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Abstract

Agents participating in different kind of organizations, usually take different positions in some network structure. Two well-known network structures are hierarchies and communication networks. We give an overview of the most common models of communication and hierarchy restrictions in cooperative games, compare different network structures with each other and discuss network structures that combine communication as well as hierarchical features. Throughout the survey, we illustrate these network structures by applying them to cooperative games with restricted cooperation.

Key words: Networks, games, communication, hierarchy, cooperative TU-game, Shapley value

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1 Introduction

A *cooperative game with transferable utility*, or simply a TU-game, consists of a finite set of players and for every subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A main question is how to allocate the worth that can be earned by all players cooperating together, over the individual players. A (single-valued) solution is a function that assigns to every game a payoff vector which components are the individual payoffs to each player. A solution is efficient if it assigns to every game a payoff vector such that the sum of the payoffs is equal to the worth of the 'grand coalition' consisting of all players. One of the most applied efficient solutions for cooperative TU-games is the *Shapley value* (Shapley, 1953), which is exposed and highlighted by Algaba, Fragnelli and Sánchez-Soriano (2019a, 2019b).

In its classical interpretation, a TU-game describes a situation in which every coalition S (i.e subset) of N can be formed and earn its worth. In the literature, various restrictions on coalition formation have been developed. Two main forms of restricted co-operation that have been studied are communication restrictions and hierarchies. Myerson (1977) introduced the well-known model of a communication graph game that consists of a TU-game and an undirected (communication) graph where it is assumed that only coalitions that are connected in the communication graph are feasible. A restricted game is defined where the worth of every feasible (i.e. connected) coalition equals its worth in the original game, while the worth of a nonconnected coalition equals the sum of the worths of its maximally connected subsets (also known as components) of the coalition. Further, he showed that the solution that assigns to every *communication graph game* the Shapley value of the restricted game is the only solution that satisfies component efficiency (meaning that every maximally connected subset of players earns its own worth) and fairness (meaning that deleting a communication link between two players has the same effect on the individual payoffs of these two players). Algaba, Bilbao, Borm and López (2001, 2002) introduce and analyze *union stable systems* being set systems or network structures that satisfy the property that the union of every pair of nondisjoint coalitions is also feasible, a property that is satisfied by the set of connected coalitions of any undirected graph. This led to a generalization of the characterization of the Shapley value for communication graph games[1].

A model that studies restrictions in cooperation arising from hierarchies is that of a *game with a permission structure*. In those games it is assumed that the players are part of a

[1]For union stable systems where all singletons are feasible Algaba, Bilbao and López (2001a) unified two important lines of restricted cooperation: the one introduced by Myerson (1977) and the one initiated by Faigle (1989). Moreover, the relationship among union stable systems and hypergraphs is established by Algaba, Bilbao and López (2004).

hierarchical organization, where some players might need permission or approval from other players before they are allowed to cooperate. Two approaches to games with a permission structure are considered. In the conjunctive approach as developed in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996), it is assumed that each player needs permission from *all* its predecessors before it is allowed to cooperate with other players. This implies that a coalition is feasible if and only if for every player in the coalition it holds that all its predecessors belong to the coalition. Alternatively, in the disjunctive approach as developed in Gilles and Owen (1994) and van den Brink (1997), it is assumed that each player (except the top-players) needs permission from *at least one* of its predecessors before it is allowed to cooperate. Consequently, a coalition is feasible if and only if every (non-top) player in the coalition has at least one predecessor who also belongs to the coalition. In Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) it is shown that the sets of feasible coalitions arising from these permission structures are *antimatroids* being well-known combinatorial structures representing hierarchies, see Dilworth (1940) and Edelman and Jamison (1985). A set of feasible coalitions is an antimatroid if it contains the empty set, satisfies *accessibility* (meaning that every nonempty feasible coalition has at least one player that can leave the coalition and the result is a feasible subcoalition) and is *union closed* (meaning that the union of two feasible coalitions is also feasible). An overview of games with a permission structure is given in van den Brink (2017)[2]

In the field of restricted cooperation, van den Brink (2012) made clear the distinction between hierarchies and communication networks by showing that the network structures that can be the set of connected coalitions in some undirected graph are exactly those network structures that, besides containing the empty set, satisfy the above mentioned union stability and *2-accessibility* (meaning that every feasible coalition with two or more players has at least two players that can leave the coalition such that the remaining set of players is still a feasible coalition). So, compared to communication feasible sets network structures that can be obtained as the set of connected coalitions in some undirected graph), antimatroids satisfy a stronger union property (since union closedness implies union stability) but a weaker accessibility property (since 2-accessibility implies accessibility).

In view of this last result, Algaba, van den Brink and Dietz (2018) considered networks that are represented by network structures that take the weaker of the two union and accessibility properties for network structures considered above, i.e. union stability

[2]An alternative to restricting the coalitions (i.e. sets of players) that are feasible, Faigle and Kern (1992) introduce games under precedence constraints where (i) the game is defined on a restricted domain that is determined by the hierarchy, and (ii) the possible orders in which coalitions can be formed is restricted. In this setting Algaba, van den Brink and Dietz (2017) define a class of new values in which the removal of certain 'irrelevant' players does not effect the payoffs of the remaining players. A comparison between the two models is given in Algaba and van den Brink (2019).

(from communication feasible sets) and accessibility (from antimatroids). They study these so-called *accessible union stable network structures*. Obviously, all sets of connected coalitions of some (undirected) communication graph as well as all antimatroids fall into this class. It is also shown that augmenting systems (Bilbao (2003)) are accessible union stable network structures, but not every accessible union stable is an augmenting system. This brings us to the conclusion that, under union stability, augmentation implies accessiblity (i.e. if a coalition can grow from the empty set to the grand coalition by letting players enter one by one, then also from the grand coalition players can leave one by one until we reach the empty set), but not the other way around.

Throughout this paper, we compare these different network structures by considering their impact on coalition formation in cooperative TU-games with restricted cooperation. We specifically consider the solution that assigns to every cooperative game on a certain network structure, the Shapley value of an associated restricted game where the worths are generated only by feasible coalitions.

This survey is organized as follows. Section 2 gives some basic definition of cooperative TU-games. Section 3 considers games with restrictions in cooperation arising from communication restrictions, while Section 4 considers restrictions arising from hierarchies. In Section 5, we combine communication and hierarchical restrictions in one type of network structure. Finally, Section 6 contains concluding remarks.

2 Cooperative TU-games and restricted cooperation

The main goal of this survey is to review and compare various network structures, specifically those modelling communication networks and hierarchies and, moreover, we will illustrate most concepts by their effect on coalition formation in cooperative games. Firstly, in this section, we give some basic preliminaries on cooperative transferable utility games.

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subseteq \mathbb{N}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For every *coalition* $S \subseteq N$, $v(S)$ is the *worth* of coalition S, meaning that the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. Since we take the player set to be fixed, we denote a TU-game (N, v) just by its characteristic function v. We denote the collection of all TU-games on player set N by $\mathcal{G}^N$.

A *payoff vector* of an n-player TU-game $v \in \mathcal{G}^N$ is an n-dimensional vector $x \in \mathbb{R}^N$ giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a mapping f that assigns to every game $v \in \mathcal{G}^N$ a payoff vector $f(v) \in \mathbb{R}^N$. One of the most well-known solutions for TU-games is the *Shapley value* (Shapley (1953)) given by

$$Sh_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!}(v(S) - v(S \setminus \{i\})) \text{ for all } i \in N.$$

Equivalently, the Shapley value can be written as the average marginal contribution vector over all permutations (orders of entrance) of the players:

$$Sh_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} (v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\})) \text{ for all } i \in N,$$

where $\Pi(N)$ is the collection of all permutations $\pi \colon N \to N$ of the player set.[3] For an updating on theoretical and applied results about the potential and versatility of this appealing value, see Algaba, Fragnelli and Sánchez-Soriano (2019a). Likewise a review of the main classical properties can be found in Algaba, Fragnelli and Sánchez-Soriano (2019b). In a TU-game any subset $S \subseteq N$ is assumed to be able to form a coalition and earn the worth $v(S)$. However, in most economic and political organizations not every set of participants can form a feasible coalition. Therefore, cooperative game theory models have been developed that take account of restrictions on coalition formation. This is modeled by considering a set of feasible coalitions $\mathcal{F} \subseteq 2^N$ that need not contain all subsets of the player set N. For a finite set N, a *set system* or *network structure* over N is a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ is a family of subsets. The sets belonging to $\mathcal{F}$ are called *feasible*. A network structure $\mathcal{F} \subseteq 2^N$ without any further requirement can be seen as the most general type of *network structure*, and is called conference structure in Myerson (1980). However, intuitively when we consider applications of a network structure, it satisfies certain properties. In this survey, we will consider some of such properties that we encounter in applications in Economics and Operations Research.

A triple $(N, v, \mathcal{F})$ with $v \in \mathcal{G}^N$ and $\mathcal{F} \subseteq 2^N$ is a *game with restricted cooperation*. Again, since we take the player set to be fixed, we denote a game with restricted cooperation $(N, v, \mathcal{F})$ by $(v, \mathcal{F})$.

3 Communication restrictions

In this section, we consider communication networks represented by undirected graphs and the more general union stable systems, and consider their role in defining cooperation networks in cooperative games.

[3]Another common expression of the Shapley value is using the so-called Harsanyi dividends (see Harsanyi (1959)) but we will not use that in this survey.

3.1 Communication graphs

One of the most well-known models of restricted cooperation are the games with communication restrictions as introduced in Myerson (1977), see also Owen (1986). In this model, a *communication network* on the set of players in a cooperative game is given, and a coalition S is feasible if and only if the players in S are connected within this communication network. This communication network is represented by an undirected graph on the set of players.

