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The α-constant-sum games

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Abstract

Given any $\alpha \in [0,1]$, an α-constant-sum game on a finite set of players, N, is a function that assigns a real number to any coalition $S \subseteq N$, such that the sum of the worth of the coalition S and the worth of its complementary coalition $N \backslash S$ is α times of the worth of the grand coalition. This class contains the constant-sum games of Khmelnitskaya (2003, [3]) (for $\alpha = 1$) and games of threats of Kohlberg and Neyman (2018, [4]) (for $\alpha = 0$) as special cases. An α-constant-sum game may not be a classical TU cooperative game as it may fail to satisfy the condition that the worth of the empty set is 0, except when $\alpha = 1$. In this paper, we will build a value theory for the class of α-constant-sum games, and mainly introduce the α-quasi-Shapley value. We characterize this value by classical axiomatizations for TU games. We show that axiomatizations of the equal division value do not work on these classes of α-constant-sum games.

Keywords: α-constant-sum game, α-quasi-Shapley value, threat game, constant-sum-game,

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1 Introduction

A *cooperative game with transferable utility (TU game)* describes situations where players can earn certain payoffs by cooperating. It assigns a worth to every subset of the player set, called coalition, which represents the value that the players in the coalition can earn by cooperating. By definition, it assigns zero worth to the empty set. A *value* for TU-games is a function that assigns a single payoff vector to each TU game. The components of this payoff vector reflect an assessment of the corresponding player's gains for participating in the game. The most widely studied value in TU games is the Shapley value (Shapley 1953, [5]), which assigns to every player its expected marginal contribution assuming that all possible orders of entrance of the players to the grand coalition occur with equal probability. In Shapley's paper, he proved that the Shapley value is the unique value satisfying efficiency, symmetry, linearity and the null player property. Young (1985, [9]) characterized the Shapley value by efficiency, symmetry and marginality, while van den Brink (2001, [6]) also provided an axiomatization of the Shapley value using efficiency, the null player property and a fairness property.

Khmelnitskaya (2003, [3]) showed that Young's axiomatization is valid on the class of *constant sum-games*, being TU games where the sum of the worths of a coalition and its complement always equals the worth of the grand coalition. Kohlberg and Neyman (2018, [4]) introduced the class of *games of threats* where the worth of any coalition and its complement equals zero. Consequently, these need not be a subclass of TU-games, since the worth of the empty set should be the negative of the worth of the grand coalition, implying that the worth of the empty set does not need to be zero. However, the structure of this class is similar to that of the class of constant-sum games.

In this paper, we show that both results mentioned above can be extended to classes of, what we call, *α-constant sum-games*. These are games where the worth of any coalition and its complement equals a fraction $\alpha \in [0, 1]$ from the worth of the grand coalition. We show this by closely following the arguments of Kohlberg and Neyman (2018, [4]). For $\alpha \neq 1$, the α-constant-sum games need not be TU games, as they may fail to satisfy the condition that the worth of the empty set is zero. In this paper, we develop a value theory for the classes of α-constant-sum games. We show that there is a unique value on these classes that satisfies Shapley's axiomatization system: efficiency, symmetry, linearity and the null player property. The value owns a similar structure as the Shapley value for TU games, and is called *α-quasi-Shapley value*.

Given any $\alpha \in [0, 1]$, we build a connection between the α-constant-sum games and the classical TU games. The α-quasi-Shapley value of any α-constant-sum game coincides with the Shapley value of the corresponding TU game. Based on this connection, Young and van den Brink's axiomatizations for the Shapley value are also applied to characterize the α-quasi-Shapley value. Further, we show that a similar analysis cannot be done for the equal division value in the sense that on a class of α-constant-sum games, the axioms of efficiency, symmetry, linearity and the nullifying player property, which characterize the equal division value for TU games in van den Brink (2007, [7]), do not give uniqueness on the class of α-constant-sum games. Finally, we introduce an alternative way to balance the power in games of threats.

The paper is organized as follows. After discussing preliminaries in Section 2, Section 3 gives the definition of α-constant-sum games, while Section 4 introduces and characterizes the α-quasi-Shapley value by efficiency, symmetry, linearity and the null player property. Section 5 builds the connection between the α-constant-sum games and the classical TU games, and additionally characterizes the α-quasi-Shapley value with Young and van den Brink's axiomatizations. In Section 6, we show that something similar cannot be done for the equal division value. In Section 7, we consider an alternative efficiency property, specifically for games where the worth of the empty set is nonzero. Section 8 introduces an alternative to balancing threats. Finally, Section 9 concludes and develops some suggestions for future research.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the set of *players*. A subset $S \subseteq N$ is called *coalition*. In particular, N is called the *grand coalition*. We denote the size of coalition S as s.

A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $v : 2^N \to \mathbb{R}$ is the *characteristic function* assigning to each coalition $S \in 2^N$ the *worth* $v(S)$, with the convention that $v(\emptyset) = 0$. For each coalition S, the real number $v(S)$ represents the reward that coalition S can guarantee by itself without the cooperation of the other players. We denote by $\mathcal{G}^N$ the game space consisting of all TU-games with player set N.

A *value* on a subclass $\mathcal{C} \subseteq \mathcal{G}^N$ is a function that assigns a single payoff vector to each TU game in $\mathcal{C}$. The most widely studied value in TU games is the Shapley value (Shapley 1953, [5]), which assigns to every player its expected marginal contribution, assuming that all possible orders of entrance of the players to the grand coalition occur with equal probability. Formally, the Shapley value Sh on $\mathcal{G}^N$ is defined by

$$Sh_i(N, v) = \sum_{S \subseteq N,\ S \ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(S\backslash\{i\})], \tag{1}$$

for any $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$.

The most egalitarian value for TU games is the *equal division value*, Ed which allocates the worth of the grand coalition equally over all players, and thus is given by

$$Ed_i(N, v) = \frac{v(N)}{n}, \text{ for any } \langle N, v\rangle \in \mathcal{G}^N \text{ and } i \in N. \quad (2)$$

Khmelnitskaya (2003, [3]) considered the class of *constant-sum games* being games $\langle N, v\rangle$ satisfying $v(S) + v(N\backslash S) = v(N)$ for all $S \subseteq N$. Kohlberg and Neyman (2018, [4]) introduced the of *games of threats*, being pairs $\langle N, d\rangle$, where $d : 2^N \to \mathbb{R}$ is a function that assigns a real number to any coalition, $S \subseteq N$, such that $d(S) = -d(N\backslash S)$, i.e., $d(S) + d(N\backslash S) = 0$. Notice that this implies that games of threats need not be TU games since the empty set can have a nonzero worth. In fact, it has nonzero worth if and only if the worth of the grand coalition is nonzero. However, the class of games of threats have a similar structure as the class of constant-sum games. Kohlberg and Neyman (2018, [4]) interpreted the amount $d(S)$ as the threat power of the coalition S. The condition $d(S) + d(N\backslash S) = 0$ implies that the threat powers of coalition S and its complementary coalition $N\backslash S$ are contrary, and offset each other. Following this interpretation, in a constant-sum game, the sum of the threat powers of coalition S and its complementary coalition $N\backslash S$ are fixed as the threat power of the grand coalition.

