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THE SHAPLEY VALUE, PROPER SHAPLEY VALUE, AND SHARING RULES FOR COOPERATIVE VENTURES

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ABSTRACT:

Moulin (1987) studies the equal and proportional sharing rule for a special class of cooperative games that he calls joint venture games. Proportionality is an important principle in allocation problems. Besides some special cases, it is not obvious how proportionality should be applied in cooperative TU-games. Such special cases, where proportionality is obvious, are inessential games and cooperative joint venture games. In this paper, we discuss an explicit axiom that shows that proper Shapley values can be seen as an appropriate way to express proportionality in value allocation in cooperative TU-games. We characterize positive proper Shapley values by affine invariance and an axiom that requires proportional allocation according to the individual singleton worths in generalized joint venture games. As a counterpart, we show that affine invariance and an axiom that requires equal allocation of the surplus in generalized joint venture games, characterize the positive part of the Shapley value among the single-valued solutions.

KEY WORDS: Equity principle, Cooperative venture game, Shapley value, proper Shapley value.
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1. INTRODUCTION

Situations where a set of agents can generate certain worths by cooperating, can be described by a *cooperative transferable utility game* (or simply *TU-game*). A TU-game consists of a (finite) set of *players* and a *characteristic function* that assigns to every subset of the player set, called *coalitions*, a real number representing the (transferable) utility that is obtained by the players in this coalition cooperate together, without the help of the players outside the coalition. A main question is how to allocate the worths that the coalitions can earn over the individual players. A *payoff vector* in a TU-game is a vector which dimension is equal to the number of players, and which components give the payoffs to individual players. A *solution* is a mapping which prescribes for every game how the worth of the grand coalition consisting of all players, should be allocated over the players. We call a solution *single-valued* if to every game it assigns a unique payoff vector, and we call it *set-valued* if to every game it assigns a set of payoff vectors (which can be empty or a singleton).

The best known single-valued solution for TU-games is the *Shapley value*, Shapley (1953b), which distributes the so-called *Harsanyi dividends* of the game equally among the players in the corresponding coalitions. These Harsanyi dividends somehow can be seen as the 'real' value added of coalitions which was not yet realized by its subcoalitions, see also Billot and Thisse (2005). The *weighted Shapley values*, see Shapley (1953a), are a modification of the Shapley value, where, besides their contributions in the game, the players have exogenous positive weights such that in every game the Harsanyi dividend of every coalition is allocated over the players proportional to these weights.[1] These weights might express some external asymmetry between the players, such as bargaining power or their influence in a network.

Whereas the Shapley value is based on the idea of equally sharing the dividends among the players in the corresponding coalition, there are also situations where *proportionality* seems a more appropriate equity principle. For example, standard business practice of dividing a firm's profit is to do this proportionally to investment (constant return per share). This is also in line with the sociopsychological equity theory of Homans (1961) and Selten (1978) that suggests proportionality rather than equality. Proportionality in TU-game solutions has been expressed in, e.g., Kalai (1977), Roth (1979), Hart and Mas-Colell (1989), Feldman (1998, 1999), Ortmann (2000) and Béal, Ferrières,

[1] Kalai and Samet (1987) consider weighted Shapley values allowing weights to be equal to zero, but to avoid dividing by zero the player set is also partitioned in certain equivalence classes which are ordered.

Rémila and Solal (2018). For games with more players there is not an obvious way how to apply proportionality. In the underlying paper, we consider the *proper Shapley value* introduced by Vorob'ev and Liapounov (1998), whose existence for monotone games is shown by van den Brink, Levínský and Zelený (2015). This solution expresses proportionality by a fixed point argument, by allocating the Harsanyi dividends proportional to endogenous weights such that the resulting payoff allocation exactly equals the weights. This is an expression of proportionality in the sense that for two-player games, it implies to allocate the worth of the grand coalition proportional to the singleton worths (if these are positive). For further motivation of this proportionality principle in TU-games, we refer to the introduction of van den Brink, Levínský and Zelený (2015).

The purpose of the underlying paper is to make a comparison between the proper Shapley value and the traditional Shapley value. A first main difference between the Shapley value and proper Shapley value is that the first is a single-valued solution, while the second is a set-valued solution. As mentioned, van den Brink, Levínský and Zelený (2015) proved existence/nonemptiness of the proper Shapley value for monotone games, but there can be more than one payoff vector being a fixed point of the 'weighted Shapley mapping'. Although possible characterizations of the Shapley value and proper Shapley value should be formulated using axioms for single-valued solutions and set-valued solutions respectively, we will introduce axioms that makes a comparison between these two solutions possible.

A second main difference between the two solutions is that the Shapley value is a linear, and therefore additive, solution while the proper Shapley value is not. The first axiom we introduce in this paper is a weaker version of additivity which is satisfied by both solutions (of course, taking account of the fact that one is a single-valued, and the other is a set-valued solution, see Section 3 for the details). This axiom, called *affine invariance*, roughly says that, having two games that have the same solution outcome, implies that also any affine combination of these two games have this same solution outcome.

