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Sharing the Surplus and Proportional Values

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Abstract

We introduce a family of proportional surplus division values for TU-games. Each value first assigns to each player a compromise between his stand-alone worth and the average stand-alone worths over all players, and then allocates the remaining worth among the players in proportion to their stand-alone worths. This family contains the proportional division value and the new egalitarian proportional surplus division value as two special cases. We provide characterizations for this family of values, as well as for each single value in this family.

Keywords: Cooperative game, proportional value, surplus sharing, axiomatization, balanced contributions
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1. Introduction

Equal and proportional division are two basic principles in allocation problems. In cooperative games with transferable utility (TU-games), usually these principles are applied to a remainder of the surplus after each individual player is assigned an individual entitlement. For two-player games, this can be formalized in axioms such as *standardness* (assigning each player its stand-alone worth and allocating the surplus equally over all players), *egalitarian standardness* (ignoring individual entitlements and allocating the full worth equally over the players), and *proportional standardness* (allocating the full surplus proportional to the stand-alone worths of the players). For example, the Shapley value (Shapley, 1953) and the equal surplus division value (Driessen and Funaki, 1991) satisfy standardness, the equal division value (axiomatized in van den Brink (2007)) satisfies egalitarian standardness, and various proportional values, such as the proportional

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value (Ortmann, 2000), the proportional Shapley value (Béal et al., 2018; Besner, 2019), and the proper Shapley value (Vorob'ev and Liapunov, 1998; van den Brink et al., 2015) satisfy proportional standardness. The values can be extended to games with more than two players by, for example reduced game consistency or balanced contributions type of axioms that relate payoffs of players in a game with their payoffs in a game on a reduced player set.

There is a large literature on 'equal sharing of the surplus' type of values. In contrast, values that appear to be 'proportional' are studied less, although proportional considerations play a central role in fair division problems as pointed out by a group of economists and academics, e.g., Brams and Taylor (1996); Chun (1988); Moulin (1987, 2004); Thomson (2015); Tijs and Driessen (1986); Young (1995). However, recently there is a growing interest in values that are based on proportionality, such as the values mentioned above.

In this paper, we provide a new family of values, called the *proportional surplus division values* which make a trade-off between a player's stand-alone worth and the average stand-alone worth, and allocate the remainder proportional to the stand-alone worths. Extreme cases of values in our family are the *proportional division value*, shortly denoted by PD value, and the *egalitarian proportional surplus division value*, shortly denoted by EPSD value. The PD value allocates the worth of the grand coalition in proportion to players' stand-alone worth. The EPSD value is a new value that assigns to each player the average stand-alone worth, and then allocates the remainder of the worth of the grand coalition in proportion to players' stand-alone worth. The EPSD value focuses on egalitarianism in allocating the stand-alone worths by first assigning to every player the average of all stand-alone worths, whereas the PD value applies an egocentric principle and first assigns to each player its own stand-alone worth. Both values apply proportionality in the allocation of the remaining surplus. Besides these two extreme values, our family consists of all convex combinations of the PD value and the EPSD value, and thus can be viewed as making a trade-off between egocentrism and egalitarianism. This family of values is in line with a recent and growing literature that combine different allocation principles by considering convex combinations of two extreme values, such as the egalitarian Shapley values (being convex combinations of the Shapley value and equal division value, see Joosten (1996) and van den Brink et al. (2013)), the consensus values (being convex combinations of the Shapley value and equal surplus division value, see Ju et al. (2007)) and the family of convex combinations of the equal division value and the equal surplus division value (axiomatized in, e.g., van den Brink and Funaki (2009); van den Brink et al. (2016); Xu et al. (2015); Ferrières (2017)). Also, our family of values is in line with a recent and growing literature on non-symmetric surplus sharing values, such as the weighted division value (Béal et al., 2015, 2016a), the weighted surplus division value (Calleja and Llerena, 2017, 2019), the weighted equal allocation of non-separable contributions value (Hou et al., 2019), and the

PD value (Zou et al., 2019).

Besides several known axioms from the literature, we introduce new axioms concerning the separatorization of a player. Separatorization[1] refers to a player's obstruction of cooperation in the sense that the worth of any coalition containing him equals the sum of the stand-alone worths of the players in this coalition, while the worth of any coalition without this player remains unchanged. This is not to be confused with dummification as introduced in Béal et al. (2018) (strengthening nullification studied in, e.g., Gómez-Rúa and Vidal-Puga (2010); Béal et al. (2016b); Ferrières (2017)), where a player becomes a dummy player. The first axiom, called *proportional loss under separatorization*, requires that if a player becomes a separator, then all other player's payoff change in proportion to their stand-alone worths. The second axiom, called *proportional balanced contributions under separatorization*, requires that, for any two players, the effects of one of them becoming a separator on the payoff of the other, are proportional to their stand-alone worths.

We identify the consequence of imposing either of the aforementioned axioms in addition to the classical axiom of efficiency. It turns out that the resulting values have all in common that they split the worth of the grand coalition in proportion to players' stand-alone worth. Moreover, any member of this family is uniquely determined by a value defined on additive games (being games where all players are separators and thus the worth of every coalition equals the sum of the stand-alone worths of the players in that coalition). Subsequently, we characterize a family of values for quasi-additive games (being games where the worth of every coalition that does not contain all the players equals the sum of the stand-alone worths of the players in that coalition) by means of known axioms of efficiency, anonymity, no advantageous reallocation, and continuity, which generalizes a remarkable result for rights problem in Chun (1988). By combining the axioms in these results and using weak linearity instead of continuity, the family of affine combinations of the PD and EPSD values is characterized. Adding superadditive monotonicity (Ferrières, 2017) and replacing anonymity by weak desirability, we derive an axiomatization of the family of convex combinations of the PD and EPSD values. Besides, we show how specific values are singled out by using a parametrized axiom which puts a certain lower bound on the payoffs of individual players.

The paper is organized as follows. Section 2 provides some notation and definitions. Section 3 introduces the concept of proportional surplus division values. Section 4 contains the results. Section 5 presents a conclusion. All proofs and the independence of the axioms in the characterization results are provided in an Appendix.

[1]We thank André Casajus for suggesting the names of separator and separatorization at the 15th European Meeting on Game Theory.

2. Preliminaries

2.1. Notation and TU-games

A *cooperative game with transferable utility*, or simply a *(TU-)game*, is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a fixed finite set of *players* and $v : 2^N \to \mathbb{R}$ is a *characteristic function* assigning to each $S \subseteq N$ a *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition*, and $v(S)$ is the reward that coalition S can guarantee itself without the cooperation of the other players. For any non-empty coalition S, let s or $|S|$ be the cardinality of S. We denote by $\mathcal{G}^N$ the set of all games with player set N.

A game (N, v) is *individually positive* if $v(\{i\}) > 0$ for all $i \in N$, and *individually negative* if $v(\{i\}) < 0$ for all $i \in N$, see Béal et al. (2018). We restrict our discussion to the class of all individually positive games and all individually negative games, and denote this class by $\mathcal{G}^N_{nz}$.

A game (N, v) is *additive* if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. A game (N, v) is *quasi-additive*[2] if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subsetneq N$, see Carreras and Owen (2013). A game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A game (N, v) is *monotone* if $v(S) \leq v(T)$ for all $S, T \subseteq N$ with $S \subseteq T$. A game (N, v) is *weakly essential* if $\sum_{i \in N} v(\{i\}) \leq v(N)$. We express the notation $\mathcal{A}^N_{nz}$ (respectively $\mathcal{QA}^N_{nz}$) for the class of all additive games (respectively quasi-additive games) in $\mathcal{G}^N_{nz}$.

Player $i \in N$ is a *separator* in (N, v) if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ with $i \in S$. For game (N, v) and permutation $\pi : N \to N$, the permuted game $(N, \pi v)$ is defined by $\pi v(S) = v(\cup_{i \in S}\{\pi(i)\})$ for all $S \subseteq N$. For $(N, v), (N, w) \in \mathcal{G}^N_{nz}$, the game $(N, v + w) \in \mathcal{G}^N_{nz}$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

2.2. Values

A (point-valued) *solution* or *value* on $\mathcal{G}^N_{nz}$ is a function ψ that assigns a single payoff vector $\psi(N, v) \in \mathbb{R}^N$ to every game $(N, v) \in \mathcal{G}^N_{nz}$. Some well-known values are the following.

The **equal division value** is the value ED on $\mathcal{G}^N_{nz}$ given by

$$ED_i(N, v) = \frac{1}{n} v(N)$$

for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

The **equal surplus division value**, also known as *Centre-of-the-Imputation-Set (CIS)-value* in Driessen and Funaki (1991), is the value ESD on $\mathcal{G}^N_{nz}$ given by

$$ESD_i(N, v) = v(\{i\}) + \frac{1}{n}[v(N) - \sum_{j \in N} v(\{j\})]$$

[2] These are closely related to joint venture situations in Moulin (1987).

for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

The **proportional division value** is the value PD on $\mathcal{G}^N_{nz}$ given by

$$PD_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N)$$

for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

2.3. Axioms

We state some known axioms of values for games. A value ψ satisfies

- **Efficiency**: if $\sum_{i \in N} \psi_i(N, v) = v(N)$ for all $(N, v) \in \mathcal{G}^N_{nz}$.
- **Anonymity**: if $\psi_i(N, v) = \psi_{\pi(i)}(N, \pi v)$ for all $(N, v) \in \mathcal{G}^N_{nz}$, all permutations $\pi : N \to N$ and all $i \in N$.
- **Weak linearity**: if $\psi(N, av + w) = a\psi(N, v) + \psi(N, w)$ for all $(N, v), (N, w) \in \mathcal{G}^N_{nz}$ and all $a \in \mathbb{R}$ such that $(N, av+w) \in \mathcal{G}^N_{nz}$ and there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$.
- **Superadditive monotonicity**: if $\psi_i(N, v) \geq 0$ for every superadditive and monotone game $(N, v) \in \mathcal{G}^N_{nz}$ and all $i \in N$.
- **Weak desirability**: if $\psi_i(N, v) \geq \psi_j(N, v)$ for all $(N, v) \in \mathcal{A}^N_{nz}$ and $i, j \in N$ such that $v(\{i\}) \geq v(\{j\})$.
- **No advantageous reallocation**: if $\sum_{i \in T} \psi_i(N, v) = \sum_{i \in T} \psi_i(N, w)$ for all quasi-additive games $(N, v), (N, w) \in \mathcal{QA}^N_{nz}$ and $T \subseteq N$ such that $v(N) = w(N)$, $\sum_{i \in T} v(\{i\}) = \sum_{i \in T} w(\{i\})$ and $v(\{i\}) = w(\{i\})$ for all $i \in N \backslash T$.
- **Continuity**: if $\psi(N, w_k) \to \psi(N, v)$ for all sequences of games $\{(N, w_k)\}$ and game (N, v) in $\mathcal{QA}^N_{nz}$ such that $(N, w_k) \to (N, v)$.

