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Naïve Learning in Social Networks with Random Communication

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Naïve Learning in Social Networks with Random Communication[§]

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Abstract

We study social learning in a social network setting where agents receive independent noisy signals about the truth. Agents naïvely update beliefs by repeatedly taking weighted averages of neighbors' opinions. The weights are fixed in the sense of representing average frequency and intensity of social interaction. However, the way people communicate is random such that agents do not update their belief in exactly the same way at every point in time. We show that even if the social network does not privilege any agent in terms of influence, a large society almost always fails to converge to the truth. We conclude that wisdom of crowds is an illusive concept and bares the danger of mistaking consensus for truth.

Keywords: Wisdom of crowds, social networks, information cascades, naïve learning

JEL classifications: D83, D85, C63

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1 Introduction

Social networks play a central role in sharing information and the formation of opinions. They carry news about products, events and job opportunities. They shape opinions and expectations, drive the spread of rumours and influence decision such as voting, smoking, education and consumption behavior. The nearly limitless set of situations in which social networks play a crucial role makes it important to understand how the architecture of the network impacts the evolution of beliefs and behavior over time.

A well established line of research studies how to extend rational learning theory into social network settings when individual decision making is based on observations from neighbors as e.g. in Bala & Goyal (1998), Gale & Kariv (2003). Since communication in a social network often involves repeated transfers of knowledge among a large number of agents, theories based on rational learning soon become infeasible even for small numbers of agents. Nonetheless, individuals may use fairly simple updating rules for their beliefs and still arrive at outcomes like those achievable under fully rational learning. In the context of social networks, such a naïve learning process is studied by Golub & Jackson (2010) based on the seminal network interaction model of DeGroot (1974). Here, the social structure of a society is described by a weighted and possibly directed network. Agents start with an individual belief, say, the probability of an outcome of an election. The belief updating mechanism is that agents communicate with neighbors in a social network. At each date, the new belief of an agent is the weighted average of her neighbors' belief from the previous period. For strongly connected networks[1] and under some weak aperiodicity condition, the updating process converges to a common belief, which is called *reaching a consensus*. Golub & Jackson (2010) study the DeGroot process for the setting when there is some true state of nature that agents are trying to learn. Each agent's initial belief is an independent zero-mean noisy signal about the true value. The question is for which social network structures this naïve updating process converges to the truth. Their main finding is that a large society converges to the truth if and only if the influence of the most influential agent vanishes as the society grows. This result is referred to as *wisdom of crowds*.

Our article challenges this result. In fact, we find the opposite: the crowd is wrong almost always even in absence of excessively influential agents. Our substantive point is that the

[1] A network is strongly connected if there is a directed path from any agent to any other agent.

wisdom of crowds result hinges on an invalid model assumption about *how* social networks orchestrate social learning. Let us elaborate. The key assumption of DeGroot influence dynamics is that agents continue to use the very same updating rules at every point in time. In particular, the weights that agents place on other's opinions are constant and used at every single time step. This is a behavioral assumption based on a bounded rationality argument which is discussed at length by DeMarzo et al. (2003). In a nutshell, the justification for constant weights is that agents fail to adjust correctly for repetitions and dependencies in information they hear multiple times.

We do not attempt to tackle the assumption of bounded rationality. Our substantive point is that the assumption of constant weights is invalid from an entirely different perspective which zooms in on the actual meaning of the weight in a social matrix. This weight is meant to describe the *frequency* or *intensity* of social interaction and as such represents an average over some time of observation. It represents the usual way social network data is collected, say, when the weight measures the observed frequency of interaction over some time or it rolls out of a questionaire attempting to quantify the importance agents assign to each others opinions. However, unlike physical networks such as fixed electrical grids, the network of social communication has an inherent variable character as it is a process of interaction with different people at different times. One may keep in touch with some friends on a monthly basis, but need not to talk to all friends every day. We bump into colleagues by chance, meetings are scheduled at different days at different time spots and we listen to the opinion of experts sequentially as opposed to all at the same time. Although the interaction patterns captured through the interaction weights might be stable as averaged over some time period, they do not reveal the actual order of interactions. The latter, however, depends on various exogenous factors and is hence of rather random nature.

As we will show in our simple variant of the DeGroot model, this sequential aspect has a fundamental impact on consensus and refutes wisdom of crowds. Instead, our finding is that the crowd is wrong almost always even if all agents have vanishing network influence as the society grows. We establish this argument by a simple model of randomization. At each date of belief updating, nature chooses randomly from a set of social networks. In expectation, however, the social network is fixed as in the setting of the DeGroot model.

The importance of the sequential aspect of belief updating is a well-known result in the field of information cascades and herding as developed in Banerjee (1992), Bikhchandani et al. (1992) and Welch (1992). Consider, for example, a hiring committee that needs to decide whether to make a job offer to candidate A or B.[2] The usual procedure is to go around the table and ask each committee member's opinion. Assume it is common knowledge that all members have roughly the same insight in the qualities of the members. Now consider the scenario that the first two members expressed their preference for candidate A, following their own private signals. Suppose the signal of the third member was in favor of B. She could argue, on rational grounds, that the two signals of the first two members outweigh her own with respect to informational content and join their opinion. Now consider the fourth member. She knows that the choice of the third member conveys no reliable information. As a result, she is in the same situation as the third member and might disregard her own signal. This will continue with all subsequent members with an ever growing committee. An information cascade has taken place. No one is under the illusion that it means that every single member received the same private signal favoring A. Still, it is rational to join the decision of the first two members. Of course, this phenomenon is even more likely under naïve updating when agents fail to recognize that the guess from agent three on conveys no reliable information. From the perspective of social networks, all members pay equal attention to all other members and hence every agent has vanishing influence in a growing committee. The crowd, however, is not wise for the simple reason that the consensus is largely determined by the noise of the first few agents. It is not the network, but the *sequential* aspect of the process that prevents averaging out the initially zero-mean noise of the private signals.