3 The α-constant-sum games

Given any $\alpha \in [0, 1]$, a *α-constant-sum game* is a pair $\langle N, \mu\rangle$, where

- $N = \{1, 2, \ldots, n\}$ is a finite set of players;
- $\mu : 2^N \to \mathbb{R}$ is a function such that $\mu(S) + \mu(N\backslash S) = \alpha\mu(N)$, for all $S \subseteq N$.

We denote by $\mathcal{C}^N_\alpha$, $\alpha \in [0, 1]$, the game space consisting of all α-constant-sum games with player set N. In particular, $\mathcal{C}^N_1$ is the class of constant-sum games, and $\mathcal{C}^N_0$ is the class of games of threats.

Example 1 *Consider $\alpha = 0.8$, $N = \{1, 2, 3\}$, and game $\langle N, \mu\rangle \in \mathcal{C}^N_{0.8}$, given by*

$$\mu(\emptyset) = -2,\ \mu(\{1\}) = 2,\ \mu(\{2\}) = 4,\ \mu(\{3\}) = 5,$$
$$\mu(\{1, 2\}) = 3,\ \mu(\{1, 3\}) = 4,\ \mu(\{2, 3\}) = 6,\ \mu(\{1, 2, 3\}) = 10.$$

By choosing, for every $S \subseteq N$, either S or $N \backslash S$, we can describe any $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ by means of 2^{n-1} numbers, thereby identifying $\mathcal{C}_\alpha^N$ with $\mathbb{R}^{2^{n-1}}$. A convenient choice is to take any $i \in N$, and consider $(\mu(S))_{S \ni i}$. The worths of the complementary coalitions $S \subseteq N \setminus \{i\}$ follow by the definition of α-constant sum game.

Example 2 *Given $\alpha = 0.8$, choosing $i = 1$, the above Example 1 is described as follows:*

$$\mu(\{1\}) = 2,\ \mu(\{1,2\}) = 3,\ \mu(\{1,3\}) = 4,\ \mu(\{1,2,3\}) = 10.$$

The worth of the other coalitions (without player 1) follow from the definition of α-constant sum game.

4 The α-quasi-Shapley value

Consider any $\alpha \in [0,1]$, and any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$. Two players $i, j \in N$ are *symmetric* in $\langle N, \mu \rangle$ if, for every coalition $S \subseteq N \backslash \{i, j\}$, $\mu(S \cup \{i\}) = \mu(S \cup \{j\})$. A player $i \in N$ is a *null player* in $\langle N, \mu \rangle$ if, for every coalition $S \subseteq N \backslash \{i\}$, $\mu(S \cup \{i\}) = \mu(S)$. For any pair of α-constant-sum games $\langle N, \mu \rangle$, $\langle N, \nu \rangle \in \mathcal{C}_\alpha^N$ and a, $b \in \mathbb{R}$, the game $a\mu + b\nu$ is given as $(a\mu + b\nu)(S) = a\mu(S) + b\nu(S)$, for all $S \subseteq N$.

A *value on $\mathcal{C}_\alpha^N$* is a function $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ that associates with each α-constant-sum game, a vector of payoffs $\phi(N, \mu) \in \mathbb{R}^N$ to the players. Following Shapley (1953, [5]), we consider the following properties.

- *Efficiency*: For any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, $\sum_{i \in N} \phi_i(N, \mu) = \mu(N)$.
- *Symmetry*: For any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, if players $i, j \in N$ are symmetric, then $\phi_i(N, \mu) = \phi_j(N, \mu)$.
- *Linearity*: For any pair of α-constant-sum games $\langle N, \mu \rangle$, $\langle N, \nu \rangle \in \mathcal{C}_\alpha^N$ and a, $b \in \mathbb{R}$, $\phi(N, a\mu + b\nu) = a\phi(N, \mu) + b\phi(N, \nu)$.
- *Null player property*: For any α-constant-sum game $\langle N, v \rangle \in \mathcal{C}_\alpha^N$, if player $i \in N$ is a null player, then $\phi_i(N, \mu) = 0$.

Notice that these are the usual efficiency, symmetry, linearity and null player axioms, except that they are defined on subclasses $\mathcal{C}_\alpha^N$.

It turns out that, for every $\alpha \in [0,1]$, there exists a unique value on $\mathcal{C}_\alpha^N$ that satisfies the above defined classical axioms on this class.

Theorem 4.1 *Take any $\alpha \in [0,1]$. There is a unique value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, symmetry, linearity and the null player property. This value* *is, for every* $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$ *and* $i \in N$*, given by*

$$SH_i^\alpha(N,\mu) = \sum_{S\subseteq N,\ S\ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha}[\mu(S) - \mu(N\backslash S)] \right). \quad (3)$$

For every $\alpha \in [0,1]$, we refer to the corresponding value SH^α on $\mathcal{C}_\alpha^N$ as the *α-quasi-Shapley value*. For $\alpha = 1$, this is the 'classical' Shapley value on the class of constant-sum games $\mathcal{C}_1^N$ as characterized in Khmelnitskaya (2003, [3]). For $\alpha = 0$, this is the Shapley type value on the class of games of threats $\mathcal{C}_0^N$ as characterized in Kohlberg and Neyman (2018, [4]). Notice also that, for $\alpha = 0$, Theorem 4.1 boils down to Theorem 1 of Kohlberg and Neyman (2018, [4]). Before proving Theorem 4.1, we introduce basis games for the classes of α-constant-sum games.

