cash flow}\\ \text{after investment}}}, \tag{42}$$

[9] The smooth-pasting condition in one dimension is classic, e.g. Moscarini and Smith (2001) and DeMarzo et al. (2012). Recent works that consider multidimensional cases include Kakhbod et al. (2021) and Chen et al. (2023).

where the function $X \mapsto f(X)$, defined in the proof, is a weakly positive sigmoid-like function (i.e. strictly increasing and bounded) that vanishes if and only if $\sigma_X = \sigma_Y = 0$.

The proof is contained in Appendix F. For the deterministic case $\sigma_X = \sigma_Y = 0$, condition (42) collapses to the first-order condition (19) in the deterministic model. For the stochastic case, the value of the additional information obtained during the infinitesimal time interval $\mathrm{d}t$ further increases the hurdle for investment. Hence, the cash flow obtained immediately after investment must now cover not only the flow cost of capital, but also the value of additional information that would have been obtained during a short delay. Even as the first-order condition (42) accounts for the stochastic nature of the state variables, it does not entirely capture the potential benefit of delay as it ignores the possibility that another, possibly sizeable, delay may yet eventuate. Thus, while condition (42) is necessary, it is generally insufficient.

For the Cobb-Douglas case, the zero-order condition (40) is automatically satisfied as $\phi = 0$, while the first-order condition (42) can be simplified by substituting equation (34) and rearranging to obtain a condition involving X but not Y:

$$\alpha\,(\beta + f(X))\,b(X) \;\le\; 1, \tag{43}$$

which generalises equation (23), itself a special case with $\sigma_X = \sigma_Y = 0$. As the left-hand side is strictly increasing in X, this first-order condition is equivalent to $X \le X^\dagger$, where $X^\dagger > 0$ is the unique solution to

$$\alpha\,(\beta + f(X^\dagger))\,b(X^\dagger) \;=\; 1, \tag{44}$$

which generalises equation (24). For our benchmark parameters (see footnote 4 and recall $\sigma_X = 0.04$, $\sigma_Y = 0.02$ and $\rho = 0$), we find $X^\dagger \approx 2.44\%$, which can be compared with $X^\dagger \approx 3.45\%$ in the deterministic case. Hence, the additional term $f(X^\dagger)$, which accounts for the information value obtained during a brief delay, makes the condition $X \le X^\dagger$ substantially more stringent.

For the Leontief and Stone-Geary cases, conditions (40)–(41) involve both X and Y. It is convenient to write the zero- and first-order conditions as $Y \ge Y_0(X)$ and $Y \ge Y_1(X)$, respectively, where $Y_0(\cdot)$ and $Y_1(\cdot)$ are critical curves to be found. For Stone Geary, this can be achieved by using equation (34) and defining

$$Y_0(X) := \left(\frac{\alpha\,\phi}{1-\alpha}\right)^{1-\alpha} \frac{1}{\alpha\,b(X)}, \tag{45}$$

$$Y_1(X) := \left(\frac{\beta\,\phi}{1-\alpha\,(\beta + f(X))\,b(X)}\right)^{1-\alpha} \left(\frac{1}{\alpha\,b(X)}\right)^{\alpha}. \tag{46}$$

The function $Y_0(X)$ is decreasing in X, indicating that the zero-order condition is more likely to be satisfied when X is increased. The function $Y_1(X)$ with $\alpha > 0$, however, is U-shaped with a vertical asymptote when the denominator equals zero, which occurs at $X = X^\dagger$. For $X > X^\dagger$, the function $Y_1(X)$ is undefined. Hence, the Cobb-Douglas finding persists that $X \le X^\dagger$ is necessary (but typically insufficient) for investment. Equations (45)–(46) remain valid for the Leontief production function by using $\alpha^{-\alpha} = 0^0 = 1$, yielding $Y_0(X) = \phi/b(X)$ and $Y_1(X) = \beta\phi$.

In sum, we have analytically derived two critical curves (45)–(46), such that (i) $Y \geq Y_0(X)$ and (ii) $Y \geq Y_1(X)$ are both necessary for investment to be optimal at (X, Y) in the case of Leontief or Stone-Geary production functions. In accordance with these results, we will later see (in Figure 4) that the optimal investment region lies as expected above both critical curves.

3.5 Analytic solution for the Cobb-Douglas case

For the Cobb-Douglas case, we established that $X \leq X^\dagger$ is necessary for investment, where $X^\dagger = 2.44\%$ for stochastic state dynamics as defined by equation (44), while $X^\dagger = 3.45\%$ for the deterministic model as defined by equation (24). This section derives a new condition $X \leq X^\star$, where $X^\star \approx 1.94\%$ for our benchmark parameters, which is both necessary and sufficient for investment in the Cobb-Douglas case. This new condition is more stringent than both the deterministic and stochastic versions of the necessary condition, i.e. $X^\star \leq X^\dagger$, with equality only when $\sigma_X = \sigma_Y = 0$.

By providing a closed-form solution for the two-dimensional option-valuation problem with a Cobb-Douglas production function, the result below offers a rare exception to the rule that analytic solutions are generally nonexistent for optimal-stopping problems in more than one dimension.

Proposition 4 (Analytic solution for Cobb-Douglas case) *Let condition* (37) *hold and let* $\rho \in [0, 1)$. *Assume* $\alpha \in (0, 1)$, $\phi = 0$ *and* $\sigma_X, \sigma_Y > 0$. *Investment is optimal if and only if* $X \leq X^\star$. *For* $X > X^\star$ *(i.e. prior to investment), Bellman's equation (38) can be solved in closed form as follows:*

$$V(X, Y) = C\, Y^{1/(1-\alpha)}\, v(X), \quad X > X^\star, Y \in \mathbb{R}_{>0}, \tag{47}$$

where $C > 0$ *is an integration constant. The function* $v(\cdot)$ *is*

$$v(X) := \exp\left(\frac{X}{\theta(1-\alpha)}\right) \mathrm{H}_{-\beta_1/\theta}\left[\frac{1}{\sqrt{\theta}\,\sigma_X}\left(X - \frac{\sigma_X^2 + \sigma_{XY}}{1-\alpha}\right)\right], \quad X \in \mathbb{R}, \tag{48}$$

where $\mathrm{H}_n(x)$ *is the generalised Hermite polynomial defined in terms of Kummer's (confluent hypergeometric) function, denoted* $\mathrm{M}(\cdot, \cdot, \cdot)$, *as follows:*

$$\mathrm{H}_n(x) := 2^n \sqrt{\pi}\left[\frac{1}{\Gamma\left(\frac{1-n}{2}\right)} \mathrm{M}\left(-\frac{n}{2}, \frac{1}{2}, x^2\right) - \frac{2\,x}{\Gamma\left(-\frac{n}{2}\right)} \mathrm{M}\left(\frac{1-n}{2}, \frac{3}{2}, x^2\right)\right].$$

The threshold $X^\star$ *(exists and) is the unique solution to*

$$\frac{b'(X^\star)}{b(X^\star)} = (1-\alpha)\frac{v'(X^\star)}{v(X^\star)}, \tag{49}$$

where primes denote derivatives. Conditional on $X^\star$, *the integration constant* C *can be expressed in closed form.*

The proof is contained in Appendix G. The necessary and sufficient condition for investment reads $X \leq X^\star \approx 1.94\%$ at our benchmark parameters, which is naturally more stringent than (i) the necessary (but insufficient) condition $X \leq X^\dagger \approx 2.44\%$ derived in the previous section as well as (ii) the necessary and sufficient condition $X \leq X^\dagger \approx 3.45\%$ for deterministic state dynamics. Adding stochasticity thus further reinforces our main finding that investment is suboptimal when growth is sufficiently high. While $X^\dagger$ defined by equation (44) accounts for stochasticity and the information value of an infinitesimal delay $\mathrm{d}t$, the optimal threshold $X^\star$ additionally accounts for the fact that after a brief delay $\mathrm{d}t$, another (possibly sizeable) delay may yet eventuate. This further increases the option value, thereby diminishing the region in which investment is optimal.

Moreover, Proposition 4 demonstrates that investments should, due to high growth, be postponed in a practically relevant number of cases. Our benchmark parameters imply that X is normally distributed around zero with standard deviation $\sim$1.10%, such that the critical threshold $X^\star \approx 1.94\%$ lies $\sim$1.77 standard deviations above the mean. In turn, this implies that $\mathbb{P}(X > X^\star) \approx 3.82\%$, i.e. high growth could stifle investment $\sim$4% of the time. This may be an underestimate, however, because the Cobb-Douglas production function (which sets $\phi = 0$) imposes no lower bound on the investment, while the more realistic Stone-Geary production function requires the investment to exceed $\phi > 0$. Indeed, as we will see in the next section, the investment region for the Stone-Geary production function is a strict subset of that for the Cobb-Douglas case.

Figure 3 uses the analytic valuation formulas in Proposition 4 to plot $V(X,Y)$ and $A(X,Y)$ evaluated at $Y = 1$ as a function of X, where $V(X,1)$ is depicted only prior to investment, i.e. for X exceeding $X^\star$. The value-matching and smooth-pasting properties at $X^\star$ are clearly visible as $V(X^\star,1)$ and $A(X^\star,1)$ are equal and share the same slope at this point. Unlike classic real-options models (e.g. those in Dixit and Pindyck, 1994), the option value $V(X,1)$ is identical to the value of immediately exercising the option $A(X,1)$ at the point at which $V(X,1)$ takes its lowest (rather than its highest) value. We have come across no other real-options models with this specific property.

3.6 Numerical solution for Stone Geary and Leontief

For the Stone-Geary and Leontief cases (i.e. with $\phi > 0$), an analytic solution to the differential equation (38) with boundary conditions (39) is no longer available due to the two-dimensional nature of the optimal-stopping problem. To find the optimal policy for these production functions, we use two numerical methods detailed in Appendix H.

The first is the Poisson optional stopping times (POST) method (Lange et al., 2020), which is based on the idea that an independent Poisson process with intensity $\lambda > 0$ generates multiple opportunities to stop, but the decision maker is permitted to stop only once. Specifically, stopping is permissible at one of the Poisson arrival times, but not in between any two arrival times; this can be viewed as a 'liquidity constraint'. In the limit $\lambda \to \infty$, opportunities to stop arrive almost continuously, such that the liquidity constraint all but vanishes.

The second method follows Compernolle et al. (2021) in directly discretising the partial differential equation (38), i.e. $(\beta - \mathrm{L})V(X,Y) = 0$, using standard finite-difference methods. While they use an upwind scheme to approximate first derivatives, we use a combination of up- and downwind schemes as the direction of the drift in our model is not fixed. Because partial differential equations can have many solutions without appropriate boundary conditions, we employ the theory of linear complementarity problems (LCPs, e.g. Schäfer, 2004 and Cottle et al., 2009) to incorporate the relevant constraint