This sequential aspect suggests to reconsider the concept of influence on collective consensus. In the DeGroot model, influence is usually measured by a concept of eigenvector centrality. It roots back to sociological measure of concept and prestige introduced by Katz (1953) and refined by Bonacich (1987). Eigenvector centrality is based on the average social network of the dynamic process which forms a sharp contrast to the result of our random communication model. We will show that the consensus level is largely determined by the first few random draws of the social network. This is easy to understand. In the process of random meetings, agents with highest possible beliefs will sooner or later have a meeting with an agent of lower belief. Similarly, agents with lowest beliefs will at some time adjust their belief by a neighbor

[2] This example is taken from Easly & Kleinberg (2010).

with higher belief. As a result, the set of possible beliefs forms a sequence of shrinking subsets over time. This implies that as time goes on, an agent has decreasing impact on neighbors' beliefs.

We will show that consensus is reached almost always under some mild conditions on connectedness and aperiodicity. This leads to the question whether consensus is close to the truth, arbitrarily close for a growing society respectively. The answer we give is negative. We show that the consensus level is determined by the sample path of random updating and that beliefs behave highly volatile in unpredictable directions during the updating process. Instead, the large variety of possible sample paths leads to a consensus distribution around the truth level with possibly large deviation and skewness.

The implication is as follows. Suppose consensus is observed in a large society or organization after some time of discussion. Also assume that all agents seem to have more of less the same influence in terms of the social network. We claim that an observed consensus level should by no means seen as representative for the truth, or even close to the truth. Instead, the consensus level contains no reliable information about the truth as it is highly susceptible to even minor changes in the dynamics of communication. A second insight is that this consensus level is largely determined by the early rounds of discussion.

The paper is structured as follows. Section 2 introduces the classic DeGroot model of updating and defines the random draw of initial beliefs that forms the starting point of all dynamics discussed in this paper. In Section 3, we introduce our randomization model which covers the DeGroot model as a special case. In Section 4, we discuss issues of convergence and develop conditions for all agents's belief to reach a consensus in our setup of random updating. We will also discuss speed of convergence. Section 5 introduces the concept of wisdom of the crowds in the randomized setup. Section 6 illustrates and discusses randomization as an obstacle to wisdom by means of simple examples. Section 7 concludes.

2 DeGroot model of social learning

Consider a society of agents $\mathcal{N} = \{1, \dots, n\}$ interacting as a social network. The interaction patterns are captured by a $n \times n$ row-stochastic matrix P. The interpretation is that $P_{ij} \geq 0$ indicates the weight or trust that agent i places on the current opinion or belief of agent j when forming i's new belief for the next period. The matrix P may be asymmetric such that such that P_{ji} can be different to P_{ij}.

In the social learning model of DeGroot (1974), each agent i forms her belief for the next period by taking averages of weighted beliefs of neighbors in the social network. In particular, let $f_i^{(t)}$ denote the belief of i at time $t \in \{0, 1, \dots\}$. Assume that each belief $f_i^{(t)}$ lies in a finite interval $[a, b] \subset \mathbb{R}$. [3] Beliefs are updated over time according to the following rule

$$\boldsymbol{f}^{(t)} = P\boldsymbol{f}^{(t-1)} = P^t \boldsymbol{f}^{(0)}, \quad t \in \{1, 2, \dots\}. \tag{1}$$

The DeGroot model is a natural starting point to understand how network structures influence the formation of opinions, where opinion can be expressed as an element in the opinion interval $[a, b]$, say, from left to right in terms of political attitudes. Another application is to let $f_i^{(t)}$ express the believed quality of a given product or the likelihood that a given individual engages in an activity. In the context of social learning of the present paper, we will interpret beliefs in the context of *information* and discuss the evolution of beliefs in terms of information diffusion. In particular, we assume that there is a 'truth', and agents only have partial information about this truth. Following DeMarzo et al. (2003) and Golub & Jackson (2010), this translates into the model as follows. At time $t = 0$, initial beliefs are given as

$$f_i^{(0)} = \mu + e_i \in [a, b] \tag{2}$$

for each $i \in \mathcal{N}$, where constant μ is said to be the true state of nature and e_i is a random noise sampled from a distribution with bounded support, zero mean and positive variance. The initial signals $f_i^{(0)}$ are independently drawn at time $t = 0$.

There are a few main questions that arise naturally about system (1):

(i) When is there convergence?

(ii) When is there *consensus* in the sense of all agents arriving at the same limiting belief?

(iii) Who has influence?

(iv) When is consensus accurate in the context of information diffusion when the initial beliefs are given by (2)?

[3] The results of this paper can be straightforwardly extended to multidimensional Euclidean spaces while maintaining the findings.

These questions have been addressed in DeGroot (1974), DeMarzo et al. (2003) and Golub & Jackson (2010). Mathematically, system (1) represents iterated multiplication by a fixed stochastic matrix. This allows to make use of the well-established analytical toolbox of Markov chains. In particular, precise conditions for convergence and closed form solutions for measures for influence on consensus can be translated directly to the context of social learning. In Section 4, we will provide a brief overview when we revisit questions (i) - (iv) in the context of our model of random communication.

