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Abstract

This work deals with the weighted excesses of players in cooperative games which are obtained by summing up all the weighted excesses of all coalitions to which they belong. We first show that lexicographically minimizing the individual weighted excesses of players gives the same minimal weighted excess for every player. Moreover, we show that the associated payoff vector is the corresponding least square value. Second, we show that minimizing the variance of the players' weighted excesses on the preimputation set, again yields the corresponding least square value. Third, we show that these results give rise to lower and upper bounds for the core payoff vectors and, using these bounds, we define the weighted super core as a polyhedron that contains the core. It turns out that the least square values can be seen as a center of this weighted super core, giving a third new characterization of the least square values. Finally, these lower and upper bounds for the core inspire us to introduce a new solution for cooperative TU games that has a strong similarity with the Shapley value.

Keywords: Individual weighted excess; Prenucleolus; Least square value; Weighted super core; Shapley value

1. Introduction

In cooperative games with transferable utility, the lexicographical framework can provide a wide variety of objectives to be minimized, and it has given rise to an entire class of solution concepts. Two of the most popular solutions, the nucleolus defined by Schmeidler [20] and the prenucleolus proposed by Sobolev [21], are the outcome of a lexicographic minimization procedure over the excess vector that can be connected with any coalition. For any payoff vector, the excess of a coalition is the difference between the coalitional value and the total coalitional allocated payoff, and thus can be seen as a measure of dissatisfaction for the coalition. Since the sum of all excesses is constant over the preimputation set, a decrease of the highest excess will definitely result in the increase of other

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excesses. Therefore, Ruiz [17] introduced the least square prenucleolus which minimizes the variance of the excesses of the coalitions under the assumption that all coalitions are equally important, i.e., all the excesses are given the same weight. Later, Ruiz [18] relaxed this assumption by allowing different weights for different coalitions. Hence, they introduced a function of coalitional weights and studied a family of symmetric values, called the LS family, obtained by minimizing the weighted variance of the excesses of all coalitions. Successively, Derks and Haller [6] considered the weighted excess obtained by multiplying the ordinary excess of each coalition with a coalition specific positive coefficient or weight, and presented the weighted nucleolus.

The solutions mentioned above are based on the excesses of all coalitions, reflecting the dissatisfaction of any coalition. Aiming to evaluate a payoff vector by means of the dissatisfaction of any player, Sakawa and Nishizaki [22] presented the excess of a player by summing up all the excesses of all coalitions which he belongs to, and defined the lexicographical solution in view of the players' excesses. Vanam and Hemachandra [24] took into account the per-capita excess-sum of any player, and proposed the per-capita excess-sum allocation in a TU cost game. Kong et al. [10] defined the concept of the general prenucleolus of cooperative games with fuzzy coalitions which is also based on the players' excesses.

The goal of the current paper is to explore the effect of allowing different weights for different coalitions. We consider the *weighted excess of a player* by summing up all the weighted excesses of coalitions to which he belongs. It can be interpreted as the weighted dissatisfaction of a player with respect to the proposed payoff. Firstly, we show that lexicographically minimizing the weighted excesses of players yields the same weighted excess for every player. Moreover, taking the same weighted excess for all players as in Sakawa and Nishizaki [22] and Molina and Tejada [13], it turns out that the corresponding solution is a least square value as proposed by Ruiz [18]. Second, by minimizing the variance of the weighted excesses of all players, we again obtain the corresponding least square value. This insight leads us to obtain an alternative axiomatic characterization of the least square (LS) family by efficiency and an equal weighted dissatisfaction property. Third, the results above give rise to an upper bound and a lower bound for the core and, using these bounds, we define the *weighted super core*. It is further shown that any least square value is obtained as some center of the corresponding weighted super core for any weight system. Inspired by the midpoint of these two bounds, a Shapley-like value is proposed by assigning to every player in any game its expected weighted marginal contribution. Moreover, this value can be characterized similar to the Shapley value by a weighted efficiency, weighted dummy player property, additivity and symmetry. However, it is not efficient. To obtain an efficient solution, we consider two different methods of normalization, an additive and a multiplicative, respectively raised by Hammer and Holzman [7] and Dubey and Shapley [5]. It turns out that this additive normalization coincides with the ESL-value defined by Ruiz [18].

The rest of this paper is arranged as follows. In Section 2, we recall some related preliminaries about cooperative game theory. Section 3 introduces the individual weighted excess of a player and shows that lexicographically minimizing these excesses yields the corresponding least square value with equal excess for every player. In Section 4, we show that minimizing the variance of the weighted excesses also yields the least square values. In Section 5, we introduce the weighted super core as a polyhedron using core lower and upper bounds that are determined using the insights from the previous sections. We show that the least square value is some kind of center of the corresponding weighted super core. Section 6 introduces a Shapley-like value based on the core bounds determined before. Section 7 concludes with a brief summary.

2. Preliminaries

A *characteristic function game with transferable utility* (TU game for short) is a pair (N, v) consisting of a set $N = \{1, 2, \cdots, n\}$ of n players, and a *characteristic function* $v : 2^N \to R$, such that $v(\emptyset) = 0$. The power set 2^N denotes the set of all subsets or *coalitions* of N. For each coalition $S \subseteq N$, $v(S)$ represents the *worth* that coalition S achieves when its members cooperate. The number of players in any coalition $S \subseteq N$ is denoted by s. The set of all TU games with player set N is denoted by G^N.

In this paper, $x \in \mathbb{R}^n$ will be called a *payoff vector*, and $x(S) = \sum_{i \in S} x_i$ for any coalition. Since the set of players is fixed, we often shortly write v instead of (N, v). For a game v , we say that a payoff vector $x \in \mathbb{R}^n$ is

- *efficient* if $x(N) = v(N)$;
- *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$;
- *coalitionally rational* if $x(S) \geq v(S)$ for all $S \subseteq N$.

A solution is a function φ that assigns to every game $v \in G^N$ a set of n-dimensional payoff vectors. A solution φ is $s$ingle-valued if $\varphi(N, v)$ consists of exactly one payoff vector for every game (N, v). In that case, we usually write it as a function $\varphi : G^N \to \mathbb{R}^n$ with $\varphi(N, v) \in \mathbb{R}^n$ being the unique payoff vector assigned to the game. A single-valued solution is also called a *value*. A payoff vector x is said to be a *preimputation* if it is efficient. A preimputation is called an *imputation* if it is also individually rational. Let $\mathscr{I}^*(N, v)$ and $\mathscr{I}(N, v)$ be the preimputation set and the imputation set, respectively. The *core* of a game (N, v) consists of the set of efficient and coalitionally rational payoff vectors, and is denoted by $\mathscr{C}(N, v)$.

For any payoff vector $x \in \mathbb{R}^n$ and any nonempty coalition S, the *excess* of S at x is

$$e(S, x) = v(S) - x(S).$$

The excess, $e(S, x)$, can be viewed as a measure of the *dissatisfaction* of coalition S with respect to the payoff vector x. The core of a game $v \in G^N$ can be written as

$$\mathscr{C}(N, v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } e(S, x) \leq 0 \;\forall\; S \subseteq N\}.$$

Let $m \in \mathbb{N}$ and consider the m-dimensional vector $\theta(x)$ whose components are arranged in nonincreasing order, that is, $\theta_i(x) \geq \theta_j(x)$, $1 \leq i \leq j \leq m$. The *lexicographic order* $\leq_L$ on $\mathbb{R}^m$ is used to compare payoff vectors as follows: for any $x, y \in \mathbb{R}^m$,

(i) $\theta(x) <_L \theta(y)$: if there exists an integer $1 \leq k \leq m$ such that $\theta_i(x) = \theta_i(y)$ for $1 \leq i < k$, and $\theta_k(x) < \theta_k(y)$.