Definition 1 *Consider any* $\alpha \in [0,1]$ *and* $T \subseteq N$, $T \neq \emptyset$*. The* ***unanimity α-constant-sum game****,* $\langle N, u_T^\alpha\rangle \in \mathcal{C}_\alpha^N$*, is defined by*

$$u_T^\alpha(S) = \begin{cases} \frac{2}{2-\alpha}, & \text{if } S \supseteq T; \\ \frac{2\alpha-2}{2-\alpha}, & \text{if } S \subseteq N\backslash T; \\ \frac{\alpha}{2-\alpha}, & \text{otherwise.} \end{cases} \quad (4)$$

For $\alpha = 1$, these are twice the type of unanimity game used in Khmelnitskaya (2003, [3]), while for $\alpha = 0$, these are $\frac{1}{|T|}$ times the type of unanimity game used by Kohlberg and Neyman (2018, [4]).

Proposition 4.2 *Consider any* $\alpha \in [0,1]$*, and take any* $i \in N$*. The unanimity α-constant-sum games,* $\{\langle N, u_T^\alpha\rangle\}_{T\subseteq N,\ T\ni i}$*, span the class of α-constant-sum games,* $\mathcal{C}_\alpha^N$*, i.e., for every* $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$*, there exist numbers* $a_T \in \mathbb{R}$*,* $i \in T \subseteq N$*, such that* $\mu = \sum_{T\subseteq N,\ T\ni i} a_T u_T^\alpha$*.*

Proof. Consider any $\alpha \in [0,1]$, and let $i_0 \in N$. It is sufficient to show that the 2^{n-1} unanimity α-constant-sum games, $\{\langle N, u_T^\alpha\rangle\}_{T\subseteq N, T\ni i_0}$, are linearly independent. On the contrary, suppose that there exist numbers a_j, $j = 1, \ldots, 2^{n-1}$, such that

$$\sum_{j=1}^{2^{n-1}} a_j u_{T_j}^\alpha = 0, \quad (5)$$

where $(T_j)_{j=1,\ldots,2^{n-1}}$, is such that $i_0 \in T_j$ and $T_j \neq T_k$ for all $j,k \in \{1,\ldots,2^{n-1}\}$, $j \neq k$.

Since, for $j \neq k$, we have $T_j \neq T_k$ and $T_j \cap T_k \supseteq \{i_0\} \neq \emptyset$, neither set is contained in the other's complement and therefore, using the fact that $T_k \supseteq$

$T_j \Leftrightarrow N \setminus T_k \subseteq N \setminus T_j$,

$$u^{\alpha}_{T_j}(T_k) = \begin{cases} \frac{2}{2-\alpha}, & T_k \supseteq T_j; \\ \frac{\alpha}{2-\alpha}, & \text{otherwise,} \end{cases} \quad \text{and} \quad u^{\alpha}_{T_j}(N \backslash T_k) = \begin{cases} \frac{2\alpha-2}{2-\alpha}, & T_k \supseteq T_j; \\ \frac{\alpha}{2-\alpha}, & \text{otherwise.} \end{cases}$$

Hence,

$$u^{\alpha}_{T_j}(T_k) - u^{\alpha}_{T_j}(N \backslash T_k) = \begin{cases} 2, & T_k \supseteq T_j; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Among the T_j choose one, say T_m, with a minimum number of players among those coalitions in the sequence $(T_j)_{j=1,\ldots,2^{n-1}}$ with $a_j \neq 0$, i.e., $a_m \neq 0$ and $T_j \subset T_m$ implies that $a_j = 0$. Then for any $j \neq m$ with $a_j \neq 0$, it holds that $T_m \not\supseteq T_j$ and therefore, by equation (6),

$$u^{\alpha}_{T_j}(T_m) - u^{\alpha}_{T_j}(N \backslash T_m) = 0. \tag{7}$$

Thus,

$$\begin{aligned} 0 &= \textstyle\sum_{j=1}^{2^{n-1}} a_j u^{\alpha}_{T_j}(T_m) - \sum_{j=0}^{2^{n-1}} a_j u^{\alpha}_{T_j}(N \backslash T_m) \\ &= \textstyle\sum_{j=1}^{2^{n-1}} a_j [u^{\alpha}_{T_j}(T_m) - u^{\alpha}_{T_j}(N \backslash T_m)] \\ &= a_m [u^{\alpha}_{T_m}(T_m) - u^{\alpha}_{T_m}(N \backslash T_m)] \\ &= 2a_m, \end{aligned}$$

where the first equality follows from Equation (5), the third equality follows from Equation (7) and the assumption that $a_j = 0$ if $T_j \subset T_m$, and the last equality follows from Equation (6). Thus, we have a contradiction with $a_m \neq 0$. $\square$

Now, we can prove the main theorem.

Proof of Theorem 4.1

We first prove the uniqueness. Given any $\alpha \in [0, 1]$, let $\phi : \mathcal{C}^N_\alpha \to \mathbb{R}^N$ be a value on $\mathcal{C}^N_\alpha$ that satisfies efficiency, symmetry, linearity and the null player property. We prove that the value ϕ is uniquely determined on $\mathcal{C}^N_\alpha$.

Let $T \subseteq N$, $T \neq \emptyset$. In the unanimity α-constant-sum game, $\langle N, u^{\alpha}_T \rangle \in \mathcal{C}^N_\alpha$, for any $i, j \in T$ and all $S \subseteq N \backslash \{i, j\}$, we have

$$u^{\alpha}_T(S \cup \{i\}) = u^{\alpha}_T(S \cup \{j\}) = \frac{\alpha}{2-\alpha},$$

showing that all players $i \in T$ are symmetric. Further, for any $i \notin T$ and all $S \subseteq N \backslash \{i\}$, we have

$$u_T^\alpha(S \cup \{i\}) = u_T^\alpha(S),$$

showing that all players $i \notin T$ are null players in $\langle N, u_T^\alpha \rangle$.

According to ϕ satisfying the null player property, the players $i \notin T$ earn zero payoff in $\langle N, u_T^\alpha \rangle$. By efficieny and symmetry of the function ϕ, and the fact that $u_T^\alpha(N) = \frac{2}{2-\alpha}$, we obtain

$$\phi_i(N, u_T^\alpha) = \begin{cases} \frac{2}{t(2-\alpha)}, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

With linearity, the function ϕ is uniquely determined on $\mathcal{C}_\alpha^N$.

For any $\alpha \in [0, 1]$, we now prove that the α-quasi-Shapley value SH^α, defined in (3), satisfies efficiency, symmetry, linearity and the null player property.