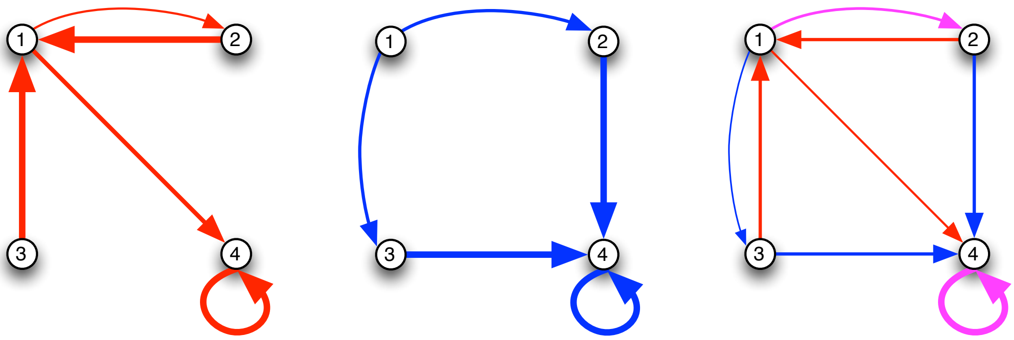
3 Randomization approach of social learning

The key assumption for (1) is that agents keep using the same updating rule throughout the entire learning process. The crude assumption of P being constant reflects the usual way social network data is collected, where the strength or weight P_{ij} of a connection between two agents i and j is an observed frequency aggregated over some time. However, it does not imply that agents update their belief on a, say, daily basis in exactly the same way. For instance, consider data collected from online communication taken over one year in order to estimate the connections in a given group of users. Alternatively, consider splitting the period into two half years or four quarter years. Obviously, the estimated networks will be different while providing the same averaged data over one year. Consequently, there is aleatoric uncertainty about the actual pattern of belief updating. In particular, the aggregated data P is more likely a (linear) combination of (different) matrices, say, X and Y such that $P = \alpha X + (1 - \alpha)Y$. Figure 1 illustrates this *superposition* of two networks. Note that the set of possible decompositions of a network P is usually infinite.[4]

To incorporate this aleatoric feature of random communication into the model, we consider a random sequence $\{\widehat{P}^{(t)} : t = 1, 2, \dots\}$ whose elements are independently drawn from a set $\mathcal{A}$ of $n \times n$ row-stochastic matrices according to a probability distribution R over $\mathcal{A}$ with $\mathrm{E}[\widehat{P}^{(t)}] = P = \sum_{A \in \mathcal{A}} R(A) \cdot A$ for every $t > 0$. The corresponding belief process $\widehat{\boldsymbol{f}}^{(t)}$ is defined by

$$\widehat{\boldsymbol{f}}^{(t)} = \widehat{P}^{(t)} \widehat{\boldsymbol{f}}^{(t-1)}, \quad \widehat{f}_i^{(0)} = \mu + e_i. \tag{3}$$

[4]Start with the trivial decomposition $X = Y = P$ and assume there exists a $P_{ij} \in (0, 1)$. It is easy to show that a nontrivial decomposition can be constructed by setting $X_{ij} = P_{ij} + \epsilon$ for any sufficiently small $\epsilon > 0$.



(a) Network X

(b) Network Y

(c) Superposition of X and Y with weights $(0.5, 0.5)$

Figure 1: Superposition of two networks: the weight associated with a link is illustrated by the thickness of the link.

Mathematically, (1) is covered by the special case of a degenerate distribution where R chooses only one element of $\mathcal{A}$ with positive probability.

Note that (3) comprises two independent random processes. The first is the draw of the initial beliefs $\widehat{\boldsymbol{f}}^{(0)}$ defined by (2). The second is the random sequence of updating $\{\widehat{P}^{(t)} : t = 1, 2, \ldots\}$. When a collection $\mathcal{A}$ and a distribution R over $\mathcal{A}$ are given, we call the corresponding average P the superposition of $\mathcal{A}$ and

$$\{\widehat{P}^{(t)} : t = 1, 2, \ldots\} \tag{4}$$

a *randomization* of $\mathcal{A}$, where $\widehat{P}^{(t)}$ is drawn from R. For notational simplicity, we use $\widehat{P}^{(t)}$ instead of (4) when there is no risk of ambiguity.

4 Reaching a consensus

We first discuss the long run behavior of the belief vectors $\boldsymbol{f}^{(t)}$ and $\widehat{\boldsymbol{f}}^{(t)}$, as defined in (1) and (3) respectively. Given the interaction matrix P, we say there exists a path from i to j if there exists some $k > 0$ such that the (i, j) element of matrix P^k is positive. The matrix P is *strongly connected* if for every ordered pair of indices (i, j) there exists a path from i to j. We say that P

is *primitive*[5] if there exists some $k > 0$ such that $(P^k)_{ij} > 0$ for all $i, j \in \mathcal{N}$. Note that primitivity implies strong connectedness, but the converse is not true. In fact, strong consistency together with aperiodicity is equivalent to primitvity. Let $\mathbf{1}$ denote a column vector with all components equal to 1.

Definition 1. *A matrix P is convergent if* $\lim_{t\to\infty} P^t \boldsymbol{f}^{(0)}$ *exists for all* $\boldsymbol{f}^{(0)} \in [a, b]^n$.

This definition requires the belief updating process to be convergent for all initial beliefs. The following result is standard in Markov chain theory.

Proposition 1. *If P be strongly connected, the following statements are equivalent:*

(i) P is convergent.

(ii) P is primitive.

(iii) There is a unique left eigenvector $\boldsymbol{\pi}$ *of P to eigenvalue* 1 *with* $\boldsymbol{\pi}^\top \mathbf{1} = 1$ *such that*

$$\lim_{t\to\infty} P^t = \mathbf{1}\boldsymbol{\pi}^\top, \tag{5}$$

where convergence is exponentially fast.

A proof can be found in Seneta (1981). Note that $\boldsymbol{\pi}$ is the stationary distribution of P.