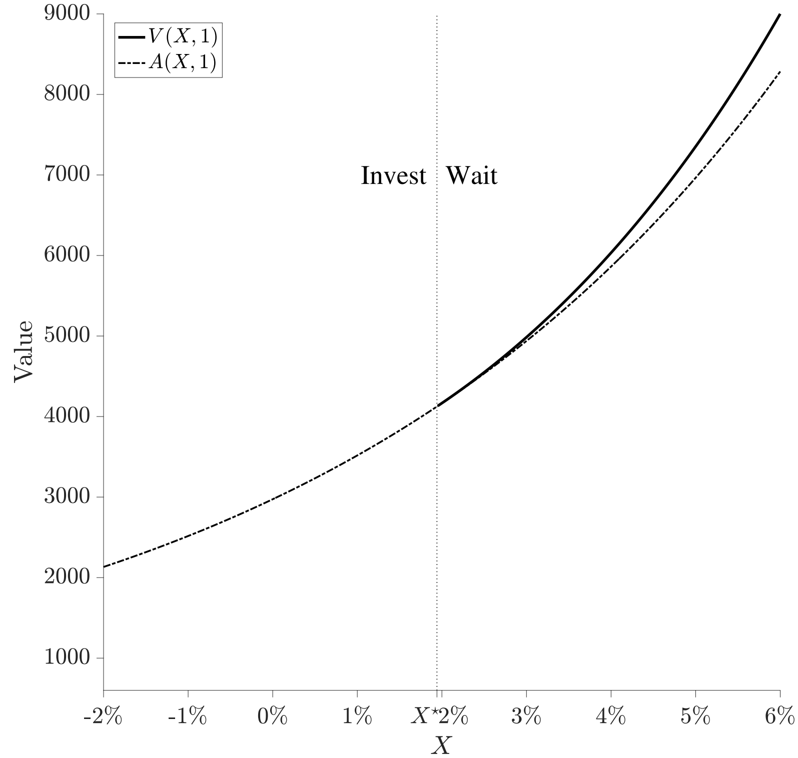


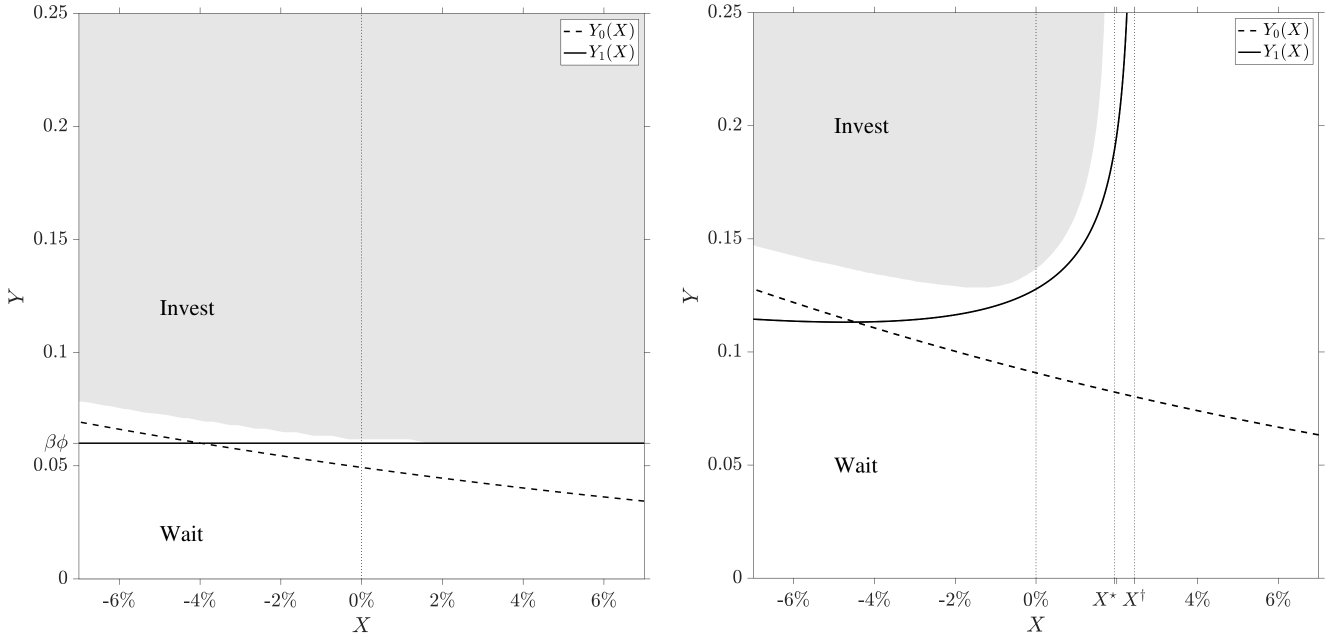
Figure 3: Value of vacant land with Cobb-Douglas production function and benchmark parameters.

$V(X,Y) \geq A(X,Y)$. The resulting LCP can be solved using standard Newton-type methods, as in e.g. Bazaraa et al. (2013).

Our confidence in the numerical solution derives from the fact that both methods (i) have solid theoretical underpinnings with guaranteed convergence properties, (ii) show a remarkable degree of agreement when solving the same problem, (iii) are highly accurate for $\sigma_X = \sigma_Y = 0$ in correctly recovering the investment policies in Figure 1, (iv) correctly produce stopping regions located just above the critical curves $Y_0(\cdot)$ and $Y_1(\cdot)$ as predicted by the theory in Section 3.4, and (v) yield near-identical results after extensive robustness checks.

Figure 4 shows the optimal investment policy in the state space for the Leontief (panel A) and Stone-Geary (panel B) production functions using benchmark parameters and stochastic state dynamics (25)–(26). Both panels can be directly compared against the analogous panels in Figure 1 for deterministic state dynamics. The panels contain the optimal investment region (shaded grey), a dotted curve that shows where the zero-order condition (40) holds with equality, and a solid curve that shows where the first-order condition (41) holds with equality. Both curves are known analytically; i.e. the dotted curve is $Y_0(X)$ in equation (45), while the solid curve is $Y_1(X)$ in equation (46). In accordance with the theory, the grey investment region lies above both curves, possibly at quite some distance.

Panel A, for the Leontief production function, is consistent with our initial hypothesis that investment should be forthcoming if Y is sufficiently high. The critical level of Y that triggers investment is relatively insensitive to X, implying a zero (or even slightly negative) correlation between X and Y at the moment of investment. Panel B, for the Stone-Geary production function, shows that investment dries up for growth rates exceeding $X^\star$, where $X^\star$ was characterised in Proposition 4—this finding



Panel A: Leontief ($\alpha = 0, \phi = 1$) Panel B: Stone Geary ($\alpha = 0.7, \phi = 1$)

Figure 4: Optimal investment policy with stochastic dynamics (25)–(26).

contradicts our initial hypothesis. For large Y, the Stone-Geary policy approaches the Cobb-Douglas policy, i.e. to invest if and only if $X \leq X^\star$. In contrast with the Leontief production function, the Stone-Geary case implies that X and Y at the moment of investment tend to be positively (rather than negatively) correlated, meaning X and Y acts as complements rather than substitutes in spurring investment.

In both panels, the necessary first-order condition (42) is highly influential in shaping the investment region. The solid curve in panel B of Figure 4 is located somewhat higher in the state space than in panel B of Figure 1, which featured deterministic state dynamics. This is because the first-order condition (42) in the stochastic case requires the cash flow after investment to exceed not only the capital cost of the investment but also the information value that would have been obtained during a brief delay. Relative to Figure 1, Figure 4 thus shows that postponing investment has become even more attractive, as uncertainty creates an additional incentive for delay.

Finally, Figure 5 points to an important policy implication of the model. It shows the value $V(X, Y)$ of the committed factor (e.g. land) at the moment of investment for the Stone-Geary case as a function of the excess growth rate X. When X moves from its long-term average to one standard deviation above this average (i.e. from $X = 0\%$ to $\sim 1.1\%$), the value increases by $> 200\%$ (from ~ 3 to ~ 10). The high value of vacant land in high-growth cities has been interpreted as a sign of regulatory inefficiency. Here, we show that this phenomenon may instead be attributed to the combination of persistent growth rates, flexible but irreversible investment and rational investor behaviour.

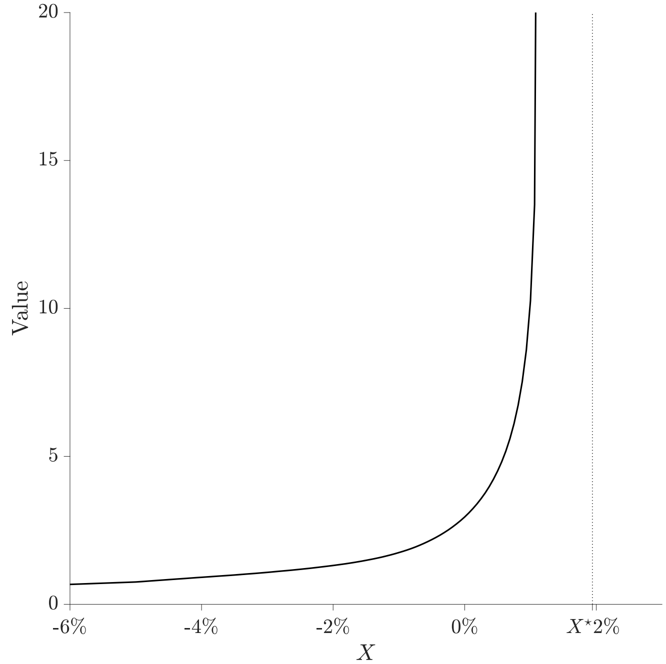


Figure 5: Value of vacant land at the moment of construction.

4 Conclusion

We have shown that in the case of a one-time, flexible but irreversible investment, the cash-flow level and its growth rate are positively correlated at the moment of investment, such that they act as complements: a higher growth rate necessitates a higher cash flow to trigger investment. Moreover, investment dries up altogether—irrespective of the cash-flow level—when growth rates are sufficiently high. While stochasticity in the cash-flow level and its growth rate is *in*essential to these findings, it does amplify them. These results contradict our initial hypotheses that (i) investment should by triggered—irrespective of the growth rate—by sufficiently high cash-flow levels, and that (ii) the cash-flow level and its growth rate should act as substitutes in spurring investment (i.e. a trade-off between them is considered acceptable), which would imply a negative correlation between these two variables at the moment of investment. Only in the case of fixed (i.e. inflexible) investment do these hypotheses appear to hold true.

Our findings rely on two critical assumptions: (i) the growth rate is at least partially (and possibly fully) predictable, (ii) the one-off investment is flexible in size, but irreversible. That is, the investor has pre-committed some fixed (i.e. unalterable) factor of production, while the second factor of production adheres to a putty-clay capital structure. Assumption (i) is satisfied in many industries, as evidenced by the heterogeneity observed in price-dividend ratios, which is due in part to substantial variation in expected growth rates. While the standard model imposes that the (constant) growth rate is exceeded by the time-preference rate, our model allows the *short-term* growth rate take any value, as long as the *long-term* growth rate is exceeded by the time-preference rate. This is relevant as the (short-term) dividend-growth rate of some stocks substantially exceeds the time-preference rate, which in the classic model would imply unbounded present values. Our analytic valuation formula may thus have wide applications in the valuation of (growth) stocks.

The real-estate market—which accounts for half the world's capital stock—is a prime example of a sector that satisfies both assumptions simultaneously. First, most cities go through prolonged periods of above- or below-average growth in terms of both their size and real-estate prices, which are highly correlated. Second, land is a fixed factor that forms an indispensable input for construction, while urban infrastructure is highly persistent, consistent with the putty-clay assumption. Manhattan's rectangular grid was laid out more than 200 years ago; likewise, Hausmann's boulevards in Paris and the canals in Amsterdam.

While the standard prediction is that land values depend solely on cash-flow levels, our model suggests that they are highly sensitive to recent growth rates. For our benchmark parameters, the value of vacant land rises by >200%, i.e. more than triples, when the growth rate rises from its long-term average by one standard deviation. Similarly, while the standard model posits that the density of construction should depend not on growth but on city size, our model suggests that land in booming cities should be developed more densely. Empirically testing these contrasting claims would serve as a litmus test to gauge the model's validity.

By building our model around the above assumptions, we have demonstrated that investors in possession of vacant land should rationally postpone construction when the growth rate of the rental price per unit of floorspace exceeds a critical threshold. This is not, we argue, merely an esoteric mathematical finding. The majority of investment occurs along the upward-sloping part of the investment boundary, implying that the growth rate and the level of the rental price per unit of floorspace act as complements rather than substitutes. By implication, an upward shock to the growth rate of a city's rental rates will stymie rather than speed up new construction. This finding is, to the best of our knowledge, novel in the real-options and optimal-investment literature.

Finally, it is our hope that this article contributes to current policy debates on why 'superstar' cities (so dubbed by Gyourko et al., 2013) attract relatively low levels of investment in construction even as housing prices soar. One may wonder whether letting expensive, high-growth locations lie vacant is a market failure. Indeed, a related strand of literature attributes the high value of vacant land in the vicinity of growing cities to regulatory inefficiencies (e.g. minimum lot size regulation) due to rent-seeking behaviour (e.g. Glaeser et al., 2005, Glaeser and Ward, 2009 and Duranton and Puga, 2023). However, our model contains no externalities; hence, the decisions of rational investors are Pareto efficient. The fact that land on the urban periphery remains vacant even as its price soars is not a market failure but an efficient, socially optimal response that optimises the option value of land. While private investors do not take account of agglomeration externalities, one may conjecture that a social planner would delay investment even further, to permit even higher-density construction at a later stage.