Besides having this axiom in common, we introduce two equity principles that each characterize each one of the two solutions. Both equity principles are stated for a special type of game, called *generalized joint venture game*, being a class of games that generalizes the ordinary *joint venture games* introduced by Moulin (1987). A joint venture game is a game such that nonzero Harsanyi dividends are only allowed for the grand coalition of all players and the singletons. In other words, a joint venture game is a linear combination of an inessential game and the unanimity game of the grand coalition. A generalized joint venture game is a game such that there is at most one nonsingleton coalition with a nonzero dividend, but this is not necessarily the grand coalition. It turns out that these games form a basis for the space of TU-games, which we will use in characterizing the Shapley and proper Shapley values. To characterize the Shapley value, we extend the equal sharing principle of Moulin (1987), and introduce a *generalized equal sharing principle* for single-valued solutions, which requires that in a generalized joint venture game, every singleton receives its own worth, and the dividend of the only nonsingleton coalition with nonnegative dividend is allocated equally over the players in this coalition. It turns out that, together with affine invariance, this characterizes the Shapley value among the single-valued solutions.

To characterize the proper Shapley value among the set-valued solutions, we extend the equal sharing principle of Moulin (1987), and introduce a *generalized proportional sharing principle* for set-valued solutions, which requires that in a generalized joint venture game, every singleton receives its own worth, and the dividend of the only nonsingleton coalition with nonnegative dividend is allocated proportionally to the singleton worths. It turns out that, together with affine invariance, this characterizes the proper Shapley value among the set-valued solutions. Besides explaining the difference between the two solutions, this also gives a stronger motivation why to consider the proper Shapley value as an expression of proportionality. Whereas in its definition, proportionality is obviously applied for two-player games, for general games proportionality is translated as a fixed point argument. With this generalized proportional sharing principle, for any game the role of the singleton worths in the proportional allocation of any Harsanyi dividend is made explicit.

2. Preliminaries

A *cooperative transferable utility game* (or simply *TU-game*) is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set consisting of n players and $v\colon 2^N \to \mathbb{R}^N$, satisfying $v(\emptyset) = 0$ is a *characteristic function* such that for any *coalition* $S \subseteq N$, the real number $v(S)$ is the *worth* of S, which the members of the coalition S can distribute among themselves. In certain applications, $v(S)$ can also be interpreted as the *cost* which has to be split among members of S. In this paper, we assume the set of players N to be fixed. The set of all TU-games on the player set N is denoted by $\mathcal{G}$. A TU-game (N, v) is *monotone* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$.

Let $(N, v) \in \mathcal{G}$. The *Harsanyi dividends* $\Delta_{N,v}(S)$, where $S \subseteq N$, are defined inductively by

$$\Delta_{N,v}(S) = \begin{cases} 0, & \text{for } S = \emptyset, \\ v(S) - \sum_{T \subsetneq S} \Delta_{N,v}(T), & \text{for } S \neq \emptyset \end{cases}$$

(see Harsanyi, 1959). Let us note that $v(S) = \sum_{T \subseteq S} \Delta_{N,v}(T)$ for every $S \subseteq N$. This formula shows that dividends uniquely determine the characteristic function. A coalition $S \subseteq N$ in the game (N, v) is called *essential* if $\Delta_{N,v}(S) \neq 0$, otherwise S is called *inessential*. Any TU-game, where all the coalitions but singletons are inessential, is called an *inessential* game.

We employ the following notation. Let $y \in \mathbb{R}^N$ and $S \subseteq N$. The symbol y_S stands for $\sum_{i \in S} y_i$. By convention, the value of any empty sum of real numbers is zero, i.e., $y_\emptyset = 0$. A *payoff vector* in a TU-game (N, v) is an n-dimensional vector whose components are the payoffs of the corresponding players. A payoff vector $x \in \mathbb{R}^N$ for a game (N, v) is *efficient* if it exactly distributes the worth $v(N)$ of the grand coalition N, i.e., if $x_N = v(N)$. In the literature, efficient payoff vectors are also called *preimputations*. The set of all efficient payoff vectors of (N, v) is denoted by $X(N, v)$, the set of all efficient payoff vectors with positive coordinates is denoted by $X_+(N, v)$ and the set of all efficient payoff vectors with nonnegative coordinates is denoted by $X_0(N, v)$.

Let $\mathcal{C} \subseteq \mathcal{G}$ be a subclass of games. A *single-valued solution* on $\mathcal{C}$ is a function f that assigns to every game $(N, v) \in \mathcal{C}$ a payoff vector $f(N, v) \in \mathbb{R}^N$. A *set-valued solution* F on $\mathcal{C}$ assigns a set of payoff vectors $F(N, v) \subset \mathbb{R}^N$ to every game $(N, v) \in \mathcal{C}$.