If an influence matrix P is primitive, then the corresponding belief $f^{(t)}$ in system (1) converges to the limit

$$\lim_{t\to\infty} \left(P^t \boldsymbol{f}^{(0)}\right)_i = \boldsymbol{\pi}^\top \boldsymbol{f}^{(0)} = \sum_j \boldsymbol{\pi}_j f_j^{(0)},$$

for all $1 \leq i \leq n$. Hence, the limiting beliefs are all equal in which case we refer to the limiting belief as the *consensus*. The latter is a weighted average of the initial beliefs, with agent i's weight given by π_i. Therefore, the weight π_i can be seen as the *influence* of agent i when the interaction matrix P is constant in each period.

The following proposition assures primitivity of the superposition if it is composed of primitive matrices. The intuition is that if a non-negative matrix A is primitive, and another non-negative matrix $\tilde{A}$ has the same dimensions as A and has positive elements in the same positions as A, it holds that $\tilde{A}$ is also primitive.

[5]For strongly connected networks P, primitivity is analogous to P being aperiodic. In graph theoretic terms, it means that the greatest common divisor of the length of P's simple cycles is 1. See e.g. Perkins (1961).

Proposition 2. *If a finite collection $\mathcal{A}$ of influence matrices contains only primitive matrices, then the corresponding superposition $P = \sum_{A \in \mathcal{A}} R(A) \cdot A$ with any distribution R on $\mathcal{A}$ is also primitive.*

A proof can be found in Seneta (1981).

We now turn to dynamics (3) under randomization. Let Ω denote the set of possible infinite draws from $\mathcal{A}$. For any $\omega \in \Omega$, let

$$B_\omega^{(t)} = \widehat{P}_\omega^{(t)} \widehat{P}_\omega^{(t-1)} \cdots \widehat{P}_\omega^{(1)} \tag{6}$$

be a partial backward product up to time t of sequence $\{\widehat{P}_\omega^{(1)}, \widehat{P}_\omega^{(2)}, \ldots\}$. The following simple example illustrates that the backward product[6] (6) does not necessarily converge even when $\mathcal{A}$ consists of primitive matrices.

Example 1. *Consider a group of* 3 *agents and two networks A and B given by*

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \times & \times & \times \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \times & \times & \times \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \times & \times & \times \end{bmatrix}}_{A \cdot B}$$

primitive *primitive* *not strongly connected*

Here, the rows $[\times, \times, \times]$ *of A and B indicate any weight vector (probability distribution, respectively) such all $\times$ are positive. Note that although A and B are primitive, the resulting network $A \cdot B$ is not. In fact, the resulting network $A \cdot B$ is not even strongly connected. It is straightforward to verify that the alternating sequence $B \cdot A \cdot B \cdot A \cdot B$ ... cycles back and forth such*

[6]Note that in each time period t, a matrix chosen from $\mathcal{A}$ is multiplied from the left to the existing matrix product $B_\omega^{(t)} = \widehat{P}_\omega^{(t)} \cdot B_\omega^{(t-1)}$. This is in contrast to the classic Markov chain models where each update of state probabilities is modeled by a matrix multiplication from the right.

that

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \times & \times & \times \end{bmatrix}}_{(A\cdot B)^k} \qquad \underbrace{\begin{bmatrix} \times & \times & \times \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{B\cdot(A\cdot B)^k}$$

for any $k > 0$.

The following theorem states that, nevertheless, consensus is reached almost always when the beliefs are updated by primitive matrices.

For $x \in \mathbb{R}^n$ we let $x^\top = \max(x_i : 1 \le i \le n)$, $x^\sharp = \min(x_i : 1 \le i \le n)$ we let $I(x) = [x^\sharp, x^\top]$ denote the range of x. To simplify notation, we put

$$I_t(\omega) = I\left(\widehat{\boldsymbol{f}}^{(t)}\right) = \left[\min(\widehat{\boldsymbol{f}}^{(t)}), \max(\widehat{\boldsymbol{f}}^{(t)})\right],$$

for all $t \ge 0$.

Theorem 1. *Let $\mathcal{A}$ be given by a set of strongly connected and primitive networks. Consider randomization with each draw given by the distribution R on $\mathcal{A}$ such that*

$$P = \sum_{A \in \mathcal{A}} R(A) \cdot A.$$

Let $\omega \in \Omega$ denote an infinite random sequence of matrices drawn from $\mathcal{A}$.

(i) The successive ranges of beliefs constitute a shrinking sequence of intervals

$$I_{t+1}(\omega) \subseteq I_t(\omega),$$

for all $t \ge 0$.

(ii) Consensus

$$\lim_{t\to\infty} \widehat{f}(\omega)^{(t)} = \widehat{f}(\omega)^{(\infty)}$$

exists with probability 1 *and is path dependent.*

(iii) For the limiting belief we have

$$\mathrm{E}[\lim_{t\to\infty} \widehat{\boldsymbol{f}}^{(t)} \mid \widehat{\boldsymbol{f}}^{(0)}] = \lim_{t\to\infty} P^t \widehat{\boldsymbol{f}}^{(0)} = \boldsymbol{\pi}^\top \cdot \widehat{\boldsymbol{f}}^{(0)} \quad \text{and} \quad \bigcap_{t\geq 0} I_t(\omega) = \left\{\widehat{f}(\omega)^{(\infty)}\right\}. \tag{7}$$

For the proof see Appendix A.

The above theorem provides a mathematical expression of our main finding: the belief range $I_t(\omega)$ is shrinking as time t is progressing. This means that once a value u falls out of $I_t(\omega)$ for some t, then there is no possibility that the belief updating process can reach u again for any t' for $t' > t$. The impact of early belief updating will be explored in more detail in the following section.