Efficiency and anonymity are well-known. Weak linearity, introduced in Béal et al. (2018), states that when taking a linear combination of two games, where the ratio $\frac{v(\{i\})}{w(\{i\})}$ is the same for all players, the payoff vector equals the linear combination of the payoff vectors of the two separate games. Superadditive monotonicity, introduced in Ferrières (2017), states that the payoff of each player should be non-negative for superadditive and monotone games. Weak desirability states that if i's contributions are greater than or equal to j's contributions in an additive game, then i should receive at least j's payoff. No advantageous reallocation states that transfers of individual productivities across a subset of players do not affect the total payoffs of this coalition. Continuity states that a small change in the parameters of the game causes only a small change in the payoff. We require the last two axioms only for quasi-additive games. Moulin (1987) and Chun (1988), respectively, used the last two axioms in surplus problems and rights problems, which can be considered as quasi-additive games.

3. The proportional surplus division values

As mentioned in the introduction, in this paper we characterize families of combinations of the PD value and a new value called EPSD value. We begin this section by defining this new value.

The **egalitarian proportional surplus division value** is the value EPSD on $\mathcal{G}_{nz}^N$ given by

$$EPSD_i(N,v) = \frac{1}{n}\sum_{j\in N} v(\{j\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v(N) - \sum_{j\in N} v(\{j\})]$$

for all $(N,v) \in \mathcal{G}_{nz}^N$ and $i \in N$.

Similar as other values mentioned before, the EPSD value is based on the idea of first assigning individual entitlements to the players, and then allocating the remainder of $v(N)$ over all players using an egalitarian or proportionality principle. In the case of the EPSD value, we first assign to every player the average stand-alone worth, and then allocate the remainder proportional to the stand-alone worths. Thus, the individual entitlements reflect egalitarianism in the sense that all stand-alone worths are equally shared among all players. However, discrimination is made in the allocation of the remainder which is allocated proportional to the stand-alone worths.

Table 1 clarifies the difference among the ED, ESD, PD and EPSD values by the way they allocate (i) the sum of all stand-alone worths $\sum_{j\in N} v(\{j\})$, and (ii) the surplus $v(N) - \sum_{j\in N} v(\{j\})$ that is left from the worth of the grand coalition. These are allocated either equally over the players (E-principle) or proportional to their stand-alone worths (P-principle). For instance, whereas the ED, respectively PD, values are the E-principle, respectively P-principle, in both aspects, the ESD value and the EPSD value reflect equal as well as proportional sharing. Specifically, the ESD value allocates the amounts of $\sum_{j\in N} v(\{j\})$ and $v(N) - \sum_{j\in N} v(\{j\})$ by respectively applying the P-principle and the E-principle, while the EPSD value does it the other way around.

In this paper, we consider combinations of the EPSD value and the PD value. For every $\alpha \in \mathbb{R}$, the corresponding value φ^α, called *α-proportional surplus division value*, is defined by

$$\varphi^\alpha(N,v) = \alpha EPSD(N,v) + (1-\alpha)PD(N,v).$$

Table 1: Values and division principles

Values	$\sum_{j\in N} v(\{j\})$		$v(N)-\sum_{j\in N} v(\{j\})$	
	E-principle	P-principle	E-principle	P-principle
ED	$\surd$		$\surd$	
ESD		$\surd$	$\surd$	
PD		$\surd$		$\surd$
$EPSD$	$\surd$			$\surd$

It is straightforward to verify that for every $(N, v) \in \mathcal{G}^N_{nz}$ and every $i \in N$, it holds that

$$\varphi^\alpha_i(N, v) = \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v(N) - \sum_{j\in N} \alpha v(\{j\})]. \tag{1}$$

The value $\varphi^\alpha(N, v)$ first assigns to every player the fraction α of the average stand-alone worth, and then allocates the remainder (which might be negative) proportional to the stand-alone worths.

Alternatively, (1) can be rewritten as follows.

$$\varphi^\alpha_i(N, v) = \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + (1-\alpha)v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}[v(N) - \sum_{j\in N} v(\{j\})]. \tag{2}$$

This formulation makes clear that an α-proportional surplus division value can also be interpreted as first assigning to every player a convex combination of the average and its own stand-alone worth, and then allocating the surplus $v(N) - \sum_{j\in N} v(\{j\})$ proportional to the stand-alone worths. The payoff $\frac{1}{n}\sum_{j\in N} v(\{j\})$ can be viewed as an egalitarian distribution, while the payoff $v(\{i\})$ can be interpreted as an egocentric distribution of the stand-alone worths. Hence, if $\alpha \in [0, 1]$ the value $\varphi^\alpha(N, v)$ can be seen as making a trade-off between egocentrism and egalitarianism, where the coefficient $\alpha \in [0, 1]$ is a measure of the social preference between egocentrism and egalitarianism. In the extreme cases, $\alpha = 1$ yields the EPSD value and reflects that the society prefers egalitarianism, while $\alpha = 0$ yields the PD value and reflects that the society prefers egocentrism.

In what follows, we refer to the class of α-proportional surplus division values as 'proportional surplus division values'.

4. Axiomatizations

4.1. Axiomatizations of the family of proportional surplus division values

In this section, we provide axiomatizations of the family of proportional surplus division values using known axioms that are mentioned in Section 2, and either one of two new axioms. These new axioms are concerned with how a value should respond to the separatorization of a player in a game. Separatorization of a player refers to the complete loss of productive potential of cooperation within any coalition containing this player. More specifically, a new game is constructed, in which the worth of any coalition containing the separator is equal to the sum of the stand-alone worths of the players in this coalition. Formally, for $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$, we denote by (N, v^i) the game in which player i becomes a separator: For every $S \subseteq N$,

$$v^i(S) = \begin{cases} \sum_{j \in S} v(\{j\}) & \text{if } i \in S, \\ v(S) & \text{otherwise.} \end{cases}$$

For $S = \{i_1, i_2, \ldots, i_s\} \subseteq N$, consider the sequence $(v^{i_1}, (v^{i_1})^{i_2}, \ldots, (v^{i_k})^{i_{k+1}}, \ldots, (v^{i_{s-1}})^{i_s})$. Note that $(v^i)^j = (v^j)^i$ for every pair $i, j \in N$, so that (N, v^S), with $v^S = (v^{i_{s-1}})^{i_s}$ in the sequence above, is well defined for every coalition $S \subseteq N$, and does not depend on the order in which the players become separators. Specifically, (N, v^N) is the corresponding additive game of (N, v) such that $v^N(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$.

4.1.1. Proportional loss under separatorization

The first new axiom is *proportional loss under separatorization* and states that, if a player becomes a separator, then any two other players are affected proportionally to their stand-alone worths. Obviously, this axiom is considered only for games with at least three players.

- **Proportional loss under separatorization**. For all $(N, v) \in \mathcal{G}^N_{nz}$, all $h \in N$, and all $i, j \in N \backslash \{h\}$,

$$\frac{\psi_i(N, v) - \psi_i(N, v^h)}{v(\{i\})} = \frac{\psi_j(N, v) - \psi_j(N, v^h)}{v(\{j\})}.$$

Notice that $(N, v^h) \in \mathcal{G}^N_{nz}$ for all $(N, v) \in \mathcal{G}^N_{nz}$ and $h \in N$, since the stand-alone worths do not change when a player becomes a separator. We begin the axiomatic study by uncovering two useful properties implied by the combination of efficiency and proportional loss under separatorization. The first property says that under these two axioms, if two values coincide on the class of quasi-additive games, then they coincide on the class of all games in $\mathcal{G}^N_{nz}$. This also shows the importance of the class of quasi-additive games.

Lemma 1. *Let $|N| \geq 3$. Consider two values ψ and φ satisfying efficiency and proportional loss under separatorization on $\mathcal{G}^N_{nz}$ such that $\psi = \varphi$ on $\mathcal{A}^N_{nz}$. Then $\psi = \varphi$ on $\mathcal{G}^N_{nz}$.*

The proof of this lemma and of all other results can be found in Appendix A.

The second property follows from this lemma and describes a relation between the payoffs of any game in $\mathcal{G}^N_{nz}$ and the game where all players become separators.

Corollary 1. Let $|N| \geq 3$. If a value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency and proportional loss under separatorization, then

$$\psi_i(N,v) - \psi_i(N,v^N) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - v(\{i\}) \tag{3}$$

for all $(N,v) \in \mathcal{G}_{nz}^N$ and $i \in N$.

Remark 1. The converse of Corollary 1 does not hold since a value with the form of (3) need not satisfy efficiency as can be illustrated by the value $\varphi = PD + a$, where $a \in \mathbb{R}^N$ is such that $\sum_{i\in N} a_i \neq 0$, which also satisfies (3) but not efficiency. However, every value of the form given in (3) satisfies proportional loss under separatorization, which follows since applying (3) to (N, v^h) and using the fact that $v^h(N) = \sum_{j\in N} v(\{j\})$, we have $\psi_i(N,v^h) - \psi_i(N,v^N) = 0$. Subtracting this equality from (3) yields $\psi_i(N,v) - \psi_i(N,v^h) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - v(\{i\})$, as desired.

The following theorem characterizes a family of values on a restrictive domain of quasi-additive games. This theorem echoes Theorem 1 in Chun (1988) that shows a similar result for the situation that the sum of all stand-alone worths is nonzero.

Theorem 1. *Let $|N| \geq 3$. A value ψ on $\mathcal{QA}_{nz}^N$ satisfies efficiency, anonymity, no advantageous reallocation, and continuity if and only if there exists a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\psi_i(N,v) = \frac{v(\{i\})v(N)}{\sum_{j\in N} v(\{j\})} - \left(\frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} - \frac{1}{n}\right) g\Big(\sum_{j\in N} v(\{j\}), v(N)\Big) \tag{4}$$

for all $(N,v) \in \mathcal{QA}_{nz}^N$ and $i \in N$.

Remark 2. In Theorem 1, if continuity is replaced by the weaker condition that ψ is continuous at least one point of its domain, then it affects only the properties of g which is no longer required to be continuous, but does not affect equation (4). We refer to Remark 1 in Chun (1988).

The ED, ESD, PD, and EPSD values are members of the family characterized by Theorem 1. They are obtained by setting $g(\sum_{j\in N} v(\{j\}), v(N))$ equal to $v(N)$, $v(N) - \sum_{j\in N} v(\{j\})$, 0, and $\sum_{j\in N} v(\{j\})$, respectively.

Among the values characterized in Theorem 1, only the affine combinations of the PD and EPSD values satisfy proportional loss under separatorization and weak linearity. This result still holds even if the domain $\mathcal{QA}_{nz}^N$ extends to the domain $\mathcal{G}_{nz}^N$.

Theorem 2. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, anonymity, no advantageous reallocation, proportional loss under separatorization, and weak linearity if and only if there is $\alpha \in \mathbb{R}$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

A subfamily of affine combinations of the PD value and the EPSD value on $\mathcal{G}_{nz}^N$ is characterized by imposing superadditive monotonicity.