(ii) $\theta(x) \leq_L \theta(y)$: if either $\theta(x) = \theta(y)$ or $\theta(x) <_L \theta(y)$.

Lexicographically minimizing the excess over the set of imputations (respectively preimputations) gives the so-called nucleolus defined by Schmeidler [20] (respectively prenucleolus proposed by Sobolev [21]) as solution. Instead of minimizing the (coalitional) excesses, in order to better reflect the dissatisfaction of the players themselves, Sakawa and Nishizaki [22] proposed the *excess of a player* at a payoff vector x by summing up all the excesses of coalitions to which he belongs,

$$w(i,x) = \sum_{\substack{S \subseteq N \\ S \ni i}} e(S,x). \tag{1}$$

On the basis of the lexicographical order $\leq_L$, Sakawa and Nishizaki [22] defined the *lexicographical solution*, which minimizes the excesses of all players.

Vanam and Hemachandra [24] took into account the *per-capita excess-sum of player i* at an imputation x, i.e.,

$$pce_i(x) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{s} e(S,x), \tag{2}$$

and proposed the *per-capita excess-sum allocation* of a TU cost game[1], which minimizes the per-capita excess-sum of any player in the lexicographical order $\leq_L$.

Given *weights* $p_S^N > 0$ for $\emptyset \subset S \subset N$ for any game $v \in G^N$ and any payoff vector $x \in \mathscr{I}(N,v)$, Derks and Haller [6] defined the *weighted excess*

$$e^p(S,x) = p_S^N e(S,x) = p_S^N (v(S) - x(S)), \tag{3}$$

and the corresponding weighted nucleolus by lexocographically minimizing the weighted excess.

Since in this paper we take the player set N to be fixed, from now on we will suppress the superindex N and write the weight of coalition S simply as p_S.

Ruiz et al. [18] also regarded games with coalitional weights and restricted their attention to *symmetric* weight systems, which assign the same weight to coalitions of the same size. In that case, a *weight system* $p = (p_S)_{S \subseteq N}$ can be written as $p = (p_s)_{1 \leq s \leq n}$, where $p_S = p_s$ for any $S \subseteq N$ with $|S| = s$. They considered the following minimization problem,

$$\min_{x \in \mathbb{R}^n} \quad \sum_{S \subseteq N} p_s(e(S,x) - \bar{e}(v))^2 \quad \text{s.t.} \quad \sum_{i \in N} x_i = v(N), \tag{4}$$

[1] A cost game is defined similar as a (profit) game, except that the interpretation of the worth of a coalition is the total cost that a coalition of players has to face jointly. Some solutions need to be redefined accordingly, for example the (anti-)core of a cost game is the set of efficient payoff vectors such that no coalition pays more than its own cost.

where

$$\bar{e}(v) = \bar{e}(v, x) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S, x)$$

is the average excess for x, which is constant for any efficient payoff vector.

Given a weight system $p = (p_s)_{1 \leq s \leq n}$, the corresponding *p-least square value* (p-LS value for short), is the value that assigns to every game (N, v) the solution of the minimization problem (4), and is given by

$$\mathrm{LS}_i^p(N, v) = \frac{v(N)}{n} + \frac{1}{n\alpha} \left(n a_i^p(v) - \sum_{j \in N} a_j^p(v) \right), \text{ for any } i \in N, \tag{5}$$

where

$$\alpha = \sum_{s=1}^{n-1} p_s \binom{n-2}{s-1} \text{ and } a_i^p(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_s v(S). \tag{6}$$

A value $\varphi : G^N \to \mathbb{R}^n$ belongs to the *least square family* (LS family for short) if there exists a weight system p such that $\varphi(N, v) = LS^p(N, v)$, for all $v \in G^N$.

In order to establish the basic properties of the solutions of the LS family, Ruiz et al. [18] restated (5) as

$$\mathrm{LS}_i^p(N, v) = \frac{v(N)}{n} + \sum_{\substack{S \subset N \\ S \ni i}} \varrho_s \frac{v(S)}{s} - \sum_{\substack{S \subset N \\ S \not\ni i}} \varrho_s \frac{v(S)}{n - s}, \tag{7}$$

where $\varrho_s = \frac{s(n-s)}{n} \frac{p_s}{\alpha}$.

We recall the following well-known axioms for a solution φ,

- *Efficiency*: For each game $v \in G^N$, $\sum_{i \in N} \varphi_i(N, v) = v(N)$.
- *Symmetry*: For each game $v \in G^N$ and each permutation $\sigma : N \to N$, let $\sigma v \in G^N$ with $\sigma v(S) = v(\sigma(S))$, $S \subseteq N$. Then, $\varphi_{\sigma(i)}(N, \sigma v) = \varphi_i(N, v)$ for all $i \in N$.
- *Linearity*: For every two games $v, w \in G^N$ and $a, b \in \mathbb{R}$, $\varphi(N, av + bw) = a\varphi(N, v) + b\varphi(N, w)$, where $(av + bw)(S) = a \cdot v(S) + b \cdot w(S)$ for all $S \subseteq N$. Particularly, this property is called *additivity* when $a = b = 1$.

Ruiz et al. [18] further showed that a value φ on G^N is efficient, linear and symmetric if, and only if, there exists a unique collection of real constants $\{\varrho_s\}_{s=1,\cdots,n-1}$ such that for every game $v \in G^N$, the payoff vector $(\varphi_i(N, v))_{i \in N}$ is given by formula (7). These values are called ESL-values.

3. p-least square values and the p-weighted excess-sum prenucleolus

In the remaining of this paper, we use a system of weights $p = (p_S)_{S \subseteq N}$ satisfying

$$p_S \geq 0 \text{ for all nonempty coalitions } S \subseteq N \text{ and } p_S > 0 \text{ for some coalition } S \neq N.$$

Inspired by the ideas of Ruiz et al. [18], but considering the individual excess as in Sakawa and Nishizaki [22] (see Eq. (1)) and using weights as in Derks and Haller [6] (see Eq. (3)), we represent the *weighted dissatisfaction* of the players by defining the *weighted excess of any player*.

Definition 1. Given a system of weights p, a game $v \in G^N$, a preimputation $x \in \mathscr{I}^*(N,v)$ and a coalition $S \subseteq N$,

$$w^p(i,x) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_S e(S,x) \tag{8}$$

is called the weighted excess of player i with respect to preimputation x.

The weighted excess of a player is the sum of all the weighted excesses of the coalitions to which he belongs, and as such it may be explained as the weighted dissatisfaction of the player towards the proposed payoff. Notice that the weight system only depends on coalition S and the player set N, and not on the worth of the coalition or the preimputation.