The best known single-valued solution for TU-games is the *Shapley value* (Shapley, 1953b) which distributes the Harsanyi dividends of the game *equally* among the players in the corresponding coalitions, i.e., the Shapley value is the function $\varphi\colon \mathcal{G} \to \mathbb{R}^N$ defined by $\varphi(N, v) = \big(\varphi_i(N, v)\big)_{i \in N}$, where

$$\varphi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{1}{|S|} \Delta_{N,v}(S), \quad i \in N.$$

The symbol $|S|$ denotes the cardinality of S.

Given a weight vector $\omega \in \mathbb{R}^N$ with positive weights $\omega_i > 0,\ i \in N$, the corresponding *weighted Shapley value* (Shapley, 1953a) is the function $\varphi^\omega\colon \mathcal{G} \to \mathbb{R}^N$ defined by

$$\varphi_i^\omega(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\omega_i}{\omega_S} \Delta_{N,v}(S), \quad i \in N.$$

The weighted Shapley value thus distributes the dividends of coalitions proportionally to the exogenously given weights of the players. Clearly, if all weights ω_i are equal to each other then the weighted Shapely value $\varphi^{\omega}(N, v)$ is equal to the Shapley value $\varphi(N, v)$. Further, observe that if ω and $\widetilde{\omega}$ are positive weight vectors with $\widetilde{\omega}_i/\widetilde{\omega}_j = \omega_i/\omega_j$ for all $i, j \in N$, then $\varphi^{\omega}(N, v) = \varphi^{\tilde{\omega}}(N, v)$.

Another solution, the proper Shapley value, was introduced by Vorob'ev and Liapounov (1998) for games with positive dividends and generalized for monotone TU-games as a set-valued solution by van den Brink, Levínský, and Zelený (2015). This solution is defined as follows. Let $(N, v) \in \mathcal{G}$. To simplify notation we denote $h(x) = \varphi^{x}(N, v)$ for $x \in X_{+}(N, v)$. We define a multi-valued mapping H assigning a subset of $\mathbb{R}^N$ to each element x of $X_0(N, v)$ by

$$H(x) = \{\alpha \in \mathbb{R}^N \,|\, \text{there exists a sequence } (x^j) \subseteq X_{+}(N, v) \text{ such that } x^j \to x \text{ and } h(x^j) \to \alpha\}.$$

Note that the multi-valued function H depends on (N, v), but we will omit this parameter for the sake of simplicity since there is no danger of confusion. The graph of H is just the closure of the graph of the mapping h.

Definition 1. Let $(N, v) \in \mathcal{G}$. A vector $x \in X_0(N, v)$ is called a *proper Shapley value* of (N, v) if $x \in H(x)$. We denote $\mathrm{PSV}(N, v) = \{x \in X_0(N, v) |\, x \text{ is a proper Shapley value of } (N, v)\}$ and $\mathcal{G}_P = \{(N, v) \in \mathcal{G} |\ \mathrm{PSV}(N, v) \neq \emptyset\}$. We refer to the solution that assigns to every game $(N, v) \in \mathcal{G}_P$ the set of all proper Shapley values $\mathrm{PSV}(N, v)$ as the *proper Shapley solution*.

3. Axiomatization and main results

We start with the following definition.

Definition 2. We say that a class $\mathcal{C} \subseteq \mathcal{G}$ is *affine* if for any TU-games $(N, v_1), (N, v_2) \in \mathcal{C}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$, we have $\lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{C}$.

The next axiom is a weakening of Shapley's additivity axiom.

Axiom 1. Let $\mathcal{C} \subseteq \mathcal{G}$ be affine. A single-valued solution f satisfies *affine invariance* on $\mathcal{C}$ if for any TU-games $(N, v_1), (N, v_2) \in \mathcal{C}$ with $f(N, v_1) = f(N, v_2)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$, we have $f(N, \lambda_1 v_1 + \lambda_2 v_2) = f(N, v_1) = f(N, v_2)$.

For set-valued solutions, affine invariance is defined as follows.

Axiom 2. Let $\mathcal{C} \subseteq \mathcal{G}$ be affine. A set-valued solution F satisfies *affine invariance* on $\mathcal{C}$ if for any TU-games $(N, v_1), (N, v_2) \in \mathcal{C}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$, and $x \in \mathbb{R}^N_+$ with $x \in F(N, v_1) \cap F(N, v_2)$, we have $x \in F(N, \lambda_1 v_1 + \lambda_2 v_2)$.