We close this section by mentioning a subclass of strongly connected, primitive matrices that generate consensus for every single path $\omega \in \Omega$. A non-negative matrix P is said to be *scrambling* if for any two rows i and j, there exists at least one column, say k, such that both $P_{ik} > 0$ and $P_{jk} > 0$. In terms of network structure, this means that any two agents in the network listen to at least one common agent who can be either of them or someone else.[7]

Proposition 3 (Anthonisse & Tijms (1977))**.** *The following statements are equivalent:*

(i) There is an integer $k \geq 1$ such that for each $t \geq k$ the partial backward product $B_\omega^{(t)}$ from (6) *is scrambling.*

(ii) There is an integer $z \geq 1$, a number β with $0 < \beta < 1$ and for any $\omega \in \Omega$ there is a vector $\boldsymbol{v}_\omega$ such that $\boldsymbol{v}_\omega^\top \mathbf{1} = 1$ and for all $j \in N$ and every element $(B_\omega^{(t)})_{ij}$ in row j,

$$\left|(B_\omega^{(t)})_{ij} - (\boldsymbol{v}_\omega)_j\right| \leq \beta^{\lfloor t/z \rfloor} \quad \text{for all} \quad t \geq 1, \tag{8}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Proposition 3 asserts that scrambling generates exponentially fast convergence. Note that a sufficient condition for (i) of Proposition 3 is each $A \in \mathcal{A}$ being scrambling.[8]

[7] The class of scrambling matrices was originally defined by Hajnal (1958).

[8] Consider the product $\tilde{A} \cdot A$ of two scrambling matrices. For any two agents i and j select k and l such that $\tilde{A}_{ik} > 0$ and $\tilde{A}_{jl} > 0$. Since A is scrambling, there exists a common r such that $A_{kr} > 0$ and $A_{lr} > 0$ and hence $(\tilde{A} \cdot A)_{ir} > 0$ as well as $(\tilde{A} \cdot A)_{jr} > 0$.

Corollary 1. *If each $A \in \mathcal{A}$ is scrambling, then for any $\omega \in \Omega$ there is a vector $\boldsymbol{v}_\omega$ such that $\boldsymbol{v}_\omega^\top \mathbf{1} = 1$ and*

$$\lim_{t\to\infty} (\widehat{\boldsymbol{f}}_\omega^{(t)})_i = \boldsymbol{v}_\omega^\top \widehat{\boldsymbol{f}}^{(0)} \quad \textit{for all } i \in N, \tag{9}$$

where convergence is exponentially fast.

5 Wisdom of crowds revisited

A central question of social learning is under what circumstances the decentralized communication of the network correctly aggregates diverse individual information. Golub & Jackson (2010) discuss this question for the DeGroot process (1) for large societies. To make this idea work at a technical level, it is necessary to be precise about what "large" means. It turns out that the cleanest way to formalize the question is to consider infinite networks.[9] To be precise, they consider a sequence of growing networks $\{P(n)\}_{n=n_0}^\infty$ where each $P(n)$ is a row-stochastic $n \times n$ matrix representing the network with associated left eigenvectors $\{\boldsymbol{\pi}(n)\}_{n=n_0}^\infty$. It is hence a setup of a double limit. For each finite n, the network reaches a consensus (or not) in the DeGroot process (1) of updating for $t \to \infty$. Subsequently, the networks $\{P(n)\}_{n=n_0}^\infty$ grow in size n with $n \to \infty$.

The following definition says that a sequence of networks is wise when the limiting beliefs converge jointly in probability to the true state μ.

Definition 2 (Golub & Jackson (2010))**.** *The sequence of networks $\{P(n)\}_{n=n_0}^\infty$ is said to be wise if*

$$\lim_{n\to\infty} \Pr\Big[\max_{i\le n} \big|f_i^{(\infty)}(n) - \mu\big| > \varepsilon\Big] = 0 \tag{10}$$

for any $\varepsilon > 0$.

[9]For finite n, every statement on consensus being close to truth or not depends in a cumbersome way on n without adding much insight.

Proposition 4 (Golub & Jackson (2010))**.** *If* $\{P(n)\}_{n=n_0}^{\infty}$ *is a sequence of primitive stochastic matrices, then it is wise if and only if the associated left eigenvectors* $\{\boldsymbol{\pi}(n)\}_{n=n_0}^{\infty}$ *are such that*

$$\max_{i\leq n} \ \pi_i(n) \to 0 \tag{11}$$

as $n \to \infty$.

The proposition states that a growing society is wise if and only if it grows in a way that for any member influence measured by $\pi_i(n)$ becomes insignificantly small for $n \to \infty$. The most straightforward examples for wise sequences are growing symmetric networks such as complete networks or circles. For symmetry reasons, $\boldsymbol{\pi}(n)$ straightforwardly follows as $(1/n, \dots, 1/n)$ for all $n \in \mathbb{N}$. Proposition 4 can be generalized by allowing non-wisdom in the growth process that disappears sufficiently fast. For details see Appendix B.

We now turn to the randomization model. Let $\mathcal{A}(n)$ denote a finite collection of influence matrices of size $n \times n$. The collection of networks $\{\widehat{P}^{(t)}(n) : t = 1, 2, \dots\}_{n=n_0}^{\infty}$ is said to be a randomization of $\{\mathcal{A}(n)\}_{n=n_0}^{\infty}$ if for each $n \geq n_0$, the outcome of $\widehat{P}^{(t)}(n)$ is independently chosen from $\mathcal{A}(n)$ for $t = 1, 2, \dots$ according to some probability distribution R_n over $\mathcal{A}(n)$. We denote by $\Omega(n)$ the collection of all infinite sequences whose members are chosen from $\mathcal{A}(n)$. The corresponding superposition $\{P(n)\}_{n=n_0}^{\infty}$ is given by $P(n) = \sum_{A\in\mathcal{A}(n)} R_n(A) \cdot A$ for $n \geq n_0$. Wisdom of crowds under randomization is defined as follows.

