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Equal Loss under Separatorization and Egalitarian Values

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Abstract

We characterize the equal division value, the equal surplus division value, and the class of their affine combinations for TU-games involving *equal loss under separatorization*. This axiom requires that, if a player becomes a dummifying player (Casajus and Huettner, in Economics Letters 122(2): 167–169, 2014), then any two other players are equally affected.

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1. Introduction

Cooperative game theory provides a mathematical framework for allocating the worth generated by a group of cooperating players. A *cooperative game with transferable utility* (TU-game henceforth) consists of a set of players and a characteristic function that specifies a worth for each coalition of players. The equal division (ED) value and the equal surplus division (ESD) value are two well-known egalitarian values for TU-games. In particular, the ED value, the ESD value, and the class of their affine combinations have been given a number of axiomatic characterizations. This paper develops new characterizations.

Our characterizations involve a new axiom relying on the separatorization due to Zou et al. (2020). Separatorization of a player refers to the complete loss of productive potential of cooperation, in the sense that the worth of any coalition containing this player equals the sum of the stand-alone worths of the players in this coalition, while the worth of any coalition without her remains unchanged. This operation is in line with 'veto-ification' introduced in van den Brink and Funaki (2009), dummification introduced in Béal et al. (2018), and nullification studied in Béal et al. (2016); Ferrières (2017); Kongo (2018, 2019, 2020). The difference among them lies in which role that a player acts as. Specifically, veto-ification, dummification, nullification, and separatorization, respectively, suppose a player becoming a veto player, a dummy player, a null player, and a separator (also known as a dummifying player in Casajus and Huettner (2014a)) in a TU-game. There exist several axioms which evaluate the consequences of the aforementioned operations in TU-games. Assuming the same change in payoff for all other players under such operation, van den Brink and Funaki (2009) suggest the veto equal loss property for the ED value, and Ferrières (2017) and Kongo (2018) independently suggest the nullified equal loss property for the ED value, the ESD value and the class of their convex combinations. Similarly, we define the axiom of equal loss under separatorization imposing the same requirement, except that a player becomes a separator.

In this paper, we show that equal loss under separatorization and efficiency yield a family of values that all have in common that they equally split the worth of the grand coalition. This family is not identical to the family implied by the axioms of the nullified equal loss property and efficiency as given by Ferrières (2017). We characterize the class of affine combinations of the ED and ESD values by using the two axioms in addition to fairness (van den Brink, 2002) and homogeneity. While Ferrières (2017) characterizes the class of affine as well as convex combinations of the ED and ESD values involving the nullified equal loss property, we highlight that replacing the nullified equal loss property by equal loss under separatorization yields a new characterization. Moreover, parallel to the axiomatic results in Kongo (2018), we provide characterizations of both the ED value and the ESD value.

This paper is organized as follows. Section 2 provides basic definitions and notation. Section 3 introduces the notion of equal loss under separatorization. Section 4 presents main results. Section 5 concludes.

2. Basic definitions and notation

Let $N = \{1, 2, \ldots, n\}$ be a finite and fixed set of *players* such that $n \geq 3$. The cardinality of any set S is denoted by $|S|$ or s. A *TU-game* is a pair (N, v) where N is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. A subset $S \subseteq N$ is a *coalition*, and $v(S)$ is the *worth* of this coalition. The class of all TU-games with player set N is denoted by $\mathcal{G}^N$.

A TU-game (N, v) is *additive* if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$. A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A TU-game (N, v) is *monotone* if $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. The *null game* is the game (N, v^0) given by $v^0(S) = 0$ for all $S \subseteq N$. For $S \subseteq N$, $S \neq \emptyset$, the *unanimity TU-game* (N, u_S) is given by $u_S(T) = 1$ if $T \supseteq S$, and $u_S(T) = 0$ otherwise. Given $(N, v), (N, w) \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, the TU-game $(N, av + bw) \in \mathcal{G}^N$ is given by $(av + bw)(S) = av(S) + bw(S)$ for all $S \subseteq N$.