The α-quasi-Shapley value SH^α satisfying symmetry, linearity and the null player property follows directly from the following equivalent representation: for any $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$,

$$\begin{aligned} SH_i^\alpha(N, \mu) &= \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(N \backslash S)] \right) \\ &= \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(S \backslash \{i\})] \right). \end{aligned} \tag{8}$$

To prove efficiency, for any $i \in N$, we define a function $I_i : 2^N \to \{0, 1\}$ by

$$I_i(S) = \begin{cases} 1, & \text{if } S \ni i; \\ 0, & \textit{otherwise}. \end{cases} \tag{9}$$

For any $S \subseteq N$, denote $r_n(S) := \frac{(s-1)!(n-s)!}{n!}$. It follows from (3) that

$$\begin{aligned} \sum_{i \in N} SH_i^\alpha(N, \mu) &= \sum_{i \in N} \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha} [\mu(S) - \mu(N \backslash S)] \right) \\ &= \frac{1}{2-\alpha} \sum_{i \in N} \sum_{S \subseteq N, S \ni i} r_n(S) [\mu(S) - \mu(N \backslash S)] \\ &= \frac{1}{2-\alpha} \sum_{i \in N} \sum_{S \subseteq N} I_i(S) r_n(S) [\mu(S) - \mu(N \backslash S)] \end{aligned}$$

$$= \frac{1}{2-\alpha} \sum_{S\subseteq N} \sum_{i\in N} I_i(S) r_n(S)[\mu(S) - \mu(N\backslash S)]$$
$$= \frac{1}{2-\alpha} \sum_{S\subseteq N} \sum_{i\in S} r_n(S)[\mu(S) - \mu(N\backslash S)]$$
$$= \frac{1}{2-\alpha} \sum_{S\subseteq N, S\neq\emptyset} s \cdot r_n(S)[\mu(S) - \mu(N\backslash S)]$$

$$= \frac{1}{2-\alpha} \sum_{S\subseteq N, S\neq\emptyset} \frac{1}{\binom{n}{s}}[\mu(S) - \mu(N\backslash S)]. \tag{10}$$

Note that, in case $S \subsetneq N$, $S \neq \emptyset$, then $N\backslash S \subseteq N$, $N\backslash S \neq \emptyset$, while $\binom{n}{s} = \binom{n}{n-s}$. Hence,

$$\sum_{i\in N} SH_i^\alpha(N,\mu) = \frac{1}{2-\alpha}[\mu(N) - \mu(\emptyset)].$$

With $\mu(\emptyset) + \mu(N) = \alpha\mu(N)$, we conclude that

$$\sum_{i\in N} SH_i^\alpha(N,\mu) = \frac{1}{2-\alpha}[\mu(N) - (\alpha-1)\mu(N)] = \mu(N).$$

This completes the proof of Theorem 4.1. □

5 The connection with classical TU games

Given $\alpha \in [0,1)$, the class of α-constant-sum games $\mathcal{C}_\alpha^N$ need not belong to the class of TU games $\mathcal{G}^N$, as such a α-constant-sum game $\langle N,\mu\rangle \in \mathcal{C}_\alpha^N$, may fail to satisfy the condition that $\mu(\emptyset) \neq 0$. In this section, we will build the connection between the class of α-constant-sum games $\mathcal{C}_\alpha^N$ and the class of TU games $\mathcal{G}^N$.

Given any $\alpha \in [0,1]$, let $K^\alpha : \mathcal{G}^N \to \mathcal{C}_\alpha^N$ and $L^\alpha : \mathcal{C}_\alpha^N \to \mathcal{G}^N$, for every $\langle N,v\rangle \in \mathcal{G}^N$ and $\langle N,\mu\rangle \in \mathcal{C}_\alpha^N$, and for all $S \subseteq N$, be defined by

$$(K^\alpha v)(S) = v(S) - v(N\backslash S) + \frac{\alpha}{2-\alpha} v(N), \tag{11}$$

and

$$(L^\alpha \mu)(S) = \frac{1}{2-\alpha}\mu(S) + \frac{1-\alpha}{2-\alpha}\mu(N). \tag{12}$$

Then, it is easy to get the following relationships between the value on $\mathcal{G}^N$ and the value on $\mathcal{C}_\alpha^N$, which we state without proof.

Proposition 5.1 *Given any $\alpha \in [0,1]$.*

(i) If φ is a value on $\mathcal{G}^N$, then $\varphi \circ L^\alpha$ is a value on $\mathcal{C}_\alpha^N$; if ψ is a value on $\mathcal{C}_\alpha^N$, then $\psi \circ K^\alpha$ is a value on $\mathcal{G}^N$.

(ii) If φ_1 and φ_2 are two different values on $\mathcal{G}^N$, then $\varphi_1 \circ L^\alpha$ and $\varphi_2 \circ L^\alpha$ are two different values on $\mathcal{C}_\alpha^N$; if ψ_1 and ψ_2 are two different values on $\mathcal{C}_\alpha^N$, then $\psi_1 \circ K^\alpha$ and $\psi_2 \circ K^\alpha$ are two different values on $\mathcal{G}^N$.

For $\alpha = 0$, the mappings K^0, L^0, and Proposition 5.1 are used in Kohlberg and Neyman (2018, [4]). The relationships, shown in Proposition 5.1, provide a new perspective to prove Theorem 4.1.

An alternative proof of the uniqueness part of Theorem 4.1 can be given using the map L^α as follows. First, we show how that the α-quasi Shapley value for α-constant sum games can be obtained as the Shapley value of a classical TU game using this map.

Proposition 5.2 *Given any $\alpha \in [0,1]$, $SH^\alpha = Sh \circ L^\alpha$, i.e., for any $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$, it holds that $SH^\alpha(N,\mu) = Sh(N, L^\alpha \mu)$.*

Proof. Consider any $\alpha \in [0,1]$. Using the definition of the Shapley value, according to (3) and Equation (8),

$$
\begin{aligned}
& SH_i^\alpha(N,\mu) \\
= & \sum_{S\subseteq N, S\ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot \frac{1}{2-\alpha}[\mu(S) - \mu(S\backslash\{i\})] \right) \\
= & \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!} \left([\frac{1}{2-\alpha}\mu(S) + \frac{1-\alpha}{2-\alpha}\mu(N)] - [\frac{1}{2-\alpha}\mu(S\backslash\{i\}) + \frac{1-\alpha}{2-\alpha}\mu(N)] \right) \\
= & \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!} [L^\alpha\mu(S) - L^\alpha\mu(S\backslash\{i\})] \\
= & Sh_i(N, L^\alpha\mu). \qquad \square
\end{aligned}
$$

Next, we can give an alternative proof of Theorem 4.1.