Definition 3. *The sequence of randomized networks* $\{\widehat{P}^{(t)}(n) : t = 1, 2, \dots\}_{n=n_0}^{\infty}$ *is said to be wise if*

(i) for all n and for every sample path $\omega(n) \in \Omega(n)$ *the limit* $(\widehat{\boldsymbol{f}}_{\omega(n)}^{(\infty)}(n))_i$ *exists with probability one for* $i = 1, \dots, n$, $n \geq n_0$, *and for any given initial beliefs; and*

(ii) for any sequence of sample paths $\{\omega(n)\}_{n=n_0}^{\infty}$

$$\lim_{n\to\infty} \Pr\Big[\max_{i\leq n} \big|(\widehat{\boldsymbol{f}}_{\omega(n)}^{(\infty)}(n))_i - \mu\big| > \varepsilon\Big] = 0 \tag{12}$$

for any $\varepsilon > 0$.

It is straightforward to provide sufficient conditions for a society to be wise under randomization, for instance, if $\{\mathcal{A}(n)\}_{n=n_0}^{\infty}$ consists of symmetric matrices for every n. However,

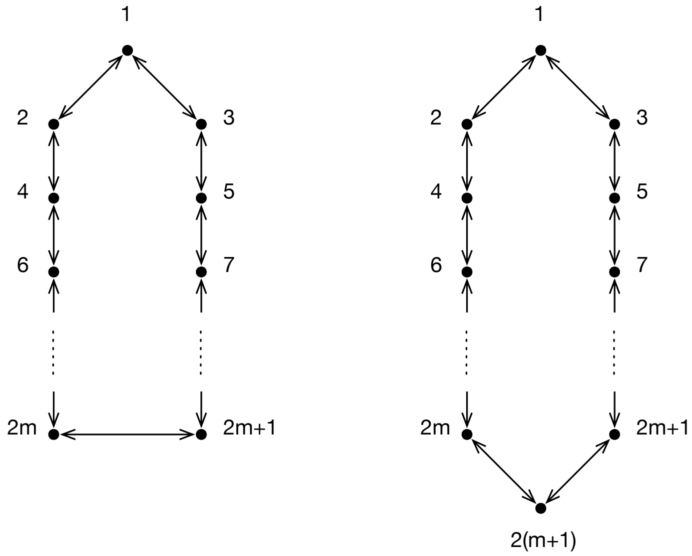


Figure 2: Ring networks with odd and even number of nodes. Self links exist but are omitted in the figures.

superposition (measured data) $P = \sum_{A \in \mathcal{A}} R(A) \cdot A$ usually stems from a plethora of all kinds of possible random communication structures $A \in \mathcal{A}$. The following section illustrates that this represents an obstacle to wisdom.

6 Superposition versus randomization

In this section, we demonstrate by a simple example how randomization prevents a a society from being wise. In our setup, the superposition $P(n)$ is symmetric and hence wise for growing n. As the following example shows, however, wisdom fails when $P(n)$ is interpreted as expectation of two non-wise networks.

Example 2. *Consider a ring network as depicted in Figure 2. Every agent has a link to herself which is omitted in the figure. The influence matrix $P(n)$ of size n (both odd and even) is given as follows:*

$$P(n) = \begin{bmatrix} 1/2 & 1/4 & 1/4 & & & & \\ 1/4 & 1/2 & 0 & 1/4 & & & \\ 1/4 & 0 & 1/2 & 0 & 1/4 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1/4 & 0 & 1/2 & 0 & 1/4 \\ & & & 1/4 & 0 & 1/2 & 1/4 \\ & & & & 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

This matrix is the superposition of two other influence matrices $X(n)$ *and* $Y(n)$ *such that* $P(n) = \alpha X(n) + (1-\alpha)Y(n)$ *for some* $\alpha \in (0, 3/4)$*, where for* $n = 2m+1$*,* $X(n)$ *and* $Y(n)$ *are given by*

$$X(2m+1) = \begin{bmatrix} 1/2 & 1/4 & 1/4 & & & & \\ 1/3 & 1/2 & 0 & 1/6 & & & \\ 1/3 & 0 & 1/2 & 0 & 1/6 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1/3 & 0 & 1/2 & 0 & 1/6 \\ & & & 1/3 & 0 & 1/2 & 1/6 \\ & & & & 1/3 & 1/6 & 1/2 \end{bmatrix},$$

and

$$Y(2m+1) = \begin{bmatrix} 1/2 & 1/4 & 1/4 & & & & \\ c & 1/2 & 0 & d & & & \\ c & 0 & 1/2 & 0 & d & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & c & 0 & 1/2 & 0 & d \\ & & & c & 0 & 1/2 & d \\ & & & & c & d & 1/2 \end{bmatrix},$$

with $c = (3-4\alpha)/(12-12\alpha)$ *and* $d = 1/2 - c = (3-2\alpha)/(12-12\alpha)$*. In network* $X(n)$*, agents attach more weight to upwards located agents than agents down the circle. In contrast, lower located agents get more weight in network* $Y(n)$*. Both* $X(n)$ *and* $Y(n)$ *are primitive and the* *corresponding left hand eigenvectors* $\boldsymbol{\pi}_X$ *and* $\boldsymbol{\pi}_Y$ *from* (5) *follow as*

$$\pi_{X:1}(2m+1) = \frac{2^m}{2^{m+2}-3} =: \pi_{X:1},$$
$$\pi_{X:2i}(2m+1) = \pi_{X:2i+1}(2m+1) = \frac{3}{2^{i+1}} \cdot \pi_{X:1} \quad \textit{for} \quad i = 1, \ldots, m \tag{13}$$

and

$$\pi_{Y:1}(2m+1) = \frac{(1-4c)(2c)^m}{(1-2c)^m - 2(2c)^{m+1}} =: \pi_{Y:1},$$
$$\pi_{Y:2i}(2m+1) = \pi_{Y:2i+1}(2m+1) = \frac{(1-2c)^{i-1}}{2 \cdot (2c)^i} \cdot \pi_{Y:1} \quad \textit{for} \quad i = 1, \ldots, m \tag{14}$$