Player $i \in N$ is a *null player* in (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$; player $i \in N$ is a *separator* or *dummifying player* in (N, v) if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ with $i \in S$; player $i \in N$ is a *nullifying player* in (N, v) if $v(S) = 0$ for all $S \subseteq N$ with $i \in S$. Players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

A *value* on $\mathcal{G}^N$ is a function ψ that assigns a payoff vector $\psi(N, v) \in \mathbb{R}^N$ to every TU-game $(N, v) \in \mathcal{G}^N$. The *equal division value* (ED value) is given by

$$ED_i(N, v) = \frac{1}{n} v(N), \quad \text{for all } (N, v) \in \mathcal{G}^N, i \in N.$$

The *equal surplus division value* (ESD value) (Driessen and Funaki, 1991) is given by

$$ESD_i(N, v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})], \quad \text{for all } (N, v) \in \mathcal{G}^N, i \in N.$$

We recall the following axioms and results.

- **Efficiency, E**. For all $(N, v) \in \mathcal{G}^N$, $\sum_{i \in N} \psi_i(N, v) = v(N)$.
- **Linearity, L**. For all $(N, v), (N, w) \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, $\psi(N, av + bw) = a\psi(N, v) + b\psi(N, w)$.
- **Additivity, A**. For all $(N, v), (N, w) \in \mathcal{G}^N$, $\psi(N, v + w) = \psi(N, v) + \psi(N, w)$.
- **Symmetry, S**. For all $(N, v) \in \mathcal{G}^N$ and all $i, j \in N$ being symmetric in (N, v), $\psi_i(N, v) = \psi_j(N, v)$.
- **Desirability, D**. For all $(N, v) \in \mathcal{G}^N$ and all $i, j \in N$ such that $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, $\psi_i(N, v) \geq \psi_j(N, v)$.
- **Superadditive monotonicity, SM**. For every superadditive and monotone TU-game $(N, v) \in \mathcal{G}^N$ and all $i \in N$, $\psi_i(N, v) \geq 0$.

- **Nullified equal loss property, NEL**. For all $(N,v) \in \mathcal{G}^N$, all $h \in N$ and all $i,j \in N\backslash\{h\}$, $\psi_i(N,v) - \psi_i(N,v_0^h) = \psi_j(N,v) - \psi_j(N,v_0^h)$, where (N,v_0^h) is given by $v_0^h(S) = v(S\backslash\{h\})$ for all $S \subseteq N$.

Theorem 1 (see Ferrières (2017)). *A value ψ on $\mathcal{G}^N$ satisfies* **E**, **NEL**, **L**, *and* **S** *if and only if there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1-\beta)ED$.*

Theorem 2 (see Ferrières (2017)). *A value ψ on $\mathcal{G}^N$ satisfies* **E**, **NEL**, **A**, **D**, *and* **SM** *if and only if there is $\beta \in [0,1]$ such that $\psi = \beta ESD + (1-\beta)ED$.*

- **Null game property, NG**. For the null game $(N,v^0) \in \mathcal{G}^N$ and all $i \in N$, $\psi_i(N,v^0) = 0$.
- **Grand coalition monotonicity, GM** (Casajus and Huettner, 2014b). For all $(N,v),(N,w) \in \mathcal{G}^N$ with $v(N) \geq w(N)$ and all $i \in N$, $\psi_i(N,v) \geq \psi_i(N,w)$.
- **Id+sur monotonicity, ISM** (Yokote and Funaki, 2017). For all $(N,v),(N,w) \in \mathcal{G}^N$ and $i \in N$ such that $v(N) - \sum_{j\in N} v(\{j\}) \geq w(N) - \sum_{j\in N} w(\{j\})$ and $v(\{i\}) \geq w(\{i\})$, $\psi_i(N,v) \geq \psi_i(N,w)$.