Alternative proof of the uniqueness part of Theorem 4.1

Given any $\alpha \in [0,1]$, for any $\langle N,\mu\rangle \in \mathcal{C}_\alpha^N$, note that the map L^α is linear, efficient (i.e., $(L^\alpha\mu)(N) = \mu(N)$), symmetric (i.e., if i and j are symmetric players in $\langle N,\mu\rangle \in \mathcal{C}_\alpha^N$, then i and j are symmetric players in $\langle N, L^\alpha\mu\rangle \in \mathcal{G}^N$), and preserves null players (i.e., if i is a null player in $\langle N,\mu\rangle \in \mathcal{C}_\alpha^N$, then i is also a null player in $\langle N, L^\alpha\mu\rangle \in \mathcal{G}^N$).

As the Shapley value Sh is the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, linearity and the null player property, according to Proposition 5.1, the value $Sh \circ L^\alpha$ should be the unique value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, symmetry, linearity and the null player property. Therefore, Proposition 5.2 completes the proof of Theorem 4.1. □

Remark 1 *Given any $\alpha \in [0,1]$, for any $\langle N, v\rangle \in \mathcal{G}^N$, note that the map K^α is linear and symmetric, but it can't preserve efficiency and null players. Hence, the value $SH^\alpha \circ K^\alpha$ is not the unique value on $\mathcal{G}^N$ that satisfies efficiency, symmetry, linearity and the null player property, i.e., $SH^\alpha \circ K^\alpha \neq Sh$.*

Young (1985, [9]) showed that the existence and uniqueness theorem for the Shapley value on $\mathcal{G}^N$ remains valid when linearity and the null player property are replaced by marginality, which requires that the value of a player $i \in N$ in a TU game $\langle N, v\rangle$ depends only on the player's marginal contributions, $v(S \cup \{i\}) - v(S)$, for all $S \subseteq N\backslash\{i\}$.

- *Marginality*: For any α-constant-sum game $\langle N, \mu\rangle \in \mathcal{C}_\alpha^N$, and $i \in N$, $\phi_i(N,\mu) = \xi_i(\{\mu(S \cup \{i\}) - \mu(S)\}_{S \subseteq N\backslash\{i\}})$, where $\xi_i : \mathbb{R}^{2^{n-1}} \to \mathbb{R}$.

The map L^α preserves marginality (i.e., if the marginal contributions of player i are the same in $\langle N, \mu\rangle$, $\langle N, \nu\rangle \in \mathcal{C}_\alpha^N$, then the marginal contributions of player i are also the same in TU games $\langle N, L^\alpha\mu\rangle$, $\langle N, L^\alpha\nu\rangle$). Therefore, the following corollary is implied.

Corollary 5.3 *Take any $\alpha \in [0,1]$. The α-quasi-Shapley value is the unique value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, symmetry and marginality.*

For $\alpha = 1$, this gives Theorem 1 of Khmelnitskaya (2003, [3]), while for $\alpha = 0$ this gives Corollary 1 of Kohlberg and Neyman (2018, [4]).

In van den Brink (2001, [6]) an axiomatization for the Shapley value on $\mathcal{G}^N$ is provided where symmetry and linearity are replaced by fairness, which states that if to a TU game we add another TU game in which two players are symmetric, then their payoffs change by the same amount.

- *Fairness*: For any α-constant-sum games $\langle N, \mu\rangle$,$\langle N, \nu\rangle \in \mathcal{C}_\alpha^N$, if $i, j \in N$ are symmetric players in $\langle N, \nu\rangle$, then $\phi_i(N, \mu+\nu) - \phi_i(N,\nu) = \phi_j(N,\mu+\nu) - \phi_j(N,\nu)$.

The map L^α preserves fairness because it can preserve linearity and symmetric players. Therefore, the following corollary is obvious.

Corollary 5.4 *Take any $\alpha \in [0,1]$. The α-quasi-Shapley value is the unique value on $\mathcal{C}_\alpha^N$ that satisfies efficiency, the null player property and fairness.*

6 δ-reducing players

In the previous sections, we showed that modified unanimity games as given by (4) allow to apply various axiomatizations of the Shapley value for TU games, to the classes of α-constant-sum games and characterize a Shapley type value. This does not work for every basis. For example, using the standard basis on the class of all TU games, van den Brink (2007, [7]) characterized the equal division value by efficiency, symmetry, linearity and the nullifying player property. Given any $\alpha \in [0,1]$, for any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, a player $i \in N$ is a *nullifying player* if, for every coalition $S \subseteq N$ with $i \in S$, $\mu(S) = 0$.

- *Nullifying player property*: For any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, if player $i \in N$ is a nullifying player, then $\phi_i(N, \mu) = 0$.

It turns out that on the classes of α-constant-sum games, the nullifying player property is implied by efficiency and symmetry. In fact, we can make a stronger statement, that the existence of a so-called δ-reducing player, $\delta \in [0,1)$, in an α-constant-sum game implies that the game is the null game. Given any $\alpha, \delta \in [0,1]$, for any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, a player $i \in N$ is a *δ-reducing player* if, for every coalition $S \subseteq N$ with $i \in S$, $\mu(S) = \delta\mu(S \setminus \{i\})$.

- *δ-reducing player property*: For any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ and $\delta \in [0,1]$, if player $i \in N$ is a δ-reducing player, then $\phi_i(N, \mu) = 0$.

The δ-reducing player property is introduced in van den Brink and Funaki (2015, [8]) who used it to characterize the corresponding *discounted Shapley value* (see Joosten (1996, [2]) and Driessen and Radzik (2002, [1])) together with efficiency, symmetry and linearity on the class of all TU games. For $\delta = 1$ this boils down to Shapley's null player property, while for $\delta = 0$ this gives the nullifying player property. It turns out that any δ-reducing player property with $\delta < 1$ has no bite for α-constant-sum games since the existence of such a player implies that the game is the null game.

Proposition 6.1 *Take any $\alpha \in [0,1]$ and $\delta \in [0,1)$. If there exists a δ-reducing player in α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, then $\mu(S) = 0$ for all $S \subseteq N$.*

Proof. For $\alpha \in [0,1]$ and $\delta \in [0,1)$, suppose that player $i \in N$ is a δ-reducing player in α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$. Then, by the δ-reducing player property,

$$\mu(N) = \delta\mu(N \setminus \{i\}). \tag{13}$$

By definition of α-constant-sum game, $\mu(\emptyset) + \mu(N) = \alpha\mu(N)$, so

$$\mu(\emptyset) = (\alpha - 1)\mu(N). \tag{14}$$

Also, by definition $\mu(\{i\}) + \mu(N \setminus \{i\}) = \alpha\mu(N)$, which by i being a δ-reducing player gives $\delta\mu(\emptyset) + \mu(N \setminus \{i\}) = \alpha\mu(N)$. Substituting (14) gives $\delta(\alpha - 1)\mu(N) + \mu(N \setminus \{i\}) = \alpha\mu(N)$. Further substituting (13) gives $\delta(\alpha - 1)\delta\mu(N\setminus\{i\})+\mu(N\setminus\{i\}) = \alpha\delta\mu(N\setminus\{i\}) \Leftrightarrow (\delta(\alpha-1)\delta + 1 - \alpha\delta)\,\mu(N\setminus\{i\}) = (\delta^2\alpha - \delta^2 + 1 - \alpha\delta)\,\mu(N \setminus \{i\}) = 0$. Since $(\delta^2\alpha - \delta^2 + 1 - \alpha\delta) \neq 0$, it holds that $\mu(N \setminus \{i\}) = 0$. By (13), then also $\mu(N) = 0$.