An *undirected graph* is a pair (N, L) where N is the set of nodes and $L \subseteq \{\{i, j\} | i, j \in N,\ i \neq j\}$ is a collection of subsets of N such that each element of L contains precisely two elements. The elements of L represent bilateral communication links and are referred to as *edges* or *links*. Since in this paper the nodes in a graph represent the positions of players in a communication network we refer to the nodes as players. If there is a link between two players, we call them *neighbours*. A sequence of k different players $(i_1, \dots, i_k)$ is a *path* in (N, L) if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \dots, k-1$. Two distinct players i and j, $i \neq j$, are *connected* in graph (N, L) if there is a path $(i_1, \dots, i_k)$ with $i_1 = i$ and $i_k = j$. A coalition $S \subseteq N$ is *connected* in graph (N, L) if every pair of players in S is connected by a path that only contains players from S, i.e. for every $i, j \in S,\ i \neq j$, there is a path $(i_1, \dots, i_k)$ such that $i_1 = i,\ i_k = j$ and $\{i_1, \dots, i_k\} \subseteq S$. In other words, a coalition S is connected in (N, L) if the *subgraph* $(S, L(S))$, with $L(S) = \{\{i, j\} \in L | \{i, j\} \subseteq S\}$ being the set of links between players in S, is connected. A maximally connected subset of coalition S in (N, L) is called a *component* of S in that graph, i.e. $T \subseteq S$ is a component of S in (N, L) if and only if (i) T is connected in $(N, L(S))$, and (ii) for every $h \in S \setminus T$ the coalition $T \cup \{h\}$ is not connected in $(N, L(S))$.

A sequence of players $(i_1, \dots, i_k)$, $k \geq 2$, is a *cycle* in (N, L) if $(i_1, \dots, i_k)$ is a path in (N, L) and $\{i_k, i_1\} \in L$. A graph (N, L) is *cycle-free* when it does not contain any cycle.

Example 3.1 Consider the communication graph (N, L) on $N = \{1, \dots, 5\}$ given by $L = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{4,5\}\}$, see Figure 1. Players 1 and 5 are connected by two paths: $(1,2,4,5)$ and $(1,3,4,5)$. Coalition $\{1,4,5\}$ has two components: $\{1\}$ and $\{4,5\}$. This communication graph has a cycle $(1,2,4,3)$. □

3.1.1 Communication graph games

A triple (N, v, L) with (N, v) a TU-game and (N, L) an undirected communication graph is called a *communication graph game*. Since, again we take the player set to be fixed, we denote a communication graph game (N, v, L) just by (v, L). In the communication graph game (v, L), players can cooperate if and only if they are able to communicate with

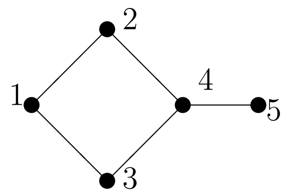

Figure 1: Communication graph (N, L) of Example 3.1

each other, i.e. a coalition S is feasible if and only if it is connected in (N, L). In other words, the set of feasible coalitions in a communication graph game (N, v, L) is the set of coalitions $\mathcal{F}_L \subseteq 2^N$ given by

$$\mathcal{F}_L = \{S \subseteq N \mid S \text{ is connected in } (N, L)\}. \tag{3.1}$$

We refer to this set as the *communication feasible set* of communication graph (N, L). Myerson (1977) introduced the *restricted game* of a communication graph game (v, L) as the TU-game v_L in which every feasible coalition S can earn its worth $v(S)$. Whenever S is not feasible it can earn the sum of the worths of its components in (N, L). Denoting the set of components of $S \subseteq N$ in (N, L) by $C_L(S)$, the restricted game v_L corresponding to communication graph game (v, L) is given by

$$v_L(S) = \sum_{T \in C_L(S)} v(T) \text{ for all } S \subseteq N. \tag{3.2}$$

Note that $C_L(S)$ is a partition of S. A solution for communication graph games assigns a payoff vector to every communication graph game. Applying any TU-game solution to the restricted game v_L gives a solution for communication graph games (v, L). The solution given by Myerson (1977) is obtained by taking for every communication graph game the Shapley value of the corresponding restricted game, a solution that was later named the *Myerson value* μ for communication graph games, i.e. $\mu(v, L) = Sh(v_L)$.

Example 3.2 *Consider the communication graph game (v, L) with L as given in Example 3.1 and game v given by*

$$v(S) = \begin{cases} 1 & \text{if } \{1, 5\} \subseteq S \\ 0 & \text{else,} \end{cases}$$

This can represent a supply chain where player 5 is a retailer who can generate worth (normalized to be equal to 1) when she has a product in store. To get the product from the factory (player 1) one of the two distributors (players 2 or 3) can be used to bring the *product to the wholesaler (player 4) who delivers the product to the retailer. The Myerson restricted game is the game* (N, v^L) *given by*

$$v^L(S) = \begin{cases} 1 & \textit{if } S \in \{\{1,2,4,5\},\{1,3,4,5\},\{1,2,3,4,5\}\} \\ 0 & \textit{else,} \end{cases}$$

So, the coalitions that generate the worth one are those coalitions that connect the retailer with the factory. Applying the Shapley value to this game gives the payoffs according to the Myerson value:

$$\mu(v, L) = \left(\frac{3}{10}, \frac{1}{20}, \frac{1}{20}, \frac{3}{10}, \frac{3}{10}\right).$$

One can argue whether the allocation of the payoffs in the example above are fair and reasonable. One way to motivate solutions is by giving axiomatic characterizations. Myerson (1977) axiomatized his value by *component efficiency* and *fairness*. Component efficiency means that every maximally connected set of players (component) earns exactly its worth. Fairness means that deleting (or adding) a link has the same effect on the payoffs of the two players on that link.[4] Other axiomatization of the Myerson value with another kind of fairness axiom, component-efficiency, a kind of null player property and additivity can be found in Selculk and Suzuki (2014).[5]

3.2 Union stable systems

Union stable systems are generalizations of communication graphs where the smallest unit of cooperation might have a size larger than 2. Let $N = \{1, \ldots, n\}$ be a finite set of players or nodes, and $\mathcal{F} \subseteq 2^N$ a set system or network structure of feasible coalitions. Union stable systems are introduced and developed in Algaba, Bilbao, Borm and López (2000, 2001) as generalizations of communication networks, where for every two nondisjoint feasible coalitions, also their union is feasible.

Definition 3.3 *A network structure* $\mathcal{F} \subseteq 2^N$ *is a union stable system if it satisfies*

(union stability) $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$ *implies that* $S \cup T \in \mathcal{F}$*.*

[4] For general TU-games, van den Brink (2001) axiomatized the Shapley value by efficiency, the null player property and a fairness axiom that requires that the payoffs of two players change by the same amount if to a game v we add another game w such that the two players are symmetric in game w. Deleting an edge from a communication graph, the two players on the deleted edge are symmetric in the difference between the two communication restricted games.

[5] Another type of motivation for a solution is by strategic implementation. A strategic implementation of the Myerson value can be found in Slikker (2007), who modified the bidding mechanism of Pérez-Castrillo and Wettstein (2001) for the Shapley value to communication graph games.

Union stable systems can be used to model restricted cooperation, where the smallest units of cooperation are not the singletons and edges as with communication graphs, but it can be that a coalition with more than two players is feasible while none of its subsets are feasible. Many real world situations find their natural framework in these structures as shown in the next example, that is taken from Algaba, Bilbao, van den Brink and López (2012).

Example 3.4 *Consider a situation where player 1 is a homeowner who wants to sell his/her house. Player* 1 *has signed a contract with a real estate agent represented by player* 2*. So, player* 1 *only can sell his/her house by means of player* 2*. There are two buyers, players* 3 *and* 4*. In this application, the family of feasible coalitions that can generate a surplus are only those which make possible that the seller can sell his/her house. Therefore, the coalitions which can trade are those coalitions that contain the homeowner, the real estate agent and at least one of the buyers:*

$$\mathcal{F} = \{\{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}. \tag{3.3}$$

Notice that for every communication graph (N, L), the set of connected coalitions $\mathcal{F}_L$ is a union stable system[6]. Obviously, if S and T are two connected coalitions, then there is a path from every player in S to every player in $S \cap T$, and there is a path from every player in $S \cap T$ to every player in T, and therefore there is a path from every player in S to every player in T, i.e. $S \cup T$ is connected. However, a union stable system cannot always be modelled by a communication graph. (In Section 5 we characterize exactly the union stable systems that can be the set of connected coalitions in a communication graph.)

3.2.1 The supports of a union stable system

Similar as the set of edges of an undirected graph determine the set of all connected coalitions, a union stable system can be fully determined by its *basis* as determined in Algaba, Bilbao, Borm and López (2000, 2001). For each union stable system $\mathcal{F}$, the following set is well-defined:

$$\mathcal{E}(\mathcal{F}) = \{G \in \mathcal{F} : G = A \cup B,\ A \neq G,\ B \neq G,\ A, B \in \mathcal{F},\ A \cap B \neq \emptyset\},$$

being the set of those coalitions in $\mathcal{F}$ that can be written as the union of two other feasible coalitions. The set $\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{F})$, is called the *basis* of $\mathcal{F}$, and the elements of $\mathcal{B}(\mathcal{F})$ are called *supports* of $\mathcal{F}$,i.e. these are the coalitions that cannot be written as the union of two other feasible coalitions.

[6]Another application of union stable systems can be found in Algaba, Béal, Remila and Solal (2019).