Theorem 3 (see Kongo (2018)). *Let ψ be a value on $\mathcal{G}^N$ that satisfies* **E**, **NEL**, *and* **NG**. *Then,*

(i) ψ satisfies **GM** *if and only if $\psi = ED$.*

(ii) ψ satisfies **ISM** *if and only if $\psi = ESD$.*

3. Equal loss under separatorization

Given a TU-game, separatorization (Zou et al., 2020) of a player means that the worth of any coalition containing this player becomes equal to the sum of the stand-alone worths of the players in this coalition. Formally, for $(N,v) \in \mathcal{G}^N$ and $h \in N$, we denote by (N,v^h) the TU-game from (N,v) if player h becomes a separator: For every $S \subseteq N$,

$$v^h(S) = \begin{cases} \sum_{j\in S} v(\{j\}) & \text{if } h \in S, \\ v(S) & \text{otherwise.} \end{cases}$$

Notice that $(v^i)^j = (v^j)^i$ for every pair $i,j \in N$. Thus, for every coalition $S \subseteq N$, (N,v^S), where the players in S became separators, is well-defined and does not depend on the order in which the players become separators.[1] Note that (N,v^N) is the corresponding additive TU-game of (N,v), namely $v^N(S) = \sum_{j\in S} v(\{j\})$ for all $S \subseteq N$.

The following new axiom imposes that if a player becomes a separator, all other players should be affected equally.

[1] Formally, $v^S(T) = \sum_{j\in T} v(\{j\})$ if $T \cap S \neq \emptyset$, and $v^S(T) = v(T)$ otherwise, is obtained by sequentially separatizing the players in S in any order.

- **Equal loss under separatorization, ELS**. For all $(N,v) \in \mathcal{G}^N$, all $h \in N$ and all $i,j \in N\backslash\{h\}$,

$$\psi_i(N,v) - \psi_i(N,v^h) = \psi_j(N,v) - \psi_j(N,v^h). \tag{1}$$

4. Main results

4.1. Axiomatizations of the class of affine combinations of ED and ESD

Before stating the characterizations, we derive a useful property implied by the combination of **E** and **ELS**.

Lemma 1. *If a value ψ on $\mathcal{G}^N$ satisfies* **E** *and* **ELS***, then for all $(N,v) \in \mathcal{G}^N$ and $i \in N$,*

$$\psi_i(N,v) - \psi_i(N,v^N) = \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})]. \tag{2}$$

Proof. The proof is divided into three steps.

Step 1. By **ELS**, (1) is satisfied for any triple of players. Taking $h \in N$ and $i \in N\backslash\{h\}$, summing (1) over $j \in N\backslash\{h\}$ and using **E** yields that for all $(N,v) \in \mathcal{G}^N$, $h \in N$ and $i \in N\backslash\{h\}$,

$$\begin{aligned}\psi_i(N,v) - \psi_i(N,v^h) &= \frac{1}{n-1}\left[\sum_{j\in N\backslash\{h\}} \psi_j(N,v) - \sum_{j\in N\backslash\{h\}} \psi_j(N,v^h)\right] \\ &= \frac{1}{n-1}[v(N) - \psi_h(N,v) - v^h(N) + \psi_h(N,v^h)]. \end{aligned}\tag{3}$$

Step 2. Next, we show that for all $(N,v) \in \mathcal{G}^N$ and $S \subseteq N$ with $1 \le |S| \le n-1$,

$$\psi(N,v^S) = \psi(N,v^N). \tag{4}$$

We derive the assertion by an induction on the number of separators.

Initialization. Since $(N,v^{N\backslash\{h\}}) = (N,v^N)$ for any $h \in N$, then $\psi(N,v^S) = \psi(N,v^N)$ for all $S \subseteq N$ with $|S| = n-1$,

Induction hypothesis (**IH**). Assume that $\psi(N,v^T) = \psi(N,v^N)$ holds for all $T \subseteq N$ with $|T| = t$, $2 \le t \le n-1$.