Remark 1. One can easily prove that a set-valued solution F satisfies affine invariance on an affine class $\mathcal{C}$ if and only if, for every $k \in \mathbb{N}$, $(N, v_1), \ldots, (N, v_k) \in \mathcal{C}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ with $\sum_{i=1}^k \lambda_i = 1$, and $x \in \mathbb{R}^N_+$ with $x \in F(N, v_i)$ for every $i \in \{1, \ldots, k\}$, we have $x \in F(N, v)$ whenever $v := \sum_{i=1}^k \lambda_i v_i$. Notice that by repeated application of condition from Definition 2, it follows that $v := \sum_{i=1}^k \lambda_i v_i \in \mathcal{C}$.

Remark 2. Notice that Axiom 1 (for single-valued solutions) is stated for all payoff vectors, while Axiom 2 (for set-valued solutions) is stated only for nonnegative payoff vectors. We will argue later, that we could also restrict Axiom 1 to nonnegative payoff vectors and have similar results.

Moulin (1987) defines a joint venture game as a TU-game where the intermediate coalitions are inessential, formally, $(N, v) \in \mathcal{G}$ is a *joint venture game*, if $\Delta_{N,v}(S) = 0$ whenever $|S| \neq 1$ and $S \neq N$. For this class of games he analyzes two sharing rules, namely the equal sharing and proportional sharing rule. We find Moulin's set-up suitable for studying the proportional sharing rule. Let us note that Moulin's proportional sharing rule is structurally identical to an older concept of Homans (1961), see also Selten(1978), who were referring to it as to an *equity principle*. Further, since any two-player game belongs to the class of joint venture games, the proportional sharing rule for two-player games corresponds to *proportional standardness for two-player games* as defined by Ortmann (2000).

Clearly, for general games (with more than two players), the proportionality principle is not obvious, because once we have several nonsingleton essential coalitions in the game, the application of proportionality becomes nontrivial. Still, to apply the proportionality principle in the broadest possible context, we can go a step further than Moulin by introducing the class of games having at most one nonsingleton essential coalition. Notice that this nonsingleton coalition can be the grand coalition N in which case it is an ordinary joint venture game.

Definition 3. A TU-game (N, v) is called a *generalized joint venture game* if there exists a set $E \subseteq N$ such that $\Delta_{N,v}(T) = 0$ whenever $|T| \neq 1$ and $T \neq E$. The set of all generalized joint venture games is denoted by $\mathcal{G}_1$. For $(N, v) \in \mathcal{G}_1$, the symbol $E(N, v)$ denotes the unique nonsingleton essential coalition if such a coalition exists, otherwise $E(N, v) = \emptyset$.

Remark 3. The symbol $\mathcal{G}_1$ is chosen to stress the fact that any game in $\mathcal{G}_1$ contains at most one nonsingleton essential coalition. The class $\mathcal{G}_1$ clearly contains all inessential games as well as joint venture games.

Now, exactly as Moulin for joint venture games, we define two possible sharing rules for general joint venture games. The generalized equal sharing principle requires that in generalized joint venture games, the surplus of the only essential nonsingleton coalition is allocated equally over the players in that coalition.

Axiom 3. A single-valued solution f satisfies the *generalized equal sharing principle* if for every $(N, v) \in \mathcal{G}_1$, we have

$$f(N,v)_i = \begin{cases} v(\{i\}) + \frac{1}{|E(N,v)|}\Delta_{N,v}\big(E(N,v)\big), & \text{for } i \in E(N,v), \\ v(\{i\}), & \text{for } i \in N \setminus E(N,v). \end{cases}$$

Alternatively, the generalized proportional sharing principle applies proportionality to the generalized joint venture games. It requires to split the surplus of the only essential nonsingleton coalition among its members proportionally to their individual worths whenever possible. Notice that although F is a set-valued solution, this principle assigns a unique payoff vector to every generalized joint venture game.

Axiom 4. A set-valued solution F satisfies the *generalized proportional sharing principle* if for every $(N, v) \in \mathcal{G}_1$ and $x \in F(N, v)$ we have

$$x_i = \begin{cases} v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in E(N,v)} v(\{j\})}\Delta_{N,v}\big(E(N,v)\big), & \text{if } \sum_{j\in E(N,v)} v(\{j\}) \neq 0 \text{ and } i \in E(N,v), \\ v(\{i\}), & \text{for } i \in N \setminus E(N,v). \end{cases}$$

Remark 4. Observe that if $(N, v) \in \mathcal{G}$ is inessential, thus $(N, v) \in \mathcal{G}_1$, then $E(N, v) = \emptyset$ and for any F that satisfies the generalized proportional sharing principle, we have that $x \in F(N, v)$ implies that $x_i = v(\{i\})$. An analogous observation can be made also for the generalized equal sharing principle.

Now, the main propositions of this paper can be stated. The first result is a characterization of the Shapley value by affine invariance and the generalized equal sharing principle.[2]

[2] It will each time be clear from the context whether we consider affine invariance for single-valued, or set-valued solutions.

Proposition 1. *The Shapley value φ is the unique single-valued solution that satisfies affine invariance and the generalized equal sharing principle.*

The proof of this and the other propositions can be found in the appendix.