We are left to show that $\mu(S) = 0$ for all $S \subseteq N$. Again, by the δ-reducing player property,

$$\mu(S) = \delta\mu(S \setminus \{i\}) \text{ for all } S \subseteq N,\ i \in S, \tag{15}$$

and by the definition of α-constant-sum game, we have for $S \subseteq N$ with $i \in S$,

$$\mu(S) + \mu(N \setminus S) = \alpha\mu(N) = \mu(S \setminus \{i\}) + \mu((N \setminus S) \cup \{i\})$$

which, with (15), implies that $\delta\mu(S\setminus\{i\})+\mu(N\setminus S) = \mu(S\setminus\{i\})+\delta\mu(N\setminus S) \Leftrightarrow (1-\delta)\mu(N\setminus S) = (1-\delta)\mu(S\setminus\{i\}) \Leftrightarrow \mu(N\setminus S) = \mu(S\setminus\{i\} \Rightarrow (\delta+1)\mu(S\setminus\{i\}) = \delta\mu(S \setminus \{i\}) + \mu(S \setminus \{i\}) = \mu(S) + \mu(S \setminus \{i\}) = \mu(S) + \mu(N \setminus S) = \alpha\mu(N) = 0 \Rightarrow \mu(S) = 0$. Then also $\mu(S) = 0$ for all $S \subseteq N \setminus \{i\}$. □

As a corollary, we can see that efficiency and symmetry imply the δ-reducing player property for any $\delta \in [0, 1)$. Specifically, this holds for the nullifying player property. Further, we see that efficiency, symmetry, linearity and the nullifying player property do not give uniqueness since, besides the equal division value, for example, also the α-quasi Shapley value satisfies these axioms on any class of α-constant sum games.

Corollary 6.2 *Take any $\alpha \in [0, 1]$.*

- *Any value on $\mathcal{C}^N_\alpha$ that satisfies efficiency and symmetry, also satisfies the δ-reducing player property on $\mathcal{C}^N_\alpha$ for any $\delta \in [0, 1)$.*
- *Specifically, any value on $\mathcal{C}^N_\alpha$ that satisfies efficiency and symmetry, also satisfies the nulliffying player property on $\mathcal{C}^N_\alpha$.*
- *For $\delta \in [0, 1)$, efficiency, symmetry, linearity, the δ-reducing player property do not give uniqueness on $\mathcal{C}^N_\alpha$.*

7 Power efficiency

In Sections 4 and 5, we followed Kohlberg and Neyman (2018, [4])'s approach, taking the Shapley axioms and showing that these characterize a unique value on the classes of α-constant-sum games. Specifically, we used the efficiency axiom that requires that the sum of all payoffs equals the worth of the grand coalition $\mu(N)$. However, for α-constant-sum games, since generally the worth of the empty set is nonzero, an alternative efficiency is to require that the sum of all payoffs equals the payoff that all players cooperating together can generate compared to the situation where there is no cooperation at all, i.e. the total sum of payoffs equals $\mu(N) - \mu(\emptyset)$.

- *Power efficiency*: For any α-constant-sum game $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$, $\sum_{i \in N} \phi_i(N, \mu) = \mu(N) - \mu(\emptyset)$.

Notice that for classical TU games, this is equivalent to efficiency, since for those games $\mu(\emptyset) = 0$. We refer to this as 'power efficiency', since it reflects that the players earn the payoff that they can generate by their ability to cooperate.

Interestingly, applying this power efficiency together with symmetry, the null player property and linearity on classes of α-constant-sum games $\mathcal{C}_\alpha^N$, characterizes a value that is given exactly by the famous Shapley value formula (without any α-term), see (1). We refer to this as the *quasi-Shapley value* since it is also defined on classes where the worth of the emptyset is nonzero.

Theorem 7.1 *Take any $\alpha \in [0, 1]$. There is a unique value on $\mathcal{C}_\alpha^N$ that satisfies power efficiency, symmetry, linearity and the null player property. This value is, for every $\langle N, \mu \rangle \in \mathcal{C}_\alpha^N$ and $i \in N$, given by*

$$SH_i(N, \mu) = \sum_{S \subseteq N,\ S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot [\mu(S) - \mu(N \backslash S)] \right). \tag{16}$$

Proof. We first prove the uniqueness. Given any $\alpha \in [0, 1]$, let $\phi : \mathcal{C}_\alpha^N \to \mathbb{R}^N$ be a value on $\mathcal{C}_\alpha^N$ that satisfies power efficiency, symmetry, linearity and the null player property. We prove that the value ϕ is uniquely determined on $\mathcal{C}_\alpha^N$.

Let $T \subseteq N$, $T \neq \emptyset$. Consider the unanimity α-constant-sum game u_T^α given by (4). As we saw in the proof of Theorem 4.1, all players $i \in T$ are symmetric, and all players $i \notin T$ are null players in u_T^α.

According to ϕ satisfying the null player property, the players $i \notin T$ earn zero payoff in $\langle N, u_T^\alpha \rangle$. By power efficieny and symmetry of the function ϕ, and the fact that $u_T^\alpha(N) - u_T^\alpha(\emptyset) = \frac{2}{2-\alpha} - \frac{2\alpha-2}{2-\alpha} = 2$, we obtain

$$\phi_i(N, u_T^\alpha) = \begin{cases} \frac{2}{t}, & \text{if } i \in T; \\ 0, & \text{if } i \notin T. \end{cases}$$

With linearity, the function ϕ is uniquely determined on $\mathcal{C}_\alpha^N$.

For any $\alpha \in [0,1]$, it is obvious that the quasi-Shapley value SH given by (16) satisfies symmetry, linearity and the null player property. We now prove that the quasi-Shapley value satisfies power efficiency.