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Appendix to "Irreversible investment under predictable growth: Why land stays vacant when housing demand is booming"

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A Elasticity of substitution of the production function

Consider a constant-returns-to-scale (CRS) Stone-Geary production function:

$$F(L,K) = L^{1-\alpha}(K-\phi L)^{\alpha},$$

where $L > 0$ and $K > \max(0, \phi L)$. This representation is an adaptation of the standard Stone-Geary production function, as the deduction ϕL from K scales with L rather than being constant. The resulting production function exhibits constant returns to scale. In our application, $L = 1$. For a two factor production function with constant returns to scale, the elasticity of substitution η is the inverse of the elasticity of complementarity (Layard and Walters, 1978, pp. 265–272):

$$\eta^{-1} = \frac{F_{KL}F}{F_K F_L} = \frac{(1-\alpha)K}{(1-\alpha)K - \phi L}.$$

Table A.1 demonstrates that production function (3) allows the entire range of elasticities of substitution between the first and second factors of production.

B Proof of Proposition 1

1. To establish that the zero-order condition is implied, the first-order condition $\beta K_t \leq Q_t Y_t$ can be written as $K_t \leq \beta^{-1} Q_t Y_t$. This can be substituted into expression (13) for $A_t = Q_t Y_t b_t - K_t$ to yield $A_t \geq Q_t Y_t (b_t - \beta^{-1}) > 0$, where the last inequality holds as $(b_t - \beta^{-1}) > (\beta - \mu)^{-1} - \beta^{-1} > 0$ for $X_t > 0$. Hence $A_t > 0$ is implied.

2. To establish uniqueness, we note that, for all three conditions (21)–(23), $X_t > 0$ implies that the right-hand sides are strictly increasing in time, while the left-hand sides are either constant (for Leontief and Cobb Douglas) or strictly decreasing (for Stone Geary). This implies that, for each condition, there can be at most one (unique) moment $t \in \mathbb{R}$ that achieves equality. Moreover, the derivative of $t \mapsto \mathrm{e}^{-\beta t} A_t$ is strictly negative (positive) strictly before (after) this unique moment.

3. To establish existence, we note that conditions (21)–(23) are satisfied asymptotically. By continuity, this means that, even if these conditions are not initially satisfied, they must hold with equality

Table A.1: Elasticities for production function (A)

	α	ϕ	η
Leontief	0	>0	0
Stone Geary ($\phi > 0$)	$(0,1)$	>0	$(0,1)$
Cobb Douglas	$(0,1)$	0	1
Stone Geary ($\phi < 0$)	$(0,1)$	<0	>1
Perfect substitution	1	<0	∞

Note: Elasticities η for different values of parameters α, ϕ in function (3).

at least once. For Leontief, Y_t grows without bound such that condition (21) must asymptotically hold. For Leontief and Stone Geary, we have $b_t^{-1} \to \beta - \mu$, such that conditions (22)–(23) become $\alpha\beta \leq \beta - \mu$ or, what is equivalent, $\mu/(1-\alpha) \leq \beta$. In turn, this is ensured by assumption (15), such that conditions (22)–(23) are satisfied asymptotically as desired. ■

C Distribution of stochastic process

Lemma 1 *1. The distribution of* $(X_t, \log Y_t)$ *for any* $t > 0$ *conditional on* $(X_0, \log Y_0)$ *is*

$$\begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} \sim \mathrm{N}(\mathbf{m}_t, \boldsymbol{\Sigma}_t), \quad \textit{where} \tag{C.1}$$

$$\mathbf{m}_t = \begin{pmatrix} \mathrm{e}^{-\theta t} X_0 \\ \log Y_0 + \mu t + \frac{1-\mathrm{e}^{-\theta t}}{\theta} \; X_0 \end{pmatrix}, \tag{C.2}$$

$$\boldsymbol{\Sigma}_t = \begin{pmatrix} \frac{1}{2}\theta\sigma_X^2(1-\mathrm{e}^{-2\theta t}) & \frac{1}{2}\sigma_X^2(1-\mathrm{e}^{-\theta t})^2 + \sigma_{XY}(1-\mathrm{e}^{-\theta t}) \\ \frac{1}{2}\sigma_X^2(1-\mathrm{e}^{-\theta t})^2 + \sigma_{XY}(1-\mathrm{e}^{-\theta t}) & \sigma^2 t - \frac{2}{\theta}(\sigma_X^2 + \sigma_{XY})(1-\mathrm{e}^{-\theta t}) + \frac{1}{2\theta}\sigma_X^2(1-\mathrm{e}^{-2\theta t}) \end{pmatrix}, \tag{C.3}$$

where σ^2 *and* σ_{XY} *are defined in equation* (29)*.*

2. *The steady-state distribution of* X*, i.e. the distribution of* X_t *for* $t \to \infty$ *reads*

$$X \sim \mathrm{N}\left(0, \frac{1}{2}\theta\sigma_X^2\right).$$

A steady-state distribution of Y *does not exist.*

3. *For* $\theta \to \infty$*, the distribution of* $\log Y_t - \log Y_0$ *is*

$$\lim_{\theta\to\infty} (\log Y_t - \log Y_0) \;\sim\; \mathrm{N}\left(\mu\, t\,,\, \sigma^2\, t\right), \quad t > 0.$$

Proof

1. Equations (25)–(26) can be written as

$$\mathrm{d}\begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \end{pmatrix}\mathrm{d}t + \begin{pmatrix} -\theta & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_t \\ \log Y_t \end{pmatrix}\mathrm{d}t + \begin{pmatrix} \theta\,\sigma_X & 0 \\ 0 & \sigma_Y \end{pmatrix}\begin{pmatrix} \mathrm{d}W_t^X \\ \mathrm{d}W_t^Y \end{pmatrix}.$$

Following Karatzas and Shreve (2012, p. 354), the solution can be written as

$$\begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} = \mathbf{M}_t \left[\begin{pmatrix} X_0 \\ \log Y_0 \end{pmatrix} + \int_0^t \mathbf{M}_s^{-1} \begin{pmatrix} 0 \\ \mu \end{pmatrix} \mathrm{d}s + \int_0^t \mathbf{M}_s^{-1} \begin{pmatrix} \theta\,\sigma_X & 0 \\ 0 & \sigma_Y \end{pmatrix} \begin{pmatrix} \mathrm{d}W_s^X \\ \mathrm{d}W_s^Y \end{pmatrix} \right],$$

where the matrix $\mathbf{M}_t$ satisfies

$$\frac{\mathrm{d}\mathbf{M}_t}{\mathrm{d}t} = \begin{pmatrix} -\theta & 0 \\ 1 & 0 \end{pmatrix} \mathbf{M}_t, \qquad \mathbf{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The solution for $\mathbf{M}_t$ and its inverse are

$$\mathbf{M}_t = \begin{pmatrix} \mathrm{e}^{-\theta t} & 0 \\ \frac{1-\mathrm{e}^{-\theta t}}{\theta} & 1 \end{pmatrix}, \quad \mathbf{M}_t^{-1} = \begin{pmatrix} \mathrm{e}^{\theta t} & 0 \\ \frac{1-\mathrm{e}^{\theta t}}{\theta} & 1 \end{pmatrix}. \tag{C.4}$$

For a fixed time $t > 0$, the expectation of $(X_t, \log Y_t)$ conditional on $(X_0, \log Y_0)$, denoted E_0, is

$$\begin{aligned} \mathrm{E}_0 \begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} &= \mathbf{M}_t \left[\begin{pmatrix} X_0 \\ \log Y_0 \end{pmatrix} + \int_0^t \mathbf{M}_s^{-1} \begin{pmatrix} 0 \\ \mu \end{pmatrix} \mathrm{d}s \right], \\ &= \begin{pmatrix} \mathrm{e}^{-\theta t} X_0 \\ \log Y_0 + \mu t + \frac{1-\mathrm{e}^{-\theta t}}{\theta}\; X_0 \end{pmatrix}. \end{aligned}$$

The covariance matrix of $(X_t, \log Y_t)$ conditional on $(X_0, \log Y_0)$ is

$$\begin{aligned} &\mathrm{E}_0 \left[\left\{ \begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} - \mathrm{E}_0 \begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} \right\} \left\{ \begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} - \mathrm{E}_0 \begin{pmatrix} X_t \\ \log Y_t \end{pmatrix} \right\}' \right], \\ &= \mathbf{M}_t \left[\int_0^t \mathbf{M}_s^{-1} \begin{pmatrix} \sigma_X \theta & 0 \\ 0 & \sigma_Y \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} \sigma_X \theta & 0 \\ 0 & \sigma_Y \end{pmatrix} \mathbf{M}_s'^{-1} \,\mathrm{d}s \right] \mathbf{M}_t', \\ &= \begin{pmatrix} \frac{1}{2}\theta\sigma_X^2(1-\mathrm{e}^{-2\theta t}) & \frac{1}{2}\sigma_X^2(1-\mathrm{e}^{-\theta t})^2 + \sigma_{XY}(1-\mathrm{e}^{-\theta t}) \\ \frac{1}{2}\sigma_X^2(1-\mathrm{e}^{-\theta t})^2 + \sigma_{XY}(1-\mathrm{e}^{-\theta t}) & \sigma^2 t - \frac{2}{\theta}(\sigma_X^2 + \sigma_{XY})(1-\mathrm{e}^{-\theta t}) + \frac{1}{2\theta}\sigma_X^2(1-\mathrm{e}^{-2\theta t}) \end{pmatrix}. \end{aligned}$$

The last line follows by using the expression for $\mathbf{M}_s$ in equation (C.4) and explicitly performing the required integrals.

2. The proofs of the other parts are straightforward, hence omitted.

■

D Proof of Proposition 2

We use E_0 and V_0 to denote the mean and variance operators conditional on the information at time zero. The expected present value of an existing unit of floorspace then equals

$$
\begin{aligned}
B(X_0, Y_0) &= \mathrm{E}_0 \int_0^\infty \mathrm{e}^{-\beta t} Y_t \,\mathrm{d}t = \int_0^\infty \mathrm{E}_0 \exp\left[-\beta t + \log Y_t\right] \mathrm{d}t, \\
&= \int_0^\infty \exp\left[-\beta t + \mathrm{E}_0(\log Y_t) + \frac{1}{2}\mathrm{V}_0(\log Y_t)\right] \mathrm{d}t, \\
&= Y_0 \int_0^\infty \exp\left[-\beta t + \mu t + X_0 \frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma^2}{2} t - (\sigma_X^2 + \sigma_{XY})\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right] \mathrm{d}t, \\
&= Y_0 \int_0^\infty \exp\left[-\beta_0\, t + X_0 \frac{1-\mathrm{e}^{-\theta t}}{\theta} - (\sigma_X^2 + \sigma_{XY})\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right] \mathrm{d}t, \qquad \text{(D.1)} \\
&= Y_0 \int_0^\infty b(X_0, t) \mathrm{d}t,
\end{aligned}
$$

where

$$
b(X, t) := \exp\left[-\beta_0\, t + (X - \sigma_X^2 - \sigma_{XY})\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]. \qquad \text{(D.2)}
$$

In the above computation, the second line uses the property $\mathrm{E}\exp(Z) = \exp(\mathrm{E}[Z] + \mathrm{V}[Z]/2)$, which holds when Z is normally distributed. Line three uses Lemma 1. Line four, which employs the definition of β_0 in equation (30), proves equation (31) in Proposition 2.

The definition of $b(X, t)$ in equation (D.2) allows us to verify that $B(X, Y)$ satisfies Bellman's equation (33) by computing

$$
(\mathrm{L} - \beta)\,[Y\, b(X, t)] \;=\; Y\, \frac{\mathrm{d}\, b(X, t)}{\mathrm{d}t},
$$

where L is the differential operator given in equation (27). This implies

$$
(\beta - \mathrm{L}) B(X, Y) \;=\; \int_0^\infty (\beta - \mathrm{L})[Y\, b(X, t)]\,\mathrm{d}t \;=\; Y\left[-\, b(X, t)\right]_{t=0}^{t=\infty} \;=\; Y,
$$

confirming Bellman's equation (33). ■

E Additional properties of the function $b(X)$

The function $b(X)$ in equation (31) of Proposition 2 can be written as

$$
b(X) = \int_0^\infty b(X, t) \mathrm{d}t, \qquad \text{(E.1)}
$$

using the definition of $b\,(X, t)$ in equation (D.2). The next Lemma states some useful properties of $b(X)$ and $b(X, t)$, which are used in the proofs of other results.