Consider the n-dimensional vector $\theta(w^p(i,x)_{i \in N})$, whose components are arranged in nonincreasing order. Just like the prenucleolus is obtained by lexicographically minimizing the coalitional excesses over all preimputations, we lexicographically minimize the individual weighted excesses over all preimputations.

Definition 2. For any weight system p and any game $v \in G^N$, the p-weighted excess-sum prenucleolus is the set of payoff vectors that lexicographically minimizes the excess $w^p(i,x)$ over the preimputation set

$$\mathscr{P}\mathscr{N}^p(N,v) = \{x \in \mathscr{I}^*(N,v) | \theta(w^p(i,x)_{i \in N}) \leq_L \theta(w^p(i,y)_{i \in N}), \forall y \in \mathscr{I}^*(N,v)\}.$$

If the weight system $p_S = 1$ for all $S \subseteq N$, then the p-weighted excess-sum prenucleolus is the lexicographical solution defined by Sakawa and Nishizaki [22] and lexicographically minimizes (1). The p-weighted excess-sum prenucleolus becomes the per-capita excess-sum allocation of a cost game as defined by Vanam and Hemachandra [24] if the weight system is given by $p_S = \frac{1}{|S|}$, see Eq. (2).

Remark 1. *According to the results in Justman [9], we have the following statements.*

(i) *If $\mathscr{I}^*(N,v)$ is nonempty and compact and if all $w^p(i,x)$, $i \in N$, are continuous, then $\mathscr{P}\mathscr{N}^p(N,v) \neq \emptyset$.*

(ii) *If $\mathscr{I}^*(N,v)$ is convex and all $w^p(i,x)$, $i \in N$, are convex, then $\mathscr{P}\mathscr{N}^p(N,v)$ is convex and $w^p(i,x) = w^p(i,y)$ for all $i \in N$ and all $x, y \in \mathscr{P}\mathscr{N}^p(N,v)$.*

Inspired by the method provided by Peleg and Sudhölter [16], let $y \in \mathscr{I}^*(N,v)$ and define

$$\mathscr{I}'(N,v) = \{x \in \mathscr{I}^*(N,v) \mid e(S,x) \leq \max_{S\subseteq N} e(S,y)\ \forall S \subseteq N\}.$$

Since $\mathscr{I}'(N,v)$ is nonempty, convex and compact, from Remark 1, we obtain that the p-weighted excess-sum prenucleolus is a singleton.

Theorem 1. *Given any weight system p, the p-weighted excess-sum prenucleolus is a singleton for every game.*

The weight system p_S has several interpretations: the probability of coalition S to form; the power of coalition S in the bargaining process; the stability degree of coalition S. Next, as in Ruiz et al. [18], we consider symmetric weight systems $p = (p_s)_{1\leq s\leq n}$ where coalitions of the same size have the same weight. It turns out that in the p-weighted excess-sum prenucleolus, the individual weighted excesses are the same for every player.

Theorem 2. *Let $p = (p_s)_{1\leq s\leq n}$ be a symmetric weight system and let $v \in G^N$. For each $x \in \mathscr{PN}^p(N,v)$ and $i,j \in N$, it holds that*

$$w^p(i,x) = w^p(j,x) = \frac{1}{n}\left(\sum_{k\in N} a_k^p(v) - (\alpha + n\beta)v(N)\right), \tag{9}$$

where $\beta = \sum\limits_{s=2}^{n} p_s\binom{n-2}{s-2}$ and α and $a_i^p(v)$ are given by (6).

Proof. We prove the theorem in four steps.

(i) Recall that $a_i^p(v) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s v(S)$, $i \in N$. For every $x \in \mathscr{I}^*(N,v)$ and $i \in N$, we have

$$\begin{aligned}
w^p(i,x) &= \sum_{\substack{S\subseteq N\\ S\ni i}} p_s(v(S) - x(S)) \\
&= \sum_{\substack{S\subseteq N\\ S\ni i}} p_s v(S) - \sum_{\substack{S\subseteq N\\ S\ni i}} p_s x(S) \\
&= a_i^p(v) - \left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s x_i + \sum_{j\in N\setminus\{i\}} \sum_{\substack{S\subseteq N\\ S\ni i,j}} p_s x_j\right) \\
&= a_i^p(v) - \left(\sum_{s=0}^{n-1}\binom{n-1}{s} p_{s+1} x_i + \sum_{j\in N\setminus\{i\}} \sum_{s=0}^{n-2}\binom{n-2}{s} p_{s+2} x_j\right) \\
&= a_i^p(v) - \sum_{s=1}^{n}\binom{n-1}{s-1} p_s x_i - \sum_{j\in N\setminus\{i\}}\left(\sum_{s=2}^{n}\binom{n-2}{s-2} p_s x_j\right)
\end{aligned}$$

$$\begin{aligned}
&= a_i^p(v) - \sum_{s=2}^{n-1}\binom{n-1}{s-1}p_s x_i - \binom{n-1}{0}p_1 x_i - \binom{n-1}{n-1}p_n x_i \\
&\quad - \sum_{j\in N\setminus\{i\}}\sum_{s=2}^{n}\binom{n-2}{s-2}p_s x_j \\
&= a_i^p(v) - \left(\sum_{s=2}^{n-1}\binom{n-1}{s-1}p_s + p_1 + p_n\right)x_i \\
&\quad - \left(\sum_{s=2}^{n}\binom{n-2}{s-2}p_s\right)\sum_{j\in N\setminus\{i\}} x_j \\
&= a_i^p(v) - \left(\sum_{s=2}^{n-1}\left(\binom{n-2}{s-1}+\binom{n-2}{s-2}\right)p_s + p_1 + p_n\right)x_i \\
&\quad - \left(\sum_{s=2}^{n}\binom{n-2}{s-2}p_s\right)(v(N)-x_i) \\
&= a_i^p(v) - \left(\sum_{s=2}^{n-1}\binom{n-2}{s-1}p_s + p_1 + p_n - \binom{n-2}{n-2}p_n\right)x_i \\
&\quad - \left(\sum_{s=2}^{n}\binom{n-2}{s-2}p_s\right)v(N) \\
&= a_i^p(v) - \left(\sum_{s=1}^{n-1}\binom{n-2}{s-1}p_s\right)x_i - \left(\sum_{s=2}^{n}\binom{n-2}{s-2}p_s\right)v(N) \\
&= a_i^p(v) - \alpha x_i - \beta v(N) \qquad (10)
\end{aligned}$$

(ii) Adding up the individual excesses over all individual players gives

$$\begin{aligned}
\sum_{i\in N} w^p(i,x) &= \sum_{i\in N}(a_i^p(v) - \alpha x_i - \beta v(N)) \\
&= \sum_{i\in N} a_i^p(v) - \alpha\sum_{i\in N} x_i - \beta\sum_{i\in N} v(N) \\
&= \sum_{i\in N} a_i^p(v) - (\alpha + n\beta)v(N). \qquad (11)
\end{aligned}$$