Induction step. Consider $(N,v^S) \in \mathcal{G}^N$ and $S \subsetneq N$ such that $|S| = t-1$. Since $v^S(N) = v^{S\cup\{h\}}(N) = \sum_{k\in N} v(\{k\})$ and $v^S(\{k\}) = v^{S\cup\{h\}}(\{k\}) = v(\{k\})$ for all $k \in N$, then by (3) applied to (N,v^S) we obtain that for all $i \neq h$,

$$\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{h\}}) = \frac{1}{n-1}[-\psi_h(N,v^S) + \psi_h(N,v^{S\cup\{h\}})]. \tag{5}$$

Pick any $j \in N \setminus S$ and $i \in N \setminus (S \cup \{j\})$ (which is possible since $|S| \le n-2$). We obtain

$$\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{j\}}) \overset{(5)}{=} \frac{1}{n-1}[-\psi_j(N,v^S) + \psi_j(N,v^{S\cup\{j\}})]$$

$$
\begin{aligned}
&\overset{\textbf{IH}}{=} \frac{1}{n-1}[-\psi_j(N, v^S) + \psi_j(N, v^N)] \\
&\overset{\textbf{IH}}{=} \frac{1}{n-1}[-\psi_j(N, v^S) + \psi_j(N, v^{S\cup\{i\}})] \\
&\overset{5}{=} \frac{1}{n-1}\Big[\frac{1}{n-1}[\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{i\}})]\Big] \\
&\overset{\textbf{IH}}{=} \frac{1}{(n-1)^2}[\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}})].
\end{aligned}
$$

Since $\frac{n^2-2n}{(n-1)^2} \neq 0$, then $\psi_i(N, v^S) = \psi_i(N, v^{S\cup\{j\}})$ for all $i \in N \setminus (S \cup \{j\})$. Pick any $k \in S$. By **ELS**, we have $\psi_k(N, v^S) - \psi_k(N, v^{S\cup\{j\}}) = \psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}}) = 0$, which implies $\psi_k(N, v^S) = \psi_k(N, v^{S\cup\{j\}})$. Since $v^S(N) = v^{S\cup\{j\}}(N)$, **E** then implies $\psi_j(N, v^S) = \psi_j(N, v^{S\cup\{j\}})$. There exists such $j \in N$ for each $S \subsetneq N$, so that $\psi(N, v^S) = \psi(N, v^{S\cup\{j\}}) \overset{\textbf{IH}}{=} \psi(N, v^N)$.

Step 3. By (4), $\psi(N, v^h) = \psi(N, v^N)$ for all $h \in N$. Then (3) implies that for two distinct players $i, h \in N$,

$$
\psi_i(N, v) - \psi_i(N, v^N) = \frac{1}{n-1}[v(N) - \psi_h(N, v) - v^N(N) + \psi_h(N, v^N)].
$$

Summing the above equality over $h \in N\backslash\{i\}$ yields

$$
\begin{aligned}
&(n-1)[\psi_i(N, v) - \psi_i(N, v^N)] \\
=&\frac{1}{n-1}\left[(n-1)[v(N) - v^N(N)] - \sum_{h\in N\backslash\{i\}} (\psi_h(N, v) - \psi_h(N, v^N))\right] \\
\overset{\textbf{E}}{=}&\frac{1}{n-1}\big[(n-2)[v(N) - v^N(N)] + [\psi_i(N, v) - \psi_i(N, v^N)]\big].
\end{aligned}
$$

It follows that $\frac{n(n-2)}{n-1}[\psi_i(N, v) - \psi_i(N, v^N)] = \frac{n-2}{n-1}[v(N) - v^N(N)]$, which implies (2) since $\frac{n-2}{n-1} \neq 0$ (by $n \geq 3$). □

Remark 1. Lemma 1 indicates that any value on $\mathcal{G}^N$ satisfying **E** and **ELS** is uniquely determined by an efficient value determined on additive TU-games since $v^N(S) = \sum_{j\in S} v(\{j\})$ for all $(N, v) \in \mathcal{G}^N$ and $S \subseteq N$. This means that, **E** and **ELS** in addition to some axiom(s) that determine the payoff allocation for additive TU-games, characterize a unique value on $\mathcal{G}^N$.

Remark 2. Any value with the form of (2) satisfies **ELS**, but need not satisfy **E**. For example, the value $\psi = ED + a$, where $a \in \mathbb{R}^N$ is such that $\sum_{j\in N} a_i \neq 0$, satisfies (2) but not **E**.

To characterize the class of affine combinations of the ED and ESD values, we introduce the well-known axioms of fairness and homogeneity.