Lemma 2 1. *The functions $b(X,t)$ and $b(X)$ satisfy differential equations as follows:*

$$\left(\mu-\beta+X+\frac{\sigma_Y^2}{2}\right)b(X,t)-\theta(X-\sigma_{XY})\frac{\mathrm{d}b(X,t)}{\mathrm{d}X}+\frac{1}{2}\theta^2\sigma_X^2\frac{\mathrm{d}^2b(X,t)}{\mathrm{d}X^2}=\frac{\mathrm{d}b(X,t)}{\mathrm{d}t}, \tag{E.2}$$

$$\left(\mu-\beta+X+\frac{\sigma_Y^2}{2}\right)b(X)-\theta(X-\sigma_{XY})b'(X)+\frac{1}{2}\theta^2\sigma_X^2b''(X)=-1. \tag{E.3}$$

2. *The left- and right-hand tails of $b(X)$ satisfy*

$$\lim_{X\to-\infty} b(X)\ \left(\beta-\mu-\sigma_Y^2/2-X\right)=1, \tag{E.4}$$

$$\lim_{X\to\infty} b(X)\ \left(\frac{X}{\theta}\right)^{\beta_0/\theta}\exp\left(-\frac{X-\sigma_X^2-\sigma_{XY}}{\theta}-\frac{\sigma_X^2}{4\theta}\right)=\frac{1}{\theta}\Gamma\left(\frac{\beta_0}{\theta}\right), \tag{E.5}$$

where $\Gamma(\cdot)$ is the Gamma function.

3. *Define*

$$b_k(X):=\theta^k b^{(k)}(X)/b(X),\quad \forall X\in\mathbb{R},\forall k\in\mathbb{N}^+, \tag{E.6}$$

where $b^{(k)}(X)$ denotes the k-th derivative of $b(X)$. The functions $b_1(X)$ and $b_2(X)$ are sigmoid functions, i.e. they are (i) strictly increasing in X, (ii) bounded between zero and one, while (iii) achieving these bounds in the limit $X\to-\infty$ and $X\to\infty$, respectively. Furthermore

$$b_1(X)>b_2(X)>0. \tag{E.7}$$

4. *$b(X)$ is strictly positive and strictly increasing with*

$$\lim_{X\to-\infty} b(X)=0,\qquad \lim_{X\to\infty} b(X)=\infty,\qquad b(0)\le\beta_0^{-1}, \tag{E.8}$$

$$b(X)\ge\left(\beta-\mu-\sigma_Y^2/2-X\right)^{-1},\quad \forall X\in\mathbb{R}_{\le 0}. \tag{E.9}$$

The inequality in equation (E.8) *holds with equality if and only if $\sigma=0$.*

5. *In the point $X=0$ and $\sigma=0$, where $\beta_0=\beta-\mu$, $b(0)$ satisfies:*

$$\begin{aligned}
b(0)_{|\sigma=0} &= \beta_0^{-1},\qquad b'(0)_{|\sigma=0}=\frac{1}{\beta_0(\beta_0+\theta)},\qquad \frac{\partial b(0)_{|\sigma=0}}{\partial\sigma_X^2}=\frac{\partial b(0)_{|\sigma=0}}{\partial\sigma_Y}=0,\\
\frac{\partial b(0)_{|\sigma=0}}{\partial\sigma_X^2} &= \frac{1}{2}\frac{\beta_0^2+3\theta\beta_0+4\theta^2}{\beta_0^2(\beta_0+\theta)(\beta_0+2\theta)},\\
\frac{\partial b'(0)_{|\sigma=0}}{\partial\sigma_X^2} &= 2\frac{\theta^2}{\beta_0^2(\beta_0+\theta)(\beta_0+2\theta)(\beta_0+3\theta)}.
\end{aligned} \tag{E.10}$$

Proof

1. (a) From definition (D.2) for $b(X,t)$, we have

$$\frac{1}{b(X,t)}\frac{\mathrm{d}b(X,t)}{\mathrm{d}X} = \frac{1-\mathrm{e}^{-\theta t}}{\theta}, \tag{E.11}$$

$$\frac{1}{b(X,t)}\frac{\mathrm{d}^2 b(X,t)}{\mathrm{d}X^2} = \left(\frac{1-\mathrm{e}^{-\theta t}}{\theta}\right)^2. \tag{E.12}$$

We use this to re-write the time-derivative of $b(X,t)$ as follows:

$$\begin{aligned}
&\frac{1}{b(X,t)}\frac{\mathrm{d}b(X,t)}{\mathrm{d}t},\\
&= -\beta_0 + (X-\sigma_X^2-\sigma_{XY})\mathrm{e}^{-\theta t} + \frac{\sigma_X^2}{2}\mathrm{e}^{-2\theta t},\\
&= -\beta_0 - (X-\sigma_X^2-\sigma_{XY})\left(1-\mathrm{e}^{-\theta t}\right) - \frac{\sigma_X^2}{2}\left(1-\mathrm{e}^{-2\theta t}\right) + X - \frac{\sigma_X^2}{2} - \sigma_{XY}, \quad \text{(rewriting)}\\
&= -\beta_0 - (X-\sigma_X^2-\sigma_{XY})\left(1-\mathrm{e}^{-\theta t}\right) - \frac{\sigma_X^2}{2}\left[2\left(1-\mathrm{e}^{-\theta t}\right) - \left(1-\mathrm{e}^{-\theta t}\right)^2\right] + X - \frac{\sigma_X^2}{2} - \sigma_{XY},\\
&= -\beta_0 - (X-\sigma_{XY})\left(1-\mathrm{e}^{-\theta t}\right) + \frac{\sigma_X^2}{2}\left(1-\mathrm{e}^{-\theta t}\right)^2 + X - \frac{\sigma_X^2}{2} - \sigma_{XY}, \quad \text{(some terms cancel)}\\
&= \mu + X + \frac{\sigma_Y^2}{2} - \beta - (X-\sigma_{XY})\left(1-\mathrm{e}^{-\theta t}\right) + \frac{\sigma_X^2}{2}\left(1-\mathrm{e}^{-\theta t}\right)^2, \quad \text{(by defintion of } \beta_0\text{)}\\
&= \mu + X + \frac{\sigma_Y^2}{2} - \beta - \theta(X-\sigma_{XY})\frac{1}{b(X,t)}\frac{\mathrm{d}b(X,t)}{\mathrm{d}X} + \frac{\theta^2\sigma_X^2}{2}\frac{1}{b(X,t)}\frac{\mathrm{d}^2 b(X,t)}{\mathrm{d}X^2}, \quad \text{(by (E.11)-(E.12))}.
\end{aligned}$$

Multiplying both sides by $b(X,t)$ confirms equation (E.2).

(b) Equation (E.2) proves equation (E.3) since

$$\left[(\mu + X + \frac{\sigma_Y^2}{2} - \beta) - \theta(X-\sigma_{XY})\frac{\mathrm{d}}{\mathrm{d}X} + \frac{\theta^2\sigma_X^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}X^2}\right] b(X) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} b(X,t)\mathrm{d}t = -1,$$

where we have used $b(X,0)=1$, thus confirming equation (E.3).

2. (a) For the left-hand tail, take X sufficiently negative to ensure $\beta-\mu-\sigma_Y^2/2-X>0$. Then, by the variable transformation $s=(\beta-\mu-\sigma_Y^2/2-X)t$, we have

$$b(X) := \frac{1}{\beta-\mu-\sigma_Y^2/2-X}\times$$
$$\int_0^\infty \exp\left[\frac{-\beta_0\, s}{\beta-\mu-\sigma_Y^2/2-X} + \left(X-\sigma_X^2-\sigma_{XY}\right)\frac{1-\mathrm{e}^{\frac{-\theta\, s}{\beta-\mu-\sigma_Y^2/2-X}}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{\frac{-2\,\theta\, s}{\beta-\mu-\sigma_Y^2/2-X}}}{2\theta}\right]\mathrm{d}s.$$

It follows that

$$\begin{aligned}
&\lim_{X\to-\infty}(\beta-\mu-\sigma_Y^2/2-X)b(X)\\
&=\lim_{X\to-\infty}\int_0^\infty \exp\left[\frac{s}{\beta-\mu-\sigma_Y^2/2-X}\left(-\beta_0+X-\sigma_X^2-\sigma_{XY}+\frac{\sigma_X^2}{2}\right)\right]\mathrm{d}s\\
&=\lim_{X\to-\infty}\int_0^\infty \exp\left[\frac{s}{\beta-\mu-\sigma_Y^2/2-X}\left(-\beta+\mu+\sigma_Y^2/2+X\right)\right]\mathrm{d}s,\\
&=\lim_{X\to-\infty}\int_0^\infty \mathrm{e}^{-s}\mathrm{d}s=1.
\end{aligned}$$

The first equality holds because $1-\mathrm{e}^{-x}=x+O(x^2)$, while the second follows from the definition of β_0.

(b) For large X, extract terms independent of t and consider the variable transformation $s=X\mathrm{e}^{-\theta t}/\theta$. This implies that $s\in[X/\theta,0]$ with $\mathrm{d}t=-(\theta s)^{-1}\mathrm{d}s$, such that

$$\begin{aligned}
b(X)&:=\int_0^\infty \exp\left[-\beta_0 t+\left(X-\sigma_X^2-\sigma_{XY}\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta}+\frac{1}{2}\sigma_X^2\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t,\\
&=G(X)\int_0^\infty \exp\left[-\beta_0 t-\left(X-\sigma_X^2-\sigma_{XY}\right)\frac{\mathrm{e}^{-\theta t}}{\theta}-\frac{\sigma_X^2}{2}\frac{\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t,\\
&=G(X)\int_0^{X/\theta}(\theta s)^{-1}\left(\frac{X}{\theta\, s}\right)^{-\beta_0/\theta}\exp\left[-s+(\sigma_X^2+\sigma_{XY})\frac{s}{X}-\frac{\theta\,\sigma_X^2}{4}\frac{s^2}{X^2}\right]\mathrm{d}s,\\
&=\frac{G(X)}{\theta}\left(\frac{X}{\theta}\right)^{-\beta_0/\theta}\int_0^{X/\theta}s^{\beta_0/\theta-1}\exp\left[-s+(\sigma_X^2+\sigma_{XY})\frac{s}{X}-\frac{\theta\,\sigma_X^2}{4}\frac{s^2}{X^2}\right]\mathrm{d}s,
\end{aligned}$$

where

$$G(X):=\exp\left(\frac{X-\sigma_X^2-\sigma_{XY}}{\theta}+\frac{\sigma_X^2}{4\theta}\right).$$

Hence

$$\lim_{X\to\infty}\frac{b(X)}{G(X)}\left(\frac{X}{\theta}\right)^{\beta_0/\theta}=\lim_{X\to\infty}\frac{1}{\theta}\int_0^{X/\theta}s^{\beta_0/\theta-1}\mathrm{e}^{-s}\mathrm{d}s=\frac{1}{\theta}\Gamma\left(\frac{\beta_0}{\theta}\right).$$

3. To prove equation (E.7) and that $b_1(X)$ and $b_2(X)$ are sigmoid functions, we must show that the following (in)equalities hold:

(a) increasing:	$b_1'(X)>0, \quad b_2'(X)>0,$
(b) bounded:	$0<b_1(X)<b_2(X)<1,$
(c) limits:	$\lim_{X\to-\infty}b_1(X)=\lim_{X\to-\infty}b_2(X)=0,$
	$\lim_{X\to\infty}b_1(X)=\lim_{X\to\infty}b_2(X)=1.$

(a) Equation (32) and (D.2) imply

$$b_k(X)=\int_0^\infty\left(1-\mathrm{e}^{-\theta t}\right)^k\frac{b(X,t)}{b(X)}\mathrm{d}t=\mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^k\right],\quad k\in\mathbb{N},$$

where $b(X,t)/b(X)$ is interpreted as a density function with associated expectation operator $\mathrm{E}_b[\cdot]$.