(iii) Next, we show that the individual weighted excess is the same for every player. On the contrary, assume that $x \in \mathscr{PN}^p(N,v)$ such that there are $i,j \in N$, $i \neq j$, with $w^p(i,x) \neq w^p(j,x)$. Without loss of generality, fix $i,j \in N$ with $w^p(i,x) > w^p(j,x)$ and $w^p(i,x) = \max_{k\in N} w^p(k,x)$. Define $c = \frac{w^p(i,x)-w^p(j,x)}{2}$ and then construct a payoff vector x'

meeting

$$w^p(k,x') = \begin{cases} w^p(i,x) - c \text{ if } k = i, \\ w^p(j,x) + c, \text{ if } k = j, \\ w^p(k,x), \text{ if } k \neq i,j. \end{cases} \tag{12}$$

Obviously, $(w^p(i,x') - w^p(i,x)) + (w^p(j,x') - w^p(j,x)) = 0$ and, by Eq. (11)

$$\sum_{k\in N} w^p(k,x') = \sum_{k\in N} w^p(k,x) = \sum_{i\in N} a_i^p(v) - (\alpha + n\beta)v(N).$$

By Eq. (10), we have

$$w^p(i,x') - w^p(i,x) = \alpha(x_i - x_i') \text{ and } w^p(j,x') - w^p(j,x) = \alpha(x_j - x_j').$$

Thus, we have $(x_i - x_i') + (x_j - x_j') = 0$. Therefore, $x' \in \mathscr{I}^*(N,v)$. However, from the construction of the payoff vector x',

$$w^p(j,x') = w^p(i,x') < w^p(i,x)$$

and, for $k \in N \setminus \{i,j\}$,

$$w^p(k,x') = w^p(k,x) \leq w^p(i,x)$$

and $\theta(w^p(k,x')_{k\in N}) <_L \theta(w^p(k,x)_{k\in N})$. This establishes a contradiction to our premise $x \in \mathscr{PN}^p(N,v)$ and, therefore, $w^p(i,x) = w^p(j,x)$ for all $i,j \in N$.

(iv) From (ii) and (iii) above, we can directly derive that the individual weighted excesses for any $x \in \mathscr{I}^*(N,v)$, $i \in N$, are given by

$$w^p(i,x) = \frac{1}{n}\sum_{i\in N} w^p(i,x) = \frac{1}{n}\sum_{j\in N} a_j^p(v) - (\alpha + n\beta)v(N). \qquad \square$$

It turns out that, for every symmetric weight system, the p-weighted excess-sum prenucleolus coincides with the corresponding least square value given by Eq. (5).

Proposition 3. *Let p be a symmetric weight system and $v \in G^N$. Then,*

$$\mathscr{PN}^p(N,v) = LS^p(N,v).$$

In the proof of Proposition 3, we use the following lemma.

Lemma 1. *For every $v \in G^N$ and $i,j \in N$, we have*

(i) $a_i^p(v) - a_j^p(v) = \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})]$.

(ii) $na_i^p(v) - \sum_{j\in N} a_j^p(v) = \sum_{j\in N\setminus\{i\}} \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})]$.

(iii) $a_{-i}^p(v) - a_{-j}^p(v) = \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{j\}) - v(S\cup\{i\})]$.

Proof.

(i)
$$
\begin{aligned}
a_i^p(v) - a_j^p(v) &= \sum_{\substack{S\subseteq N\\ S\ni i}} p_s v(S) - \sum_{\substack{S\subseteq N\\ S\ni j}} p_s v(S)\\
&= \sum_{S\subseteq N\setminus\{i,j\}} [p_{s+1}v(S\cup\{i\}) + p_{s+2}v(S\cup\{i,j\})]\\
&\quad - \sum_{S\subseteq N\setminus\{i,j\}} [p_{s+1}v(S\cup\{j\}) + p_{s+2}v(S\cup\{i,j\})]\\
&= \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})].
\end{aligned}
$$

(ii)
$$
\begin{aligned}
na_i^p(v) - \sum_{j\in N} a_j^p(v) &= \sum_{j\in N}[a_i^p(v) - a_j^p(v)]\\
&= \sum_{j\in N\setminus\{i\}} [a_i^p(v) - a_j^p(v)]\\
&= \sum_{j\in N\setminus\{i\}} \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})],
\end{aligned}
$$

where the last equality follows from part (i).

(iii)
$$
\begin{aligned}
a_{-i}^p(v) - a_{-j}^p(v) &= \sum_{k\in N\setminus\{i\}} a_k^p(v) - \sum_{k\in N\setminus\{j\}} a_k^p(v)\\
&= a_j^p(v) - a_i^p(v)\\
&= \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{j\}) - v(S\cup\{i\})],
\end{aligned}
$$

where the last equality follows from part (i). □

Proof of Proposition 3. For any $x \in \mathscr{I}^*(N,v)$, by Eq. (10), the weighted excess of player i with respect to x,

$$
w^p(i,x) = a_i^p(v) - \alpha x_i - \beta v(N), \tag{13}
$$

is a constant. Let x be a preimputation meeting Eq. (9). Then,

$$
x_i - x_j = \frac{1}{\alpha}[a_i^p(v) - a_j^p(v)] = \frac{1}{\alpha} \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})], \text{ for all } i,j\in N,
$$

where the first equality follows from Eq. (10) and the second equality follows from Lemma 1 (i).

Let us consider the constants $d_{ij}^p = \frac{1}{\alpha}[a_i^p(v) - a_j^p(v)]$, for all $i, j \in N$. It is easily seen that the system $\{d_{ij}^p\}_{i,j\in N}$ satisfies $d_{ii}^p = 0$, $d_{ij}^p = -d_{ji}^p$ and $d_{ij}^p + d_{jk}^p = d_{ik}^p$, for all $i, j, k \in N$. Furthermore, x preserves differences according to $\{d_{ij}^p\}_{i,j\in N}$, i.e., $x^i - x^j = d_{ij}^p$ for all $i, j \in N$. Thus, by Hart and Mas-Colell [8] (Theorem 3.4), there exists a unique efficient payoff vector x that preserves $\{d_{ij}^p\}_{i,j\in N}$ and it is given by

$$x_i = \frac{1}{n}\left(v(N) + \sum_{j\in N} d_{ij}^p\right), \quad \text{and } x_j = x_i - d_{ij}^p.$$

That is, for any $i \in N$,

$$x_i = \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right). \qquad \square$$

As a direct consequence of Lemma 1 (ii), the p-LS value can be written as

$$\mathrm{LS}_i^p(N, v) = \frac{v(N)}{n} + \frac{1}{n\sum\limits_{s=0}^{n-2} p_{s+1}\binom{n-2}{s}} \sum_{j\in N\setminus\{i\}} \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})] \tag{14}$$

This expression makes clear that a least square value assigns to every player i an equal share in the worth of the grand coalition, but corrects this by the average weighted difference in contributions of player i and any other player j to coalitions they do not belong to. (Here, $\sum_{s=0}^{n-2} p_{s+1}\binom{n-2}{s}$ is the sum of the weights put on all coalitions containing i and any other player $j \neq i$.)

We conclude this section with the following remark. Given any weight system p, the solution satisfying that all individual weighted excesses are equal is defined as follows

$$\mathscr{E}\mathscr{S}^p(N, v) = \{x \in \mathscr{I}^*(N, v) | w^p(1, x) = \cdots = w^p(n, x)\}. \tag{15}$$

If the weight system $p_S = 1$ for all $S \subseteq N$, this solution becomes the equalizer solution of a crisp game defined by Molina and Tejada [13].