Theorem 3. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, anonymity, no advantageous reallocation, proportional loss under separatorization, weak linearity, and superadditive monotonicity if and only if there is $\alpha \in [0, \frac{n}{n-1}]$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

Remark 3. Define the following modification of the $EPSD$ value:

$$EPSD'_i(N,v) = \frac{1}{n-1} \sum_{j \in N \setminus \{i\}} v(\{j\}) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} [v(N) - \sum_{j \in N} v(\{j\})].$$

The difference between the $EPSD$ value and $EPSD'$ is that in the last one, each player $i \in N$ first gets the average stand-alone worth over all other players $j \in N \setminus \{i\}$ instead of the average stand-alone worth over all players as in the $EPSD$ value. Then, the family of values characterized in Theorem 3 can also be expressed as $\{\alpha' EPSD' + (1-\alpha')PD \mid \alpha' \in [0,1]\}$ with $\alpha' = \frac{n-1}{n}\alpha$.

We identify the family of convex combinations of the PD value and the EPSD value on $\mathcal{G}_{nz}^N$ by replacing anonymity in Theorem 3 by weak desirability.

Theorem 4. *Let $|N| \geq 3$. A value ψ on $\mathcal{G}_{nz}^N$ satisfies efficiency, no advantageous reallocation, proportional loss under separatorization, weak linearity, superadditive monotonicity, and weak desirability if and only if there is $\alpha \in [0,1]$ such that $\psi = \alpha EPSD + (1-\alpha)PD$.*

The proof uses the following lemma, which reveals that weak desirability together with some of the axioms in Theorem 3 imply anonymity.

Lemma 2. *On $\mathcal{G}_{nz}^N$ with $|N| \geq 3$, efficiency, no advantageous reallocation, proportional loss under separatorization, and weak desirability imply anonymity.*

4.1.2. Proportional balanced contributions under separatorization

Notice that in the results of Section 4.1.1, we had to exclude two-player games. The reason is that proportional loss under separatorization compares the effect on the payoffs of two distinct players by separatorization of yet another (third) player, and thus involves three players. In contrast, we introduce *proportional balanced contributions under separatorization* which states that any two players are affected proportionally to their stand-alone worths if the other becomes a separator.

- **Proportional balanced contributions under separatorization**. For all $(N,v) \in \mathcal{G}_{nz}^N$ and all $i,j \in N$,

$$\frac{\psi_i(N,v) - \psi_i(N,v^j)}{v(\{i\})} = \frac{\psi_j(N,v) - \psi_j(N,v^i)}{v(\{j\})}.$$

Since proportional balanced contributions under separatorization only compares the effect on the payoffs of two players by mutually becoming a separator, and thus involves only two players, it turns out that using this axiom instead of proportional loss under separatorization, Lemma 1 and Corollary 1 can be stated also for two-player games.

Lemma 3. *Consider two values ψ and φ satisfying efficiency and proportional balanced contributions under separatorization on $\mathcal{G}^N_{nz}$ such that $\psi = \varphi$ on $\mathcal{A}^N_{nz}$. Then $\psi = \varphi$ on $\mathcal{G}^N_{nz}$.*

Corollary 2. If a value ψ on $\mathcal{G}^N_{nz}$ satisfies efficiency and proportional balanced contributions under separatorization, then

$$\psi_i(N, v) - \psi_i(N, v^N) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) - v(\{i\})$$

for all $(N, v) \in \mathcal{G}^N_{nz}$ and $i \in N$.

Comparing Corollary 1 and Corollary 2, efficiency together with either proportional loss under separatorization or proportional balanced contributions under separatorization generate the family of values with the same formula. Thus, we can adopt proportional balanced contributions under separatorization instead of proportional loss under separatorization for the axiomatic results given in Section 4.1.1.

Theorem 5. *Let $|N| \geq 3$. Let ψ be a value on $\mathcal{G}^N_{nz}$ that satisfies efficiency, no advantageous reallocation, proportional balanced contributions under separatorization, and weak linearity. Then,*

(i) ψ satisfies anonymity if and only if there is $\alpha \in \mathbb{R}$ such that $\psi = \alpha EPSD + (1 - \alpha)PD$.

(ii) ψ satisfies anonymity and superadditive monotonicity if and only if there is $\alpha \in [0, \frac{n}{n-1}]$ such that $\psi = \alpha EPSD + (1 - \alpha)PD$.

(iii) ψ satisfies weak desirability and superadditive monotonicity if and only if there is $\alpha \in [0, 1]$ such that $\psi = \alpha EPSD + (1 - \alpha)PD$.

Remark 4. Although Lemma 3 and Corollary 2 are valid also for two-player games, in Theorem 5, the restriction $|N| \neq 2$ cannot be omitted since no advantageous reallocation is just a restatement of efficiency if $|N| = 2$. Specifically, if $|N| = 2$ then, for example, the value defined by

$$\psi_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) - v(\{i\}) + \frac{(v(\{i\}))^2}{\sum_{j \in N} (v(\{j\}))^2} \sum_{j \in N} v(\{j\}) \tag{5}$$

satisfies all axioms, but it does not coincide with $\alpha EPSD + (1 - \alpha)PD$ for any $\alpha \in \mathbb{R}$.

Remark 5. In Theorems 2–5, no advantageous reallocation can be replaced by a stronger axiom, called **transfer rationality**, requiring that an additive game is constructed from the initial additive game by transfering individual productivities across the players, then the difference in payoffs for any two players is proportional to the difference in their stand-alone worths.[3] In this way, the restriction $|N| \neq 2$ can be taken out in Theorem 5.

4.2. Axiomatizations of specific proportional surplus division values

We now provide characterizations for specific values from the family of proportional surplus division values. For this, we use a parametrized axiom, depending on $\alpha \in [0,1]$, such that it singles out the corresponding value φ^α from the class $\{\alpha EPSD + (1-\alpha)PD \mid \alpha \in [0,1]\}$.

The *α-egalitarian inessential game property* makes a trade-off between egalitarianism and egocentrism in additive games, by requiring that in such games a fraction α of the worth of the grand coalition is allocated equally over the players, and the players additionally keep the complementary fraction $(1-\alpha)$ of their own stand-alone worth.

- **α-egalitarian inessential game property**. Let $\alpha \in [0,1]$. For every additive game $(N,v) \in \mathcal{A}^N_{nz}$ and all $i \in N$, $\psi_i(N,v) = (1-\alpha)v(\{i\}) + \alpha\frac{v(N)}{n}$.

When $\alpha = 0$ this yields the well-known inessential game property, while $\alpha = 1$ yields equal division for inessential games as introduced in Ferrières (2017). Further, a higher (respectively lower) α reflects a more egalitarian (respectively egocentric) society. Adding this axiom to the axioms of efficiency and proportional loss under separatorization characterizes the corresponding α-proportional surplus division value (except for two-player games).

Theorem 6. *Let $\alpha \in [0,1]$ and $|N| \neq 2$. A value ψ on $\mathcal{G}^N_{nz}$ satisfies efficiency, proportional loss under separatorization, and the α-egalitarian inessential game property if and only if $\psi = \varphi^\alpha$.*

Next, we provide an alternative characterization of a specific proportional surplus division value using another parameterized axiom. For $\alpha \in [0,1]$, we call a game *α-essential* if $\sum_{i\in N} \alpha v(\{i\}) \leq v(N)$. Clearly, for $\alpha = 0$ this boils down to $v(N) \geq 0$, while for $\alpha = 1$ this is weak essentiality. The following axiom imposes a lower bound on the payoffs of players in α-essential games between zero and the average stand-alone worth. Specifically, it requires that each player receives at least a fraction $\alpha \in [0,1]$ of the average stand-alone worth if it is feasible to do so.

[3]**Transfer rationality**. For any additive games $(N,v),(N,w) \in \mathcal{A}^N_{nz}$ such that $\sum_{j\in N} v(\{j\}) = \sum_{j\in N} w(\{j\})$, it holds that $\psi_i(N,v) - \psi_i(N,w) = \beta[v(\{i\}) - w(\{i\})]$ for some $\beta \in \mathbb{R}$ and all $i \in N$.

- **α-reasonable lower bound**. Let $\alpha \in [0,1]$. For every α-essential game $(N,v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, $\psi_i(N,v) \geq \frac{\alpha}{n} \sum_{j\in N} v(\{j\})$.

We compare this axiom with a known lower bound axiom for α-essential games which requires that in such games, every player earns at least a fraction α of its stand-alone worth, see van den Brink et al. (2016).[4]

- **α-individual rationality**. Let $\alpha \in [0,1]$. For every α-essential game $(N,v) \in \mathcal{G}^N_{nz}$ and all $i \in N$, $\psi_i(N,v) \geq \alpha v(\{i\})$.

Notice that α-individual rationality relies on egocentrism and α-reasonable lower bound rests on egalitarianism. In both cases, α can be seen as a social selfish coefficient balancing the preference between egalitarianism and egocentrism. For $\alpha = 0$ both boil down to *nonnegativity*, requiring that $\psi_i(N,v) \geq 0$ for all $i \in N$ and every game $(N,v) \in \mathcal{G}^N_{nz}$ with $v(N) \geq 0$. For $\alpha = 1$, 1-individual rationality is the usual *individual rationality* axiom requiring that in a weakly essential game every player earns at least its stand-alone worth, while 1-reasonable lower bound guarantees every player at least the average stand-alone worth. It turns out that adding α-reasonable lower bound to efficiency and proportional loss under separatorization characterizes the corresponding φ^α, while adding α-individual rationality yields *only* the PD value.

Theorem 7. *Let $\alpha \in [0,1]$ and $|N| \neq 2$. A value ψ on $\mathcal{G}^N_{nz}$ satisfies efficiency, proportional loss under separatorization, and α-reasonable lower bound if and only if $\psi = \varphi^\alpha$.*

Corollary 3. Let $\alpha \in [0,1]$ and $|N| \neq 2$. A value ψ on $\mathcal{G}^N_{nz}$ satisfies efficiency, proportional loss under separatorization, and α-individual rationality if and only if $\psi = PD$.

Theorems 6 and 7 immediately imply axiomatic characterizations of the PD value as well as the EPSD value by taking φ^α with $\alpha = 0, 1$ respectively.

All characterization results in this section still hold by replacing proportional loss under separatorization with proportional balanced contributions under separatorization. In this way, the restriction $|N| \neq 2$ can be taken out.

Remark 6. The difference between the PD value and the proportional Shapley value[5] is pinpointed to one axiom. With Theorem 6, it immediately follows that the PD value is

[4]Recall that α-individual rationality is used to characterize the convex combinations of the ED and ESD values in van den Brink et al. (2016) and Xu et al. (2015).

[5]The *proportional Shapley value* is defined as: $\psi^{PSh}_i(N,v) = \sum\limits_{S\subseteq N, i\in S} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})} \Delta_v(S)$ for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$, where $\Delta_v(S) = v(S) - \sum\limits_{T\subseteq S, T\neq\emptyset} \Delta_v(T)$ is the Harsanyi dividend of S.

characterized by efficiency, proportional balanced contributions under separatorization, and the inessential game property. Béal et al. (2018) offer a characterization of the proportional Shapley value on $\mathcal{G}_{nz}^N$ by employing efficiency, proportional balanced contributions under dummification[6], and the inessential game property.