Inductively, the following families are defined[7]

$$\mathcal{G}^{(0)} = \mathcal{B}(\mathcal{F}), \;\; \mathcal{G}^{(m)} = \left\{ S \cup T : S, T \in \mathcal{G}^{(m-1)}, \; S \cap T \neq \emptyset \right\}, \;\; (m = 1, 2, \dots).$$

Notice that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(m-1)} \subseteq \mathcal{G}^{(m)} \subseteq \mathcal{F}$, since $\mathcal{F}$ is union stable. So, starting with the basis $\mathcal{G}^0 = \mathcal{B}(\mathcal{F})$, the collection $\mathcal{G}^1$ is obtained by adding to $\mathcal{G}^0$ all coalitions that can be obtained as the union of any pair of nondisjoint coalitions in $\mathcal{G}$, $\mathcal{G}^2$ is obtained by adding to $\mathcal{G}^1$ all coalitions that can be obtained as the union of any pair of nondisjoint coalitions in $\mathcal{G}^1$, etc, continuing untill all unions of nondisjoint pairs of coalitions in the set also belong to the set. In fact, Algaba, Bilbao, Borm and López (2000) define $\overline{\mathcal{G}}$ by $\overline{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$. We remark that the basis $B(\mathcal{F})$ is the minimal subset of the union stable system $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$.

As mentioned, the set of connected coalitions in an undirected graph or communication feasible set, is a union stable system. The basis of a communication feasible set is exactly the set of edges (feasible coalitions of size two) and singletons (feasible coalitions of size one). The components of an undirected graph can be generalized to union stable systems as follows. Let $\mathcal{F} \subseteq 2^N$ be a network structure and let $S \subseteq N$. A set $T \subseteq S$ is called a $\mathcal{F}$-*component* of S if (i) $T \in \mathcal{F}$, and (ii) there exists no $T' \in \mathcal{F}$ such that $T \subset T' \subseteq S$. Therefore, the $\mathcal{F}$-components of S are the maximal feasible coalitions that belong to $\mathcal{F}$ and are contained in S. We denote by $C_{\mathcal{F}}(S)$ the collection of the $\mathcal{F}$-components of S. Union stable systems can be characterized in terms of the $\mathcal{F}$-components of a coalition in the following way: The network structure $\mathcal{F} \subseteq 2^N$ is union stable if and only if for every $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S are a collection of pairwise disjoint subsets of S, see Algaba, Bilbao, Borm and López (2000). So, if $\mathcal{F}$ is a union stable system, such that for every $i \in N$, there is an $S \in \mathcal{F}$ with $i \in S$, then the $\mathcal{F}$-components form a partition of the player set N.

3.2.2 Cooperative games on a union stable system

A *union stable cooperation structure* or a *game on a union stable system* is a triple $(N, v, \mathcal{F})$ where $N = \{1, \dots, n\}$ is the set of players, (N, v) is a TU-game and $\mathcal{F}$ is a union stable system. Again, we assume that the player set is fixed and denote game on union stable system $(N, v, \mathcal{F})$ just as $(v, \mathcal{F})$. For convenience, we assume from now on that the underlying game (N, v) is zero-normalized, i.e., $v(\{i\}) = 0$, for all $i \in N$.

Let $\mathcal{B}(\mathcal{F})$ be the basis of $\mathcal{F}$ and $\mathcal{C}(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : |B| \geq 2\}$. If there is no confusion we will just write $\mathcal{B}$ and $\mathcal{C}$. Extending the approach of Myerson (1977) for communication graph games to games on a union stable system, the $\mathcal{F}$-*restricted game,*

[7] Algaba, Bilbao, Borm and López (2000) inductively apply the union stability operator to any network structure $\mathcal{G} \subseteq 2^N$ which always ends up in a union stable system.

$v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined on the player set N and is given by $v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T)$, where $v^{\mathcal{F}}(S) = 0$, if $C_{\mathcal{F}}(S) = \emptyset$.

A solution for games on a union stable system is a map that assigns a payoff vector to each game on a union stable system. The Myerson value for communication graph games can be extended straightforward by applying the Shapley value to the associated $\mathcal{F}$-restricted game, see van den Nouweland, Borm and Tijs (1992) and Algaba, Bilbao, Borm and López (2001). Given a game on a union stable system $(v, \mathcal{F})$, the *Myerson value,*[8] denoted by $\mu(v, \mathcal{F}) \in \mathbb{R}^N$, is defined by

$$\mu(v, \mathcal{F}) = Sh\left(v^{\mathcal{F}}\right).$$

In order to characterize the Myerson value using a fairness axiom, we introduce the following axioms in this framework.

Component efficiency of a solution on a class of games on union stable systems states that for every game with restricted cooperation in this class, the total payoff to every component equals its worth. A player $i \in N$ is called a *component dummy* in union stable system $\mathcal{F}$ if this player does not belong to any maximal component of the grand coalition, i.e., $i \notin \bigcup_{M \in C_{\mathcal{F}}(N)} M$. A solution satisfies *component dummy* if every component dummy is assigned a zero payoff.

For games on a union stable system, Algaba, Bilbao, Borm and López (2001) presented a generalized version of the fairness axiom presented in Myerson (1977), requiring that for union stable systems all players in a support B lose or gain the same amount if the support B and all coalitions that are obtained by union stability using support B are deleted. They characterized the Myerson value for games on a union stable system by component efficiency, component dummy and fairness. Notice that this generalizes the axiomatization of Myerson (1977) for communication graph games since component efficiency and fairness boil down to the corresponding axioms for communication graph games, while component dummy becomes void since in a communication graph all singletons are connected/feasible.

Another axiom that is used in characterizing the Shapley value for TU-games, but also on various network restrictions is *balanced contributions* and variations. We briefly

[8]The Myerson value for union stable systems is a particular case of the class of Harsanyi power solutions for union stable systems, as introduced by Algaba, Bilbao and van den Brink (2015), which generalizes the class of Harsanyi power solutions for communication graph games presented by van den Brink, van der Laan and Pruzhansky (2011b). Moreover, the class of Harsanyi power solutions has been studied, for a particular case of union stable systems derived from the family of winning coalitions associated with a voting game, in Algaba, Béal, Rémila and Solal (2019). In fact, this setting allows for studying situations in which there exists a feedback between the economic influence of each coalition of agents and its political power

get back to this in Section 5.

4 Hierarchies

In this section, we review some models of hierarchical cooperation structures, specifically permission structures and the more general antimatroids, and consider their role in defining cooperation networks in cooperative games.

4.1 Permission structures

A model that studies restrictions in coalition formation arising from hierarchies is that of a *game with a permission structure*. In those games, it is assumed that players are part of a hierarchical organization in which there are players that need permission or approval from certain other players before they are allowed to cooperate. For a finite set of players N such a hierarchical organization is represented by an irreflexive directed graph (N, D) with $D \subseteq N \times N$ such that $(i, i) \notin D$ for all $i \in N$, referred to as a *permission structure* on N. Since, again we take the player set to be fixed, we denote a permission structure (N, D) just by its binary relation D. The directed links $(i, j) \in D$ are called *arcs*. The players in $F_D(i) = \{j \in N \mid (i, j) \in D\}$ are called the successors or *followers* of player i, while the players in $P_D(i) = \{j \in N \mid (j, i) \in D\}$ are called the *predecessors* of i. A sequence of different players $(i_1, \dots, i_k)$ is a *directed path* between players i and j, $i \neq j$, in a permission structure D if $i_1 = i$, $i_k = j$ and $(i_h, i_{h+1}) \in D$ for all $1 \leq h \leq k-1$. A permission structure D is *acyclic* if there exists no directed path $(i_1, \dots, i_k)$ with $(i_k, i_1) \in D$. Note that in an acyclic permission structure there can be more than one directed path from player i to player $j \neq i$. Also note that in an acyclic permission structure D, there always exists at least one player with no predecessors, i.e. $TOP(D) = \{i \in N \mid P_D(i) = \emptyset\} \neq \emptyset$. We refer to these players as the *top-players* in the permission structure.

Two approaches to games with a permission structure have been considered. In the conjunctive approach as developed in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996), it is assumed that each player needs permission from *all* its predecessors in order to cooperate. This implies that a coalition $S \subseteq N$ is feasible if and only if for every player in S all its predecessors belong to S. The set of feasible coalitions in this approach is therefore given by

$$\Phi_D^c = \{S \subseteq N \mid P_D(i) \subset S \text{ for all } i \in S\},$$

which we refer to as the *conjunctive feasible set* of D.

Alternatively, in the disjunctive approach as developed in Gilles and Owen (1994) and van den Brink (1997), it is assumed that each player (except the top-players) needs

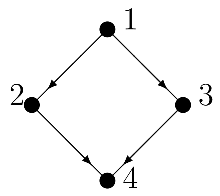

Figure 2: Permission structure D of Example 4.1

permission from *at least one* of its predecessors before it is allowed to cooperate with other players. Consequently, a coalition is feasible if and only if every player in the coalition (except the top-players) has at least one predecessor who also belongs to the coalition. Thus, the feasible coalitions are the ones in the set

$$\Phi_D^d = \{S \subseteq N \mid P_D(i) \cap S \neq \emptyset \text{ for all } i \in S \setminus TOP(D)\},$$

which we refer to as the *disjunctive feasible set* of D.