4. Minimizing the variance of individual weighted excesses

In the previous section, we gave a characterization of the least square values by lexicographically minimizing the (weighted) individual player excesses. In this section, we consider a least square method, but using the individual weighted excesses as considered in Section 3.

Ruiz et al. [18] minimize the sum of squared differences from the coalitional excesses and average excess to obtain the least square values. We now minimize the sum of squared diffferences of the individual weighted excesses and the per capita weighted excess

$\bar{w}(v) = \frac{1}{n}\sum_{i\in N} w^p(i,x) = \frac{1}{n}\left(\sum_{i\in N} a_i^p(v) - (\alpha + n\beta)v(N)\right)$.

Given a symmetric weight system p, we consider the following problem for a game $v \in G^N$:

Problem 1:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x) - \bar{w}(v)\right)^2 \\ \text{s.t.} \quad & \sum_{i\in N} x_i = v(N). \end{aligned}$$

Notice that for $c \in \mathbb{R}$,

$$\sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x) - c\right)^2 = \sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x)\right)^2 + nc^2 - 2c\sum_{i\in N}\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x),$$

where the last summation is constant over the preimputation set since

$$\sum_{i\in N}\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x) = \sum_{i\in N} w^p(i,x) = \sum_{i\in N} a_i^p(v) - (\alpha + n\beta)v(N).$$

As a consequence, substituting $\bar{w}(v)$ in the objective function in Problem 1 by any constant c, the resulting objective function differs only in a constant, and the optimal solution remains unchanged. Particularly, for $c = 0$, the optimal solution of Problem 1 is that of the following problem.

Problem 2:

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x)\right)^2 \\ \text{s.t.} \quad & \sum_{i\in N} x_i = v(N). \end{aligned}$$

This gives another characterization of the least square values.

Theorem 4. *For each symmetric weight system p and for each game $v \in G^N$, the unique solution of Problem 1 is $LS^p(N,v)$.*

Proof. By working out the Hessian matrix, it can easily be checked that the objective function in Problem 2 is strictly convex in $\mathbb{R}^n$. Moreover, it is obvious that the objective function is continuous. Since the feasible set is convex and determined by an equality constraint, there is at most one optimal solution, and the Lagrange conditions are necessary and sufficient for a point to be the optimal solution. Applying Lagrange, it follows that the unique point x satisfying these conditions is given by

$$x_i^p = \frac{v(N)}{n} + \frac{\gamma}{n\alpha^2}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right) + \frac{\beta}{n\alpha^2}\left(na_{-i}^p(v) - \sum_{j\in N} a_{-j}^p(v)\right),\ i \in N, \quad (16)$$

where $\gamma = \sum_{s=1}^{n} p_s\binom{n-1}{s-1}$ and, as previously defined, $\alpha = \sum_{s=1}^{n-1} p_s\binom{n-2}{s-1}$, $a_i^p(v) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s v(S)$, $\beta = \sum_{s=2}^{n} p_s\binom{n-2}{s-2}$ and $a_{-i}^p(v) = \sum_{j\in N\setminus i} a_j^p(v)$. From Lemma 1 (iii) and $\gamma - \beta = \alpha$, Eq. (16) becomes

$$\begin{aligned} x_i &= \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right) \\ &= \frac{v(N)}{n} + \frac{1}{n\sum_{s=0}^{n-2} p_{s+1}\binom{n-2}{s}} \sum_{j\in N\setminus\{i\}} \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})]. \end{aligned}$$

This coincides with Equation (5) of the p-LS value. □

So far, we have seen that p-least square values can be obtained both as the allocation that lexicographically minimizes the individual weighted excesses and as the allocation that minimizes the variance of the individual weighted excesses. Using this, we propose a new axiomatic characterization of the p-LS values which requires equal individual weighted excesses for each player.

- *Equal p-weighted dissatisfaction property*: Let p be a symmetric weight system. The solution φ satisfies equal p-weighted dissatisfaction if for every game $v \in G^N$, $w^p(i, \varphi(v)) = w^p(j, \varphi(v))$ for every $i, j \in N$ with $i \neq j$.

Together with efficiency, this property characterizes the corresponding p-least square value.

Theorem 5. *Let p be a symmetric weight system. A value $\varphi : G^N \to \mathbb{R}^n$ satisfies efficiency and the equal p-weighted dissatisfaction property if, and only if, φ is the p-LS value.*

Proof. It can easily be checked that any value defined by (5) satisfies the two axioms with the corresponding weight system p. To see the converse, let φ be a value satisfying the two axioms for some symmetric weight system p. On the contrary, suppose that there are two different values $\varphi^1(v), \varphi^2(v) \in \mathbb{R}^n$ that verify the two properties. On account of the equal p-weighted dissatisfaction property, it is true that

$$w^p(i, \varphi^1(v)) = w^p(j, \varphi^1(v)) \text{ and } w^p(i, \varphi^2(v)) = w^p(j, \varphi^2(v)) \text{ for any } i, j \in N.$$

Since the sum of the weighted excesses of all players is constant, (see Theorem 2), it holds that

$$w^p(i,\varphi^1(v)) = \frac{1}{n}\left(\sum_{k\in N} a_k^p(v) - (\alpha+n\beta)v(N)\right) = w^p(i,\varphi^2(v)).$$

Moreover, $w^p(i,\varphi^1(v)) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s(v(S)-\varphi^1(S))$ and $w^p(i,\varphi^2(v)) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s(v(S)-\varphi^2(S))$ imply $\sum_{\substack{S\subseteq N\\ S\ni i}} p_s\varphi^1(S) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s\varphi^2(S)$. Since $p_s \geq 0$ for all $1\leq s\leq n$, and $p_s>0$ for at least one $1\leq s<n$, and the equality should hold for every game, it must be $\varphi_i^1(v) = \varphi_i^2(v)$ for any $i\in N$. □

5. The p-LS value as center of the weighted super core

In this section, we consider balanced games, i.e. games $v\in G^N$ with a nonempty core. We denote by G_B^N the class of balanced games on player set N. Let $v\in G_B^N$ and $x\in\mathscr{C}(N,v)$. It is obvious that $e(S,x)\leq 0$ for every $S\subseteq N$. Consequently,

$$w^p(i,x) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x) \leq 0,\ i\in N,$$

which allows us to define lower bounds $lo_i^p(v)$, $i\in N$, for core elements. From (13) in the proof of Proposition 3, it follows that

$$x_i \geq \frac{a_i^p(v)-\beta v(N)}{\alpha} \equiv lo_i^p(v).$$

Besides, summing over all core constraints with $i\notin S$, it holds that

$$\sum_{S\subseteq N\setminus i} p_s x(S) \geq \sum_{S\subseteq N\setminus i} p_s v(S). \tag{17}$$

Since

$$\begin{aligned}
\sum_{S\subseteq N\setminus\{i\}} p_s x(S) &= \sum_{j\in N\setminus\{i\}}\sum_{\substack{S\subseteq N\setminus\{i\}\\ S\ni j}} p_s x_j = \sum_{j\in N\setminus\{i\}}\left(\sum_{s=1}^{n-1} p_s \sum_{\substack{S\subseteq N\setminus\{i\}\\ S\ni j,\ |S|=s}} x_j\right)\\
&= \sum_{j\in N\setminus\{i\}}\sum_{s=1}^{n-1}\binom{n-2}{s-1}p_s x_j = \sum_{s=1}^{n-1}\binom{n-2}{s-1}p_s\sum_{j\in N\setminus\{i\}} x_j\\
&= \alpha\sum_{j\in N\setminus\{i\}} x_j = \alpha(v(N)-x_i),
\end{aligned}$$

by (17), we obtain upper bounds $up_i^p(v)$, $i\in N$, given by

$$x_i \leq \frac{\alpha v(N) - \sum_{S\subseteq N\setminus i} p_s v(S)}{\alpha} \equiv up_i^p(v).$$

This inspires us to define the following set valued solution that contains the core.