Monotonicity of $b_1(X)$ requires $b_2(X) > b_1(X)^2$. Then

$$b_2(X) = \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^2\right] > \left(\mathrm{E}_b\left[1-\mathrm{e}^{-\theta t}\right]\right)^2 = b_1(X)^2.$$

Monotonicity of $b_2(X)$ requires $b_3(X) > b_2(X)b_1(X)$. For a positive random variables $x \in \mathbb{R}_{>0}$, it holds

$$\mathrm{Cov}(x^2, x) = \mathrm{E}[x^3] - \mathrm{E}[x^2]\mathrm{E}[x] > 0.$$

Thus we obtain

$$b_3(X) = \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^3\right] > \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^2\right] \times \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)\right] = b_2(X)b_1(X).$$

(b) To show $0 < b_2(X) < b_1(X) < 1$ or, what is equivalent, $0 < \theta^2 b''(X) < \theta b'(X) < b(X)$, we compute

$$\begin{aligned} 0 &< \theta b'(X) = \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right) b(X,t)\mathrm{d}t < \int_0^\infty b(X,t)\mathrm{d}t = b(X), \\ 0 &< \theta^2 b''(X) = \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)^2 b(X,t)\mathrm{d}t < \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right) b(X,t)\mathrm{d}t = \theta b'(X). \end{aligned}$$

(c) Let $b(X,t;\beta_0)$ denote the function $b(X,t)$ from equation (D.2), now with β_0 added as an explicit argument, and similarly for $b(X;\beta_0)$. Hence, $b(X,t;\beta_0) = b(X,t)$ and $b(X;\beta_0) = b(X)$. The derivative of $b(X)$ satisfies

$$\begin{aligned} \theta\, b'(X) &= \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right) b(X,t)\mathrm{d}t = b(X) - \int_0^\infty \mathrm{e}^{-\theta t} b(X,t;\beta_0)\,\mathrm{d}t \\ &= b(X) - b(X;\beta_0+\theta). \end{aligned}$$

Hence, $b_1(X)$ can be written as

$$b_1(X) := \frac{\theta\, b'(X)}{b(X)} = 1 - \frac{b(X;\beta_0+\theta)}{b(X)}.$$

Using the limits of $b(X;\beta_0)$ for $X \to +\infty$ and $X \to -\infty$ in equations (E.4)–(E.5), we obtain

$$\frac{b(X;\beta_0+\theta)}{b(X)} \approx \begin{cases} \dfrac{\Gamma\left(\frac{\beta_0}{\theta}+1\right)}{\Gamma\left(\frac{\beta_0}{\theta}\right)}\dfrac{\theta}{X} = \dfrac{\beta_0}{X} & \to 0 \text{ as } X \to \infty, \\ \dfrac{\beta_0 + \frac{\sigma_X^2}{2} - X}{\beta_0 + \theta + \frac{\sigma_X^2}{2} - X} & \to 1 \text{ as } X \to -\infty, \end{cases}$$

using $\Gamma(x+1) = x\Gamma(x)$ in the second equality in the first line. The desired result then follows.

The second derivative of $b(X)$ satisfies

$$\theta^2\, b''(X) = \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)^2 b(X,t)\,\mathrm{d}t = b(X) - 2b(X;\beta_0+\theta) + b(X;\beta_0+2\theta).$$

Hence, $b_2(X)$ can be written as

$$b_2(X) := \frac{\theta^2\, b''(X)}{b(X)} = 1 - 2\frac{b(X;\beta_0+\theta)}{b(X;\beta_0)} + \frac{b(X;\beta_0+2\theta)}{b(X;\beta_0)}.$$

Using the limits of $b(X;\beta_0)$ for $X \to +\infty$ and $X \to -\infty$ in equations (E.4)–(E.5), we obtain

$$\begin{aligned}
\frac{b(X;\beta_0+\theta)}{b(X;\beta_0)} &\to \begin{cases} 0, & \text{as } X \to +\infty, \\ 1, & \text{as } X \to -\infty, \end{cases} \\
\frac{b(X;\beta_0+2\theta)}{b(X;\beta_0)} &\to \begin{cases} 0, & \text{as } X \to +\infty, \\ 1, & \text{as } X \to -\infty, \end{cases}
\end{aligned}$$

such that the desired result follows.

4. For $\theta > 0$,

$$\frac{1-\mathrm{e}^{-\theta t}}{\theta} - \frac{1-\mathrm{e}^{-2\theta t}}{2\theta} = \frac{1}{2\theta}\left(1-\mathrm{e}^{-\theta t}\right)^2 \geq 0.$$

Using $\sigma_{XY} \geq 0$ (because $\rho \geq 0$), we have

$$\begin{aligned}
b(0) &= \int_0^\infty \exp\left[-\beta_0\, t - \left(\sigma_X^2 + \sigma_{XY}\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t, \\
&\leq \int_0^\infty \exp\left[-\beta_0\, t - \sigma_X^2\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t, \quad \text{dropping one term} \\
&\leq \int_0^\infty \exp\left[-\beta_0\, t - \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t, \quad \text{halving one term} \\
&= \int_0^\infty \exp\left[-\beta_0\, t - \frac{\sigma_X^2}{2}\left\{\frac{1-\mathrm{e}^{-\theta t}}{\theta} - \frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right\}\right]\mathrm{d}t, \quad \text{combining two terms} \\
&\leq \int_0^\infty \exp\left[-\beta_0\, t\right]\mathrm{d}t = 1/\beta_0.
\end{aligned}$$

For the bound on $b(X)$ for $X \leq 0$, we have

$$\begin{aligned}
b(X) &= \int_0^\infty \exp\left[-\beta_0\, t + \left(X - \sigma_X^2 - \sigma_{XY}\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma_X^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right] \mathrm{d}t, \text{ by definition} \\
&= \int_0^\infty \exp\left[-\beta_0\, t + (X - \sigma_{XY})\frac{1-\mathrm{e}^{-\theta t}}{\theta} - \sigma_X^2\left\{\frac{1-\mathrm{e}^{-\theta t}}{\theta} - \frac{1-\mathrm{e}^{-2\theta t}}{4\theta}\right\}\right] \mathrm{d}t, \text{ by rewriting} \\
&\geq \int_0^\infty \exp\left[-\beta_0\, t + (X - \sigma_{XY})\, t - \frac{\sigma_X^2}{2} t\right] \mathrm{d}t, \quad \text{if } \sigma_{XY} \geq 0, \text{ using linear bounds shown below} \\
&= \int_0^\infty \exp\left[-(\beta_0 - X + \sigma_{XY} + \sigma_X^2/2)\, t\right] \mathrm{d}t, \quad \text{rewriting} \\
&= \int_0^\infty \exp\left[-(\beta_0 - X + \sigma^2/2 - \sigma_Y^2/2)\, t\right] \mathrm{d}t, \quad \text{by definition of } \sigma^2 \\
&= \int_0^\infty \exp\left[-(\beta - \mu - \sigma^2/2 - X + \sigma^2/2 - \sigma_Y^2/2)\, t\right] \mathrm{d}t, \quad \text{by definition of } \beta_0 \\
&= \int_0^\infty \exp\left[-(\beta - \mu - X - \sigma_Y^2/2)\, t\right] \mathrm{d}t, \quad \text{by cancellations} \\
&= \left(\beta - \mu - X - \sigma_Y^2/2\right)^{-1}, \qquad \forall X \in \mathbb{R}_{\leq 0}.
\end{aligned}$$

The linear bounds that drive the inequality follow from

$$\begin{aligned}
\frac{1-\mathrm{e}^{-\theta t}}{\theta} &< t, \\
\frac{1-\mathrm{e}^{-\theta t}}{\theta} - \frac{1-\mathrm{e}^{-2\theta t}}{2\theta} &= \frac{1}{2}\frac{1-\mathrm{e}^{-\theta t}}{\theta}\left(1-\mathrm{e}^{-\theta t}\right) < \frac{1}{2}\frac{1-\mathrm{e}^{-\theta t}}{\theta} < \frac{1}{2}t.
\end{aligned}$$

5. Equation (E.10) follows from differentiating equation (32). For $X = 0$ and $\sigma = 0$, the integral can solved analytically.

■

F Proof of Proposition 3

To compute $\mathrm{L}A(X,Y)$, we must compute $\mathrm{L}B(X,Y)^{1/(1-\alpha)}$. Using $B(X,Y) = Yb(X)$ and L given in equation (27), the quantity $\mathrm{L}B(X,Y)^{1/(1-\alpha)}$ contains five terms as follows:

$$\begin{aligned}
-\theta X \frac{\mathrm{d}}{\mathrm{d}X} B(X,Y)^{1/(1-\alpha)} &= -\theta X \frac{1}{1-\alpha} B(X,Y)^{1/(1-\alpha)} \frac{b'(X)}{b(X)}, \\
\frac{\theta^2\sigma_X^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}X^2} B(X,Y)^{1/(1-\alpha)} &= \frac{\theta^2\sigma_X^2}{2} B(X,Y)^{1/(1-\alpha)}\left[\frac{1}{1-\alpha}\frac{b''(X)}{b(X)} + \frac{\alpha}{(1-\alpha)^2}\left(\frac{b'(X)}{b(X)}\right)^2\right], \\
\left(X + \mu + \frac{\sigma_Y^2}{2}\right) Y \frac{\mathrm{d}}{\mathrm{d}Y} B(X,Y)^{1/(1-\alpha)} &= \left(X + \mu + \frac{\sigma_Y^2}{2}\right)\frac{1}{1-\alpha} B(X,Y)^{1/(1-\alpha)}, \\
\frac{\sigma_Y^2}{2} Y^2 \frac{\mathrm{d}^2}{\mathrm{d}Y^2} B(X,Y)^{1/(1-\alpha)} &= \frac{\sigma_Y^2}{2}\frac{\alpha}{(1-\alpha)^2} B(X,Y)^{1/(1-\alpha)}, \\
\theta\sigma_{XY} Y \frac{\mathrm{d}^2}{\mathrm{d}X\mathrm{d}Y} B(X,Y)^{1/(1-\alpha)} &= \theta\sigma_{XY}\frac{1}{(1-\alpha)^2} B(X,Y)^{1/(1-\alpha)}\frac{b'(X)}{b(X)}.
\end{aligned}$$

Adding up all five lines and multiplying by $(1-\alpha)B(X,Y)^{-1/(1-\alpha)}$, we obtain

$$\begin{aligned}
&(1-\alpha)B(X,Y)^{-1/(1-\alpha)} \times \mathrm{L}B(X,Y)^{1/(1-\alpha)} \\
&= -\theta X\frac{b'(X)}{b(X)} + \frac{\theta^2\sigma_X^2}{2}\frac{b''(X)}{b(X)} + \frac{\theta^2\sigma_X^2}{2}\frac{\alpha}{1-\alpha}\left(\frac{b'(X)}{b(X)}\right)^2 + X + \mu + \frac{\sigma_Y^2}{2} + \frac{\alpha}{1-\alpha}\frac{\sigma_Y^2}{2} \\
&\qquad + \theta\sigma_{XY}\frac{1}{1-\alpha}\frac{b'(X)}{b(X)}, \\
&= \underbrace{-\theta(X-\sigma_{XY})\frac{b'(X)}{b(X)} + \frac{\theta^2\sigma_X^2}{2}\frac{b''(X)}{b(X)}}_{=\beta-1/b(X)-\mu-X-\sigma_Y^2/2 \text{ by (E.3)}} + \frac{\theta^2\sigma_X^2}{2}\frac{\alpha}{1-\alpha}\left(\frac{b'(X)}{b(X)}\right)^2 + X + \mu + \frac{\sigma_Y^2}{2} + \frac{\alpha}{1-\alpha}\frac{\sigma_Y^2}{2} \\
&\qquad + \theta\sigma_{XY}\frac{\alpha}{1-\alpha}\frac{b'(X)}{b(X)}, \\
&= \beta - \frac{1}{b(X)} + \alpha f(X),
\end{aligned} \tag{F.1}$$

where

$$f(X) := \frac{1}{1-\alpha}\left[\frac{\sigma_X^2}{2}b_1(X)^2 + \frac{\sigma_Y^2}{2} + \sigma_{XY}b_1(X)\right] \leq \frac{1}{1-\alpha}\frac{\sigma^2}{2}, \tag{F.2}$$

and $b_1(X)$ is the sigmoid function defined in equation (E.6). The function $f(X)$ is nonnegative, because it equals the expectation of a square, i.e.

$$f(X) = \frac{1}{2}\frac{1}{1-\alpha}\mathrm{E}[(b_1(X)\sigma_X \mathrm{d}W_t^X + \sigma_Y \mathrm{d}W_t^Y)^2] \geq 0.$$

For $\sigma_{XY} \geq 0$, $f(X)$ is also increasing in X as $b_1(X)$ is increasing in X and all coefficients of $b_1(X)$ are nonnegative. As $X \to \infty$, we have $b_1(X) \to 1$, such that $f(X)$ achieves the bound stated above.