Example 4.1 Consider the permission structure D on $N = \{1,2,3,4\}$ given by $D = \{(1,2),(1,3),(2,4),(3,4)\}$, see Figure 2. Then $\Phi_D^c = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,2,3,4\}\}$ and $\Phi_D^d = \Phi_D^c \cup \{\{1,2,4\},\{1,3,4\}\}$. □

Both the conjunctive and disjunctive feasible sets are *union closed*, i.e. for every two feasible sets, also the union is feasible. This gives rise to an interesting difference between the communication feasible sets in Myerson (1977) as mentioned in Section 3, and the conjunctive and disjunctive feasible sets arising from permission structures. Whereas in a communication graph every coalition can be partitioned in maximally connected subsets (components), in the conjunctive and disjunctive feasible sets every coalition has a unique largest feasible subset.[9]

4.1.1 Games with a permission structure

A *game with a permission structure* is a triple (N, v, D), where $N = \{1, \ldots, n\}$ is the set of players, (N, v) is a TU-game and (N, D) is a permission structure. Again, we assume that the player set is fixed and denote game with a permission structure (N, v, D) just as the pair (v, D).

[9] In van den Brink, Katsev and van der Laan (2011) union closed systems are considered where the only requirement of a set of feasible coalitions is that it is union closed. They exploit this property that every feasible coalition has a unique largest feasible subset.

As mentioned above, union closedness of the conjunctive and disjunctive feasible sets implies that every coalition has a unique largest feasible subset in each of the two approaches. An approach using restricted games similar to the approach of Myerson (1977) for communication graph games (see Section 3), assigns to every coalition in a game with a permission structure the worth of its largest feasible subset. structure every coalition has a unique largest feasible subset.

This gives two restricted games. The *conjunctive restriction* $r^c_{v,D}$ of v on D assigns to every coalition the worth of its largest conjunctive feasible subset. Similar, the *disjunctive restriction* $r^d_{v,D}$ assigns to every coalition the worth of its largest disjunctive feasible subset. A solution for games with a permission structure assigns a payoff vector to every game with permission structure. Similar as in Myerson (1977)'s approach to communication graph games, we can apply any TU-game solution to the restricted games. Here, we focus on the Shapley value[10] , yielding two solutions, the conjunctive, respectively, disjunctive (Shapley) *permission value*:

$$\varphi^c(v, D) = Sh(r^c_{v,D}) \text{ and } \varphi^d(v, D) = Sh(r^d_{v,D}).$$

Example 4.2 *Consider the game with permission structure (N, v, D) on $N = \{1, 2, 3, 4\}$ with permission structure D as given in Example 4.1 and game v given by*

$$v(S) = \begin{cases} 1 & \text{if } S \ni 4 \\ 0 & \text{else.} \end{cases}$$

This models a situation where player 4 needs to be activated in order to earn a worth of 1, but 4 needs approval from its predecessors. The conjunctive restricted game is given by

$$r^c_{v,D}(S) = \begin{cases} 1 & \text{if } S = \{1, 2, 3, 4\} \\ 0 & \text{else,} \end{cases}$$

with corresponding conjunctive permission value payoffs

$$\varphi^c(v, D) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

The disjunctive restricted game is given by

$$r^d_{v,D}(S) = \begin{cases} 1 & \text{if } S \in \{\{1, 2, 4\},\ \{1, 3, 4\},\ \{1, 2, 3, 4\}\} \\ 0 & \text{else,} \end{cases}$$

with corresponding disjunctive permission value payoffs

$$\varphi^d(v, D) = \left(\frac{5}{12}, \frac{1}{12}, \frac{1}{12}, \frac{5}{12}\right).$$

[10] Core properties are considered in Derks and Gilles (1995).

Several axiomatizations of the conjunctive and disjunctive permission value can be found in the literature. Here, we just focus on a difference between the two permission values by an axiom similar to Myerson's fairness for communication graph games as mentioned in Section 3.[11]

In van den Brink (1992) an axiomatization of the disjunctive permission value is given showing that the disjunctive permission value satisfies *disjunctive fairness* which implies that deleting (or adding) an arc (such that the successor on the arc has at least one other predecessor) has the same effect on the payoffs of the two players on that arc.[12]. On the other hand, in van den Brink (1999) an axiomatization of the conjunctive permission value is given showing that the conjunctive permission value satisfies *conjunctive fairness* implying that deleting (or adding) an arc has the same effect on the payoffs of the successor on the arc and every 'other' predecessor of this successor[13].

4.2 Antimatroids

Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) show that the conjunctive and disjunctive feasible sets in acyclic permission structures are *antimatroids*. Antimatroids were introduced by Dilworth (1940) as particular examples of semimodular lattices. Since then, several authors have obtained the same concept by abstracting various combinatorial situations (see Korte, Lovász, and Schrader (1991) and Edelman and Jamison (1985)).

Definition 4.3 *A network structure $\mathcal{A} \subseteq 2^N$ is an antimatroid if it satisfies the following properties*

(feasible empty set) $\emptyset \in \mathcal{A}$,

(union closedness) $S, T \in \mathcal{A}$ *implies that* $S \cup T \in \mathcal{A}$,

(accessibility) $S \in \mathcal{A}$ *with* $S \neq \emptyset$, *implies that there exists* $i \in S$ *such that* $S \setminus \{i\} \in \mathcal{A}$.

[11] Axiomatizations of the two permission values using conjunctive, respectively disjunctive, fairness together with efficiency, additivity, the inessential player property, the necessary player property and (weak) structural monotonicity can be found in van den Brink (1997, 1999). These axioms have a natural interpretation in several applications such as, for example, polluted river problems of Ni and Wang (2007) and Dong, Ni and Wang (2012), see van den Brink, He and Huang (2018).

[12] The disjunctive fairness axiom is stronger and also requires an equal change in the payoff of every 'complete superior' of the predecessor on the arc, being those superiors that are on every path from a top player to this predecessor.

[13] Similar as mentioned for disjunctive fairness in the previous footnote, the full axiom also requires equal changes on the payoffs of every 'complete superior' of these other predecessors.

Union closedness means that the union of two feasible coalitions is also feasible. Accessibility means that every nonempty feasible coalition has at least one player that can leave such that the set of remaining players is a feasible subcoalition.

Besides these characterizing properties, in a game theory context the network structure is usually such that every player/node belongs to at least one feasible coalition.

(normality) for every $i \in N$ there exists an $S \in \mathcal{A}$ such that $i \in S$.

Note that normality and union closedness imply that $N \in \mathcal{A}$. In the following we refer to normal antimatroids simply as antimatroids. The conjunctive and disjunctive feasible sets corresponding to an acyclic permission structure are antimatroids.

Theorem 4.4 *(Algaba, Bilbao, van den Brink and Jiménez-Losada, 2004) If D is an acyclic permission structure on N then Φ_D^c and Φ_D^d are antimatroids on N.*

Next question is if antimatroids are really more general than permission structures. First, we exactly characterize those antimatroids that can be the conjunctive or disjunctive feasible set of some permission structure. It turns out that conjunctive feasible sets are exactly those that are closed under intersection. These are well known structures, also known as *poset antimatroids*.[14]

Theorem 4.5 *(Algaba, Bilbao, van den Brink and Jiménez-Losada, 2004) Let $\mathcal{A}$ be an antimatroid. There is an acyclic permission structure D such that $\mathcal{A} = \Phi_D^c$ if and only if $\mathcal{A}$ satisfies*

(intersection closedness) $S, T \in \mathcal{A}$ implies that $S \cap T \in \mathcal{A}$.

An alternative way to characterize poset antimatroids is by using paths. An *extreme player* of $S \in \mathcal{A}$ is a player $i \in S$ such that $S \setminus \{i\} \in \mathcal{A}$. So, extreme players are those players that can leave a feasible coalition S keeping feasibility. By accessibility, every feasible coalition has at least one extreme player. Coalition $S \in \mathcal{A}$ is a *path* in $\mathcal{A}$ if it has a unique extreme player. The path $S \in \mathcal{A}$ is a *i-path* in $\mathcal{A}$ if it has $i \in N$ as unique extreme player.

[14]Games on intersection closed set systems are studied in Béal, Moyouwou, Rémila and Solal (2020). Intersection closedness is one of the characterizing properties of the important network structures called convex geometries. Convex geometries are a combinatorial abstraction of convex sets introduced by Edelman and Jamison (1985). A network structure $\mathcal{G} \subseteq 2^N$ is a convex geometry if it satisfies the following properties: (i) *(feasible empty set)* $\emptyset \in \mathcal{G}$, (ii) *(intersection closed)* $S, T \in \mathcal{G}$ implies that $S \cap T \in \mathcal{G}$, and (iii) *(augmentation')* $S \in \mathcal{G}$ with $S \neq N$, implies that there exists $i \in N \setminus S$ such that $S \cup \{i\} \in \mathcal{G}$. (In Section 5, we consider another augmentation property.)

The paths form the basis of an antimatroid in the sense that every feasible coalition in an antimatroid is either a path, or can be written as the union of other feasible coalitions. So, if we know the paths, then we generate the full antimatroid by applying the union operator.

Example 4.6 *Consider the permission structure of Example 4.1, whose conjunctive and disjunctive feasible set was given in Example 4.2. The paths of the conjunctive feasible set* Φ^c_D *are* $\{1\}, \{1,2\}, \{1,3\}$ *and* $\{1,2,3,4\}$*. The paths in the disjunctive feasible set* Φ^d_D *are* $\{1\}, \{1,2\}, \{1,3\}, \{1,2,4\}$ *and* $\{1,3,4\}$*.*

In Example 4.6, we see that the conjunctive feasible set has exactly four paths, one for each player. This property characterizes the conjunctive feasible sets among all antimatroids.