5. Conclusion

In this paper, we have introduced the family of proportional division surplus values, being the convex combinations of the EPSD and PD values. These values make a trade-off between egalitarianism and egocentrism. Therefore, this is similar in spirit to the literature that combines diffferent economic allocation principles, such as also, for example, the egalitarian Shapley values, the consensus values, or the convex combinations of the equal surplus division value and the equal division value. We provided characterizations for this family of values as well as any member belonging to this family using two parallel axioms on a fixed player set based on player separatorization. The study of other characterizations for the family of proportional surplus division values is left for future research.

Appendix A: Proofs

Let us denote $K(v) = \sum_{j \in N} v(\{j\})$ for all $(N, v) \in \mathcal{G}_{nz}^N$. If no ambiguity is possible, we use K instead of $K(v)$.

Proof of Lemma 1. Suppose that $(N, v) \in \mathcal{G}_{nz}^N$ with $|N| \geq 3$. Denote $D(N, v) = \{i \in N \mid i \text{ is a separator in } (N, v)\}$. We proceed by descending induction on $|D(N, v)|$.

Initialization. For $|D(N, v)| = n$, i.e. all players are separators, (N, v) is an additive game. Then $\psi = \varphi$ by hypothesis. There is no game in which $|D(N, v)| = n - 1$, because if $n-1$ players are separators then the nth one is also a separator. Therefore, $\psi = \varphi$ holds for $|D(N, v)| \geq n - 1$.

Induction hypothesis. Suppose that $\psi(N, v) = \varphi(N, v)$ for all games $(N, v) \in \mathcal{G}_{nz}^N$ such that $|D(N, v)| \geq d$, for $0 < d \leq n - 1$.

Induction step. Consider any game $(N, v) \in \mathcal{G}_{nz}^N$ such that $|D(N, v)| = d - 1$. Since $d \leq n-1$, and thus $d-1 = |D(N, v)| \leq n-2$, then $|N \backslash D(N, v)| \geq 2$. Let h, l be two distinct players in $N \backslash D(N, v)$. For any $i, j \in N \backslash \{h\}$, by proportional loss under separatorization

[6] **Proportional balanced contributions under dummification.** For all $(N, v) \in \mathcal{G}_{nz}^N$ and $i \in N$, it holds that $\frac{\psi_i(N,v) - \psi_i(N,v_d^j)}{v(\{i\})} = \frac{\psi_j(N,v) - \psi_j(N,v_d^i)}{v(\{j\})}$, where $(N, v_d^i) \in \mathcal{G}_{nz}^N$ is the game in which i is *dummified*: $v_d^i(S) = v(S \backslash \{i\}) + v(\{i\})$ for all $S \subseteq N$ with $i \in S$, and $v_d^i(S) = v(S)$ for all $S \not\ni i$.

of ψ and φ,

$$\frac{\psi_i(N,v)-\psi_i(N,v^h)}{v(\{i\})}=\frac{\psi_j(N,v)-\psi_j(N,v^h)}{v(\{j\})}, \tag{6}$$

and

$$\frac{\varphi_i(N,v)-\varphi_i(N,v^h)}{v(\{i\})}=\frac{\varphi_j(N,v)-\varphi_j(N,v^h)}{v(\{j\})}. \tag{7}$$

Since h is a separator in (N,v^h) and not a separator in (N,v), and $D(N,v)\subset D(N,v^h)$, then $|D(N,v^h)|\geq |D(N,v)|+1=d$. The induction hypothesis then implies that

$$\psi_k(N,v^h)=\varphi_k(N,v^h), \quad \text{for all } k\in N. \tag{8}$$

Subtracting (7) from (6) and using (8) yields

$$\psi_i(N,v)-\varphi_i(N,v)=\frac{v(\{i\})}{v(\{j\})}[\psi_j(N,v)-\varphi_j(N,v)].$$

The above equality similarly holds for all $i,j\in N\backslash\{l\}$. Since $|N|\geq 3$, this equality holds for all $i,j\in N$. Then, summing this equality over $i\in N$ and using efficiency, we obtain

$$v(N)-v(N)=\frac{\sum_{i\in N}v(\{i\})}{v(\{j\})}[\psi_j(N,v)-\varphi_j(N,v)].$$

Since $\frac{\sum_{i\in N}v(\{i\})}{v(\{j\})}\neq 0$ for all $(N,v)\in\mathcal{G}^N_{nz}$, it immediately follows that $\psi_j(N,v)=\varphi_j(N,v)$ for all $j\in N$. $\square$

Proof of Corollary 1. Let ψ be a value on $\mathcal{G}^N_{nz}$, $|N|\geq 3$, satisfying efficiency and proportional loss under separatorization. We first present two claims on ψ.

Claim 1. For any $h\in N$, $i\in N\backslash\{h\}$ and any non-empty $S\subseteq N\backslash\{i,h\}$,

$$\psi_i(N,v)-\psi_i(N,v^h)=\tfrac{v(\{i\})}{K-v(\{h\})}[v(N)-\psi_h(N,v)-v^h(N)+\psi_h(N,v^h)], \tag{9}$$

$$\psi_i(N,v^S)-\psi_i(N,v^{S\cup\{h\}})=\tfrac{v(\{i\})}{K-v(\{h\})}[-\psi_h(N,v^S)+\psi_h(N,v^{S\cup\{h\}})]. \tag{10}$$

Proof. Let $(N,v)\in\mathcal{G}^N_{nz}$, $h\in N$ and $i,j\in N\backslash\{h\}$. By proportional loss under separatorization, we have

$$\psi_j(N,v)-\psi_j(N,v^h)=\frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v)-\psi_i(N,v^h)].$$

Summing this equality over $j \in N\backslash\{h\}$ and using efficiency, we have

$$v(N) - \psi_h(N,v) - [v^h(N) - \psi_h(N,v^h)] = \frac{\sum_{j\in N\backslash\{h\}} v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^h)],$$

which implies (9).

Pick any non-empty $S \subseteq N$, and consider $(N, v^S) \in \mathcal{G}^N_{nz}$. Since $v^S(N) = \sum_{k\in N} v(\{k\}) = K$ and $v^S(\{k\}) = v(\{k\})$ for all $k \in N$, then (10) is implied by (9) applied to (N, v^S). □

Claim 2. For all $S \subseteq N$ with $1 \leq |S| \leq n-1$, $\psi(N, v^S) = \psi(N, v^N)$.

Proof. The assertion is obtained by an induction with respect to the number of separators.

Initialization. Since $(N, v^{N\backslash\{h\}}) = (N, v^N)$ for all $h \in N$, we obtain that $\psi(N, v^S) = \psi(N, v^N)$ for all $S \subseteq N$ with $|S| = n-1$,

Induction hypothesis. Assume that $\psi(N, v^T) = \psi(N, v^N)$ holds for all $T \subseteq N$ with $|T| = t$, for some $2 \leq t \leq n-1$.

Induction step. Consider $(N, v^S) \in \mathcal{G}^N_{nz}$ and $S \subsetneq N$ such that $|S| = t-1$. Take $j \in N\backslash S$ and $i \in N \setminus (S \cup \{j\})$ (It is possible since $|S| \leq n-2$). We have

$$\begin{aligned}
&\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}}) \\
\overset{(10)}{=}& \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}[-\psi_j(N, v^S) + \psi_j(N, v^{S\cup\{j\}})] \\
\overset{\text{(IH)}}{=}& \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}[-\psi_j(N, v^S) + \psi_j(N, v^N)] \\
\overset{\text{(IH)}}{=}& \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}[-\psi_j(N, v^S) + \psi_j(N, v^{S\cup\{i\}})] \\
\overset{(10)}{=}& \frac{v(\{i\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})}\Big[\frac{v(\{j\})}{\sum_{k\in N\backslash\{i\}} v(\{k\})}[\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{i\}})]\Big] \\
\overset{\text{(IH)}}{=}& \frac{v(\{i\})v(\{j\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})\sum_{k\in N\backslash\{i\}} v(\{k\})}[\psi_i(N, v^S) - \psi_i(N, v^{S\cup\{j\}})],
\end{aligned}$$

where (IH) represents that the equality holds by the induction hypothesis.

Since $\frac{v(\{i\})v(\{j\})}{\sum_{k\in N\backslash\{j\}} v(\{k\})\sum_{k\in N\backslash\{i\}} v(\{k\})} \neq 1$, we have $\psi_i(N, v^S) = \psi_i(N, v^{S\cup\{j\}})$ for every $i \in N \setminus (S \cup \{j\})$. For any $k \in S$, again by proportional loss under separatorization, we have $\frac{\psi_k(N,v^S)-\psi_k(N,v^{S\cup\{j\}})}{v(\{j\})} = \frac{\psi_i(N,v^S)-\psi_i(N,v^{S\cup\{j\}})}{v(\{i\})}$, which yields $\psi_k(N, v^S) = \psi_k(N, v^{S\cup\{j\}})$. Efficiency then implies that $\psi_j(N, v^S) = \psi_j(N, v^{S\cup\{j\}})$. Since there exists such a j for all $S \subsetneq N$, we conclude that $\psi(N, v^S) = \psi(N, v^{S\cup\{j\}}) \overset{\text{(IH)}}{=} \psi(N, v^N)$. □

Based on Claims 1 and 2, we prove Corollary 1 as follows.

Proof of Corollary 1. For any $i \in N$ and $j \in N \backslash \{i\}$, Claim 2 together with (9) imply that

$$\psi_i(N,v) - \psi_i(N,v^N) = \frac{v(\{i\})}{\sum_{k \in N\backslash\{j\}} v(\{k\})}[v(N) - v^N(N) - \psi_j(N,v) + \psi_j(N,v^N)],$$

which can be rewritten as:

$$\begin{aligned} &[K - v(\{j\})][\psi_i(N,v) - \psi_i(N,v^N)] \\ =&v(\{i\})[v(N) - v^N(N) - (\psi_j(N,v) - \psi_j(N,v^N))]. \end{aligned}$$

Summing the above equality over $j \in N\backslash\{i\}$ yields

$$\begin{aligned} &[(n-1)K - \sum_{j\in N\backslash\{i\}} v(\{j\})][\psi_i(N,v) - \psi_i(N,v^N)] \\ =&v(\{i\})\Bigg[(n-1)[v(N) - v^N(N)] - \sum_{j \in N\backslash\{i\}}(\psi_j(N,v) - \psi_j(N,v^N))\Bigg]. \end{aligned}$$

Using $\sum_{j\in N\backslash\{i\}}(\psi_j(N,v) - \psi_j(N,v^N)) = v(N) - \psi_i(N,v) - v^N(N) + \psi_i(N,v^N)$, which follows from efficiency, we have

$$\begin{aligned} &[(n-2)K + v(\{i\})][\psi_i(N,v) - \psi_i(N,v^N)] \\ =&v(\{i\})\big[(n-2)[v(N) - v^N(N)] + [\psi_i(N,v) - \psi_i(N,v^N)]\big]. \end{aligned}$$

Since $n - 2 \neq 0$, it follows that

$$K[\psi_i(N,v) - \psi_i(N,v^N)] = v(\{i\})[v(N) - v^N(N)],$$

as desired. □

Proof of Theorem 1. It can easily be checked that any value of the form given in (4) satisfies the four axioms on $\mathcal{QA}^N_{nz}$. To prove the 'only if' part, let $(N,v) \in \mathcal{QA}^N_{nz}$ be any game with $|N| \geq 3$. Without loss of generality, we assume that (N,v) is individually positive. Let ψ be a value on $\mathcal{QA}^N_{nz}$ satisfying the four axioms. Also let $i,j \in N$ be two fixed players, and let $\varepsilon \in \mathbb{R}_+$ be any number such that $0 < \varepsilon < \min_{i\in N}\{v(\{i\})\}$.