Definition 3. For any game $v \in G^N$ and any weight system p, the weighted super core of a game is given by

$$\mathscr{SC}^p(N,v) = \left\{ x \in \mathbb{R}^n \mid \frac{a_i^p(v) - \beta v(N)}{\alpha} \le x_i \le \frac{\alpha v(N) - \sum_{S\subseteq N\setminus i} p_s v(S)}{\alpha},\, i \in N \right\}.$$

Observe that these bounds of the weighted super core have the following properties:

(i) The midpoint of each of these bounds of the weighted super core for $v \in G^N$ is[2]

$$\frac{lo_i^p(v) + up_i^p(v)}{2} = \frac{\sum_{S\subseteq N\setminus\{i\}}(p_{s+1}v(S\cup\{i\}) - p_s v(S)) + (\alpha-\beta)v(N)}{2\alpha}, \text{ for any } i \in N.$$

(ii) The difference between these bounds of the weighted super core of $v \in G^N$ is the same for every player, and is given by[3]

$$lo_i^p(v) - up_i^p(v) = \frac{\sum_{S\subseteq N} p_s v(S) - (\alpha+\beta)v(N)}{\alpha}, \text{ for any } i \in N$$

For the symmetric weight system $p_s = 1$, that is, $\alpha = \beta = 2^{n-2}$, these bounds coincide with those in Vanam and Hemachandra [24]. In this case, the midpoint of each of these bounds gives rise to the Banzhaf value defined by Banzhaf [3].

It is easily seen that the core is contained in the weighted super core of a game (N, v). It turns out that, for every weight system, the vector that equalizes the difference between the realized payoff and the lower bound payoff over all players, is the payoff vector assigned by the corresponding least square value. In this sense, the p-LS value can be seen as the center of the weighted super core.

Theorem 6. *Let $v \in G^N$ and p a weight system. If $x \in \mathbb{R}^n$ with $x_i - lo_i^p(v) = x_j - lo_j^p(v)$ for each $i, j \in N$, then $x = LS^p(N,v)$.*

Proof. Let $i \in N$. Obviously, for $j \in N\setminus\{i\}$, $x_i - lo_i^p(v) = x_j - lo_j^p(v)$ implies

$$x_i - x_j = lo_i^p(v) - lo_j(v) = \frac{a_i^p(v) - \beta v(N)}{\alpha} - \frac{a_j^p(v) - \beta v(N)}{\alpha} = \frac{a_i^p(v) - a_j^p(v)}{\alpha}.$$

Adding over all $j \in N$ and by efficiency of x, $nx_i - v(N) = \frac{na_i^p(v) - \sum_{j\in N} a_j^p(v)}{\alpha}$. Then,

$$x_i = \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right).$$

From (5), we conclude that $x = \mathrm{LS}^p(N,v)$. □

[2]This follows from substituting $a_i^p(v)$ in $lo_i^p(v) + up_i^p(v) = \frac{a_i^p(v)-\beta v(N)}{\alpha} + \frac{\alpha v(N) - \sum_{S\subseteq N\setminus\{i\}} p_s v(S)}{\alpha} = \frac{\sum_{S\subseteq N\setminus\{i\}} p_s v(S) + (\alpha-\beta)v(N) - \sum_{S\subseteq N, S\ni i} p_s v(S)}{\alpha} = \frac{\sum_{S\subseteq N\setminus\{i\}}(p_{s+1}v(S\cup\{i\}) - p_s v(S)) + (\alpha-\beta)v(N)}{\alpha}$.

[3]This follows from substituting $a_i^p(v)$ in $lo_i^p(v) - up_i^p(v) = \frac{a_i^p(v)-\beta v(N)}{\alpha} - \frac{\alpha v(N) - \sum_{S\subseteq N\setminus\{i\}} p_s v(S)}{\alpha} = \frac{\sum_{S\subseteq N\setminus\{i\}} p_s v(S) - (\alpha+\beta)v(N) + \sum_{S\subseteq N, S\ni i} p_s v(S)}{\alpha} = \frac{\sum_{S\subseteq N} p_s v(S) - (\alpha+\beta)v(N)}{\alpha}$.

6. A weighted Shapley-like value

Inspired by the midpoint of the two bounds of the payoff in the *p-weighted super core*, we can define the *p-weighted Shapley value* for a weight system p as

$$\mathscr{SH}_i^p(N,v) = \sum_{S\subseteq N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}(p_{s+1}v(S\cup\{i\}) - p_s v(S)),\ i \in N.$$

Here $\frac{s!(n-s-1)!}{n!}$ is the probability that player i joins coalition S and that the *weighted marginal contribution* $p_{s+1}v(S\cup\{i\}) - p_s v(S)$ is paid to player i for joining coalition S. Hence, the p-weighted Shapley value is the expected weighted contribution of player i in the game $v \in G^N$. Unfortunately, the p-weighted Shapley value need not be efficient. Next, we characterize the p-weighted Shapley value for a symmetric weight system. For this, we need to introduce new properties.

Let p be a symmetric weight system. Player $i \in N$ is called a *p-weighted dummy* in the game $v \in G^N$ if

$$p_{s+1}v(S\cup\{i\}) - p_s v(S) = p_1 v(i).$$

- *p-weighted dummy player property*: For every $v \in G^N$ and every p-weighted dummy player $i \in N$, it holds that $\varphi_i(v) = p_1 v(i)$.
- *p-weighted efficiency*: For every game $v \in G^N$, $\sum_{j\in N}\varphi_j(N,v) = p_n v(N)$.

The following result follows straightforward from the definition of the weighted Shapley value. The proof is therefore omitted.

Proposition 7. $\sum_{j\in N}\mathscr{SH}_j^p(N,v) = p_n v(N)$ *for every* $v \in G^N$ *and every symmetric weight system* $p = (p_s)_{1\le s\le n}$.

Similar as the axiomatization of the Shapley value by efficiency, symmetry, the dummy player property and additivity, we can prove the following.

Theorem 8. *Let p be a symmetric weight system. The p-weighted Shapley value $\mathscr{SH}^p$: $G^N \to \mathbb{R}^n$ is the unique value on G^N with the following four properties: symmetry, weighted dummy player property, additivity and weighted efficiency.*

The proof follows the same lines as the original proof in Shapley [19]. The only difference is that, instead of using unanimity games as a basis for the class of games with player set N, we need to use p-weighted unanimity games, u_T^p, $\emptyset \neq T \subseteq N$, defined as $u_T^p(S) = \frac{1}{p_s}$ if $T \subseteq S$ and $u_T^p(S) = 0$ otherwise. The proof is, therefore, omitted.