Theorem 4.7 *(Algaba, Bilbao, van den Brink and Jiménez-Losada, 2004) Let* $\mathcal{A}$ *be an antimatroid. There is an acyclic permission structure* D *such that* $\mathcal{A} = \Phi^c_D$ *if and only if for every player* $i \in N$ *there is a unique* i*-path in* $\mathcal{A}$*.*

Obviously, the disjunctive feasible set does not satisfy this property. Looking at Example 4.6, we see that $\{1,2,4\}$ and $\{1,3,4\}$ are both paths of player 4. On the other hand, in the disjunctive feasible set Φ^d_D we see that, given a path, leaving out the unique extreme player, we have again a path, see for example the sequence of coalitions $\{1,2,4\}$, $\{1,2\}$, $\{1\}, \emptyset$ in Examples 4.2 and 4.6. In this example, this is not satisfied by the conjunctive feasible set Φ^c_D since deleting the unique extreme player from the path $\{1,2,3,4\}$, we are left with $\{1,2,3\}$ which is not a path since both players 2 and 3 are extreme players (it is the union of the feasible coalitions $\{1,2\}$ and $\{1,3\}$). It turns out that this 'path property' is typical for disjunctive feasible sets. In fact, we need a stronger property to characterize them.

Theorem 4.8 *(Algaba, Bilbao, van den Brink and Jiménez-Losada, 2004) Let* $\mathcal{A}$ *be an antimatroid. There is an acyclic permission structure* D *such that* $\mathcal{A} = \Phi^d_D$ *if and only if*

1. *Every path* S *has a unique feasible ordering, i.e.* $S = (i_1 > \cdots > i_t)$ *such that* $\{i_1, \dots, i_k\} \in \mathcal{A}$ *for all* $1 \leq k \leq t$*. Furthermore, the union of these orderings for all paths is a partial ordering of* N*.*

2. *If* S, T *and* $S \setminus \{i\}$ *are paths such that the extreme player of* T *equals the extreme player of* $S \setminus \{i\}$*, then* $T \cup \{i\} \in \mathcal{A}$*.*

Next we show that antimatroids are really more general than permission structures by giving an example of an antimatroid that does not satisfy the properties of Theorems 4.7 and 4.8 that characterize the conjunctive and disjunctive feasible sets.

Example 4.9 Ordered partition voting *Consider player set $N = \{1,2,3,4,5\}$. Suppose that the player set is partitioned in two levels: Level 1 consists of players 1, 2 and 3, while Level 2 consists of players 4 and 5. Suppose that all subsets of Level 1 are feasible, but every subset of Level 2 needs approval of a majority (two-player) coalition of Level 1. So, the set of feasible coalitions is*

$$\mathcal{A} = \left\{ \begin{array}{l} \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \\ \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \\ \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\} \end{array} \right\}.$$

This is an antimatroid. However, it is not a conjunctive feasible set (poset antimatroid) since $\{1,2,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$ are all paths of player 4. It is also not a disjunctive feasible set since taking out the unique extreme player (4) from the path $\{1,2,4\}$ gives coalition $\{1,2\}$ which is not a path. □

It is not difficult to prove that the conjunctive and disjunctive approach coincide if and only if the permission structure is a forest.

Theorem 4.10 *(Algaba, Bilbao, van den Brink and Jiménez-Losada, 2004) Let D be an acyclic permission structure. Then $\Phi^c_D = \Phi^d_D$ if and only if $|P_D(i)| \leq 1$ for all $i \in N$.*

Games with a permission tree (i.e. a connected forest) are studied in van den Brink, Herings, van der Laan and Talman (2015), Alvarez-Mozos, van den Brink, van der Laan and Tejada (2017) and van den Brink, Dietz, van der Laan and Xu (2017)[15]. Peer group games where, moreover, the original game is an additive or inessential game, are studied in Brânzei, Solymosi and Tijs (2005) and in Brânzei, Fragneli and Tijs (2002) for the special case of a rooted line-graph. These games have many applications in Economics and Operations research, as we will mention in our concluding remarks.

4.2.1 Games on an antimatroid

A *game on an antimatroid* is a triple $(N, v, \mathcal{A})$ where (N, v) is a TU-game, and $\mathcal{A}$ is an antimatroid on player set N. Since we take the player set to be fixed, we denote a game

[15] Although van den Brink, Dietz, van der Laan and Xu (2017) consider permission tree games, they distinguish solutions that are based on a communication or hierarchy approach. For cycle-free communication graph games, Demange (2004) introduces the so-called hierarchical outcomes, that are extreme points of the core of the restricted game if the game is superadditive and the graph is cycle-free. Nonemptines of the core of a superadditive, cycle-free graph game was shown, independently, in Demange (1992) and Le Breton, Owen and Weber (1992). Herings, van der Laan and Talman (2008) consider the average of the hierarchical outcomes.

on an antimatroid just as a pair $(v, \mathcal{A})$. The antimatroid is the set of feasible coalitions in the game, and thus reflects the restricted cooperation possibilities. Since the conjunctive and disjunctive feasible sets derived from an acyclic permission structure are antimatroids, this model generalizes the games with an acyclic permission structure.

As mentioned before, by union closedness every coalition has a unique largest feasible subset. For antimatroids, Korte, Lóvasz and Schrader (1991) introduce the *interior operator* $int_{\mathcal{A}} \colon 2^N \to \mathcal{A}$ that assigns to every set its largest feasible subset, i.e.

$$int_{\mathcal{A}}(S) = \bigcup_{\{T \in \mathcal{A} | T \subseteq S\}} T \text{ for all } S \subseteq N. \tag{4.4}$$

Using this operator[16] we can easily generalize the definition of the conjunctive and disjunctive restricted game for games with a permission structure to games on antimatroids. The *restriction* of game v on antimatroid $\mathcal{A}$ is the game $v_{\mathcal{A}}$ that assigns to every coalition the worth of its largest feasible subset, and thus is given by

$$v_{\mathcal{A}}(S) = v(int_{\mathcal{A}}(S)) \text{ for all } S \subseteq N.$$

A solution f for games on antimatroids assigns a payoff vector to every game on an antimatroid $(v, \mathcal{A})$ on N. We consider the solution *Sh* that assigns to every game on antimatroid $(v, \mathcal{A})$, the Shapley value of the restricted game, i.e.

$$Sh(v, \mathcal{A}) = Sh(v_{\mathcal{A}}).$$

Algaba, Bilbao, van den Brink and Jiménez-Losada (2004) introduce a fairness axiom for games on an antimatroid that generalizes both conjunctive as well as disjunctive fairness, by requiring that deleting a feasible coalition from an antimatroid, such that what is left is still an antimatroid, has the same effect on the payoffs of all players in the coalition that is deleted.[17] [18][19]

[16]The interior operator is characterized by: (i) $int_{\mathcal{A}}(\emptyset) = \emptyset$, (ii) $int_{\mathcal{A}}(S) \subseteq S$, (iii) if $S \subseteq T$ then $int_{\mathcal{A}}(S) \subseteq int_{\mathcal{A}}(T)$, (iv) $int_{\mathcal{A}}(int_{\mathcal{A}}(S)) = int_{\mathcal{A}}(S)$, and (v) if $i, j \in int_{\mathcal{A}}(S)$ and $j \in int_{\mathcal{A}}(S \setminus \{i\})$ then $i \notin int_{\mathcal{A}}(S \setminus \{j\})$.

[17]In antimatroid $\mathcal{A}$, the path S is covered by path T if $S \subset T$ with $|T| = |S| + 1$, and the unique extreme player of T is the player in $T \setminus S$. For a coalition S to be deleted leaving behind an antimatroid, the deleted coalition should be a path (otherwise union closedness will be violated) that is not covered by a path (otherwise accessibility will be violated, since if paths S is covered by path $T \supset S$ with $|T| = |S| + 1$, after deleting S from the antimatroid, T has no extreme player).

[18]One can see that applying this fairness to the sets Φ_D^c, respectively Φ_D^d of a game with permission structure gives the corresponding conjunctive, respectively disjunctive, fairness as follows. Deleting an arc (i, j), with $|P_D(j)| \geq 2$, in a permission structure leads to more (respectively less) feasible coalitions in Φ_D^c (respectively Φ_D^d) and every coalition that is 'gained' (respectively 'lost') contains player j and all other predecessors $h \in P_D(j) \setminus \{i\}$ (respectively player i).

[19]They characterize the Shapley value for games on an antimatroid by this fairness axiom together with axioms generalizing efficiency, additivity, the inessential player property and the necessary player property for games with a permission structure as mentioned in Footnote 11.

Finally, notice that an antimatroid is a union stable system, since union closedness implies union stability. In the next section, we compare different structures.[20]

5 Communication and hierarchies

In this section, we explore the possibilities to combine communication features with hierarchy.

5.1 Comparing communication and hierarchies

Let $\mathcal{F} \subseteq 2^N$ be an arbitrary set of feasible coalitions. Since all singletons in a communication graph are connected, it follows that communication feasible sets arising from communication graphs contain the empty set and satisfy normality, i.e. every player belongs to at least one feasible coalition. Further, communication feasible sets also satisfy accessibility. They even satisfy the stronger *2-accessibility* meaning that every feasible coalition with two or more players has at least two players that can leave the coalition such that the set of remaining players is a feasible coalition. Communication feasible sets are not union closed (as is illustrated by the two connected coalitions $\{1, 2\}$ and $\{5\}$ in Example 3.1 whose union is not connected). However, as mentioned before (see Section 3), communication feasible sets satisfy the weaker *union stability*. In van den Brink (2012) it is shown that a network structure is a communication feasible set if and only if it contains the empty set and satisfies normality, union stability and 2-accessibility.