- **Fairness, F** (van den Brink, 2002). For all $(N, v), (N, w) \in \mathcal{G}^N$ and all $i, j \in N$ such that i and j are symmetric in (N, w), $\psi_i(N, v + w) - \psi_i(N, v) = \psi_j(N, v + w) - \psi_j(N, v)$.

- **Homogeneity, H**. For all $(N, v) \in \mathcal{G}^N$ and all $c \in \mathbb{R}$, $\psi(N, cv) = c\psi(N, v)$.

Theorem 4. *A value ψ on $\mathcal{G}^N$ satisfies* **E**, **ELS**, **F**, *and* **H** *if and only if there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1-\beta)ED$.*

Proof. Existence is obvious. For the uniqueness part, let ψ be a value on $\mathcal{G}^N$ that satisfies the four axioms. By Lemma 1 and Remark 1, we have to show that $\psi(N,v) = \beta ESD(N,v) + (1-\beta)ED(N,v)$, $\beta \in \mathbb{R}$, for all additive games (N,v). Let $D(N,v) = \{i \in N \mid v(\{i\}) \neq 0\}$. We prove uniqueness by induction on $d(N,v) = |D(N,v)|$.

Initialization. If $d(N,v^0) = 0$, i.e. (N,v^0) is the null game, then **H** implies that $\psi_i(N,v^0) = 0$ for all $i \in N$.

Suppose that $d(N,v) = 1$, i.e. $v = v(\{i\})u_{\{i\}}$. Since any $j,k \in N\backslash\{i\}$ are symmetric in $(N,u_{\{i\}})$, **F** implies that $\psi_j(N,v^0+u_{\{i\}}) - \psi_j(N,v^0) = \psi_k(N,v^0+u_{\{i\}}) - \psi_k(N,v^0)$, and thus $\psi_j(N,u_{\{i\}}) = \psi_k(N,u_{\{i\}})$. **E** then implies that for all $j \in N\backslash\{i\}$,

$$\psi_j(N,u_{\{i\}}) = \frac{1-\psi_i(N,u_{\{i\}})}{n-1}. \tag{6}$$

Next, pick any $i,j \in N$ with $i \neq j$, and consider $(N,-u_{\{i\}})$ and $(N,u_{\{i\}}+u_{\{j\}})$. Since i and j are symmetric in $(N,u_{\{i\}}+u_{\{j\}})$, **F** implies that

$$\psi_i(N,-u_{\{i\}}+u_{\{i\}}+u_{\{j\}}) - \psi_i(N,-u_{\{i\}}) = \psi_j(N,-u_{\{i\}}+u_{\{i\}}+u_{\{j\}}) - \psi_j(N,-u_{\{i\}}).$$

By **H**,

$$\psi_i(N,u_{\{j\}}) + \psi_i(N,u_{\{i\}}) = \psi_j(N,u_{\{j\}}) + \psi_j(N,u_{\{i\}}). \tag{7}$$

Combining (6) with (7) yields

$$\frac{1-\psi_j(N,u_{\{j\}})}{n-1} + \psi_i(N,u_{\{i\}}) = \psi_j(N,u_{\{j\}}) + \frac{1-\psi_i(N,u_{\{i\}})}{n-1}.$$

Since

$$\psi_i(N,u_{\{i\}}) - \frac{1-\psi_i(N,u_{\{i\}})}{n-1} = \frac{(n-1)\psi_i(N,u_{\{i\}}) - 1 + \psi_i(N,u_{\{i\}})}{n-1} = \frac{n.\psi_i(N,u_{\{i\}}) - 1}{n-1},$$

and similar for j, it follows that

$$\psi_i(N,u_{\{i\}}) = \psi_j(N,u_{\{j\}}). \tag{8}$$

According to (8), setting $a = \psi_i(N,u_{\{i\}})$ for all $i \in N$, and $\beta = \frac{na-1}{n-1}$, for $v = v(\{i\})u_{\{i\}}$, we have

$$\begin{aligned}
&\beta\ ESD_i(N,v) + (1-\beta)ED_i(N,v)\\
&= \frac{na-1}{n-1}\left(v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j\in N} v(\{j\})\right)\right) + \frac{n(1-a)}{n-1}\cdot\frac{v(N)}{n}\\
&= \frac{na-1}{n-1}v(\{i\}) + 0 + \frac{1-a}{n-1}v(\{i\})
\end{aligned}$$

$$= v(\{i\})a = v(\{i\}\psi_i(N, u_{\{i\}}) \overset{\mathbf{H}}{=} \psi_i(N, v).$$