Efficiency is a crucial requirement if one is looking for a solution that can be accepted by all the players. This leads us to consider an "efficient normalization" of the p-weighted Shapley value. One can obtain an efficient normalization by adding the same constant to all its components as in Hammer and Holzman [7]. Consequently, the *additive normalized weighted Shapley value* $\widehat{\mathscr{S}\mathscr{H}}_i^p(N,v)$ in G^N is given by

$$\widehat{\mathscr{S}\mathscr{H}}_i^p(N,v) = \mathscr{S}\mathscr{H}_i^p(N,v) + \frac{1}{n}\left(v(N) - \sum_{j\in N}\mathscr{S}\mathscr{H}_j^p(N,v)\right), \text{ for any } i \in N.$$

Actually, this normalized p-weighted Shapley value is the ESL-value proposed by Ruiz et al. [18].

Another possible normalization is to multiply all components by the same constant as in Dubey and Shapley [5], and obtain the multiplicative normalized p-weighted Shapley value

$$\overline{\mathscr{S}\mathscr{H}}_i^p(N,v) = \frac{\mathscr{S}\mathscr{H}_i^p(N,v)}{\sum_{i\in N}\mathscr{S}\mathscr{H}_i^p(N,v)}v(N) = \frac{1}{p_n}\mathscr{S}\mathscr{H}_i^p(N,v),\ i \in N,$$

which is only possible when $p_n > 0$.

We illustrate these solutions with two well-known examples. First, we study a bankruptcy game as introduced in O'Neill [14].

Example 1. A bankruptcy problem is described by a tuple (N, E, d) where a set N of agents have rightful demands, given by $d \in \mathbb{R}_+^N$, over the scarce estate E, that is, $E \leq d(N)$. The associated bankruptcy game, $(N, v_{E,d})$, is defined, for $S \subseteq N$, as $v_{E,d}(S) = \max\{0, E - c(N \setminus S)\}$.

Let (N, E, d) be a bankruptcy problem with $N = \{1, 2, 3\}$, estate $E = 80$, and three claims $d_1 = 30$, $d_2 = 40$, $d_3 = 60$, and consider the associated bankruptcy game $(N, v_{E,d})$ given by $v_{E,d}(\{1\}) = v_{E,d}(\{2\}) = 0$, $v_{E,d}(\{3\}) = 10$, $v_{E,d}(\{1, 2\}) = 20$, $v_{E,d}(\{1, 3\}) = 40$, $v_{E,d}(\{2, 3\}) = 50$, $v_{E,d}(\{1, 2, 3\}) = 80$. Let $p = (p_1, p_2, p_3)$ be a symmetric weight system. Then,

$$\mathrm{LS}^p(N,v) = \left(\frac{80}{3} - \frac{10}{3}\frac{p_1 + 4p_2}{p_1 + p_2}, \frac{70}{3}, \frac{80}{3} + \frac{10}{3}\frac{2p_1 + 5p_2}{p_1 + p_2}\right)$$

and

$$\overline{\mathscr{S}\mathscr{H}}_i^p(N,v) = \left(\frac{80}{3} - \frac{5}{3}\frac{p_1 + 4p_2}{p_3}, \frac{80}{3} - \frac{5}{3}\frac{p_1 + p_2}{p_3}, \frac{80}{3} + \frac{5}{3}\frac{2p_1 + 5p_2}{p_3}\right).$$

Next, we compare these solutions with the allocations proposed by some well-known bankruptcy rules: the constrained equal awards, constrained equal losses, and Talmud rules (cf. Auman and Maschler [1], for Talmud rule see contested garment consistent rule); the random arrival rule (cf. O'Neill [14] as recursive completion); and the adjusted proportional rule (cf. Curiel et al. [4]).[4]

$$\mathrm{CEA}(N,E,d)=\left(\frac{80}{3},\frac{80}{3},\frac{80}{3}\right),\ \mathrm{CEL}(N,E,d)=\left(\frac{40}{3},\frac{70}{3},\frac{130}{3}\right),$$

$$\mathrm{Tal}(N,E,d)=\left(15,\frac{45}{2},\frac{85}{2}\right),\ \mathrm{RA}(N,E,d)=\left(\frac{55}{3},\frac{70}{3},\frac{115}{3}\right),$$

$$\text{and } \mathrm{AP}(N,E,d)=\left(\frac{35}{2},\frac{70}{3},\frac{235}{6}\right).$$

Moreover, for the 3-person bankruptcy game with the weight system p, it holds that

$$\mathrm{LS}^p(N,v)=\begin{cases}\mathrm{CEL}(N,E,d) & \text{if } p_1=0,\\ \mathrm{RA}(N,E,d) & \text{if } p_1=p_2,\ p_1\neq 0,\\ \mathrm{AP}(N,E,d) & \text{if } p_2=\frac{7}{5}p_1,\ p_1\neq 0,\end{cases}$$

and

$$\overline{\mathscr{SH}}^p_i(N,v)=\begin{cases}\mathrm{CEA}(N,E,d) & \text{if } p_1=p_2=0,\\ \mathrm{CEL}(N,E,d) & \text{if } p_3=\frac{1}{2}p_2,\ p_2\neq 0,\ p_1=0,\\ \mathrm{Tal}(N,E,d) & \text{if } p_3=p_1=\frac{2}{3}p_2,\ p_2\neq 0,\\ \mathrm{RA}(N,E,d) & \text{if } p_3=p_1=p_2,\ p_2\neq 0,\\ \mathrm{AP}(N,E,d) & \text{if } p_3=\frac{6}{5}p_1,\ p_2=\frac{7}{5}p_1,\ p_1\neq 0.\end{cases}$$

Next, we consider an airport game as introduced in Littlechild and Owen [11].

Example 2. In an airport problem, a group of aircrafts need different landing lengths which have different associated costs. Smaller aircrafts can use the same runway as bigger aircrafts, but not the other way around. Let $C_1,\ldots,C_n$ represent the costs associated to the different types of aircrafts, with $C_1\leq C_2\leq\ldots\leq C_n$. Littlechild and Owen [11] modelled the corresponding allocation cost problem using an associated cost game defined by $c(S)=\max\{C_i|i\in S\}$ for each $S\subseteq N$.