First, we consider the following quasi-additive games such that the worth of the grand coalition and the sum of all stand-alone worths are identical to those of (N,v):

(i) Consider the game $(N,v_{ij}) \in \mathcal{QA}^N_{nz}$ defined by $v_{ij}(\{i\}) = v(\{i\}) + v(\{j\}) - \varepsilon$, $v_{ij}(\{j\}) = \varepsilon$, $v_{ij}(\{k\}) = v(\{k\})$ for all $k \in N\backslash\{i,j\}$ and $v_{ij}(N) = v(N)$. This involves a transfer from j to i. By no advantageous reallocation we have

$$\psi_i(N,v) + \psi_j(N,v) = \psi_i(N,v_{ij}) + \psi_j(N,v_{ij}). \qquad (11)$$

(ii) Consider the game $(N, v_i') \in \mathcal{QA}_{nz}^N$ defined by $v_i'(\{i\}) = v(\{i\})$, $v_i'(\{j\}) = \sum_{k\in N\setminus\{i\}} v(\{k\}) - (n-2)\varepsilon$, $v_i'(\{k\}) = \varepsilon$ for all $k \in N\setminus\{i,j\}$, and $v_i'(N) = v(N)$. This involves a transfer from the players in $N\setminus\{i,j\}$ to player j. By no advantageous reallocation applied to (N, v) and (N, v_i'), we obtain

$$\sum_{k\in N\setminus\{i\}} \psi_k(N, v) = \sum_{k\in N\setminus\{i\}} \psi_k(N, v_i').$$

Efficiency then implies

$$\psi_i(N, v) = \psi_i(N, v_i'). \tag{12}$$

(iii) Consider the game $(N, v_{\overline{ij}}') \in \mathcal{QA}_{nz}^N$ defined by $v_{\overline{ij}}'(\{i\}) = v(\{j\})$, $v_{\overline{ij}}'(\{j\}) = \sum_{k\in N\setminus\{j\}} v(\{k\}) - (n-2)\varepsilon$, $v_{\overline{ij}}'(\{k\}) = \varepsilon$ for all $k \in N\setminus\{i,j\}$, and $v_{\overline{ij}}'(N) = v(N)$. This game is obtained by first switching roles between i and j in game (N, v) and then making a transfer similar to the one in case (ii). Let π be a permutation such that $\pi(i) = j$, $\pi(j) = i$, and $\pi(k) = k$ for all $k \in N\setminus\{i,j\}$.

Define the game $(N, v_j') \in \mathcal{QA}_{nz}^N$ by $v_j'(\{i\}) = \sum_{k\in N\setminus\{j\}} v(\{k\}) - (n-2)\varepsilon$, $v_j'(\{j\}) = v(\{j\})$, $v_j'(\{k\}) = \varepsilon$ for all $k \in N\setminus\{i,j\}$, and $v_j'(N) = v(N)$. By no advantageous reallocation applied to (N, v) and (N, v_j'), we obtain $\sum_{k\in N\setminus\{j\}} \psi_k(N, v) = \sum_{k\in N\setminus\{j\}} \psi_k(N, v_j')$. Efficiency then implies $\psi_j(N, v) = \psi_j(N, v_j')$. Notice that $(N, v_j') = (N, \pi v_{\overline{ij}}')$. By anonymity, $\psi_j(N, v_j') = \psi_{\pi(i)}(N, \pi v_{\overline{ij}}') = \psi_i(N, v_{\overline{ij}}')$. Therefore,

$$\psi_j(N, v) = \psi_i(N, v_{\overline{ij}}'). \tag{13}$$

(iv) Consider the game $(N, v_{ij}') \in \mathcal{QA}_{nz}^N$ defined by $v_{ij}'(\{i\}) = v(\{i\}) + v(\{j\}) - \varepsilon$, $v_{ij}'(\{j\}) = \sum_{k\in N\setminus\{i,j\}} v(\{k\}) - (n-3)\varepsilon$, $v_{ij}'(\{k\}) = \varepsilon$ for all $k \in N\setminus\{i,j\}$, and $v_{ij}'(N) = v(N)$. This involves a transfer from the players in $N\setminus\{i,j\}$ to player j given game (N, v_{ij}). By no advantageous reallocation applied to (N, v_{ij}) and (N, v_{ij}'), we obtain $\sum_{k\in N\setminus\{i\}} \psi_k(N, v_{ij}) = \sum_{k\in N\setminus\{i\}} \psi_k(N, v_{ij}')$. Efficiency then implies

$$\psi_i(N, v_{ij}) = \psi_i(N, v_{ij}'). \tag{14}$$

(v) Consider the game $(N, v') \in \mathcal{QA}_{nz}^N$ defined by $v'(\{j\}) = \sum_{k\in N} v(\{k\}) - (n-1)\varepsilon$, $v'(\{k\}) = \varepsilon$ for all $k \in N\setminus\{j\}$, and $v'(N) = v(N)$. This involves a transfer from i to j given game (N, v'_{ij}). Let $(N, v'') \in \mathcal{QA}_{nz}^N$ be the game defined by $v''(\{i\}) = \sum_{k\in N} v(\{k\}) - (n-1)\varepsilon$, $v''(\{k\}) = \varepsilon$ for all $k \in N\setminus\{i\}$, and $v''(N) = v(N)$. Clearly, $(N, v'') = (N, \pi v')$ for the permutation such that $\pi(i) = j$, $\pi(j) = i$, and $\pi(k) = k$ for all $k \in N\setminus\{i,j\}$. By anonymity, we obtain $\psi_i(N, v') = \psi_j(N, v'')$. On the other hand, applying no advantageous reallocation to (N, v'') and (N, v_{ij}), and then using efficiency, we obtain $\psi_j(N, v'') = \psi_j(N, v_{ij})$. Therefore,

$$\psi_j(N, v_{ij}) = \psi_i(N, v'). \tag{15}$$

Next, based on (11)-(15), we derive the formula of $\psi_i(N,v)$. Substituting (12)-(15) into (11), we have

$$\psi_i(N,v'_i)+\psi_i(N,v'_{\overline{ij}})=\psi_i(N,v'_{ij})+\psi_i(N,v'). \tag{16}$$

Notice that each game used in (16) is uniquely determined by four parameters: the worth of $\{i\}$, the number ε (which determined the stand-alone worths of players $k\in N\setminus\{i,j\}$), the sum of stand-alone worths $\sum_{k\in N}v(\{k\})$ (which, with ε, determines the stand-alone worth of j), and the worth of the grand coalition $v(N)$. For such game (N,v_0), let $F_i:\mathbb{R}^4\to\mathbb{R}$ be such that $\psi_i(N,v_0)=F_i(v_0(\{i\}),\varepsilon,\sum_{k\in N}v_0(\{k\}),v_0(N))$. Clearly,

$$\begin{cases}\psi_i(N,v'_i) &= F_i(v(\{i\}),\varepsilon,K,v(N)),\\ \psi_i(N,v'_{\overline{ij}}) &= F_i(v(\{j\}),\varepsilon,K,v(N)),\\ \psi_i(N,v'_{ij}) &= F_i(v(\{i\})+v(\{j\})-\varepsilon,\varepsilon,K,v(N)),\\ \psi_i(N,v') &= F_i(\varepsilon,\varepsilon,K,v(N)).\end{cases} \tag{17}$$

Let $c\in\mathbb{R}$ be such that $c-\varepsilon>0$. Let $f:\mathbb{R}^4\to\mathbb{R}$ and $h:\mathbb{R}^3\to\mathbb{R}$ be defined by

$$f(c-\varepsilon,\varepsilon,K,v(N))=F_i(c,\varepsilon,K,v(N))-F_i(\varepsilon,\varepsilon,K,v(N)), \tag{18}$$

and

$$h(\varepsilon,K,v(N))=F_i(\varepsilon,\varepsilon,K,v(N)). \tag{19}$$

(Notice that we surpress the index i at the functions f and h. In fact, as we see later, these functions are the same for every $i\in N$.)

Notice that (16) can be rewritten as

$$\psi_i(N,v'_i)-\psi_i(N,v')+\psi_i(N,v'_{\overline{ij}})-\psi_i(N,v')=\psi_i(N,v'_{ij})-\psi_i(N,v').$$

Taking (17) and (18) into account, we can then write

$$\begin{aligned}&\big[F_i(v(\{i\}),\varepsilon,K,v(N))-F_i(\varepsilon,\varepsilon,K,v(N))\big]+\big[F_i(v(\{j\}),\varepsilon,K,v(N))-F_i(\varepsilon,\varepsilon,K,v(N))\big]\\ =&F_i(v(\{i\})+v(\{j\})-\varepsilon,\varepsilon,K,v(N))-F_i(\varepsilon,\varepsilon,K,v(N)),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&f(v(\{i\})-\varepsilon,\varepsilon,K,v(N))+f(v(\{j\})-\varepsilon,\varepsilon,K,v(N))\\ =&f(v(\{i\})+v(\{j\})-2\varepsilon,\varepsilon,K,v(N))\end{aligned}$$

Since $v(\{i\})+v(\{j\})-2\varepsilon=[v(\{i\})-\varepsilon]+[v(\{j\})-\varepsilon]$, f is additive with respect to its first argument for each ε, K and $v(N)$. By continuity of ψ, f is continuous. Therefore, the theorem on Cauchy's equation (see Corollary 3.1.9, p.51, Eichhorn (1978)) applied to f implies that there exists a continuous function $f_0 : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(c-\varepsilon, \varepsilon, K, v(N)) = (c-\varepsilon) f_0(\varepsilon, K, v(N)). \tag{20}$$

Substituting (20) into (18) and taking $c = v(\{i\})$, it follows that

$$(v(\{i\})-\varepsilon) f_0(\varepsilon, K, v(N)) = F_i(v(\{i\}), \varepsilon, K, v(N)) - F_i(\varepsilon, \varepsilon, K, v(N)).$$

We obtain

$$\begin{aligned} \psi_i(N,v) &= \psi_i(N,v_i') \\ &= F_i(v(\{i\}, \varepsilon, K, v(N)) \\ &= f(v(\{i\})-\varepsilon, \varepsilon, K, v(N)) + F_i(\varepsilon, \varepsilon, K, v(N)) \\ &= (v(\{i\})-\varepsilon) f_0(\varepsilon, K, v(N)) + h(\varepsilon, K, v(N)), \end{aligned} \tag{21}$$

where the first equality follows from (12), the second from (17), the third from (18), and the last from (19) and (20).