Obviously, the proper Shapley solution does not satisfy the generalized equal sharing principle.[3] Instead, restricting ourselves to nonnegative payoff vectors, the generalized proportional sharing principle implies that we allocate according to a proper Shapley value.

Proposition 2. *Let F be a set-valued solution satisfying affine invariance on its affine domain* $\operatorname{dom} F$ *and the generalized proportional sharing principle. Then $F(N,v) \cap \mathbb{R}^N_+ = \mathrm{PSV}(N,v) \cap \mathbb{R}^N_+$ for every $(N,v) \in \mathcal{G}_P$.*

Proposition 2 would be meaningless if there does not exist a solution satisfying affine invariance and the generalized proportional sharing principle. It turns out that the following solution satisfies these axioms. For $(N,v) \in \mathcal{G}$, define $g_{N,v} \colon \mathbb{R}^N \to \mathbb{R}^N$ by

$$g_{N,v}(x)_i = \sum_{\substack{S \ni i \\ x_S \neq 0}} \frac{x_i}{x_S} \Delta_{N,v}(S) + \sum_{\substack{S \ni i \\ x_S = 0 \\ \sum_{j \in S} v(\{j\}) \neq 0}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_{N,v}(S) + \sum_{\substack{S \ni i \\ x_S = 0 \\ \sum_{j \in S} v(\{j\}) = 0}} \frac{1}{|S|} \Delta_{N,v}(S), \qquad i \in N,$$

and consider the set valued solution G defined by

$$G(N,v) = \{x \in \mathbb{R}^N \mid g_{N,v}(x) = x\}.$$

The weight mapping $g_{N,v}$ is constructed in such a way that, if possible, the corresponding dividend is split proportionally to weights that are equal to the final payoffs. If this is not possible (i.e., the sum of the payoffs/weights of the players is zero), then, if possible, the dividend is split proportionally to the individual worths of the players. If this is not possible either (i.e., also the sum of the singleton worths of the players in the coalition is zero), then the dividend is split equally.

Proposition 3. *The set-valued solution G satisfies affine invariance and the generalized proportional sharing principle.*

[3] Similar as the generalized equal sharing principle, the generalized proportional sharing principle for set-valued solutions requires to assign a unique payoff vector to every generalized joint venture game.

4. Concluding remarks

In this paper, we compared the Shapley value with the proper Shapley value (solution) by characterizing them using affine invariance and generalizations of Moulin (1987)'s equity, respectively, proportionality principles, applied to generalized joint venture games. Besides making the Shapley and proper Shapley value comparable, using the generalized proportionality principle, we explicitly stated the proportionality feature of the proper Shapley values by applying proportionality to generalized joint venture games, and extending to other games by affine invariance, being a weak additivity axiom which is satisfied by both solutions.

Whereas the Shapley value is well-defined for all TU-games, proper Shapley values is not. Therefore, we characterized the nonnegative proper Shapley values in Propositions 2 and 3. In Proposition 1, we characterized the Shapley value without the restriction to nonnegative payoff vectors. For single-valued solutions, we could redefine affine invariance restricting it only to the case of nonnegative payoff vectors in a similar way as Proposition 1, and characterize the Shapley value payoff vectors in case these are nonnegative.

5. Proofs

5.1. **Proof of Proposition 1.** Since the Shapley value obviously satisfies affine invariance and the generalized proportional sharing principle, it remains to prove uniqueness.

Let $(N, v) \in \mathcal{G}_1$ and take any $x \in \mathbb{R}^N$. Now, for $S \subseteq N, |S| \geq 2$, we define the game (N, v_S^x) by

$$\Delta_{N,v_S^x}(T) = \begin{cases} -\Delta_{N,v}(S), & \text{for } T = S, \\ x_i + \frac{1}{|S|}\Delta_{N,v}(S), & \text{for } T = \{i\},\ i \in S, \\ x_i, & \text{for } T = \{i\},\ i \in N \setminus S, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have $(N, v_S^x) \in \mathcal{G}_1$. Further, we define the inessential game $(N, \nu^x) \in \mathcal{G}_1$ by

$$\Delta_{N,\nu^x}(T) = \begin{cases} x_i, & \text{for } T = \{i\},\ i \in N, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Finally, let us define the game (N, w^x) as a sum

$$w^x = v - (2^n - n - 1)\nu^x + \sum_{S \subseteq N, |S| \geq 2} v_S^x. \tag{2}$$

One can easily check that

$$\Delta_{N,w^x}(T) = \begin{cases} 0, & \text{for } |T| \geq 2, \\ v(\{i\}) + \sum_{S \ni i, |S| \geq 2} \frac{1}{|S|} \Delta_{N,v}(S) = \sum_{S \ni i} \frac{1}{|S|} \Delta_{N,v}(S), & \text{for } T = \{i\}. \end{cases}$$