By (6) and **H**, $\psi_j(N,v) = \frac{1-a}{n-1} v(\{i\}) = \beta\ ESD_j(N,v) + (1-\beta)ED_j(N,v)$ for all $j \in N \backslash \{i\}$.

Induction hypothesis. Assume that $\psi(N, v')$ is uniquely determined whenever $d(N, v') = k$, $1 \leq k \leq n-1$.

Induction step. Let $(N, v) \in \mathcal{G}^N$ be an additive game such that $d(N, v) = k+1$. Take $h \in D(N, v)$, and consider game (N, v') given by $v' = v - v(\{h\})u_{\{h\}}$. Take a $j \in N \setminus \{h\}$. Then, for all $i \in N \setminus \{j, h\}$, **F** implies that

$$\psi_i(N, v) - \psi_j(N, v) = \psi_i(N, v') - \psi_j(N, v'), \tag{9}$$

where the right-hand side is determined by the induction hypothesis.

Take $g \in D(N, v) \setminus \{h\}$ (which exists since $d(N, v) \geq 2$) and $j \in N \setminus \{g, h\}$ (which exists since $n \geq 3$), and consider $v'' = v - v(\{g\})u_{\{g\}}$. Then **F** implies

$$\psi_h(N, v) - \psi_j(N, v) = \psi_h(N, v'') - \psi_j(N, v''), \tag{10}$$

where the right-hand side is determined by the induction hypothesis.

Finally, **E** implies that

$$\sum_{i \in N} \psi_i(N, v) = v(N). \tag{11}$$

Since the $(n-2)+1+1 = n$ equations (9), (10) and (11) are linearly independent in the n unkown payoffs $\psi_i(N, v)$, these payoffs are uniquely determined.

Thus, the payoffs in any additive game $(N, v) \in \mathcal{G}^N$ are uniquely determined for any choice of $a = \psi_i(N, u_{\{i\}}),\ i \in N$, and thus for any choice of β. Since the corresponding affine combination of the ESD and ED values satisfies the axioms, it must be that $\psi = \beta ESD + (1-\beta)ED$. □

Notice that **L** implies **H**, and **L** and **S** together imply **F**. The following corollary is a direct consequence of Theorem 4.

Corollary 1. A value ψ on $\mathcal{G}^N$ satisfies **E**, **ELS**, **L**, and **S** if and only if there is $\beta \in \mathbb{R}$ such that $\psi = \beta ESD + (1-\beta)ED$.

Remark 3. Under **E**, **L**, and **S**, the nullifying player property (**NFP**) [2] and the dummifying player property (**DFP**) [3] characterize the ED value and the ESD value, respectively. Therefore, the difference among the ED value, the ESD value, and the class of their affine combinations is pinpointed to one axiom.

[2] **Nullifying player property, NFP** (van den Brink, 2007). For all $(N, v) \in \mathcal{G}^N$ and $i \in N$ being a nullifying player in (N, v), $\psi_i(N, v) = 0$.

[3] **Dummifying player property, DFP** (Casajus and Huettner, 2014a). For all $(N, v) \in \mathcal{G}^N$ and $i \in N$ being a dummifying player in (N, v), $\psi_i(N, v) = v(\{i\})$.

Mind that, from Remark 1, under **E** and **ELS**, **DFP** characterizes the ESD value, whereas **NFP** does not characterize the ED value. Consider, for example, $\psi_i(N,v) = \frac{v(N)}{n} + a_i(N,v)v(\{i\})$ for all $i \in N$, where $a\colon \mathcal{G}^N \to \mathbb{R}^N$ is a function such that (i) $a(N,v) = a(N,w)$ of $v(\{i\}) = w(\{i\})$ for all $i \in N$, and (ii) $\sum_{i\in N} a_i(N,v)v(\{i\}) = 0$ for all $(N,v) \in \mathcal{G}^N$. This value also satisfies the three axioms.