Note that (21) holds for all $i \in N$. Summing up these equations over all $i \in N$ and using efficiency, we obtain

$$v(N) = (K - n\varepsilon) f_0(\varepsilon, K, v(N)) + nh(\varepsilon, K, v(N)).$$

It follows that

$$h(\varepsilon, K, v(N)) = \frac{v(N)}{n} - \frac{K}{n} f_0(\varepsilon, K, v(N)) + \varepsilon f_0(\varepsilon, K, v(N)).$$

The above equation and (21) yield

$$\psi_i(N,v) = v(\{i\}) f_0(\varepsilon, K, v(N)) + \frac{v(N)}{n} - \frac{K}{n} f_0(\varepsilon, K, v(N)). \tag{22}$$

Taking any two positive numbers $\varepsilon_1, \varepsilon_2 < \min_{i \in N}\{v(\{i\})\}$, (22) yields $v(\{i\}) f_0(\varepsilon_1, K, v(N)) + \frac{v(N)}{n} - \frac{K}{n} f_0(\varepsilon_1, K, v(N)) = \psi_i(N,v) = v(\{i\}) f_0(\varepsilon_2, K, v(N)) + \frac{v(N)}{n} - \frac{K}{n} f_0(\varepsilon_2, K, v(N))$, and thus it must be that $(v(\{i\}) - \frac{K}{n})(f_0(\varepsilon_1, K, v(N)) - f_0(\varepsilon_2, K, v(N))) = 0$. Notice that f_0 is a function with respect to $\varepsilon, K, v(N)$, then it is possible to take a game with $v(\{i\}) - \frac{K}{n} \neq 0$, and thus $f_0(\varepsilon_1, K, v(N)) = f_0(\varepsilon_2, K, v(N))$. This means that the number $f_0(\varepsilon, K, v(N))$ does not depend on ε (belonging to its domain). Then, let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined by $f_0(\varepsilon, K, v(N)) = \frac{v(N)}{K} - \frac{1}{K} g(K, v(N))$. Using this function, (22) can be rewritten as

$$\begin{aligned} \psi_i(N,v) &= \frac{v(\{i\})v(N)}{K} - \frac{v(\{i\})}{K} g(K, v(N)) + \frac{v(N)}{|N|} - \frac{Kv(N)}{nK} + \frac{Kg(K, v(N))}{nK} \\ &= \frac{v(\{i\})}{K} v(N) - (\frac{v(\{i\})}{K} - \frac{1}{n}) g(K, v(N)), \end{aligned}$$

as desired. □

Proof of Theorem 2. Since it is obvious that $\psi^\alpha = \alpha EPSD + (1-\alpha)PD$, $\alpha \in \mathbb{R}$, satisfies efficiency, anonymity, and weak linearity, we only show that ψ satisfies proportional loss under separatorization and no advantageous reallocation. For any $(N, v) \in \mathcal{G}^N_{nz}$ and $h \in N$, using (1) and the definition of (N, v^h), we have

$$\begin{aligned}\varphi_i^\alpha(N, v^h) =& \frac{\alpha}{n}\sum_{j\in N} v^h(\{j\}) + \frac{v^h(\{i\})}{\sum_{j\in N} v^h(\{j\})}[v^h(N) - \sum_{j\in N} \alpha v^h(\{j\})] \\ =& \frac{\alpha}{n}K(v) + \frac{v(\{i\})}{K(v)}[K(v) - \alpha K(v)]\end{aligned}$$

for all $i \in N \backslash \{h\}$.

Subtracting the above equation from (1) for arbitrary $(N, v) \in \mathcal{G}^N_{nz}$, we have

$$\varphi_i^\alpha(N, v) - \varphi_i^\alpha(N, v^h) = \frac{v(\{i\})}{K(v)}[v(N) - K(v)], \quad \text{ for all } i \in N \setminus \{h\}.$$

It follows that

$$\frac{\varphi_i^\alpha(N, v) - \varphi_i^\alpha(N, v^h)}{v(\{i\})} = \frac{1}{K(v)}[v(N) - K(v)], \quad \text{ for all } i \in N \setminus \{h\},$$

which immediately shows that proportional loss under separatorization is satisfied.

To show that φ^α satisfies no advantageous reallocation, let $(N, v), (N, w) \in \mathcal{QA}^N_{nz}$ and $T \subseteq N$ be such that $v(N) = w(N)$, $\sum_{i\in T} v(\{i\}) = \sum_{i\in T} w(\{i\})$, and $v(\{i\}) = w(\{i\})$ for all $i \in N \backslash T$. Clearly, $K(v) = K(w)$. Then, using (1),

$$\begin{aligned}\sum_{i\in T}\varphi_i^\alpha(N, v) =& \sum_{i\in T}\big[\frac{\alpha}{n}K(v) + \frac{v(\{i\})}{K(v)}[v(N) - \alpha K(v)]\big] \\ =& \frac{\alpha t}{n}K(v) + \frac{\sum_{i\in T} v(\{i\})}{K(v)}[v(N) - \alpha K(v)] \\ =& \frac{\alpha t}{n}K(w) + \frac{\sum_{i\in T} w(\{i\})}{K(w)}[w(N) - \alpha K(w)] \\ =& \sum_{i\in T}\varphi_i^\alpha(N, w),\end{aligned}$$

which shows that no advantageous reallocation is satisfied.

It remains to prove the 'only if' part. Let ψ be a value on $\mathcal{G}^N_{nz}$ that satisfies the five axioms.

First, consider any game $(N, v) \in \mathcal{QA}^N_{nz}$ and $(N, v^N) \in \mathcal{A}^N_{nz}$. From Corollary 1, we have

$$\psi_i(N, v) - \psi_i(N, v^N) = \frac{v(\{i\})}{K}v(N) - v(\{i\}), \quad \text{for all } i \in N. \tag{23}$$

Since (N, v^N) is an additive game, this implies that $\psi_i(N, v^N)$ doesn't have the terms of $v(S)$, $S \subseteq N$, $|S| \neq 1$. Moreover, since the right-hand side of (23) only has the terms of $v(S)$ with $|S| = 1, n$, we obtain from (23) that $\psi_i(N, v)$ has the term $\frac{v(\{i\})}{K} v(N)$, but no terms of $v(S)$, $S \subseteq N$, $1 < |S| < n$. This implies that $\psi_i(N, v)$ is continuous with respect to $v(S)$ with $S \subseteq N$, $|S| \neq 1$. Hence, from Remark 2 and Theorem 1, $\psi_i(N, v)$ and $\psi_i(N, v^N)$ have the form of (4). Substituting them into (23), we obtain for every $i \in N$,

$$
\begin{aligned}
& \psi_i(N, v) - \psi_i(N, v^N) - \frac{v(\{i\})}{K} v(N) + v(\{i\}) \\
= \; & \frac{v(\{i\}) v(N)}{K} - \left(\frac{v(\{i\})}{K} - \frac{1}{n} \right) g(K, v(N)) - \frac{v(\{i\}) v^N(N)}{K} + \\
& + \left(\frac{v(\{i\})}{K} - \frac{1}{n} \right) g(K, v^N(N)) - \frac{v(\{i\}) v(N)}{K} + v(\{i\}) \\
= \; & - \left(\frac{v(\{i\})}{K} - \frac{1}{n} \right) (g(K, v(N)) - g(K, v^N(N)) = 0, \qquad (24)
\end{aligned}
$$

where in the second equality we use $v^N(N) = K$, and the last equality follows from (23).

To obtain the formula of $\psi_i(N, v)$, $(N, v) \in \mathcal{QA}^N_{nz}$, we consider two cases:

(i) Suppose that $(N, v) \in \mathcal{QA}^N_{nz}$ is such that $v(\{i\}) \neq v(\{j\})$ for some $i, j \in N$. It must be that $\frac{v(\{h\})}{K} \neq \frac{1}{n}$ for some $h \in N$. Then, from (24) we obtain $g(K, v(N)) = g(K, v^N(N))$. This means that $g : \mathbb{R}\backslash\{0\} \times \mathbb{R} \to \mathbb{R}$ is a constant function with respect to its second argument for each K since $v^N(N) = K$. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be such that $f(x) = g(x, y)$ for all $x \in \mathbb{R}\backslash\{0\}$ and $y \in \mathbb{R}$. Then (4) can be written as

$$
\psi_i(N, v) = \frac{v(\{i\}) v(N)}{K} - \left(\frac{v(\{i\})}{K} - \frac{1}{n} \right) f(K). \qquad (25)
$$

Consider any $(N, v), (N, w) \in \mathcal{QA}^N_{nz}$ and $a \in \mathbb{R}$ such that $(N, av + w) \in \mathcal{QA}^N_{nz}$ and there exists $c \in \mathbb{R}$ with $w(\{i\}) = cv(\{i\})$ for all $i \in N$. By weak linearity, $\psi_i(N, av + w) = a\psi_i(N, v) + \psi_i(N, w)$ for all $i \in N$. Using (25), this yields that $f(K(av + w)) = af(K(v)) + f(K(w))$, which implies that f satisfies linearity on $\mathbb{R}\backslash\{0\}$. Hence, $f(K) = \alpha K$, where α is an arbitrary constant. Therefore, using (25) we have

$$
\begin{aligned}
\psi_i(N, v) &= \frac{v(\{i\}) v(N)}{K} - \left(\frac{v(\{i\})}{K} - \frac{1}{n} \right) \alpha K \\
&= \frac{v(\{i\}) v(N)}{K} - \alpha v(\{i\}) + \frac{1}{n} \alpha K \\
&= \frac{1}{n} \sum_{j \in N} \alpha v(\{j\}) + \frac{v(\{i\})}{K} [v(N) - \sum_{j \in N} \alpha v(\{j\})],
\end{aligned}
$$

which equals to Formule (1) of $\varphi^\alpha(N, v)$.

(ii) Suppose that $(N, v) \in \mathcal{QA}_{nz}^N$ is such that $v(\{i\}) = v(\{j\})$ for all $i, j \in N$. Then, by (4) we have $\psi_i(N, v) = \frac{v(N)}{n}$, which also satisfies (1).

Second, consider any game $(N, v) \in \mathcal{G}_{nz}^N$. Since (N, v^N) is an additive game, by (1) applied to (N, v^N), we have $\psi_i(N, v^N) = \frac{\alpha}{n}K + \frac{v(\{i\})}{K}(K - \alpha K) = \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + (1 - \alpha)v(\{i\})$. Substituting this equation into (3) from Corollary 1, we obtain $\psi_i(N, v) = \psi_i(N, v^N) + \frac{v(\{i\})}{K}v(N) - v(\{i\}) = \frac{\alpha}{n}K + (1-\alpha)v(\{i\}) + \frac{v(\{i\})}{K}v(N) - v(\{i\})$, which coincides with (2), and thus $\psi(N, v) = \alpha EPSD(N, v) + (1 - \alpha)PD(N, v)$. □

Proof of Theorem 3. For the 'if' part, we already know that $\varphi^\alpha = \alpha EPSD + (1-\alpha)PD$ satisfies efficiency, anonymity, no advantageous reallocation, proportional loss under separatorization, and weak linearity. We show that φ^α also satisfies superadditive monotonicity if $\alpha \in [0, \frac{n}{n-1}]$. Let $(N, v) \in \mathcal{G}_{nz}^N$ be an arbitrary superadditive and monotone game. Since $v(N) \geq \sum_{j\in N} v(\{j\})$, by (2) we have $\psi_i(N, v) \geq \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + (1 - \alpha)v(\{i\}) > \frac{\alpha}{n}v(\{i\}) + (1 - \alpha)v(\{i\}) = \left(1 - \frac{n-1}{n}\alpha\right) v(\{i\}) \geq 0$. Hence, ψ satisfies superadditive monotonicity.