Using equation (F.1), we compute the desired result as follows:

$$\begin{aligned}
&(\mathrm{L}-\beta)A(X,Y) = (\mathrm{L}-\beta)\left\{\frac{1-\alpha}{\alpha}[\alpha B(X,Y)]^{1/(1-\alpha)} - \phi\right\}, \quad \text{(by (35))} \\
&= \alpha^{\alpha/(1-\alpha)}(1-\alpha)\mathrm{L}B(X,Y)^{1/(1-\alpha)} - \alpha^{\alpha/(1-\alpha)}(1-\alpha)\beta B(X,Y)^{1/(1-\alpha)} + \beta\phi, \quad \text{(expanding)} \\
&= \alpha^{\alpha/(1-\alpha)}\left\{\cancel{\beta} - \frac{1}{b(X)} + \alpha f(X)\right\}B(X,Y)^{1/(1-\alpha)} - \alpha^{\alpha/(1-\alpha)}(\cancel{1}-\alpha)\beta B(X,Y)^{1/(1-\alpha)} + \beta\phi, \text{ (by (F.1))} \\
&= \beta\underbrace{\left(\phi + [\alpha B(X,Y)]^{1/(1-\alpha)}\right)}_{=K(X,Y) \text{ by (34)}} - Y\underbrace{[\alpha B(X,Y)]^{\alpha/(1-\alpha)}}_{=(K(X,Y)-\phi)^\alpha \text{ by (34)}} + f(X)[\alpha B(X,Y)]^{1/(1-\alpha)}, \text{ (two terms cancel)}
\end{aligned}$$

using $b(X)^{-1}B(X,Y)^{1/(1-\alpha)} = YB(X,Y)^{\alpha/(1-\alpha)}$ in the last line. This confirms condition (42) in the main text. ■

G Proof of Proposition 4

First, we focus on proving equations (47)–(48) in Proposition 4. Substituting conjecture (47) into Bellman's equation (38), we find that function $v(\cdot)$ satisfies the ordinary differential equation (ODE):

$$0 = \left(\frac{\mu+X}{1-\alpha} + \frac{\sigma_Y^2/2}{(1-\alpha)^2} - \beta\right)v(X) - \theta\left(X - \frac{\sigma_{XY}}{1-\alpha}\right)v'(X) + \frac{1}{2}\sigma_X^2\theta^2 v''(X), \quad X \in \mathbb{R}. \tag{G.1}$$

Next, we define

$$\overline{v}(X) := v(X)\,\mathrm{e}^{-X/\theta/(1-\alpha)}\ .$$

Equation (G.1) implies that $\overline{v}(\cdot)$ must satisfy

$$0 = -\beta_1\,\overline{v}(X) - \theta\left(X - \frac{\sigma_X^2 + \sigma_{XY}}{1-\alpha}\right)\overline{v}'(X) + \frac{1}{2}\theta^2\sigma_X^2\overline{v}''(X), \quad \forall X \in \mathbb{R}, \tag{G.2}$$

where β_1 is defined in Assumption (37). We define Z as a linear transformation of X, i.e.

$$\begin{aligned} Z(X) &:= \frac{1}{\sigma_X\sqrt{\theta}}\left(X - \frac{\sigma_X^2 + \sigma_{XY}}{1-\alpha}\right), \\ w\left[Z(X)\right] &:= \overline{v}(X). \end{aligned} \tag{G.3}$$

Under this coordinate transformation, derivatives of $\overline{v}(\cdot)$ with respect to X can be written as

$$\overline{v}'(X) = \frac{w'(Z)}{\sigma_X\sqrt{\theta}}, \qquad \left(X - \frac{\sigma_X^2 + \sigma_{XY}}{1-\alpha}\right)\overline{v}'(X) = Zw'(Z), \qquad \overline{v}''(X) = \frac{w''(Z)}{\sigma_X^2\theta}.$$

It follows that equation (G.2) can be written in terms of $w(\cdot)$ and Z as

$$0 = \omega\,w(Z) - Z\,w'(Z) + \frac{1}{2}w''(Z), \quad \forall Z \in \mathbb{R}, \tag{G.4}$$

where $\omega := -\beta_1/\theta < 0$. The resulting ODE (G.4) is known as Hermite's ODE. If ω were a positive integer, the solution would be $\mathrm{H}_\omega(Z)$, where H_ω is the Hermite polynomial of order ω. As we will show below, our solution can still be written as $\mathrm{H}_\omega(Z)$ if the Hermite polynomial is interpreted in a generalised sense, which allows for negative values of ω.

To solve ODE (G.4), we investigate the following series expansion as our candidate solution:

$$\begin{aligned} w(Z) &= \sum_{i=0}^{\infty} c_i Z^i, \text{ such that} \\ -Zw'(Z) &= -Z\frac{\mathrm{d}}{\mathrm{d}Z}\sum_{i=0}^{\infty} c_i Z^i = -\sum_{i=1}^{\infty} i\,c_i\,Z^i, \\ \frac{1}{2}w''(Z) &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\sum_{i=0}^{\infty} c_i Z^i = c_2 + \frac{1}{2}\sum_{i=1}^{\infty} c_{i+2}\,(i+2)\,(i+1)\,Z^i. \end{aligned}$$

Using these equalities, Hermite's ODE (G.4) becomes

$$\begin{aligned} 0 &= c_2 + \omega\,c_0 + \sum_{i=1}^{\infty}\left[\frac{1}{2}c_{i+2}(i+2)(i+1) - (i-\omega)\,c_i\right]Z^i \\ &= \sum_{i=0}^{\infty}\left[\frac{1}{2}c_{i+2}(i+2)(i+1) - (i-\omega)\,c_i\right]Z^i, \qquad \forall Z \in \mathbb{R}. \end{aligned}$$

This equation holds only if the coefficient in square brackets is zero for *every* single value of $i =$

$0, 1, 2, 3, \cdots$. Hence we need

$$c_{i+2} = \frac{2(i-\omega)}{(i+2)(i+1)} c_i, \qquad \forall i = 0, 1, 2, 3, \cdots$$

This recursive equation relates c_{i+2} to c_i. Two independent solutions $w_k(Z)$ for $k = 1, 2$ may be obtained by starting with an arbitrary value of c_0 (or c_1) and considering only even (or odd) powers as follows:

$$w_1(Z) = c_0 \left[1 + 2\tfrac{-\omega}{2\times1} Z^2 + 2^2 \tfrac{-\omega}{2\times1}\tfrac{2-\omega}{4\times3} Z^4 + \cdots\right] =: c_0\, \mathrm{M}\left(-\frac{\omega}{2}, \frac{1}{2}, Z^2\right),$$
$$w_2(Z) = c_1 \left[Z + 2\tfrac{1-\omega}{3\times2} Z^3 + 2^2 \tfrac{1-\omega}{3\times2}\tfrac{3-\omega}{5\times4} Z^5 + \cdots\right] =: \frac{c_1\, Z}{\sigma_X}\, \mathrm{M}\left(\frac{1-\omega}{2}, \frac{3}{2}, Z^2\right),$$

where, on the far right-hand side, we use the definition of the confluent hypergeometric function of the first kind, denoted by $\mathrm{M}(\cdot,\cdot,\cdot)$, see e.g. Abramovich and Stegun (1972, p. 504, equation 13.1.2).

In the limit where $Z \to \infty$, these functions behave like

$$\begin{aligned} c_0\, \mathrm{M}\left(-\frac{\omega}{2}, \frac{1}{2}, Z^2\right) &\approx c_0 \frac{\sqrt{\pi}}{\Gamma\left(-\frac{\omega}{2}\right)} 1/Z^{1+\omega} \exp(Z^2), \quad \text{as } Z \to \infty, \\ c_1\, Z\, \mathrm{M}\left(\frac{1-\omega}{2}, \frac{3}{2}, Z^2\right) &\approx c_1 \frac{\sqrt{\pi}}{2\Gamma\left(\frac{1-\omega}{2}\right)} 1/Z^{1+\omega} \exp(Z^2), \quad \text{as } Z \to \infty, \end{aligned} \tag{G.5}$$

see e.g. Abramovich and Stegun (1972, p. 504, equation 13.1.4). By the approximation sign "$\approx$", we mean that the ratio of the quantities on the left- and right-hand sides approaches to unity as $Z \to \infty$.

We recall from Lemma 1 that the steady-state distribution of X is normal with a variance of $\theta\sigma_X^2/2$. Since Z is a linear transformation of X with 'slope' coefficient $1/(\sigma_X\sqrt{\theta})$, see equation (G.3), the steady-state distribution of Z is normal with variance $1/2$. The steady-state probability density of Z decays therefore proportional to $\exp(-Z^2)$ in the limit where $Z \to \infty$. Hence, if we multiply the steady-state density by $w_1(Z)$ or $w_2(Z)$, then as $Z \to \infty$ the product is proportional to $1/Z^{1+\omega}$, which is not an integrable function (recall that $\omega < 0$). Hence, for $c_0, c_1 \geq 0$, we have

$$\mathrm{E}\left[w_1(Z)|Z > 0\right] = \mathrm{E}\left[w_2(Z)|Z > 0\right] = \infty.$$

In such circumstances, Dixit and Pindyck (1994, pp. 181-2) use a 'no-bubble argument' to rule out a solution with undesirable asymptotic properties. In our case, however, this rules out both our candidate solutions. Hence, we must pick c_0 and c_1 so that the combination combination $w(Z) = w_1(Z) + w_2(Z)$ contains only terms behave appropriately as $Z \to \infty$ and, in particular, are integrable with respect to the density $\exp(-Z^2/2)$. From (G.5), this can be achieved by choosing

$$c_0 = 2^{\omega}\sqrt{\pi}\frac{1}{\Gamma\left(\frac{1-\omega}{2}\right)}, \qquad c_1 = -2^{\omega}\sqrt{\pi}\frac{2}{\Gamma\left(-\frac{\omega}{2}\right)},$$

where the factor $2^{\omega}\sqrt{\pi}$, which we include in both c_0 and c_1, is introduced for later convenience. The full solution then reads

$$
\begin{aligned}
w\,(Z) &= w_1(Z) + w_2(Z), \\
&= 2^{\omega}\sqrt{\pi}\left[\frac{1}{\Gamma\left(\frac{1-\omega}{2}\right)}\,\mathrm{M}\left(-\frac{\omega}{2},\frac{1}{2},Z^2\right) - \frac{2\,Z}{\Gamma\left(-\frac{\omega}{2}\right)}\,\mathrm{M}\left(\frac{1-\omega}{2},\frac{3}{2},Z^2\right)\right], \\
&= \mathrm{H}_{\omega}\,(Z)\,,
\end{aligned} \tag{G.6}
$$

where the third equality holds only if the Hermite polynomial is understood in a generalised sense, in which case it is defined as in the second line. Equations (47)–(48) in Proposition 4 are obtained by noting that

$$
V(X,Y) = CY^{1/(1-\alpha)}v(X) = CY^{1/(1-\alpha)}\exp\left(\frac{X}{\theta(1-\alpha)}\right)\overline{v}(X),
$$

where $\overline{v}(X) = w(Z) = \mathrm{H}_{\omega}(Z)$ and Z is given in equation (G.3), i.e.

$$
V(X,Y) = C\,Y^{1/(1-\alpha)}\,\exp\left(\frac{X}{\theta(1-\alpha)}\right)\mathrm{H}_{-\beta_1/\theta}\left[\frac{1}{\sqrt{\theta}\,\sigma_X}\left(X - \frac{\sigma_X^2 + \sigma_{XY}}{1-\alpha}\right)\right],
$$

thus confirming equations (47)–(48) in Proposition 4.