Let (N,C) be an airport problem with $N=\{1,2,3\}$, three different needs on runways, and three costs C_1, C_2, C_3, $C_1\leq C_2\leq C_3$, and consider the associated airport game (N,c) given by $c(\{1\})=C_1$, $c(\{2\})=c(\{1,2\})=C_2$, $c(\{3\})=c(\{1,3\})=c(\{2,3\})=c(\{1,2,3\})=C_3$. Let $p=(p_1,p_2,p_3)$ be a symmetric weight system. Then, the components of the p-LS value are

$$x_1=\frac{C_3}{3}+\frac{1}{3(p_1+p_2)}\left[p_1(2C_1-C_3-C_2)+p_2(C_2-C_3)\right],$$

[4]Let (N,E,d) be a bankruptcy problem. The constrained equal awards rule is defined by $\mathrm{CEA}_i(N,E,d)=\min\{\alpha,d_i\}$ for each $i\in N$ with α such that $\sum_{i\in N}\mathrm{CEA}_i(N,E,d)=E$; the constrained equal losses rule is defined by $\mathrm{CEL}_i(N,E,d)=\max\{0,d_i-\beta\}$ for each $i\in N$ with β such that $\sum_{i\in N}\mathrm{CEL}_i(N,E,d)=E$; the Talmud rule is defined by $\mathrm{Tal}(N,E,d)=\mathrm{CEA}(N,E,\frac{1}{2}d)$ if $\sum_{i\in N}\frac{d_i}{2}\geq E$ and $\mathrm{Tal}(N,E,d)=\frac{1}{2}d+\mathrm{CEL}(N,E-\frac{1}{2}d(N),\frac{1}{2}d)$ otherwise; the random arrival rule is defined by $\mathrm{RA}_i(N,E,d)=\sum_{S\subseteq N\setminus\{i\}}\frac{s!(n-s-1)!}{n!}\min\{d_i,\max\{0,E-d(S)\}\}$ for each $i\in N$.

$$x_2 = \frac{C_3}{3} + \frac{1}{3(p_1+p_2)}\left[p_1(2C_2 - C_3 - C_1) + p_2(C_2 - C_3)\right],$$
$$x_3 = \frac{C_3}{3} + \frac{1}{3(p_1+p_2)}\left[p_1(2C_3 - C_2 - C_1) + 2p_2(C_3 - C_2)\right],$$

and the components of the p-weighted Shapley value are

$$x_1 = \frac{C_3}{3} + \frac{p_1(2C_1 - C_2 - C_3) + p_2(C_2 - C_3)}{6p_3},$$
$$x_2 = \frac{C_3}{3} + \frac{p_1(2C_2 - C_1 - C_3) + p_2(C_2 - C_3)}{6p_3},$$
$$x_3 = \frac{C_3}{3} + \frac{p_1(2C_3 - C_1 - C_2) + p_2(2C_3 - 2C_2)}{6p_3}.$$

Next, we compare these solutions with the allocation proposed by some well-known rules: the sequential equal contributions rule (cf. Littlechild and Owen [11], for the sequential equal contributions rule see the Shapley value of the airport problem), the slack maximizer rule (cf. Littlechild [12], for slack maximizer rule see the nucleolus of the airport problem proposed by Albizuri et al. [2]), and the constrained equal benefits rule[5] (cf. Potters [15]).

$$\mathrm{SEC}(N,C) = \left(\frac{C_1}{3}, \frac{C_1}{3} + \frac{C_2 - C_1}{2}, \frac{C_1}{3} + \frac{C_2 - C_1}{2} + C_3 - C_2\right),$$

$$\mathrm{SM}(N,C) = \left(\min\{\frac{C_1}{2}, \frac{C_2}{3}\}, \frac{C_2}{2} - \min\{\frac{C_1}{4}, \frac{C_2}{6}\}, C_3 - \frac{C_2}{2} - \min\{\frac{C_1}{4}, \frac{C_2}{6}\}\right),$$

and

$$\mathrm{CEB}(N,C) = \left(\frac{2C_1 - C_2}{3}, \frac{2C_2 - C_1}{3}, \frac{3C_3 - C_2 - C_1}{3}\right).$$

Furthermore, for the 3-person airport game with the weight system p, it holds that

$$\mathrm{LS}^p(N,v) = \begin{cases} \mathrm{SEC}(N,C) & \text{if } p_1 = p_2 \neq 0, \\ \mathrm{CEB}(N,C) & \text{if } p_2 = 0,\ p_1 \neq 0, \end{cases}$$

$$\mathrm{LS}^p(N,v) = \begin{cases} \mathrm{SM}(N,C) & \text{if } p_1 = \frac{2C_2 - 3C_1}{2C_2 - C_1} p_2 \neq 0, \text{ and } C_1 \leq \frac{2}{3}C_2, \\ \mathrm{SM}(N,C) & \text{if } p_1 p_2 \neq 0, \text{ and } C_1 = C_2, \end{cases}$$

[5]Let (N,C) be an airport problem. The sequential equal contributions rule is defined by $\mathrm{SEC}_i(N,C) = \sum_{k=1}^{i} \frac{C_k - C_{k-1}}{n+1-k}$, for any $i \in N$; the slack maximizer rule with $n \geq 2$ is given inductively by $\mathrm{SM}_i(N,C) = \min_{l=i}^{n-1} \frac{C_l - \sum_{k=0}^{i-1} \mathrm{SM}_k(N,C)}{l-i+2}$, for any $i = 1, \cdots, n-1$ and $\mathrm{SM}_n(N,C) = \mathrm{SM}_{n-1}(N,C) + C_n - C_{n-1}$ beginning with $\mathrm{SM}_0(N,C) = C_0 = 0$; the constrained equal benefits rule is defined by $\mathrm{CEB}_i(N,C) = \max\{c_i - \beta, 0\}$ for each $i \in N$ with $\beta \in \mathbb{R}_+$ such that $\sum_{i\in N} \mathrm{CEB}_i(N,C) = C_n$.

and

$$\overline{\mathscr{S}\mathscr{H}}^p(N,v) = \begin{cases} \mathrm{SEC}(N,C) & \text{if } p_1 = p_2 = p_3 \neq 0, \\ \mathrm{CEB}(N,C) & \text{if } p_1 = 2p_3 \neq 0,\ p_2 = 0, \end{cases}$$

$$\overline{\mathscr{S}\mathscr{H}}^p(N,v) = \begin{cases} \mathrm{SM}(N,C) & \text{if } p_1 = \frac{2C_2-3C_1}{2C_2-2C_1}p_3,\ p_2 = \frac{2C_2-C_1}{2C_2-2C_1}p_3,\ p_3 \neq 0, \\ & \quad \text{and } C_1 \leq \frac{2}{3}C_2, \\ \mathrm{SM}(N,C) & \text{if } p_1p_2 \neq 0, \text{ and } C_1 = C_2. \end{cases}$$

7. Conclusions

In this paper, we gave three characterizations of the least square values for cooperative TU games: (i) by lexicographically minimizing the individual weighted excesses of players, (ii) by minimizing the variance of the players' weighted excesses on the preimputation set, and (iii) by showing that they are a kind of center of the weighted super core defined by certain lower and upper bounds for the core payoff vectors. Based on these lower and upper bounds, we presented a new solution similar to the Shapley value for cooperative TU games. Finally, we illustrate these solutions in two well-known examples that are studied in the literature: bankruptcy games and airport games.

These results not only give more insight in the least square values, specifically regarding the effect of weights assigned to individuals instead of coalitional weights, but also provide inspiration for new solutions such as the p-weighted super core and the p-weighted Shapley value. Some ideas for further investigation are the following. First, we can consider other solutions by dividing the weighted marginal contribution $(p_{s+1}v(S\cup i) - p_s v(S))$ according to other different ratios, such as dividing them equally. Second, since the p-weighted Shapley value is not efficient, we can consider to characterize the (additive or multiplicative) normalized p-weighted Shapley value. Third, we might consider the efficient point on the segment between the lower and upper bounds, similar as the τ-value proposed by Tijs [23] is defined as the efficient point between the minimal right vector and the utopia vector.

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