Theorem 5.1 *[van den Brink (2012)] Let $\mathcal{F} \subseteq 2^N$ be a normal network structure on $N \subset \mathbb{N}$. Then $\mathcal{F}$ is the communication feasible set of some communication graph if and only if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(union stability) $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$ *implies that* $S \cup T \in \mathcal{F}$,

(2-accessibility) $S \in \mathcal{F}$ *with* $|S| \geq 2$ *implies that there exist* $i, j \in S$, $i \neq j$, *such that* $S \setminus \{i\} \in \mathcal{F}$ *and* $S \setminus \{j\} \in \mathcal{F}$.

Usually the set of links in an undirected communication graph/communication feasible set, being coalitions of size two, is considered as the basis of a communication network.

[20]For results of games on antimatroids we refer to Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004). For antimatroids that are not normal, similar results can be stated restricted to the class of players that belong to at least one feasible coalition.

Note that by repeated application of 2-accessibility untill we are left with coalitions of size two and one, we can generate these bilateral links from any communication feasible set.

Also note that given 2-accessibility, normality implies that $\{i\} \in \mathcal{F}$ for all $i \in N$ as is the case for communication feasible sets.

Adding other properties characterizes the sets of connected coalitions in special graphs. For example adding *closedness under intersection* yields those communication feasible sets arising from *cycle-complete communication graphs*.[21] Other special types of graphs, such as cycle-free graphs and line-graphs, are characterized in van den Brink (2012). For example,

(i) a communication feasible set is the set of connected coalitions in some communication line-graph if and only if it satisfies path union stability (meaning that the union of every pair of paths that have a nonempty intersection is also a path[22]);

(ii) a communication feasible set is the set of connected coalitions in some communication cycle-free communication graph if and only if it satisfies weak path union stability (meaning that the union of two feasible paths that have an endpoint in common, is also a path);

(iii) a communication feasible set is the set of connected coalitions in some communication tree (i.e. connected cycle-free graph) if and only if it satisfies connectedness (meaning that for every pair of players, there is a feasible coalition containing both players).

Comparing Theorem 5.1 with Definition 4.3, we conclude that communication feasible sets are characterized by properties similar to the ones used in defining antimatroids. Besides normality and feasibility of the empty set, both satisfy an accessibility and a union property. Obviously, 2-accessibility implies accessibility and therefore communication feasible sets satisfy a stronger accessibility property. However, antimatroids satisfy union closedness instead of union stability, and therefore antimatroids satisfy a stronger union property.

5.2 Accessible union stable network structures

Considering Theorem 5.1, a natural network structure that combines hierarchy with communication restrictions, is to consider network structures that satisfy the weaker union and accessibility properties that characterize antimatroids and communication feasible sets, and

[21]A graph is cycle-complete if, whenever there is a cycle, the subgraph on that cycle is complete, see Figure 3.

[22]The context makes clear if we consider paths in a communication graph or paths in an antimatroid.

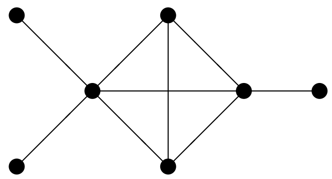

Figure 3: A cycle-complete communication graph

Table 1. This gives the following network structure/set system that is introduced in Algaba, van den Brink and Dietz (2018).

Antimatroids		Communication
$\emptyset$ is feasible		$\emptyset$ is feasible
union closed	$\Rightarrow$	union stable
accessible	$\Leftarrow$	2-accessible

Table 1: Comparing communication with hierarchy

Definition 5.2 *A network structure $\mathcal{F} \subseteq 2^N$ is an accessible union stable network structure if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(union stability) $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$, *implies that* $S \cup T \in \mathcal{F}$,

(accessibility) $S \in \mathcal{F}$ *with* $S \neq \emptyset$, *implies that there exists* $i \in S$ *such that* $S \setminus \{i\} \in \mathcal{F}$.

Obviously, antimatroids and communication feasible sets are accessible union stable network structures. In these network structures, union stability reflects communication in the sense that players that belong to the intersection of two coalitions can generate communication through the full (i.e. union) coalition. Accessibility reflects asymmetry between the players, specifically between the players that can, and those that cannot, leave the coalition keeping feasibility. Thus, accessible union stable network structures seem to be a natural network structure to model organizations with communication as well as hierarchy features. By definition, we have the following obvious proposition.

Proposition 5.3 *(i) An accessible union stable network structureis an antimatroid if and only if it is union closed. (ii) A normal accessible union stable network structure is a communication feasible set if and only if it satisfies 2-accessibility.*

Since accessible union stable network structures generalize communication feasible sets as well as antimatroids, they can help us to study organizations that have hierarchical as well as communication features. A class that contains communication feasible sets and antimatroids are the augmenting systems introduced by Bilbao (2003).

Definition 5.4 *A network structure $\mathcal{F} \subseteq 2^N$ is an augmenting system if it satisfies the following properties:*

(feasible empty set) $\emptyset \in \mathcal{F}$*,*

(union stability) $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$*, implies that* $S \cup T \in \mathcal{F}$*,*

(augmentation) $S, T \in \mathcal{F}$ *with* $S \subset T$*, implies that there exists* $i \in T \setminus S$ *such that* $S \cup \{i\} \in \mathcal{F}$*.*

Algaba, Bilbao and Slikker (2010) establishes that an augmenting system, where all singletons are feasible, is a communication feasible set.

Augmentation[23] establishes that, whenever there are two feasible coalitions such that one is contained in the other, we can keep adding players from the 'bigger' coalition to the 'smaller' coalition one by one, such that after each addition the new coalition is feasible. This property can be used in defining solutions for games that are based on marginal vectors, such as the Shapley value. Assuming that the 'grand coalition' N is feasible one can, starting with the empty set and adding one player at each step, define a sequence of feasible coalitions ending up in the 'grand coalition'. This means that we can always define a permutation $\pi\colon N \longrightarrow N$ such that $\{\pi(1), \ldots, \pi(k)\}$ is a feasible coalition for every $k \in \{1, \ldots, n\}$. We will see later that the same cannot be done for arbitrary accessible union stable network structures.

Players who can be joined to a feasible coalition $S \in \mathcal{F}$ keeping feasibility are called *augmentation players*, i.e. a player $i \in N \setminus S$, with $S \cup \{i\} \in \mathcal{F} \subseteq 2^N$, is called an augmentation player of coalition S in $\mathcal{F}$.

Note that augmentation implies accessibility. Indeed, for $T \in \mathcal{F}$ with $T \neq \emptyset$, by the augmentation property, we have that there exists a sequence of coalitions $T_0, T_1, \ldots, T_t$, with $T_h \in \mathcal{F}$, $|T_h| = h$, $0 \leq h \leq t$, such that $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_{t-1} \subset T_t = T$. Therefore, there exists a player $i \in T$ such that $T \setminus \{i\} = T_{t-1} \in \mathcal{F}$. This shows that augmenting systems satisfy accessibility, and therefore they are accessible union stable network structures.

[23]Notice the difference with the augmentation property that is used as a defining property of a convex geometry, see Footnote 6.

Proposition 5.5 *(Algaba, van den Brink and Dietz (2018)) If $\mathcal{F} \subseteq 2^N$ is an augmenting system then $\mathcal{F}$ is an accessible union stable network structure.*

However, accessibility does not imply augmentation, and therefore not every accessible union stable network structure is an augmenting system, see Example 5.6.

So, interpreting augmentation as the possibility to grow and accessibility as the possibility to shrink, the possibility to grow from the empty set to the grand coalition by players entering one by one, implies the we can shrink from the grand coalition to the empty set by players leaving one by one. But the possibility to shrink does not imply the possibility to grow. This has an important impact on solutions that are defined using marginal vectors such as the Shapley value.

The next example discusses an application of accessible union stable network structures that are neither an augmenting system nor an antimatroid nor a communication feasible set.

Example 5.6 (Exploring and careful societies) *Let $K = \{1, 2\}$ be a society of explorers, and $M = \{3, 4, 5\}$ a society of careful players. Assume that all coalitions within each society are feasible, but a coalition containing players from both societies can only be formed if it contains all players of M (and any subset of K). Therefore, we can consider society K as a society of 'explorers' who each can go individually or with any group to the 'outside world', and society M as a 'careful' society consisting of players who can only go out together.*

The corresponding set of feasible coalitions is

$$\begin{aligned}\mathcal{F} \;=\; & \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{3,4,5\},\\ & \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\},\end{aligned}$$

(where the first row of $\mathcal{F}$ contains K, M and all their subsets, while in the second row are the coalitions that contain players from both societies.)

This network structure is union stable since (i) any union of two coalitions that are either a subset of K or a subset of M are feasible, and (ii) the union of two nondisjoint feasible coalitions containing players from K and M must contain all players from M (since at least) one of these two coalitions must contain players from K and M, and thus the union must contain all players from M), and therefore is feasible.

The network structure is accessible since (i) for each non-empty feasible coalition that is a subset of K or M every player in this coalition is an extreme player (i.e., it can be deleted keeping feasibility), and (ii) for every feasible coalition that contains players from both K and M each player from K can be deleted. The resulting coalition still contains all players from M, and therefore is feasible.

To show that the network structure $\mathcal{F}$ is not a communication feasible set, consider for example feasible coalition $\{2,3,4,5\}$. This coalition contains one player from K, player 2, and all players from M. But since none of the players of M can be deleted, player 2 is the only extreme player, so the network structure does not satisfy 2-accessibility.

To show that the network structure $\mathcal{F}$ is not an antimatroid, consider for example coalitions $\{1\}$ and $\{3\}$. These are proper subsets of K, respectively M, so their union contains a player from K and a player from M but does not contain all players from M, and therefore is not feasible, showing that $\mathcal{F}$ does not satisfy union closedness.