We provide a characterization of the class of convex combinations of the ED and ESD values, whose proof is omitted since it is similar to that of Theorem 1 in Ferrières (2017).

Theorem 5. *A value ψ on $\mathcal{G}^N$ satisfies* **E**, **ELS**, **A**, **D**, *and* **SM** *if and only if there is $\beta \in [0,1]$ such that $\psi = \beta ESD + (1-\beta)ED$.*

Corollary 1 and Theorem 5 show that Theorems 1 and 2 are still valid if **NEL** is replaced by **ELS**, although (2) does not coincide with the formula of values satisfying **E** and **NEL** (see Formula (3), Ferrières (2017)).

Remark 4. The axioms invoked in Theorem 4 and Corollary 1 are logically independent:

(i) The value given by $\psi_i(N,v) = 0$ for all $i \in N$, satisfies all axioms except **E**.

(ii) The Shapley value satisfies all axioms except **ELS**.

(iii) The value given by

$$\psi_i(N,v) = \frac{i}{\sum_{j\in N} j}\sum_{j\in N} v(\{j\}) + \frac{1}{n}[v(N) - \sum_{j\in N} v(\{j\})], \quad \text{for all } i \in N, \tag{12}$$

satisfies all axioms except **S** and **F**.

(iv) Let $a \in \mathbb{R}^N$ be such that $\sum_{i\in N} a_i = 0$ and $a \neq 0$. The value given by

$$\psi_i(N,v) = \frac{v(N)}{n} + a_i, \quad \text{for all } i \in N, \tag{13}$$

satisfies all axioms of Theorem 4 except **H**.

(v) The value given by

$$\psi(N,v) = \begin{cases} ED(N,v) & \text{if } v(\{i\}) > 0 \ \text{ for all } \ i \in N; \\ ESD(N,v) & \text{otherwise,} \end{cases} \tag{14}$$

satisfies all axioms of Corollary 1 except **L**.

Remark 5. The axioms invoked in Theorem 5 are logically independent:

(i) The value given by $\psi_i(N,v) = 0$ for all $i \in N$, satisfies all axioms except **E**.

(ii) The Shapley value satisfies all axioms except **ELS**.

(iii) The value defined by (14) satisfies all axioms except **A**.

(iv) The value $\psi = 2ED - ESD$ satisfies all axioms except **D**.

(v) The value $\psi = 2ESD - ED$ satisfies all axioms except **SM**.

4.2. Axiomatizations of the ED value and the ESD value

Notice that nullification of all players in a TU-game leads to the null game, whereas separatorization of all players leads to the corresponding additive TU-game. **NG** requires that all players gain zero for any null game. This axiom is well adapted to the representation of a special allocation among players under nullification, but not separatorization. Thus, **NG** is used in Theorem 3, as well as other axiomatic results in Kongo (2018, 2019). Interestingly, Theorem 3 is still valid when we use **ELS** instead of **NEL**. To show this, we first characterize the ED value using the axiom of nonnegativity.

- **Nonnegativity, N**. For all $(N, v) \in \mathcal{G}^N$ with $v(N) \geq 0$ and all $i \in N$, $\psi_i(N, v) \geq 0$.

Lemma 2. *A value ψ on $\mathcal{G}^N$ satisfies* **E**, **ELS**, *and* **N** *if and only if $\psi = ED$.*