The linear combination in (2) is affine, since we have

$$1 - (2^n - n - 1) + \sum_{S \subseteq N, |S| \geq 2} 1 = 1 - 2^n + \sum_{S \subseteq N} 1 = 1 - 2^n + 2^n = 1. \tag{3}$$

Now, we show that for $x \in \mathbb{R}^N$ we have $x = f(N, v)$ if and only if $x = \varphi(N, v)$. First, assume that $x = f(N, v)$. Since f satisfies the generalized equal sharing principle, we get $x = f(N, v^x_S)$ for every $S \subseteq N, |S| \geq 2$, and $x = f(N, \nu^x)$. Thus, by affine invariance of f we obtain $x = f(N, w^x)$. The game (N, w^x) is inessential and therefore

$$x_i = w^x(\{i\}) = \sum_{S \ni i} \frac{1}{|S|} \Delta_{N,v}(S).$$

This shows that x is the Shapley value payoff of (N, v).

Now suppose that x is the Shapley value payoff vector of (N, v), i.e., $x = \varphi(N, v)$. Using (2) we can write

$$v = w^x + (2^n - n - 1)\nu^x - \sum_{S \subseteq N, |S| \geq 2} v^x_S. \tag{4}$$

Thus, v is an affine combination of games from $\mathcal{G}_1$, $x = f(N, \nu^x)$, and $x = f(N, v^x_S)$ for $S \subseteq N, |S| \geq 2$, by the generalized equal sharing principle. Since x is the Shapley value payoff vector of (N, v), we get $x = f(N, w^x)$ by the generalized equal sharing principle. Now, affine invariance of f yields $x = f(N, v)$ and we are done.

5.2. **Proof of Proposition 2.**

Claim. *Let $(N, v) \in \mathcal{G}$, $x \in \mathbb{R}^N_+$. Then there exist games $(N, v^0), (N, v^1), \ldots, (N, v^k) \in \mathcal{G}_1$ such that*

(a) *v is an affine combination of $v^0, v^1, \ldots, v^k$ of the form $v = v^0 + \sum_{j=1}^k \gamma_j v^j$ with $\sum_{j=1}^k \gamma_j = 0$,*

(b) *$x \in F(N, v^j)$, $j = 1, \ldots, k$,*

(c) *we have*

$$\Delta_{N,v^0}(T) = \begin{cases} 0, & \textit{for } T \subseteq N, |T| \geq 2, \\ \sum_{S \ni i} \frac{x_i}{x_S} \Delta_{N,v}(S), & \textit{for } T = \{i\}, i \in N. \end{cases}$$

Proof of Claim. We set $\mathcal{A} = \{S \subseteq N \,|\, |S| \geq 2, \Delta_{N,v}(S) \neq 0\}$. For $S \in \mathcal{A}$ and $\alpha \in \mathbb{R}$, we define a game $(N, w_S^{x,\alpha})$ as follows

$$\Delta_{N,w_S^{x,\alpha}}(T) = \begin{cases} -\alpha\Delta_{N,v}(S), & \text{for } T = S, \\ x_i + \frac{x_i}{x_S}\alpha\Delta_{N,v}(S), & \text{for } T = \{i\}, i \in S, \\ x_i, & \text{for } T = \{i\}, i \notin S, \\ 0, & \text{otherwise.} \end{cases}$$

We have clearly $(N, w_S^{x,\alpha}) \in \mathcal{G}_1$. We define a game (N, v^0) as the following sum

$$v^0 = v - 2|\mathcal{A}|\nu^x + \sum_{S \in \mathcal{A}} (w_S^{x,\alpha_S} + w_S^{x,\beta_S}), \tag{5}$$

where ν^x is defined by (1), α_S and β_S are chosen in such a way that $\alpha_S \neq 0$, $\beta_S \neq 0$, $\alpha_S + \beta_S = 1$, $x_S + \alpha_S\Delta_{N,v}(S) \neq 0$ and $x_S + \beta_S\Delta_{N,v}(S) \neq 0$. Now we verify (a)–(c).

(a) Using (5) it is easy to choose $\gamma_1, \ldots, \gamma_k$ and to denote appropriately the games ν^x, $w_S^{x,\alpha_S}, w_S^{x,\beta_S}$, $S \in \mathcal{A}$, by $v^1, \ldots, v^k$ to satisfy (a).