Again, we use the function $I_i : 2^N \to \{0,1\}$ defined in (9) of the proof of of Theorem 4.1. For any $S \subseteq N$, denoting $r_n(S) := \frac{(s-1)!(n-s)!}{n!}$, it follows from (16) that $\sum_{i\in N} Sh_i(N,\mu) = \sum_{S\subseteq N, S\neq\emptyset} \frac{1}{\binom{n}{s}}[\mu(S) - \mu(N\backslash S)]$, in the same way as in (10) in the proof of Theorem 4.1, just deleting the term $\frac{1}{2-\alpha}$ everywhere.

Note that, in case $S \subsetneq N$, $S \neq \emptyset$, then $N\backslash S \subseteq N$, $N\backslash S \neq \emptyset$, while $\binom{n}{s} = \binom{n}{n-s}$. Hence, after cancelling out terms, we are left with

$$\sum_{i\in N} Sh_i(N,\mu) = \mu(N) - \mu(\emptyset). \qquad \square$$

8 The β-constant sum games

In the previous sections, given any $\alpha \in [0,1]$, we considered the class of α-constant-sum games in which the sum of threat powers of coalition S, $\mu(S)$, and its complementary coalition $N\backslash S$, $\mu(N\backslash S)$, is a fixed fraction between 0 and $\mu(N)$. We showed that axiomatizations of the Shapley value on the class of TU games are also valid on the classes of α-constant-sum games, but this is not the case for the equal division value.

The threat power of the empty set may not be 0, because it can represent the latent power in nature. Therefore, one can, as an alternative, consider the class of games, where the sum of the threat powers of coalition S and its complementary coalition $N\backslash S$ will be in between the threat power of the empty set and the threat power of the grand coalition.

Given any $\beta \in [0,1]$, a *β-constant-sum game* is a pair $\langle N, \eta\rangle$, where

- $N = \{1, 2, \ldots, n\}$ is a finite set of players;
- $\eta : 2^N \to \mathbb{R}$ is a function such that $\eta(S) + \eta(N\backslash S) = \beta\eta(\emptyset) + (1-\beta)\eta(N)$, for all $S \subseteq N$.

Denote by $\mathcal{C}_\beta^N$ the game space consisting of all β-constant-sum games with player set N. In particular, when $\beta = 0$, it is the class of constant-sum games. However, the class of β-constant-sum games does not cover the class of threat of games. Notice that, for $\beta = 1$, it does contain the threat games with $\eta(\emptyset) = 0$ (i.e. the intersection of the class of threat games and TU games). We can do a similar analysis as for α-constant sum games. Given any $\beta \in [0, 1]$. Let $T \subseteq N$, $T \neq \emptyset$. The *unanimity β-constant-sum game*, $\langle N, u_T^\beta \rangle \in \mathcal{C}_\beta^N$, is given as

$$u_T^\beta(S) = \begin{cases} 2 - 2\beta, & \text{if } S \supseteq T; \\ -2\beta, & \text{if } S \subseteq N \backslash T; \\ 1 - 2\beta, & \text{otherwise.} \end{cases}$$

Taking any $i \in N$, the unanimity β-constant-sum games, $\{\langle N, u_T^\beta \rangle\}_{T \subseteq N, T \ni i}$, span the class of β-constant-sum games, $\mathcal{C}_\beta^N$.

Given any $\beta \in [0, 1)$, let $K^\beta : \mathcal{G}^N \to \mathcal{C}_\beta^N$ and $L^\beta : \mathcal{C}_\beta^N \to \mathcal{G}^N$ be defined by, for all $S \subseteq N$,

$$(K^\beta v)(S) = \eta(S) - \eta(N \backslash S) + (1 - 2\beta)\eta(N), \tag{17}$$

and

$$(L^\beta \eta)(S) = (1 - \beta)\eta(S) + \beta\eta(N), \tag{18}$$

where $\langle N, v \rangle \in \mathcal{G}^N$ and $\langle N, \eta \rangle \in \mathcal{C}_\beta^N$.

Remark 2 *Notice that, if $\beta = 1$, then (18) would boil down to $(L^\beta \eta)(S) = \eta(N)$ for all $S \subseteq N$, and thus be a constant game where all coalitional worths are the same. Therefore, we exclude $\beta = 1$ from now on.*

A *value on $\mathcal{C}_\beta^N$* is a function that associates with each β-constant-sum game a vector of payoffs to the players. The relationships between the value on $\mathcal{G}^N$ and the value on $\mathcal{C}_\beta^N$ is as follows.

Proposition 8.1 *Given any $\beta \in [0, 1)$.*

(i) If φ is a value on $\mathcal{G}^N$, then $\varphi \circ L^\beta$ is a value on $\mathcal{C}_\beta^N$; if ψ is a value on $\mathcal{C}_\beta^N$, then $\psi \circ K^\beta$ is a value on $\mathcal{G}^N$.

(ii) If φ_1 and φ_2 are two different values on $\mathcal{G}^N$, then $\varphi_1 \circ L^\beta$ and $\varphi_2 \circ L^\beta$ are two different values on $\mathcal{C}_\beta^N$; if ψ_1 and ψ_2 are two different values on $\mathcal{C}_\beta^N$, then $\psi_1 \circ K^\beta$ and $\psi_2 \circ K^\beta$ are two different values on $\mathcal{G}^N$.

Given any $\beta \in [0, 1)$, the map L^β preserves efficiency, symmetry, linearity, marginality, fairness, and the null players. With Proposition 8.1, we can build the following results for values in the class of β-constant-sum games.

First, we define the following solutions. Given any $\beta \in [0,1)$, the **β-quasi-Shapley value**, is given by

$$SH_i^{\beta}(N,\eta) = \sum_{S \subseteq N, S \ni i} \left(\frac{(s-1)!(n-s)!}{n!} \cdot (1-\beta)[\eta(S) - \eta(N \backslash S)] \right),$$

for any β-constant-sum game $\langle N, \eta \rangle \in \mathcal{C}_{\beta}^{N}$, and $i \in N$.

Corollary 8.2 *Consider any $\beta \in [0,1)$. Then,*

(i) *The β-quasi-Shapley value is the unique value that satisfies efficiency, symmetry, linearity and the null player property.*
(ii) *The β-quasi-Shapley value is the unique value on $\mathcal{C}_{\beta}^{N}$ that satisfies efficiency, symmetry and marginality.*
(iii) *The β-quasi-Shapley value is the unique value on $\mathcal{C}_{\beta}^{N}$ that satisfies efficiency, the null player property and fairness.*

As with the class of α-constant-sum games, on the class of β-constant-sum games, $\beta \in [0,1)$, a game having δ-reducing players, $\delta \in [0,1)$, also implies that the game is a null game.