Finally, to show that the network structure $\mathcal{F}$ is not an augmenting system, consider coalitions $K = \{1,2\}$ and $K \cup M = \{1,2,3,4,5\}$. Then no single player from M can be added to K to get a feasible coalition since the players of M only join K as group, showing that it does not satisfy augmentation.

5.2.1 The supports of an accessible union stable network structures

Since accessible union stable network structures are union stable systems, they can be described by the supports as defined in Section 3.

Example 5.7 *Consider the accessible union stable network structure $\mathcal{F}$ on $N = \{1,2,3,4,5\}$ given by*

$$\mathcal{F} = \left\{ \begin{array}{c} \emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \{3,4\}, \{4,5\}, \{2,3,4\}, \\ \{3,4,5\}, \{1,2,3,4\}, \{2,3,4,5\}, N \end{array} \right\}.$$

The supports of $\mathcal{F}$ are given by

$$B(\mathcal{F}) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \{3,4\}, \{4,5\}, \{2,3,4\}\}.$$

We introduced accessible union stable network structures as a model that generalizes communication feasible sets as well as antimatroids in such a way that two defining properties reflect communication (union stability), respectively, hierarchy (accessibility). An interesting question is to see how these features influence the basis of the system. As mentioned in Section 3, the supports of a communication feasible set are exactly those elements that have cardinality one or two, the first type being the singletons and the second type being the links or edges of the communication graph. The full communication feasible set is obtained from the supports by repeatedly applying the union stability operator. An antimatroid can be fully described by its paths, being those feasible coalitions that have exactly one extreme player, and applying the union operator.

Applying accessibility, it holds that every nonempty feasible coalition in an accessible union stable network structure has an extreme player. For accessible union stable network structures it turns out that every support either has cardinality at most two or is a path. (In the accessible union stable network structure of Example 5.7, the supports with cardinality at most two are those in $B(\mathcal{F}) \setminus \{\{2,3,4\}\}$, while support $\{2,3,4\}$ is a path.)

Proposition 5.8 *(Algaba, van den Brink and Dietz (2018)) Let $\mathcal{F} \subseteq 2^N$ be an accessible union stable network structure. If $B \in B(\mathcal{F})$ with $|B| > 2$ then B is a path.*

The reverse is not true, i.e., not every path with more than two players is a support.

Example 5.9 *Consider the set $N = \{1,2,3,4\}$ and the accessible union stable network structure given by $\mathcal{F} = \{\{1\},\{2\},\{4\},\{1,2\},\{3,4\},\{2,3,4\},N\}$.*

Its basis is $B(\mathcal{F}) = \{\{1\},\{2\},\{4\},\{1,2\},\{3,4\},\{2,3,4\}\}$. Since the only extreme player of the 'grand coalition' N is player 1, the grand coalition is a path but it is not a support, since it is the union of $\{1,2\}$ and $\{2,3,4\}$.

This 'hybrid' form of the basis of accessible union stable network structures make these structures in some sense more difficult than the other structures mentioned before. Specifically, (i) the basis of a union stable system is formed by the supports being the coalitions that are not the union of two nondisjoint feasible coalitions, (ii) communication feasible sets are determined by the singletons and the links, while (iii) antimatroids are described by their paths. This is useful, specifically for the fairness type of axioms for solutions for games with restricted cooperation which compare two different structures where one of these is obtained by deleting certain feasible coalitions from the other. In the case of union stable systems, deleting a support, what is left is still a union stable system. For antimatroids, taking out any path that is not covered by another path. In the mentioned literature, it is shown that in these structures, we have enough possibilities to delete coalitions and apply the fairness axiom mentioned earlier in this survey to get uniqueness with some other axioms. In the next subsection, we mention some problems that we encounter when applying fairness for games on accessible union network structures.

5.2.2 Cooperative games on accessible union stable network structures

A *game on an accessible union stable network structures* is a triple $(N, v, \mathcal{F})$ where (N, v) is a TU-game, and $\mathcal{F} \subseteq 2^N$ is an accessible union stable network structure. Since, again we take the player set to be fixed, we denote a game on an accessible union stable network structure $(N, v, \mathcal{F})$ by $(v, \mathcal{F})$.

Since accesible union stable systems are union stable systems, we can directly apply the approach of Section 3, and define a restricted game associated to games on an accessible union stable network structure as follows. Let $v : 2^N \to \mathbb{R}$ be a cooperative game and let $\mathcal{F} \subseteq 2^N$ be an accessible union stable network structure. The restricted game $v^{\mathcal{F}} : 2^N \to \mathbb{R}$, is defined by

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \text{ for all } S \subseteq N.$$

Notice that, if $\mathcal{F}$ is an accessible union stable network structure, then for every $S \subseteq N$ such that $C_{\mathcal{F}}(S) = \emptyset$, we have $v^{\mathcal{F}}(S) = 0$. If $\mathcal{F}$ is a communication feasible set, then the game $v^{\mathcal{F}}$ is the graph-restricted game of Myerson (1977) and Owen (1986), see Section 3. Since an antimatroid $\mathcal{A}$ is union closed, every subset $S \subseteq N$ has a unique component given by the interior operator $int\,(S)$, see (4.4). The restricted game $v^{\mathcal{A}} : 2^N \to \mathbb{R}$, is the game defined by $v^{\mathcal{A}}\,(S) = v\,(int(S))$, see Section 4.

A solution for games on an accessible union stable structure is a function that assigns a payoff vector to every game on an accessible union stable network structure. Following the previous sections, we consider the solution that assigns to every game on an accessible union stable network structure the Shapley value of the corresponding restricted game: $\varphi\,(v, \mathcal{F}) = Sh(v^{\mathcal{F}})$ for every game on accessible union stable network structure $(v, \mathcal{F})$.

Clearly, the value φ for games on accessible union stable network structures generalizes the Myerson value for games restricted by communication graphs and the (conjunctive and disjunctive) permission value for games with a permission structure.

As mentioned before, if $\mathcal{F}$ is a union stable system then the components of N in $\mathcal{F}$ form a partition of a subset of N. (If $\mathcal{F}$ is, moreover, normal then the components form a partition of N).

Notice that we cannot just apply fairness since, besides union stability, we now also need to take care that, after deleting a coalition/support, the remaining network structure still satisfies accessibility.

Before considering a fairness type of axiom for games on an accessible union stable network structure, we consider a balanced contributions type of axiom. Balanced contributions axioms balance mutual dependence of two players on each other in the sense that they equalize the effect of the removal of one player on the payoff of the other player. Given a network structure $\mathcal{F} \subseteq 2^N$ and a player $i \in N$, the network structure

$$\mathcal{F}_{-i} = \{S \in \mathcal{F} \mid i \notin S\}$$

consisting all those feasible coalitions in $\mathcal{F}$ which do not contain player i. This operation has the nice feature that the reduced structure $\mathcal{F}_{-i}$ is still an accessible union stable network structure.

Proposition 5.10 *(Algaba, van den Brink and Dietz (2018)) If $\mathcal{F} \subseteq 2^N$ is an accessible union stable network structure and $i \in N$, then $\mathcal{F}_{-i}$ is an accessible union stable network structure.*

This proposition allows us to define the following axiom. A solution f for games on an accessible union stable network structure has balanced contributions if for every game on accessible union stable network structure $(v, \mathcal{F})$ and any two players $i, j \in N$ with $i \neq j$, we have

$$f_i(v, \mathcal{F}) - f_i(v, \mathcal{F}_{-j}) = f_j(v, \mathcal{F}) - f_j(v, \mathcal{F}_{-i}).$$

The 'freedom' to 'isolate' any player in an accessible union stable network structure gives us the positive characterization result that the Shapley value is the unique value for games on an accessible union stable network structures that satisfies component efficiency, component dummy, and has balanced contributions, see Algaba, van den Brink and Dietz (2018).

Above, using balanced contributions, we considered the effects of deleting all coalitions containing one particular player from the set of feasible coalitions, on the payoffs of another player. For accessible union stable network structures, we can define various types of fairness axioms. Here, we consider the folllowing. Comparing payoffs of two players, we can, similar as balanced contributions, consider the effect on their payoffs when we delete all coalitions containing both players. So, for an accessible union stable network structure $\mathcal{F}$ and two players $i, j \in N$, we consider the network structure

$$\mathcal{F}_{-ij} = \{S \in \mathcal{F} \mid \{i, j\} \not\subseteq S\}$$

being the collection of feasible coalitions in $\mathcal{F}$ that do not contain both players i and j. Next, we define a version of fairness where we delete all coalitions containing two particular players, and require the payoffs of these two players to change by the same amount. Then a solution f for games on an accessible union stable network structure satisfies fairness if

$$f_i(v, \mathcal{F}) - f_i(v, \mathcal{F}_{-ij}) = f_j(v, \mathcal{F}) - f_j(v, \mathcal{F}_{-ij})$$

for all games on accessible union stable network structure $(v, \mathcal{F})$ and $i, j \in N$ such that $\mathcal{F}_{-ij}$ is an accessible union stable network structure.

The restriction that $\mathcal{F}_{-ij}$ is an accessible union stable network structure implies that not all feasible coalitions can be deleted. It turns out that $\mathcal{F}$ being accessible implies that $\mathcal{F}_{-ij}$ is accessible.

Proposition 5.11 *(Algaba, van den Brink and Dietz (2013)) If $\mathcal{F} \subseteq 2^N$ is an accessible network structure then $\mathcal{F}_{-ij}$ is accessible.*

However, for an arbitrary accessible union stable network structure $\mathcal{F}$ the network structure $\mathcal{F}_{-ij}$ need not be union stable as the following example shows.