For even $n = 2(m+1)$*,* $X(n)$ *and* $Y(n)$ *are defined in a similar way with slightly modified* (13) *and* (14).[10]

For symmetry reasons, the sequence $\{P(n)\}_{n=n_0}^{\infty}$ in Example 2 is wise according to Definition 2 (and the more general Definition 3). For the other two networks we get

$$\lim_{m\to\infty} \pi_{X:1}(2m+1) = \frac{1}{4},$$

and

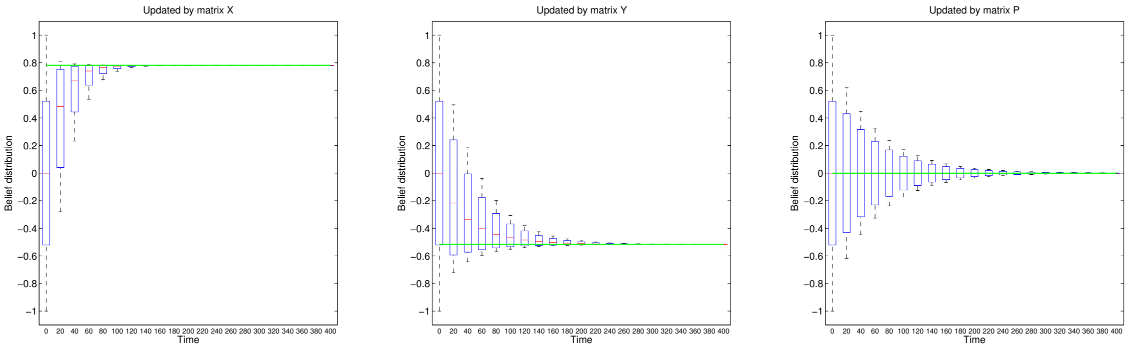
$$\lim_{m\to\infty} \pi_{Y:2m}(2m+1) = \lim_{m\to\infty} \pi_{Y:2m+1}(2m+1) = \frac{1-4c}{2-4c} = \frac{\alpha}{3-2\alpha} > 0.$$

From (11), we conclude that $\{X(n)\}_{n=n_0}^{\infty}$ and $\{Y(n)\}_{n=n_0}^{\infty}$ are non-wise.

We now turn to the corresponding randomization R_n of Example 2. At each time $t > 0$, the belief updating (3) is a random draw such that $X(n)$ is chosen with probability α, $Y(n)$ with probability $1-\alpha$, respectively. It is easy to see that for any $t \geq \lfloor n/2 \rfloor$, the backward product (6) has at least one column (e.g. column 1) with all positive elements for all $\omega \in \Omega(n)$. This assures condition (i) of Proposition 3 such that consensus is reached for every path ω. Due to a lack of closed form solutions we will test wisdom of crowds by simulation.

Let network size $n = 25$, and the probability of choosing $X(n)$ be 0.3, i.e. $\alpha = 0.3$. We

[10] For even $n = 2(m+1)$, all $\pi_{X:k}$ and $\pi_{Y:k}$ are similar to (13) and (14) for $k = 1, ..., 2m+1$. For the last element follows $\pi_{X:2(m+1)} = 2^{-m}\,\pi_{X:1}$ and $\pi_{Y:2(m+1)} = (1-2c)^m\,\pi_{Y:1}/(2c)^m$.



(a) Update by $X(25)$ (b) Update by $Y(25)$ (c) Update by $P(25)$

Figure 3: Convergence of beliefs under fixed influence matrices with $n = 25$ and $\alpha = 0.3$.

generate 5 sample paths:

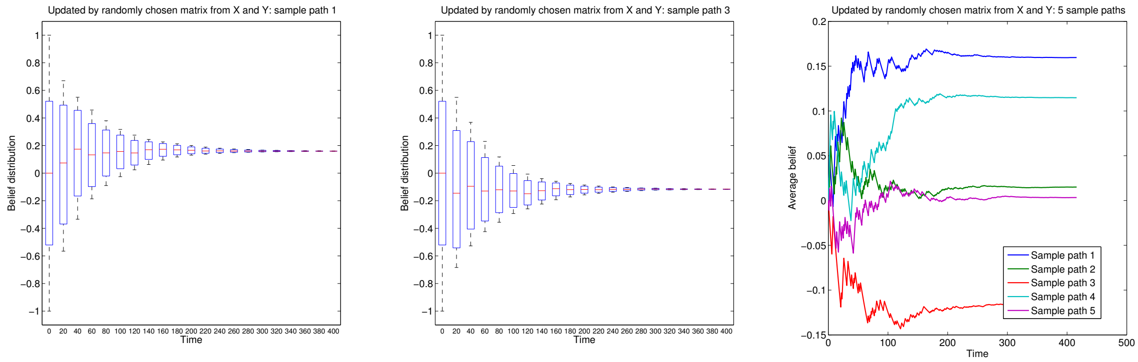
$$\begin{aligned}
&\text{Sample path 1:}\ \ X\ X\ Y\ Y\ Y\ Y\ Y\ Y\ X\ X\ X\ X\ Y\ Y\ X\ Y\ X\ Y\ Y\ Y\ \ldots,\\
&\text{Sample path 2:}\ \ X\ Y\ X\ X\ Y\ Y\ Y\ Y\ Y\ Y\ X\ Y\ X\ Y\ X\ X\ Y\ Y\ Y\ X\ \ldots,\\
&\text{Sample path 3:}\ \ Y\ Y\ Y\ Y\ Y\ Y\ X\ X\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ \ldots,\\
&\text{Sample path 4:}\ \ X\ X\ X\ X\ Y\ Y\ Y\ X\ Y\ X\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ \ldots,\\
&\text{Sample path 5:}\ \ Y\ X\ Y\ Y\ X\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ Y\ X\ Y\ Y\ Y\ X\ X\ Y\ \ldots.
\end{aligned}$$