It remains to prove the 'only if' part. Let ψ be a value on $\mathcal{G}_{nz}^N$ satisfying the six axioms. From Theorem 2, there exists $\alpha \in \mathbb{R}$ such that $\psi = \alpha EPSD + (1 - \alpha)PD$. We must show that α belongs to $[0, \frac{n}{n-1}]$. Suppose, by contradiction, that $\alpha \notin [0, \frac{n}{n-1}]$. We distinguish the following two cases.

(i) Suppose that $\alpha < 0$. Consider an additive game $(N, v) \in \mathcal{A}_{nz}^N$, where $v(\{i\}) = 1$ and $v(\{j\}) = 1 - \frac{\alpha}{n-1} - \frac{n}{(n-1)\alpha}$ for all $j \in N\backslash\{i\}$. Clearly, this game is superadditive and monotone since $\frac{\alpha}{n-1} + \frac{n}{(n-1)\alpha} = \frac{\alpha^2+n}{(n-1)\alpha} < 0$. By Theorem 2 and (N, v) being additive, $\psi_i(N, v) = \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + (1-\alpha)v(\{i\}) = \frac{\alpha}{n}(1+(n-1) - \alpha - \frac{n}{\alpha}) + 1 - \alpha = \frac{\alpha+(n-1)\alpha)-\alpha^2}{n} - 1 + 1 - \alpha = -\frac{\alpha^2}{n} < 0$, which contradicts superadditive monotonicity.

(ii) Suppose that $\alpha > \frac{n}{n-1}$. Consider an additive game $(N, v) \in \mathcal{A}_{nz}^N$ such that $v(\{i\}) = 1 + \frac{2n}{(n-1)\alpha-n}$ and $v(\{j\}) = 1$ for all $j \in N\backslash\{i\}$. Also this game is superadditive and monotone. In this case, $\psi_i(N, v) = \frac{\alpha}{n}(n + \frac{2n}{(n-1)\alpha-n}) + (1 - \alpha)(1 + \frac{2n}{(n-1)\alpha-n}) = \alpha + \frac{2\alpha}{(n-1)\alpha-n} + 1 - \alpha + \frac{2n(1-\alpha)}{(n-1)\alpha-n} = 1 + \frac{2\alpha+2n(1-\alpha)}{(n-1)\alpha-n} = -1 < 0$, which contradicts superadditive monotonicity.

□

Proof of Theorem 4. It is easy to check that $\psi = \alpha EPSD + (1 - \alpha)PD$, $\alpha \in [0, 1]$, satisfies the six aioms. For the uniqueness, together with Theorem 3 and Lemma 2, we have to show $\alpha \leq 1$, which follows immediately from (2) and weak desirability. □

Proof of Lemma 2. Let ψ be a value on $\mathcal{G}_{nz}^N$ satisfying efficiency, proportional loss under separatorization, no advantageous reallocation, and weak desirability. Let π be a permutation on N. First, consider any two games $(N,v),(N,w) \in \mathcal{A}_{nz}^N$ such that $(N,w)=(N,\pi v)$. Without loss of generality, we assume that (N,v) is individually positive. We distinguish the following three cases with respect to the players:

(i) $\pi(i)=i$. In this case, $v(\{i\})=w(\{i\})$ and $\sum_{k\in N\setminus\{i\}} v(\{k\}) = \sum_{k\in N\setminus\{i\}} w(\{k\})$. By no advantageous reallocation, $\sum_{k\in N\setminus\{i\}} \psi_k(N,v) = \sum_{k\in N\setminus\{i\}} \psi_k(N,w)$. Efficiency then implies that $\psi_i(N,v)=\psi_i(N,w)=\psi_{\pi(i)}(N,\pi v)$.

(ii) $\pi(i)\neq i$ and $v(\{i\}) < \frac{K}{2}$. We consider a game $(N,v') \in \mathcal{A}_{nz}^N$ such that $v'(\{i\}) = v'(\{j\}) = v(\{i\})$ and $\sum_{k\in N\setminus\{i,j\}} v'(\{k\}) = K - 2v(\{i\})$, where $j=\pi(i)$. By no advantageous reallocation applied to $(N,v),(N,v')$ and $N\setminus\{i\}$, we have $\sum_{k\in N\setminus\{i\}} \psi_k(N,v) = \sum_{k\in N\setminus\{i\}} \psi_k(N,v')$. This together with efficiency imply that $\psi_i(N,v)=\psi_i(N,v')$. On the other hand, by no advantageous reallocation applied to $(N,w),(N,v')$ and $N\setminus\{j\}$, we have $\sum_{k\in N\setminus\{j\}} \psi_k(N,w) = \sum_{k\in N\setminus\{j\}} \psi_k(N,v')$. Efficiency then implies that $\psi_j(N,w)=\psi_j(N,v')$. Moreover, since $v'(S\cup\{i\}) = v'(S\cup\{j\})$ for all $S \subseteq N\setminus\{i,j\}$, weak desirability implies that $\psi_i(N,v')=\psi_j(N,v')$. Therefore, $\psi_i(N,v) = \psi_j(N,w) = \psi_{\pi(i)}(N,\pi v)$.

(iii) $\pi(i)\neq i$ and $v(\{i\}) \geq \frac{K}{2}$. Since $|N|\geq 3$, there exists at most one such player. Applying cases (i) and (ii) to all other players $j\in N\setminus\{i\}$, we have that $\psi_j(N,v) = \psi_{\pi(j)}(N,\pi v)$ for all $j \in N\setminus\{i\}$. Efficiency then implies that $\psi_i(N,v)=\psi_{\pi(i)}(N,\pi v)$.

The above three cases show that if a value ψ on $\mathcal{A}_{nz}^N$ satisfies efficiency, no advantageous reallocation, and weak desirability, then it also satisfies anonymity. From Corollary 1, efficiency and proportional loss under separatorization imply (3), and thus $\psi_i(N,v) = \psi_i(N,v^N) + \frac{v(\{i\})}{K} v(N) - v(\{i\}) = \psi_{\pi(i)}(N,\pi v^N) + \frac{\pi v(\{\pi(i)\})}{K} \pi v(N) - \pi v(\{\pi(i)\}) = \psi_{\pi(i)}(N,\pi v)$ since $\pi v^N = (\pi v)^N$. Thus, ψ satisfies anonymity on $\mathcal{G}_{nz}^N$. □

Proof of Lemma 3. It is easy to check that the assertion holds for $|N|=2$. For $|N|\geq 3$, the proof is similar to the proof of Lemma 1 except the induction step, which now is as follows.

Induction step. Consider any game $(N,v)\in\mathcal{G}_{nz}^N$ such that $|D(N,v)| = d-1$. Since $d < n-1$, then $|N\setminus D(N,v)| \geq 2$.

First, consider any $i \in N\setminus D(N,v)$ and any $j\in D(N,v)$. Obviously, $|D(N,v^i)| \geq |D(N,v)|+1 = d$ and $(N,v)=(N,v^j)$. Proportional balanced contributions under separatorization and the induction hypothesis imply that

$$\begin{aligned}
\psi_j(N,v) &= \psi_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^j)] \\
&= \psi_j(N,v^i) \overset{\text{(IH)}}{=} \varphi_j(N,v^i) \\
&= \varphi_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})}[\varphi_i(N,v) - \varphi_i(N,v^j)] \\
&= \varphi_j(N,v), \qquad (26)
\end{aligned}$$

where the first and the last equalities follow from proportional balanced contributions under separatorization.

Next, consider two distinct players $i, k \in N \backslash D(N,v)$. Again, proportional balanced contributions under separatorization and the induction hypothesis imply that

$$\begin{aligned}
\psi_k(N,v) =& \psi_k(N,v^i) + \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^k)] \\
\overset{\text{(IH)}}{=}& \varphi_k(N,v^i) + \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v^k)] \\
=& \varphi_k(N,v) + \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)],
\end{aligned}$$

where again the first and the last equalities follow from proportional balanced contributions under separatorization.

Thus

$$\psi_k(N,v) - \varphi_k(N,v) = \frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)].$$

Summing the above equality over $k \in N \backslash D(N,v)$, together with (26), and then using efficiency, we obtain that

$$\sum_{k \in N \backslash D(N,v)} (\psi_k(N,v) - \varphi_k(N,v)) = \sum_{k \in N \backslash D(N,v)} \left(\frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)] \right)$$

$$\Leftrightarrow$$

$$v(N) - \sum_{j \in D(N,v)} \psi_j(N,v) - v(N) + \sum_{j \in D(N,v)} \varphi_j(N,v)$$
$$= \sum_{k \in N \backslash D(N,v)} \left(\frac{v(\{k\})}{v(\{i\})}[\psi_i(N,v) - \varphi_i(N,v)] \right)$$

$$\Leftrightarrow$$

$$0 = \frac{\psi_i(N,v) - \varphi_i(N,v)}{v(\{i\})} \sum_{k \in N \backslash D(N,v)} v(\{k\}),$$

where the second equivalence follows from (26). Thus, since $v(\{k\}) \neq 0$ for all $k \in N$, $\psi_i(N,v) = \varphi_i(N,v)$ for any $i \in N \backslash D(N,v)$. $\square$

Proof of Corollary 2. Let ψ be a value on $\mathcal{G}^N_{nz}$ satisfying the two axioms. For any $(N,v) \in \mathcal{G}^N_{nz}$ and any $i,j \in N$, by proportional balanced contributions under separatorization we have $\psi_j(N,v) - \psi_j(N,v^i) = \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v) - \psi_i(N,v^j)]$. Summing this equality over $j \in N\backslash\{i\}$ and using efficiency, we have

$$\begin{aligned}&v(N) - \psi_i(N,v) - [v^i(N) - \psi_i(N,v^i)]\\ =&\frac{\sum_{j\in N\backslash\{i\}} v(\{j\})}{v(\{i\})}\psi_i(N,v) - \frac{1}{v(\{i\})}\sum_{j\in N\backslash\{i\}} v(\{j\})\psi_i(N,v^j).\end{aligned}$$

It follows that

$$\psi_i(N,v)\left(\frac{K}{v(\{i\})}\right) = v(N) - v^i(N) + \psi_i(N,v^i) + \frac{1}{v(\{i\})}\sum_{j\in N\backslash\{i\}} v(\{j\})\psi_i(N,v^j)$$

$\Leftrightarrow$

$$\psi_i(N,v) = \frac{v(\{i\})}{K}[v(N) - v^i(N)] + \sum_{j\in N}\frac{v(\{j\})}{K}\psi_i(N,v^j). \tag{27}$$

Next, we show that $\psi(N,v^S) = \psi(N,v^N)$ for all $S \subseteq N$ with $1 \leq |S| \leq n-1$. We use an induction on the number of separators.