(b) Because of the choice of α_S and β_S, $S \in \mathcal{A}$, the corresponding games satisfy $E(N, w_S^{x,\alpha_S}) = E(N, w_S^{x,\beta_S}) = S$,

$$\sum_{i \in S} w_S^{x,\alpha_S}(\{i\}) = x_S + \alpha_S\Delta_{N,v}(S) \neq 0 \quad \text{and}$$
$$\sum_{i \in S} w_S^{x,\beta_S}(\{i\}) = x_S + \beta_S\Delta_{N,v}(S) \neq 0.$$

Thus, we can apply the generalized proportional sharing principle for F to infer $x \in F(N, w_S^{x,\alpha_S})$ and $x \in F(N, w_S^{x,\beta_S})$. To see this, notice that obviously $w_S^{x,\alpha_S}(\{i\}) = x_i$ for every $i \notin S$ and, for every $i \in S$, we have

$$\begin{aligned} w_S^{x,\alpha_S}(\{i\}) &+ \frac{w_S^{x,\alpha_S}(\{i\})}{\sum_{j \in S} w_S^{x,\alpha_S}(\{j\})}\alpha_S\Delta_{N,v}(S) \\ &= x_i + \frac{x_i}{x_S}\alpha_S\Delta_{N,v}(S) - \frac{x_i + \frac{x_i}{x_S}\alpha_S\Delta_{N,v}(S)}{x_S + \alpha_S\Delta_{N,v}(S)}\alpha_S\Delta_{N,v}(S) \\ &= x_i + \frac{x_i}{x_S}\alpha_S\Delta_{N,v}(S) - \frac{x_i}{x_S}\alpha_S\Delta_{N,v}(S) = x_i \end{aligned}$$

and similarly for $x \in F(N, w_S^{x,\beta_S})$. Since $x \in F(N, \nu^x)$ and ν^x is inessential, by Remark 4 we get (b).

(c) Consider $T \subseteq N$ with $|T| \geq 2$. Then we have

$$\begin{aligned}\Delta_{N,v^0}(T) &= \Delta_{N,v}(T) - 2|\mathcal{A}|\Delta_{N,\nu^x}(T) + \sum_{S\in\mathcal{A}}\big(\Delta_{N,v_S^{x,\alpha_S}}(T) + \Delta_{N,v_S^{x,\beta_S}}(T)\big)\\ &= \Delta_{N,v}(T) + \Delta_{N,v_T^{x,\alpha_T}}(T) + \Delta_{N,v_T^{x,\beta_T}}(T)\\ &= \Delta_{N,v}(T) - \alpha_T\Delta_{N,v}(T) - \beta_T\Delta_{N,v}(T) = \Delta_{N,v}(T)(1-\alpha_T-\beta_T) = 0.\end{aligned}$$

Now choose $T \subseteq N$ with $T = \{i\}, i \in N$. Then we have

$$\begin{aligned}\Delta_{N,v^0}(\{i\}) &= \Delta_{N,v}(\{i\}) - 2|\mathcal{A}|\Delta_{N,\nu^x}(\{i\}) + \sum_{S\in\mathcal{A}}\big(\Delta_{N,v_S^{x,\alpha_S}}(\{i\}) + \Delta_{N,v_S^{x,\beta_S}}(\{i\})\big)\\ &= \Delta_{N,v}(\{i\}) - 2|\mathcal{A}|x_i\\ &\quad + \sum_{S\in\mathcal{A},i\in S}\big(x_i + \frac{x_i}{x_S}\alpha_S\Delta_{N,v}(S) + x_i + \frac{x_i}{x_S}\beta_S\Delta_{N,v}(S)\big) + \sum_{S\in\mathcal{A},i\notin S}(x_i+x_i)\\ &= \Delta_{N,v}(\{i\}) + \sum_{S\in\mathcal{A},i\in S}\frac{x_i}{x_S}\Delta_{N,v}(S)\\ &= \sum_{S\ni i}\frac{x_i}{x_S}\Delta_{N,v}(S). \qquad \text{(by the definition of } \mathcal{A})\end{aligned}$$

This ends the proof of the claim. □

Now, we show that $F(N,v) \cap \mathbb{R}_+^N = \mathrm{PSV}(N,v) \cap \mathbb{R}_+^N$.

Proof of the inclusion $F(N,v) \cap \mathbb{R}_+^N \subseteq \mathrm{PSV}(N,v)$. Suppose that $x \in F(N,v) \cap \mathbb{R}_+^N$. Take $v^0,\dots,v^k$ for (N,v) and x according to the Claim. By (b), we have $x \in F(N,v^j)$ for $j \in \{1,\dots,k\}$ and (N,v^0) is an affine combination of (N,v) and $(N,v^1),\dots,(N,v^k)$. By affine invariance of F, we get $x \in F(N,v^0)$. Since the game (N,v^0) is inessential, we have for $i \in N$

$$x_i = v^0(\{i\}) = \sum_{S\ni i}\frac{x_i}{x_S}\Delta_{N,v}(S).$$

Thus, we have $x = h(x)$ and, consequently, $x \in H(x)$. This shows that $x \in \mathrm{PSV}(N,v)$. □