Proposition 8.3 *Take any $\beta \in [0,1)$ and $\delta \in [0,1)$. If there exists a δ-reducing player in β-constant-sum game $\langle N, \eta \rangle \in \mathcal{C}_{\beta}^{N}$, then $\eta(S) = 0$ for all $S \subseteq N$.*

Proof. For $\beta \in [0,1)$ and $\delta \in [0,1)$, suppose that player $i \in N$ is a δ-reducing player in β-constant-sum game $\langle N, \eta \rangle \in \mathcal{C}_{\beta}^{N}$. Then, by the δ-reducing player property,

$$\eta(N) = \delta \eta(N \setminus \{i\}). \tag{19}$$

By definition of β-constant-sum game, $\eta(\emptyset) + \eta(N) = \beta\eta(\emptyset) + (1-\beta)\eta(N)$, so

$$\eta(\emptyset) = -\frac{\beta}{1-\beta}\eta(N). \tag{20}$$

Also, by definition $\eta(\{i\}) + \eta(N \setminus \{i\}) = \beta\eta(\emptyset) + (1-\beta)\eta(N)$, which by i being a δ-reducing player gives $\delta\eta(\emptyset) + \eta(N \setminus \{i\}) = \beta\eta(\emptyset) + (1-\beta)\eta(N)$. Substituting (20) gives $-\frac{\delta\beta}{1-\beta}\eta(N) + \eta(N \setminus \{i\}) = -\frac{\beta^2}{1-\beta}\eta(N) + (1-\beta)\eta(N)$. Further substituting (19) gives $-\frac{\delta^2\beta}{1-\beta}\eta(N \backslash \{i\}) + \eta(N \backslash \{i\}) = -\frac{\delta\beta^2}{1-\beta}\eta(N \backslash \{i\}) + \delta(1-\beta)\eta(N \backslash \{i\}) \Leftrightarrow \frac{(1-\delta)[1-\beta(1-\delta)]}{1-\beta}\eta(N \backslash \{i\}) = 0$. Since $(1-\delta)[1-\beta(1-\delta)] \neq 0$, it holds that $\eta(N \backslash \{i\}) = 0$. By (19) and (20), then also $\eta(N) = 0$ and $\eta(\emptyset) = 0$.

We are left to show that $\eta(S) = 0$ for all $S \subseteq N$. Again, by the δ-reducing player property,

$$\eta(S) = \delta\eta(S \setminus \{i\}) \text{ for all } S \subseteq N,\ i \in S, \tag{21}$$

and by the definition of β-constant-sum game, we have for $S \subseteq N$ with $i \in S$,

$$\eta(S) + \eta(N \setminus S) = \beta\eta(\emptyset) + (1-\beta)\eta(N) = \eta(S \setminus \{i\}) + \eta((N \setminus S) \cup \{i\})$$

which, with (21), implies that $\delta\eta(S\setminus\{i\})+\eta(N\setminus S) = \eta(S\setminus\{i\})+\delta\eta(N\setminus S) \Leftrightarrow (1-\delta)\eta(N \setminus S) = (1-\delta)\eta(S \setminus \{i\}) \Leftrightarrow \eta(N \setminus S) = \eta(S \setminus \{i\} \Rightarrow (\delta+1)\eta(S \setminus \{i\}) = \delta\eta(S \setminus \{i\}) + \eta(S \setminus \{i\}) = \eta(S) + \eta(S \setminus \{i\}) = \eta(S) + \eta(N \setminus S) = \beta\eta(\emptyset) + (1-\beta)\eta(N) = 0 \Rightarrow \eta(S \setminus \{i\}) = 0$. Then also $\eta(S) = 0$ for all $S \subseteq N$ with $i \in S$. □

From this, on the class of β-constant-sum games, we can also see that efficiency and symmetry imply the δ-reducing player property for any $\delta \in [0, 1)$. Specifically, this holds for the nullifying player property. Further, we see that efficiency, symmetry, linearity and the nullifying player property do not give uniqueness since, besides the equal division value, for example, also the β-quasi Shapley value satisfies these axioms on any class of β-constant sum games.

Corollary 8.4 *Take any $\beta \in [0, 1)$.*

- *Any value on $\mathcal{C}^N_\beta$ that satisfies efficiency and symmetry, also satisfies the δ-reducing player property on $\mathcal{C}^N_\beta$ for any $\delta \in [0, 1)$.*
- *Specifically, any value on $\mathcal{C}^N_\beta$ that satisfies efficiency and symmetry, also satisfies the nulliffying player property on $\mathcal{C}^N_\beta$.*
- *For $\delta \in [0, 1)$, efficiency, symmetry, linearity, the δ-reducing player property do not give uniqueness on $\mathcal{C}^N_\beta$.*

9 Conclusions

Khmelnitskaya (2003, [3]) and Kohlberg and Neyman (2018, [4]) showed that classical axiomatizations of the Shapley value hold for constant sum games, respectively games of threats. In this paper, we showed that this can be extended to any class of α-constant sum games, where the sum of the worth of a coalition and its complement always equals the same fraction of the worth of the grand coalition. This is an interesting feature of these classes of games since mostly, when considering a subclass of TU-games, a classical axiomatization of the Shapley value does not give uniqueness since some axioms (in particular axioms that compare different games such as linearity, marginality and fairness) have less bite when we consider subclasses of games.

We showed that a simlar analysis cannot be done for the equal division value. We also introduced an alternative efficiency, called power efficiency, which reflects that the players allocate among themselves what they can earn with their ability to cooperate, and showed that replacing efficiency with this power efficiency characterizes on any class of α-constant-sum games, a value that is defined with exactly the same formula as the Shapley value and does not depend on α. Finally, we introduced another way to combine the threats of power of coalitions and their complements, leading to the class of β-constant sum games. Future research will be devoted to extending the anaylsis of this last class of games.

We remark that in this paper the parameter α determined a class of games, and on these classes we characterized a Shapley type value by classical (non-parametrized) axioms. In the literature on classical TU games, there exist classes of (parametrized) solutions, such as the classes of egalitarian Shapley values and discounted Shapley values, that can be characterized by parametrized axioms. We want to stress the difference with the underlying paper, where we only consider one solution but apply it to different classes of games. Another future goal is to consider classes of (parametrized) solutions, such as the ones mentioned above, for classes of α-constant sum games.

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