Example 5.12 *Consider the accessible union stable network structure $\mathcal{F}$ of Example 5.6. Take a player from N and one from M, for example players 2 and 4.*

Then $\mathcal{F}_{-24} = \mathcal{F} \setminus \{\{2,3,4,5\},\{1,2,3,4,5\}\}$, which is not union stable since $\{1,2\}$ and $\{1,3,4,5\}$ both belong to $\mathcal{F}_{-24}$ but their union does not.

This creates problems for axiomatizing the Shapley value for games on an accessible union stable network structure, since we are restricted in the coalitions that can be deleted from an accessible union stable network structure.

Notice that fairness as defined above for the class of accessible union stable network structures is not the same as Myerson's fairness for communication graph games, since if $\mathcal{F} = \mathcal{F}_L$ for some communication graph L then $\mathcal{F}_{-ij}$, in general, is not the set of connected coalitions in $L \setminus \{\{i,j\}\}$. Consider, for example, the communication graph L on $N = \{1,2,3,4\}$ given by $L = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$. Then $\{1,2,3,4\} \in \mathcal{F}_{L\setminus\{\{1,2\}\}} \setminus (\mathcal{F}_L)_{-12}$. Therefore, on the class of communication graph games this axiom is not the same as Myerson's fairness. However, it can be shown that for (sets of connected coalitions in) cycle-free graphs the two fairness axioms are the same. Therefore, it is known that for these set systems, fairness and component efficiency characterize the Shapley value. Similar, it follows that for the class games on cycle-free accessible union stable network structures, the Shapley value is characterized by component efficiency, component dummy and fairness.

6 Concluding remarks

The main goal of this survey is to review and compare various network structures, specifically those that model communication networks and hierarchies. We illustrated most concepts by their effect on coalition formation in cooperative (transferable utility) games.

We discussed (undirected) communication graphs and the more general union stable systems as models of communication networks, and we discussed permission structures and the more general antimatroids as models of hierarchical structures. Also, we discussed the model of an accessible union stable network structure as a model that combines communication with hierarchy. In the survey, we already mentioned several other network structures and their relation with the structures reviewed her, such as union closed network structures, convex geometries, intersection closed network structures and augmenting systems. Other models in the literature are, for example, the games in coalition structure of Aumann and Drèze (1974) and Owen (1977), where the player set is partitioned into a-priori unions (which can also be modeled as an undirected graph where there is a link between two players if and only if they belong to the same a-priori union), or its generalization to level structures in Winter (1989), where there is a sequence of coalition structures, the next one finer than the previous one.

It is interesting to observe that cooperative games in itself are a generalization of undirected graphs. In fact, cooperative games are a generalization of hypergraphs. To be precise, a hypergraph is a simple cooperative game, being a game (N, v) with $v(S) \in \{0, 1\}$ for all $S \subseteq N$. A simple cooperative game then represents hypergraph $(N, \mathcal{F})$, $\mathcal{F} \subseteq 2^N$, if $v(S) = 1$ if $S \in \mathcal{F}$. In game terminology, S is called a coalition, while in graph terminology it is called a hyperlink. Undirected graphs are a special type of hypergraphs where the hyperlinks have size one or two. Representing an undirected graph (N, L) by its collection $\mathcal{F}_L$ of connected coalitions, this can also be represented by the so-called 2-additive game (N, v) given by (i) $v(S) = 1$ if $|S| = 2$ and $S \in L$, (ii) $v(S) = 0$ if $|S| = 2$ and $S \notin L$, or $|S| \leq 1$, and (iii) $v(S) = \sum_{T \subset S, |T|=2} v(T)$ if $|S| > 2$, see Deng and Papadimitriou(1994). In this context, undirected graphs are usually called bilateral graphs or networks. Summarizing, the class of undirected bilateral graphs is contained in the class of hypergraphs which is contained in the class of cooperative games. Specifically, hypergraphs are simple cooperative games, while undirected bilateral graphs are 2-additive simple cooperative games. Obviously, 2-additive cooperative games (that are not simple) coincide with the so-called weighted undirected graphs where the links have weights expressing their importance.[24]

A very general approach to games with restricted cooperation is followed by Derks and Peters (1993) who consider a restriction as a mapping $\rho \colon 2^N \to 2^N$ that assigns to every coalition an associated 'feasible' coalition. One of the characterizing properties of a restriction σ is that $\rho(\rho(S)) = \rho(S)$ for all $S \subseteq N$.[25] This approach contains the games on a union closed system, and thus games on an antimatroid and permission structure, where to every coalition it assigns the largest feasible subset. It does not contain the communication graph games but, for example, it also does not contain the games with a local permission structure of van den Brink and Dietz (2014), where the role of a player as value generator is separated from its role as authorizer in the sense that a player can give permission to its subordinates to act, but still needs permission from its own predecessors to be active.[26]

[24] Applications of weighted graphs, considering weights on links as well as on nodes can be found in, for example, Lindelauf, Hamers, and Husslage (2013), who measure the importance of terrorists based on their centrality in a terrorist network.

[25] The other characterizing properties are that $\rho(S) \subseteq S$ for all $S \subseteq N$, and $\rho(S) \subseteq \rho(T)$ for all $S \subseteq T \subseteq N$.

[26] For example, in the line permission tree $D = \{(1, 2), (2, 3)\}$ on $N = \{1, 2, 3\}$ the feasible part of coalition $\{2, 3\}$ is singleton $\{3\}$ since its direct predecessor is in the coalition, while the predecessor of 2 is not in the coalition. In this case $\rho(\{1, 2, 3\}) = \{1, 2, 3\}$, $\rho(\{1, 2\} = \rho(\{1, 3\}) = \rho(\{1\}) = \{1\}, \rho(\{2, 3\}) = \{3\}$ and $\rho(\{2\}) = \rho(\{3\}) = \emptyset$. Thus $\rho(\rho(\{2, 3\})) = \rho(\{3\}) = \emptyset \neq \{3\} = \rho(\{2, 3\})$.

In this survey, we focussed on comparing different network structures. We want to mention that in many applications in Economics and Operations Research, also looking at more specific network structures gives insight into these Economics and OR problems. Without going into detail, we mention some of these applications. Regarding communication networks, van den Brink, van der Laan and Vasil'ev (2007) show that, for example the river games of Ambec and Sprumont (2002), and the sequencing games of Curiel, Potters, Rajendra Prasad, Tijs and Veltman (1993, 1994), can be seen as communication graph games where the underlying communication graph is a line graph. Looking at it in this way, we can see that, for example, the downstream incremental solution for river games (Ambec and Sprumont (2002)) and the drop-out monotonic solution for sequencing games (Fernández, Borm, Hendrickx and Tijs (2005)) boil down to the same solution, being the marginal vector where players enter consecutive in the order of the line. Regarding applications of hierarchies, Brânzei, Fragnelli and Tijs (2002) and Brânzei, Solymosi and Tijs (2005)) show several applications of peer group games, being games with a permission strcutrure where the permission structure is a rooted tree and the game is an additive game, such as the auction games of Graham, Marshal and Richard (1990) and the ATM-games of Bjorndal, Hamers and Koster (2004). Other appplications are, for example, the polluted river problems of Ni and Wang (2007) and Dong, Ni and Wang (2012), mentioned in Footnote 11.[27] Other approaches to additive games on a rooted tree can be found in, e.g the hierarchical ventures in Hougaard, Moreno-Ternero, Tvede and Osterdal (2017).

A way to generalize peer group games is keeping a rooted permission tree but allowing any game giving the permission tree games as mentioned in the paragraph after Theorem 4.10. A special case of a permission tree game that is not a peer group game, are the hierarchically structured firms in van den Brink (2008) being games with a permission structure where the permission structure is a rooted tree, and the game is any convex game on the 'lowest level' of the hierarchy (i.e. all players that are not on the lowest level are null players in the game), with the interpretation that the lowest level players are workers that generate value, while the other players are managers who do not physically produce, but organize the production process. The above mentioned applications are just some of the applications of games on communication or hierarchy networks that one can find in the literature.

The usefulness of the Myerson graph game approach is also shown in van den Brink and Pintér (2015) who show that axiomatizations of the Shapley value for TU-games are

[27]Although they are not games with a permission structure, the airport games of Littlechild and Owen (1973), respectively, the joint liability problems of Dehez and Ferey (2013) are the duals of auction games, respectively polluted river games. This also makes the graph game approach to these applications useful, in particular for self-dual solutions such as the Shapley value. A comparison between these four applications, based on anti-duality relations, can be found in Oishi, Nakayama, Hokari and Funaki (2016).

invalid on the class of assignment games (see Shapley and Shubik (1972)) in the sense that they do not give uniqueness, but Myerson's component efficiency and fairness do characterize the Shapley value for assignment games (when we consider these games as restriction to the bipartite graph where two players are linked if and only if one is a seller and the other is a buyer).

An important relation between network structures that we did not consider in this survey is the duality relation. The *dual structure* of a network structure $\mathcal{F} \subseteq 2^N$ is the set system $\mathcal{F}^d$ given by $\mathcal{F}^d = \{S \subseteq N \mid N \setminus S \in \mathcal{F}\}$. It is, for example, well known that the dual of an antimatroid is a convex geometry, see Footnote 6. Algaba, van den Brink and Dietz (2018) show that the dual structures of accessible union stable network structures form a class of network structures that contains all convex geometries.[28]

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[28]Specifically, they satisfy feasible empty set, augmentation', and a weaker intersection property requiring that $S, T \in \mathcal{F}$ with $S \cup T \neq N$, implies that $S \cap T \in \mathcal{F}$.

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