The resulting solution is well behaved as $Z \to \infty$, because $\mathrm{H}_{\omega}(Z) \approx (2Z)^{\omega}$ as $Z \to \infty$, which is decreasing in Z (recall $\omega := -\beta_1/\theta < 0$). As such, we have

$$
V(X,Y) \;\approx\; C\,Y^{1/(1-\alpha)}\,\exp\left(\frac{X}{\theta(1-\alpha)}\right)\left(\frac{2X}{\sigma_X\sqrt{\theta}} - \frac{2(\sigma_X + \rho\,\sigma_Y)}{(1-\alpha)\sqrt{\theta}}\right)^{-\beta_1/\theta}, \quad X \to \infty,
$$

ensuring the right-hand tail is integrable with respect to the unconditional density of X, as desired. Hence, the derived solution satisfies Bellman's equation (38) as well as the required transversality condition, ensuring integrability with respect to the relevant density.

Some computer packages, such as Wolfram's `Mathematica`, automatically compute $\mathrm{H}_{\omega}(\cdot)$ for negative values of ω by using the second line in (G.6) as the definition of the third.[1] Other software packages, notably `Matlab`, return an error message, in which case the second rather than the third line of equation (G.6) must be used.

Second, we focus on proving that $X^{\star}$ is uniquely determined by equation (49), which can be solved numerically, after which the integration constant C can be expressed in closed form. Equations (31), (35) and (47) for $B\,(X,Y)$, $A\,(X,Y)$ and $V\,(X,Y)$, respectively, imply that the value-matching condition and smooth-pasting conditions (39) can be written as

$$
\begin{aligned}
V(X^{\star},Y^{\star}) = C\,Y^{\star 1/(1-\alpha)}v(X^{\star}) &= \frac{1-\alpha}{\alpha}\left[\alpha Y^{\star} b\,(X^{\star})\right]^{1/(1-\alpha)} = A(X^{\star},Y^{\star}), \\
C\,Y^{\star 1/(1-\alpha)}v'\,(X^{\star}) &= Y^{\star 1/(1-\alpha)}\left[\alpha b\,(X^{\star})\right]^{\alpha/(1-\alpha)} b'\,(X^{\star})\,.
\end{aligned}
$$

Dividing both equations by $Y^{\star 1/(1-\alpha)}$, the dependence on the variable $Y^{\star}$ is eliminated. Moreover, the integration constant C is eliminated by dividing the second equation by the first, in which case we obtain equation (49) in the Proposition, which can be solved numerically for the critical value $X^{\star}$. Conditional on $X^{\star}$, the integration constant C can be expressed in closed form using either condition

[1] See http://functions.wolfram.com/05.01.26.0002.01

above, e.g. by using the first we have

$$C = (1-\alpha)\,\alpha^{\alpha/(1-\alpha)}\,\frac{b(X^\star)^{1/(1-\alpha)}}{v(X^\star)} > 0.$$

What remains is to show the existence and uniqueness of $X^\star$. To this end, we introduce some notation:

$$b_k(X) := \theta^k \frac{b^{(k)}(X)}{b(X)}, \qquad v_k(X) := \theta^k \frac{v^{(k)}(X)}{v(X)}, \tag{G.7}$$

where $\cdot^{(k)}$ denotes the k-th derivative. From Lemma 2, we know that $b_1(X)$ and $b_2(X)$ are sigmoid functions: i.e. they are non-negative and increasing. This is useful because it means that $\theta b_1'(X) > 0$, which using definition (G.7) can be written as $b_2(X) - b_1(X)^2 > 0$. Similarly, $\theta b_2'(X) = b_3(X) - b_2(X)b_1(X) > 0$. Then we introduce the function $g(\cdot)$ on $\mathbb{R}$ as

$$\begin{aligned} g(X) &:= (1-\alpha)\, v_1(X) - b_1(X), \\ \theta g'(X) &= (1-\alpha)\left[v_2(X) - v_1(X)^2\right] - \left[b_2(X) - b_1(X)^2\right], \\ \theta^2 g''(X) &= (1-\alpha)\left[v_3(X) - 3v_2(X)v_1(X) + 2v_1(X)^3\right] - \left[b_3(X) - 3b_2(X)b_1(X) + 2b_1(X)^3\right], \end{aligned} \tag{G.8}$$

where we provide the first to derivatives of $g(X)$ for future reference. The critical value $X^\star$ is defined in equation (49) as the intersection of $g(\cdot)$ with the horizontal axis. We must show this intersection exists and is unique.

Existence. We note that $g(\cdot)$ and its derivatives are continuous on $\mathbb{R}$, as both $b(\cdot)$ and $v(\cdot)$ and their derivatives are continuous on $\mathbb{R}$. Further, using the explicit formulas for $b(\cdot)$ and $v(\cdot)$, it can be shown that $g(X)$ is strictly negative (positive) as X goes to negative (positive) infinity, which means that $g(\cdot)$ must change sign at least once, such that existence of at least one intersection is established.

Uniqueness. Given that at least one intersection with the horizontal axis exists, our strategy will be to *assume* that $g(X) = 0$ for some $X \in \mathbb{R}$. If we can show that $g'(X)$ is strictly positive (negative), then we know that the point under consideration represents an up-crossing (down-crossing) of the horizontal axis. Below, we will show that up-crossings can only occur strictly to the left of $X^\dagger$. Conversely, down-crossings, if they exist, can only occur weakly to the right of $X^\dagger$. Because the function $g(\cdot)$ approaches the horizontal axis from below while ending above the horizontal axis, this argument establishes that there must be exactly one up-crossing, which must occur strictly to the left of $X^\dagger$, while down-crossings are ruled out.

To operationalise the above argument, we use the following implication proved below:

$$g(X) = 0 \qquad \Rightarrow \qquad g'(X) \begin{cases} > 0 & \text{if } X < X^\dagger : \text{an `up-crossing' occurs,} \\ = 0 & \text{if } X = X^\dagger : \text{indeterminate,} \\ < 0 & \text{if } X > X^\dagger : \text{a `down-crossing' occurs.} \end{cases} \tag{G.9}$$

Implication (G.9) says that any intersection of $g(\cdot)$ with the horizontal axis, if it exists, is guaranteed to be an up-crossing (down-crossing) when it occurs strictly to the left (right) of $X^\dagger$, in which case the slope of $g(\cdot)$ is strictly positive (negative). The first-derivative test is indeterminate at $X^\dagger$, such that we cannot establish whether an intersection at $X^\dagger$, if it exists, is an up-crossing, down-crossing, tangent from above, or tangent from below.

Next, we show that case left indeterminate by the first-derivative test, must be a tangent from below, because

$$g(X) = 0 \text{ and } g'(X) = 0 \qquad \Rightarrow \qquad g''(X) < 0. \tag{G.10}$$

Jointly, implications (G.9) and (G.10) imply that up-crossings can only occur strictly to the left of $X^\dagger$, while no down-crossing weakly to the right of $X^\dagger$ can exist if the function $g(\cdot)$ is to remain positive. Since $g(\cdot)$ changes sign from negative to positive at least once, we must have at least one up-crossing, which must occur strictly to the left of $X^\dagger$ if $\sigma > 0$.

Proof of implication (G.9). The derivative $g'(X)$ from equation (G.8) can be written as

$$\theta\, g'(X) = (1-\alpha)\left[v_2(X) - v_1(X)^2\right] - \left[b_2(X) - b_1(X)^2\right], \tag{G.11}$$

$$= (1-\alpha)v_2(X) - b_2(X) - \frac{\alpha}{1-\alpha} b_1(X)^2. \tag{G.12}$$

where the second line follows from the assumption $g(X) = 0$, i.e. $v_1(X) = (1-\alpha)^{-1}\, b_1(X)$. To compute the first term on the right-hand side, equation (G.1) can be written as

$$(1-\alpha)\frac{1}{2}\sigma_X^2 v_2(X) = -\mu - X + (1-\alpha)\beta - \frac{\sigma_Y^2/2}{1-\alpha} + (1-\alpha)\left(X - \frac{\sigma_{XY}}{1-\alpha}\right) v_1(X), \tag{G.13}$$

$$= -\mu - X + (1-\alpha)\beta - \frac{\sigma_Y^2/2}{1-\alpha} + \left(X - \frac{\sigma_{XY}}{1-\alpha}\right) b_1(X), \tag{G.14}$$

where the second line holds by the assumption $g(X) = 0$ (i.e. $(1-\alpha)\, v_1(X) = b_1(X)$). To compute the second term on the right-hand side of equation (G.12), equation (E.3) can be written as

$$\frac{1}{2}\sigma_X^2 b_2(X) = -\frac{1}{b(X)} - \mu - X - \frac{\sigma_Y^2}{2} + \beta + (X - \sigma_{XY})b_1(X). \tag{G.15}$$

Multiplying equation (G.12) by $\sigma_X^2/2$ and substituting equations (G.14) and (G.15), we obtain

$$\frac{\sigma_X^2}{2}\theta g'(X) = \frac{1}{b(X)} - \alpha\beta - \frac{\sigma_Y^2/2}{1-\alpha} - \frac{\sigma_{XY}}{1-\alpha} b_1(X) + \frac{\sigma_Y^2}{2} + \sigma_{XY} b_1(X) - \frac{\alpha}{1-\alpha}\frac{\sigma_X^2}{2} b_1(X)^2,$$

$$= \frac{1}{b(X)} - \alpha\beta - \frac{\alpha}{1-\alpha}\left[\frac{1}{2}\sigma_X^2 b_1(X)^2 + \sigma_{XY} b_1(X) + \frac{1}{2}\sigma_Y^2\right],$$

$$= \frac{1}{b(X)} - \alpha[\beta + f(X)] \quad \begin{cases} > 0 \text{ if } X < X^\dagger : \text{an `up-crossing' occurs,} \\ = 0 \text{ if } X = X^\dagger : \text{indeterminate,} \\ < 0 \text{ if } X > X^\dagger : \text{a `down-crossing' occurs.} \end{cases} \tag{G.16}$$

We have used definition (F.2) for $f(X)$ in the last line. The final set of two inequalities and one equality after the curly bracket follows from the fact that $X \mapsto \alpha\, [\beta + f(X)]\, b(X)$ is strictly increasing from zero (as $X \to -\infty$) to infinity (as $X \to \infty$). Hence the level 1 is crossed exactly once, while $\alpha\, [\beta + f(X)]\, b(X)$ exceeds (is exceeded by) unity to the right (left) of this crossing. Strict monotonicity of the left-hand side implies that

$$\alpha\,[\beta + f(X)]b(X) \quad \begin{cases} < 1, & X < X^\dagger, \\ = 1, & X = X^\dagger, \\ > 1, & X > X^\dagger. \end{cases} \tag{G.17}$$

Rewriting the resulting (in)equalities yields the desired result in equation (G.16). This completes the proof of implication(G.9).