Proof of the inclusion $\mathrm{PSV}(N,v) \cap \mathbb{R}_+^N \subseteq F(N,v)$. Consider $x \in \mathrm{PSV}(N,v) \cap \mathbb{R}_+^N$. Again, take $v^0,\dots,v^k$ for (N,v) and x according to the Claim. Since $x \in \mathrm{PSV}(N,v) \cap \mathbb{R}_+^N$, we have $x = h(x)$, i.e.,

$$x_i = \sum_{S\ni i}\frac{x_i}{x_S}\Delta_{N,v}(S), \qquad i \in N.$$

This shows that $x \in F(N,v^0)$. Since $x \in F(N,v^j), j = 1,\dots,k$, by (b) of Claim, we obtain $x \in F(N,v)$ using affine invariance of F. □

5.3. **Proof of Proposition 3.** *Affine invariance on $\mathcal{G}$.* Let $(N, v^1), (N, v^2) \in \mathcal{G}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$. Denote $v = \lambda_1 v^1 + \lambda_2 v^2$. Suppose that $x \in \mathbb{R}^N_+$ satisfies $x \in G(N, v^1) \cap G(N, v^2)$. Then, we have

$$x_i = g_{N,v^l}(x)_i = \sum_{S \ni i} \frac{x_i}{x_S} \Delta_{N,v^l}(S) \qquad \text{for } l \in \{1, 2\},\ i \in N.$$

This implies that

$$\begin{aligned} x_i &= \lambda_1 x_i + \lambda_2 x_i = \lambda_1 g_{N,v^1}(x)_i + \lambda_2 g_{N,v^2}(x)_i \\ &= \sum_{S \ni i} \frac{\lambda_1 x_i}{x_S} \Delta_{N,v^1}(S) + \sum_{S \ni i} \frac{\lambda_2 x_i}{x_S} \Delta_{N,v^2}(S) \\ &= \sum_{S \ni i} \frac{x_i}{x_S} \big(\lambda_1 \Delta_{N,v^1}(S) + \lambda_2 \Delta_{N,v^2}(S)\big) \\ &= \sum_{S \ni i} \frac{x_i}{x_S} \Delta_{N,v}(S) = g_{N,v}(x)_i, \qquad i \in N. \end{aligned}$$

Thus, we have $x \in G(N, v)$.

Generalized proportional sharing principle. Let $(N, v) \in \mathcal{G}_1$ and $x \in G(N, v)$, i.e., $g_{N,v}(x) = x$. To simplify the notation we denote $E = E(N, v)$. We distinguish several possibilities.

(a) Assume that $x_E \neq 0$. Then the equality $x = g_{N,v}(x)$ can be written as

$$x_i = \begin{cases} \Delta_{N,v}(\{i\}) + \frac{x_i}{x_E} \Delta_{N,v}(E), & \text{for } i \in E, \\ \Delta_{N,v}(\{i\}), & \text{for } i \in N \setminus E. \end{cases} \tag{6}$$

For $i \in N \setminus E$, we have the desired equality $x_i = v(\{i\})$. If $\sum_{i \in E} \Delta_{N,v}(\{i\}) = \sum_{i \in E} v(\{i\}) = 0$, then there is nothing to verify for $i \in E$ since the generalized proportional sharing principle does not require anything in this case. So, assume that $\sum_{i \in E} \Delta_{N,v}(\{i\}) \neq 0$. Summing up x_i over $i \in E$, we infer from (6)

$$x_E = \sum_{i \in E} \Delta_{N,v}(\{i\}) + \Delta_{N,v}(E) = \sum_{i \in E} v(\{i\}) + \Delta_{N,v}(E). \tag{7}$$

Further, using (6), we infer for $i \in E$,

$$x_i = \frac{v(\{i\})}{1 - \frac{\Delta_{N,v}(E)}{x_E}}$$

and using (7) we get

$$x_i = \frac{v(\{i\})}{1 - \frac{\Delta_{N,v}(E)}{x_E}} = \frac{v(\{i\})}{1 - \frac{\Delta_{N,v}(E)}{\sum_{j\in E} v(\{j\}) + \Delta_{N,v}(E)}}$$
$$= v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in E} v(\{j\})}\Delta_{N,v}(E).$$

Thus, we verified the generalized proportional sharing principle if $x_E \neq 0$.

(b) Assume that $x_E = 0$ and $\sum_{i\in E} \Delta_{N,v}(\{i\}) \neq 0$. Then the equality $x = g_{N,v}(x)$ can be written as

$$x_i = \begin{cases} \Delta_{N,v}(\{i\}) + \frac{v(\{i\})}{\sum_{i\in E} v(\{i\})}\Delta_{N,v}(E), & \text{for } i \in E, \\ \Delta_{N,v}(\{i\}), & \text{for } i \in N \setminus E. \end{cases}$$

This is exactly the generalized proportional sharing condition.

(c) Assume that $x_E = 0$ and $\sum_{i\in E} \Delta_{N,v}(\{i\}) = 0$. Then from the equality $x = g_{N,v}(x)$ we get $x_i = v(\{i\})$ for $i \in N \setminus E$, and the proof is finished.

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