Proof. It is clear that ED satisfies **E**, **ELS**, and **N**. Conversely, suppose that ψ is a value on $\mathcal{G}^N$ that satisfies the three axioms. For any $(N, v) \in \mathcal{G}^N$, consider $(N, w) \in \mathcal{G}^N$ such that $w(\{i\}) = v(\{i\})$ for all $i \in N$ and $w(N) = 0$. By (2) (see Lemma 1) applied to (N, w) and (N, w^N), we have $\psi_i(N, w) - \psi_i(N, w^N) = -\frac{1}{n}\sum_{j \in N} w(\{j\})$ for all $i \in N$. It follows that $\psi_i(N, w^N) = \psi_i(N, w) + \frac{1}{n}\sum_{j \in N} w(\{j\}) \geq \frac{1}{n}\sum_{j \in N} w(\{j\})$, where the last inequality holds from **N**. Then, **E** implies that $\psi_i(N, w^N) = \frac{1}{n}\sum_{j \in N} w(\{j\})$ for all $i \in N$. Since $(N, v^N) = (N, w^N)$, then $\psi_i(N, v^N) = \frac{1}{n}\sum_{j \in N} v(\{j\})$. Again, by (2) but now applied to (N, v) and (N, v^N), we have $\psi_i(N, v) = \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})] + \psi_i(N, v^N) = \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})] + \frac{1}{n}\sum_{j \in N} v(\{j\}) = \frac{1}{n}v(N)$. □

Theorem 6. *Let ψ be a value on $\mathcal{G}^N$ that satisfies* **E**, **ELS**, *and* **NG**. *Then,*

(i) ψ satisfies **GM** *if and only if $\psi = ED$.*

(ii) ψ satisfies **ISM** *if and only if $\psi = ESD$.*

Proof. (i) Existence is obvious. Uniqueness follows from Lemma 2 and the fact that **NG** and **GM** imply **N**.

(ii) Existence is obvious. For the uniqueness part, let ψ be a value on $\mathcal{G}^N$ that satisfies the four axioms. Consider two additive TU-games $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(\{i\}) = w(\{i\})$. By **ISM**, $\psi_i(N, v) = \psi_i(N, w)$, which means that i's payoff depends only on her stand-alone worth. Next, consider the additive TU-game $(N, v') \in \mathcal{G}^N$ such that $v'(\{i\}) = v(\{i\})$ and $v'(\{j\}) = 0$ for all $j \in N \backslash \{i\}$, and let $(N, v^0) \in \mathcal{G}^N$ be the null game. It holds that $\psi_i(N, v) = \psi_i(N, v') \overset{\mathbf{E}}{=} v(\{i\}) - \sum_{j \in N \backslash \{i\}} \psi_j(N, v') = v(\{i\}) - \sum_{j \in N \backslash \{i\}} \psi_j(N, v^0) \overset{\mathbf{NG}}{=} v(\{i\})$. The assertion immediately follows from Remark 1. □

Remark 6. Theorem 6(i) is still valid if **GM** is replaced by coalitional monotonicity in van den Brink (2007), which states that $\psi_i(N, v) \geq \psi_i(N, w)$ for two games $(N, v), (N, w) \in \mathcal{G}^N$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in N$.

Remark 7. The axioms invoked in Theorem 6 are logically independent:

(i) The value given by $\psi_i(N,v) = 0$ for all $i \in N$, satisfies all axioms except **E**.

(ii) The value $\psi_i(N,v) = \frac{i}{\sum_{j\in N} j} v(N)$ for all $i \in N$ satisfies **E**, **NG**, and **GM**, but not **ELS**.

(iii) The value given by $\psi_i(N,v) = v(\{i\}) + \frac{i}{\sum_{j\in N} j}[v(N) - \sum_{j\in N} v(\{j\})]$ for all $i \in N$ satisfies **E**, **NG**, and **ISM**, but not **ELS**.

(iv) The value defined by (13) satisfies **E**, **ELS**, and **GM**, but not **NG**.

(v) The value $\psi = ESD + a$, where $a \in \mathbb{R}^N$ is such that $\sum_{j\in N} a_j = 0$ and $a \neq 0$, satisfies **E**, **ELS**, and **ISM**, but not **NG**.

(vi) The value defined by (12) satisfies all axioms, but neither **GM** nor **ISM**.

5. Conclusion

In this paper, we have proposed the axiom of equal loss under separatorization, and have formulized the family of values satisfying equal loss under separatorization and efficiency. After that, we added other well-known axioms to characterize (i) the class of affine combinations of the ESD and ED values, (ii) the class of convex combinations of the ESD and ED values, (iii) the ED value, and (iv) the ESD value.

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