Proof of implication (G.10). Assuming $g(X) = g'(X) = 0$, by equation (G.8) we have

$$(1-\alpha)\, v_1(X) = b_1(X), \tag{G.18}$$
$$(1-\alpha)\left[v_2(X) - v_1(X)^2\right] = b_2(X) - b_1(X)^2. \tag{G.19}$$

Equation (G.8) for $\theta^2 g''(X)$ then implies

$$\begin{aligned}
\theta^2\, g''(X) &= (1-\alpha)\left[v_3(X) - v_2(X)v_1(X) - 2v_1(X)\left\{v_2(X) - v_1(X)^2\right\}\right], \\
&\qquad - \left[b_3(X) - b_2(X)b_1(X) - 2b_1(X)\left\{b_2(X) - b_1(X)^2\right\}\right] \\
&= (1-\alpha)[v_3(X) - v_2(X)v_1(X)] - [b_3(X) - b_2(X)b_1(X)] \\
&\qquad - 2(1-\alpha)v_1(X)\left[v_2(X) - v_1(X)^2\right] + 2b_1(X)\left[b_2(X) - b_1(X)^2\right], \\
&= (1-\alpha)[v_3(X) - v_2(X)v_1(X)] - [b_3(X) - b_2(X)b_1(X)] - 2\frac{\alpha}{1-\alpha}\underbrace{b_1(X)}_{>0}\underbrace{\left\{b_2(X) - b_1(X)^2\right\}}_{=\theta b_1'(X)>0}.
\end{aligned}$$

The second equality follows by reordering terms. The third equality follows by equations (G.18) and (G.19) above. The last term is strictly negative, as desired. To show that the collection terms in front of this term is also negative, we subtract equation (G.15) from equation (G.13), we get

$$\begin{aligned}
&\frac{1}{2}\sigma_X^2\left[(1-\alpha)v_2(X) - b_2(X)\right] \\
&= -\alpha\beta + \frac{1}{b(X)} + X\left[(1-\alpha)v_1(X) - b_1(X)\right] - \sigma_{XY}[v_1(X) - b_1(X)] - \frac{\alpha}{1-\alpha}\frac{1}{2}\sigma_Y^2.
\end{aligned} \tag{G.20}$$

Then we differentiate equation (G.20) with respect to X, where we use that $(\mathrm{d}/\mathrm{d}X)[(1-\alpha)v_1(X) - b_1(X)] = 0$, which holds because $(1-\alpha)v_1(X) - b_1(X) = 0$ and $(1-\alpha)v_1'(X) - b_1'(X) = 0$ by assumption (see equations (G.18) and (G.19) above). Multiplying the result by θ, and evaluating this equation at the point $g(X) = g'(X) = 0$ yields

$$\begin{aligned}
&\frac{1}{2}\sigma_X^2\left[(1-\alpha)v_3(X) - (1-\alpha)v_2(X)v_1(X) - b_3(X) + b_2(X)b_1(X)\right], \\
&= -\theta\frac{b'(X)}{b(X)^2} - \sigma_{XY}[v_2(X) - v_1(X)^2 - b_2(X) + b_1(X)^2], \\
&= -\frac{b_1(X)}{b(X)} - \sigma_{XY}\frac{\alpha}{1-\alpha}[b_2(X) - b_1(X)^2] < 0,
\end{aligned} \tag{G.21}$$

where we have used equations (G.18) and (G.19) to simplify expressions. The result is negative as $\sigma_{XY} \geq 0$ because we assume $\rho \in [0,1)$, $\alpha \in (0,1)$ and $b_1(X) > 0$ and $\theta b_1'(X) = b_2(X) - b_1(X) > 0$. This completes the proof of implication (G.10). ■

H Details of numerical methods

For our **first numerical approach**, we use the Poisson optional stopping times (POST) method (Lange et al., 2020), which is based on the idea that an independent Poisson process with intensity $\lambda > 0$ generates opportunities to stop. The decision maker is permitted to stop only once, exactly at a Poisson arrival time, but not in between any two arrival times. In the limit $\lambda \to \infty$, opportunities to stop arrive almost continuously. The case $\lambda < \infty$ can be viewed as a 'finite-liquidity' constraint, which prohibits the decision maker from exercising the option at what would otherwise be the optimal time. The resulting POST value function $V_\lambda(X,Y)$ will, due to the restriction, be dominated by the classic value function $V(X,Y)$, for which stopping is possible at any time, i.e. $V_\lambda(X,Y) \leq V(X,Y)$.

For a fixed Poisson intensity $\lambda > 0$, the POST algorithm provides a *fixed-point contraction* in function space as long as the exponential growth rate of the stopping gain (i.e. $\mu + \sigma^2/2$) is strictly exceeded by the discount rate β, as guaranteed in our case by assumption (37). The discretised version of the POST algorithm similarly provides a geometric contraction in discrete (vector) space as long as the the matrix that describes the discretised version of the generator L is weakly diagonally dominant with nonpositive (nonnegative) diagonal (off-diagonal) entries (for details, see Lange et al., 2020, App. C). In our case, a finite-difference stencil that satisfies these conditions is presented below, such that geometric convergence is ensured. For the infinitesimal generator L defined in equation (27), the five-point stencil below (which takes $\rho = \sigma_{XY} = 0$ as for our benchmark parameters) generates such a discretisation:

	$\frac{1}{\mathrm{d}Y}\left(\mu + X + \frac{\sigma_Y^2}{2}\right)^+ Y + \frac{1}{\mathrm{d}Y^2}\frac{\sigma_Y^2}{2}$	
$\frac{1}{\mathrm{d}X^2}\frac{\theta^2\sigma_X^2}{2} + \frac{1}{\mathrm{d}X}\theta X^+$	$-\frac{1}{\mathrm{d}X^2}\theta^2\sigma_X^2 - \frac{1}{\mathrm{d}Y^2}\sigma_Y^2 - \frac{1}{\mathrm{d}X}\theta\lvert X\rvert - \frac{1}{\mathrm{d}Y}\left\lvert \mu + X + \frac{\sigma_Y^2}{2}\right\rvert Y$	$\frac{1}{\mathrm{d}X^2}\frac{\theta^2\sigma_X^2}{2} + \frac{1}{\mathrm{d}X}\theta X^-$
	$\frac{1}{\mathrm{d}Y}\left(\mu + X + \frac{\sigma_Y^2}{2}\right)^- Y + \frac{1}{\mathrm{d}Y^2}\frac{\sigma_Y^2}{2}$	

where $X \in \mathbb{R}$, $Y \in \mathbb{R}_{\geq 0}$, $(\cdot)^+ = \max\{0, \cdot\}$, $(\cdot)^- = \max\{0, -\cdot\}$, while $\mathrm{d}X$ and $\mathrm{d}Y$ denote the horizontal and vertical spacing of the grid in the X and Y directions, respectively. This stencil satisfies the assumptions above, since the center value, which is placed on the diagonal of $\mathbf{L}$, is constructed to be non-positive, while the values corresponding to its four neighbours, which are placed on the offdiagonal elements of $\mathbf{L}$, are constructed to be non-negative. Diagonal dominance of $\mathbf{L}$ follows trivially, since the value in the center, which ends up on the diagonal of $\mathbf{L}$, is not exceeded in absolute value by sum of the other values in the stencil.

In the approximation of second derivatives with respect to X and Y, we have used a central difference scheme, which uses two neighbouring grid points. In the approximation of first derivatives with respect to either X or Y, in contrast, we use either a 'forward' or 'backward' approximation. This means that, in addition to the center point, we use only the neighbour in the direction of the drift. This leads to the desirable result that negative values are guaranteed to end up at the center of the stencil. While forward and backward approximations of derivatives are only first-order accurate in the grid spacing, the resulting up- and downwind scheme guarantees numerical stability, which is our main concern here.

We must also consider boundary conditions. When we reach the edge of our grid, some points in our stencil may not be 'available'. One method for dealing with such 'ghost points' besides the grid is simply to ignore the stencil value corresponding to the non-existent neighbour, which leads to Dirichlet boundary conditions. Alternatively, the stencil value corresponding to the non-existent neighbour may be re-assigned to the center value, leading to Neumann boundary conditions. In our numerical analysis, using Dirichlet or Neumann boundary conditions makes no noticeable difference to the optimal policy. Experiments with finer grids, larger grids and different boundary conditions gave nearly identical results.

For the stencil above, the discrete POST algorithm is geometric contraction in vector space. Numerical methods based on fixed-point contractions are attractive as they deliver existence and uniqueness of the discretised solution. Naturally, this solution may be inaccurate due to e.g. a 'bad' discretisation. Hence, the discretised solution will be accurate in approximating the function-space solution only if (i) the finite-difference stencil is consistent with the stochastic process (as the stencil above is), (ii) the grid spacing is fine enough, (iii) the computational domain is large enough, and (iv) the boundary conditions on the edge of this domain are reasonable. We experimented with variations in these aspects and found no noticeable difference in the optimal policy, suggesting the POST solution to be quite accurate. The same robustness checks were performed for the second numerical method, discussed below, for which the results were similarly positive.

As the Poisson intensity λ is increased without bound, such that opportunities to stop arrive almost continuously, it is obvious that the POST value function approaches the classical value function, for which stopping is permissable at any time, if two sufficient conditions are satisfied: (i) the stochastic process is continuous and (ii) the stopping gain is continuous. These conditions rule out pathological cases in which the POST method fails to recover the classic value function.[2] Both conditions are clearly satisfied for the investment problem considered in the current article, as (i) the continuity of the process is guaranteed by the stochastic differential equations (25)–(26), while (ii) continuity of $A(X, Y)$ is ensured by equation (31). Hence we may increase the Poisson intensity λ to closely approximate the case where stopping is permissible at any time. The Poisson intensity $\lambda = 1024$, used in our numerical computations, implies that the investor can expect 1024 investment opportunities per annum, i.e. on average ∼4 few per business day, which is sufficiently high to approximate the limit $\lambda \to \infty$. Consistent with the theory, increasing λ further does not noticeably increase the solution.

Finally, a note on numerical efficiency. Because the POST contraction rate is determined by $\lambda/(\beta+\lambda)$, which may be close to unity when λ is large, it is advisable to start with a low value of λ, e.g. $\lambda = 1$, and gradually update the value of λ, e.g. by considering the sequence $\lambda = 2^k$ for $k = 0, 1, 2 \ldots$. Each time λ is doubled, the (final) value function corresponding to the previous problem can be used to initialise the value function for the next problem. The resulting method is numerically stable and converges quickly. After ten doublings, the final Poisson intensity equals $2^{10} = 1024$, which is sufficiently high to closely approximate the solution corresponding to $\lambda = \infty$.

For our **second numerical approach**, we follow Compernolle et al. (2021) in directly discretising the partial differential equation (38), i.e. $(\beta - \mathrm{L})V(X, Y) = 0$, using standard finite-difference methods (see their equation 35). They use an upwind scheme for the first derivatives in both spatial directions, which is appropriate as the drift of both geometric Brownian motions in their application is positive. In our more complicated setup (25)–(26), the drift in both directions may take either sign; hence, we use a combination of up- and downwind schemes. As in the stencil used for the POST method, above, this means that first derivatives are approximated either in the forward or backward direction depending on the local sign (i.e. 'direction') of the drift. We then obtain a linear system of equations similar to equation (37) in Compernolle et al. (2021).

Partial differential equations can have many solutions without the appropriate boundary conditions; in our case, we require $V(X, Y) \geq A(X, Y)$ on the entire state space. To impose this restriction in practice, we employ the theory of linear complementarity problems (LCPs, e.g. Schäfer, 2004 and Cottle et al., 2009). This literature has recognised that while $(\beta - \mathrm{L})V(X, Y) = 0$ holds on the continuation region, we have $(\beta - \mathrm{L})V(X, Y) = (\beta - \mathrm{L})A(X, Y) \geq 0$ on the stopping region, where the inequality derives from the necessary condition (41). It follows that $(\beta - \mathrm{L})V(X, Y) \geq 0$ on the entire state space,

[2] Hobson (2021, Example 2.5) notes that the the solution under the POST method may fail to converge to the classic value function as $\lambda \to \infty$ if the stopping gain is strictly positive *at a single point of measure zero* and zero elsewhere.

which is 'complementary' with $V(X,Y) \geq A(X,Y)$, i.e.

$$0 \leq V(X,Y) - A(X,Y) \quad \perp \quad (\beta - \mathrm{L})V(X,Y) \geq 0.$$

Here, the perpendicular sign '⊥' means that at each point in the state space at most one of the two inequalities is strict, such that the 'inner product' of the (discretised versions of the) two nonnegative quantities on either side is zero. Having formulated the discretised version of the LCP, we can apply standard Newton-type methods as in Bazaraa et al. (2013) to obtain the solution. The solution obtained by the LCP method was subjected to the same robustness check as the POST method with respect to the grid, computational domain and boundary conditions, with similarly reassuring results.

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