Initialization. Since $(N,v^{N\backslash\{h\}}) = (N,v^N)$ for all $h \in N$, then $\psi(N,v^S) = \psi(N,v^N)$ for all $S \subseteq N$ with $|S| = n-1$.

Induction hypothesis. Assume that $\psi(N,v^T) = \psi(N,v^N)$ holds for all $T \subseteq N$ with $|T| = t$ for some $2 \leq t \leq n-1$.

Induction step. Consider $(N,v^S) \in \mathcal{G}^N$ and $S \subsetneq N$ such that $|S| = t-1$. Let $i,k \in N\backslash S$ be two distinct players. We have

$$\begin{aligned}&\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{k\}})\\ \overset{(27)}{=}&\sum_{j\in N}\frac{v(\{j\})}{K}\psi_i(N,v^{S\cup\{j\}}) - \sum_{j\in N}\frac{v(\{j\})}{K}\psi_i(N,v^{S\cup\{k,j\}})\\ =&\sum_{j\in N}\frac{v(\{j\})}{K}[\psi_i(N,v^{S\cup\{j\}}) - \psi_i(N,v^{S\cup\{k,j\}})]\\ \overset{(IH)}{=}&\sum_{j\in N}\frac{v(\{j\})}{K}[\psi_i(N,v^N) - \psi_i(N,v^N)]\\ =&0.\end{aligned}$$

Hence,

$$\psi_i(N,v^S) = \psi_i(N,v^{S\cup\{k\}}) \overset{\text{(IH)}}{=} \psi_i(N,v^N), \quad \text{for all } i \in N\backslash S. \tag{28}$$

To prove this equality also for all $j \in S$, pick $i \in N\backslash S$ and $j \in S$. Proportional balanced contributions under separatorization implies that $\psi_j(N,v^S) - \psi_j(N,v^{S\cup\{i\}}) = \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v^S) - \psi_i(N,v^{S\cup\{j\}})] \overset{(28)}{=} \frac{v(\{j\})}{v(\{i\})}[\psi_i(N,v^S) - \psi_i(N,v^S)] = 0$. Hence, $\psi_j(N,v^S) = \psi_j(N,v^{S\cup\{i\}}) \overset{\text{(IH)}}{=} \psi_j(N,v^N)$ for all $j \in S$.

Therefore, $\psi(N,v^S) = \psi(N,v^N)$ holds for all $S \subseteq N$ with $1 \leq |S| \leq n-1$. This, together with (27), yields the desired formula. □

Proof of Theorem 6. It is clear that φ^α satisfies the three axioms. Conversely, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying efficiency, proportional loss under separatorization, and the α-egalitarian inessential game property for some $\alpha \in [0,1]$. For $|N| = 1$, $\psi = \varphi^\alpha$ holds from efficiency. Next, suppose that $|N| \geq 3$. By Corollary 1, efficiency and proportional loss under separatorization imply that ψ satisfies (3). Moreover, the α-egalitarian inessential game property implies that $\psi_i(N,v^N) = (1-\alpha)v(\{i\}) + \alpha\frac{v^N(N)}{n} = (1-\alpha)v(\{i\}) + \frac{\alpha}{n}\sum_{j\in N} v(\{j\})$. These two equations together imply $\psi = \varphi^\alpha$. □

Proof of Theorem 7. It is clear that φ^α satisfies the three axioms. Conversely, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying efficiency, proportional loss under separatorization, and α-reasonable lower bound for some $\alpha \in [0,1]$. For any $(N,v) \in \mathcal{G}^N_{nz}$, consider a game $(N,w) \in \mathcal{G}^N_{nz}$ such that $w(\{i\}) = v(\{i\})$ for all $i \in N$, and $w(N) = \alpha\sum_{j\in N} w(\{j\}) = \alpha\sum_{j\in N} v(\{j\})$. From Corollary 1, efficiency and proportional loss under separatorization imply that $\psi_i(N,w) = \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})}w(N) - w(\{i\}) + \psi_i(N,w^N) = (\alpha-1)w(\{i\}) + \psi_i(N,w^N)$ for all $i \in N$. Since (N,w) is an α-essential game, α-reasonable lower bound gives that $\psi_i(N,w) \geq \frac{\alpha}{n}\sum_{j\in N} w(\{j\})$. Hence,

$$(\alpha-1)w(\{i\}) + \psi_i(N,w^N) = \psi_i(N,w) \geq \frac{\alpha}{n}\sum_{j\in N} w(\{j\}), \quad \text{for all } i \in N,$$

and thus $\psi_i(N,w^N) \geq \frac{\alpha}{n}\sum_{j\in N} w(\{j\}) + (1-\alpha)w(\{i\})$ for all $i \in N$. Efficiency applied to (N,w^N) implies that $w^N(N) = \sum_{i\in N}\psi_i(N,w^N) \geq \alpha\sum_{j\in N} w(\{j\}) + (1-\alpha)\sum_{j\in N} w(\{j\}) = \sum_{j\in N} w(\{j\}) = w^N(N)$, and thus these inequalities are equalities. Thus,

$$\psi_i(N,w^N) = \frac{\alpha}{n}\sum_{j\in N} w(\{j\}) + (1-\alpha)w(\{i\}).$$

Since $(N,v^N) = (N,w^N)$, then $\psi_i(N,v^N) = \frac{\alpha}{n}\sum_{j\in N} v(\{j\}) + (1-\alpha)v(\{i\})$. Again, by Corollary 1, $\psi_i(N,v) = \frac{v(\{i\})}{K}v(N) - v(\{i\}) + \psi_i(N,v^N) = \frac{v(\{i\})}{K}v(N) - v(\{i\}) + \frac{\alpha}{n}K + (1-\alpha)v(\{i\}) = \frac{v(\{i\})}{K}v(N) + \frac{\alpha}{n}K - \alpha v(\{i\}) = \varphi^\alpha_i(N,v)$. □

Proof of Corollary 3. It is clear that the PD value satisfies the three axioms. Conversely, suppose that ψ is a value on $\mathcal{G}^N_{nz}$ satisfying efficiency, proportional loss under separatorization, and α-individual rationality for some $\alpha \in [0,1]$. From Corollary 1, ψ has the form given in (3). For any $(N,v) \in \mathcal{G}^N_{nz}$, similar as in the proof of Theorem 7, consider a game $(N,w) \in \mathcal{G}^N_{nz}$ such that $w(\{i\}) = v(\{i\})$ for all $i \in N$ and $w(N) = \alpha \sum_{j\in N} v(\{j\})$. Since (N,w) is an α-essential game, α-individual rationality implies that $\psi_i(N,w) \geq \alpha w(\{i\})$ for all $i \in N$. By (3) applied to (N,w) and (N,w^N), we have $\psi_i(N,w) - \psi_i(N,w^N) = \frac{w(\{i\})}{\sum_{j\in N} w(\{j\})} w(N) - w(\{i\}) = (\alpha-1)w(\{i\})$. Hence, $\psi_i(N,w^N) = \psi_i(N,w) - (\alpha-1)w(\{i\}) \geq \alpha w(\{i\}) - (\alpha-1)w(\{i\}) = w(\{i\})$. Efficiency then implies that it must be $\psi_i(N,w^N) = w(\{i\})$ for all $i \in N$, since $w^N(N) = \sum_{j\in N} w(\{j\})$.

Since $(N,v^N) = (N,w^N)$, then $\psi_i(N,v^N) = w(\{i\}) = v(\{i\})$. Again, by (3) applied to (N,v) and (N,v^N), we have $\psi_i(N,v) = \frac{v(\{i\})}{K} v(N) - v(\{i\}) + \psi_i(N,v^N) = \frac{v(\{i\})}{K} v(N) - v(\{i\}) + v(\{i\}) = \frac{v(\{i\})}{K} v(N) = PD_i(N,v)$. □

Appendix B: Logical independence of the axioms

Logical independence of the axioms used in the characterization results can be shown by the following alternative values.

Theorem 2:

(i) The value $\psi(N,v) = 0$ for all $(N,v) \in \mathcal{G}^N_{nz}$ satisfies all axioms except efficiency.

(ii) The value on $\mathcal{G}^N_{nz}$ defined for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - v(\{i\}) + \frac{i}{\sum_{j\in N} j} \sum_{j\in N} v(\{j\}) \tag{29}$$

satisfies all axioms except anonymity.

(iii) The value defined by (5) satisfies all axioms except no advantageous reallocation.

(iv) The ED value satisfies all axioms except proportional loss under separatorization.

(v) The value on $\mathcal{G}^N_{nz}$ defined for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - \left(v(\{i\}) - \frac{1}{n} \sum_{j\in N} v(\{j\}) \right) \left(\frac{1}{2}\right)^{\sum_{j\in N} v(\{j\})}$$

satisfies all axioms except weak linearity.

Theorems 3 and 4:

(i) The value $\psi(N,v) = 0$ for all $(N,v) \in \mathcal{G}^N_{nz}$ satisfies all axioms except efficiency.

(ii) The value defined by (5) satisfies all axioms except no advantageous reallocation.

(iii) The value defined by (29) satisfies all axioms except anonymity and weak desirability.

(iv) The ED value satisfies all axioms except proportional loss under separatorization.

(v) The value on $\mathcal{G}^N_{nz}$ defined for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$, by

$$\psi_i(N,v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) - \left(v(\{i\}) - \frac{1}{n}\sum_{j\in N} v(\{j\}) \right) \left(\frac{1}{2}\right)^{|\sum\limits_{j\in N} v(\{j\})|}$$

satisfies all axioms except weak linearity.

(vi) The value $\psi(N,v) = 2EPSD(N,v) - PD(N,v)$ for all $(N,v) \in \mathcal{G}^N_{nz}$ satisfies all axioms except superadditive monotonicity.

Theorem 6:

(i) The value $\psi_i(N,v) = (1-\alpha)v(\{i\}) + \frac{\alpha}{n}\sum_{j\in N} v(\{j\})$ for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The value $\psi(N,v) = \alpha ED(N,v) + (1-\alpha)ESD(N,v)$ for all $(N,v) \in \mathcal{G}^N_{nz}$, satisfies all axioms except proportional loss under separatorization.

(iii) The value defined by (29) satisfies all axioms except the α-egalitarian inessential game property.

Theorem 7:

(i) The value $\psi_i(N,v) = \frac{1}{n}\sum_{j\in N} v(\{j\})$ for all $(N,v) \in \mathcal{G}^N_{nz}$ and $i \in N$, satisfies all axioms except efficiency.

(ii) The ED value satisfies all axioms except proportional loss under separatorization.

(iii) The value defined by (29) satisfies all axioms except α-reasonable lower bound.

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