In order to focus on the impact of random network structures, the initial beliefs are

$$\widehat{f}_i^{(0)}(n) = 1 - 2(i-1)/(n-1)$$

for $i \in \mathcal{N}$ for all sample paths. Under this construction, the initial beliefs are equally distributed on $[-1, 1]$ with mean 0. Agent 1 has the highest belief, agent n the lowest respectively.

We start with illustrating convergence to consensus under $X(n)$, as well as $Y(n)$, and the corresponding superposition $P(n)$. Figure 3 depicts the distributions of individual beliefs at points in time using box plots. Although consensus is reached for every network, the actual levels differ substantially. Recall that by design of $X(n)$, agents with low index i receive more weight than agents with higher index. The opposite holds for $Y(n)$. The comparatively high consensus level of (a) reflects that agents with low index i push belief updating towards the positive direction of their initial beliefs. In contrast, beliefs in (b) is dragged down to a negative



(a) Sample path 1 (b) Sample path 3 (c) Average beliefs

Figure 4: Path wise convergence of beliefs under randomization with $n = 25$ and $\alpha = 0.3$.

consensus level. In (c), these different tendencies are averaged out in the belief updating of superposition $P(n)$.

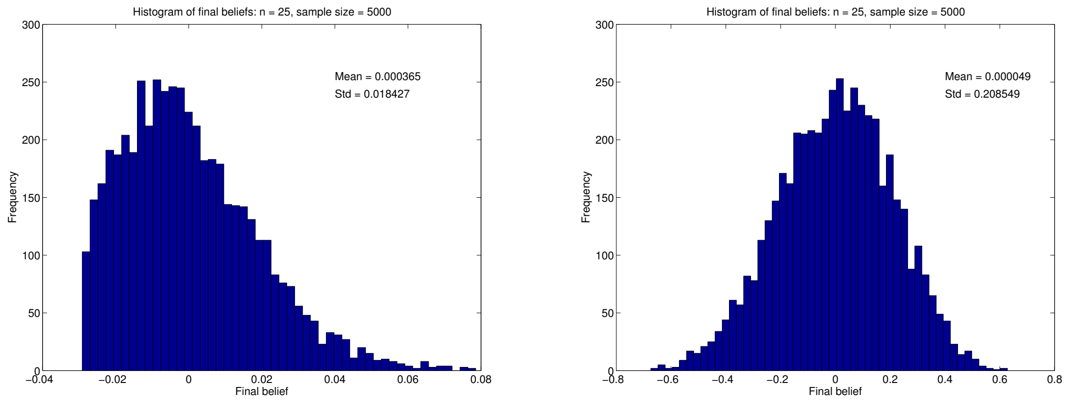
Figure 4 (a) and (b) illustrate belief evolution under sample paths 1 and 3 of the randomization model. Convergence to consensus is verified as expected. The consensus levels, on the other hand, are path dependent. Average beliefs of five sample paths are plotted on Figure 4 (c). Here, average beliefs seem to behave in a somehow erratic way at early times of updating. This illustrates the random draw feature as $X(n)$ and $Y(n)$ push beliefs into different directions. The dynamics, however, shows decreasing volatility over time before the beliefs settle down at consensus level (at around $t = 200$).

The dynamics suggests that early updates have a larger impact on the consensus value than later updates. Recall from Theorem 1.i that the range of possible beliefs is shrinking over time which is illustrated by the shrinking boxplots of Figure 3 and 4. The speed of this convergence process is largely determined by the amount of the second eigenvalue of each stochastic matrix.[11]

This volatile shrinking process sheds a new light on measuring an agent's influence. Recall that agent 1 is the most influential agent in $X(n)$ and least influential in $Y(n)$ as measured by $\boldsymbol{\pi}_X$ and $\boldsymbol{\pi}_Y$. If the path ω starts with sufficiently many draws of $Y(n)$, however, agent 1 might only have a negligible effect consensus. This holds in particular if the range of possible beliefs $I_t(\omega)$ shrinks fast over time.

Figure 4 shows that wisdom of crowds cannot be confirmed in the randomization model,

[11]For an overview on techniques see Jackson (2010). A technique for understanding rates of convergence that is particularly relevant to the setting of social networks has recently been developed in Golub & Jackson (2012).



(a) $\alpha = 0.02$ (b) $\alpha = 0.73$

Figure 5: Histograms of consensus levels under randomization with different values of α.

even if the superposition is wise. The consensus level is path dependent and deviation from the truth level 0 largely depends on early draws. Figure 5 illustrates histograms of consensus levels of 5000 sample paths with the randomization model, where $n = 25$ and α is 0.02 for subfigure (a) and 0.73 for subfigure (b). A small α means that network $X(n)$ is relatively rare. Since the beliefs are influenced by $Y(n)$ almost all the time, the consensus level is more likely to be negative which coincides with the findings of Figure 3 (b). A large α affects the consensus distribution in the opposite way.

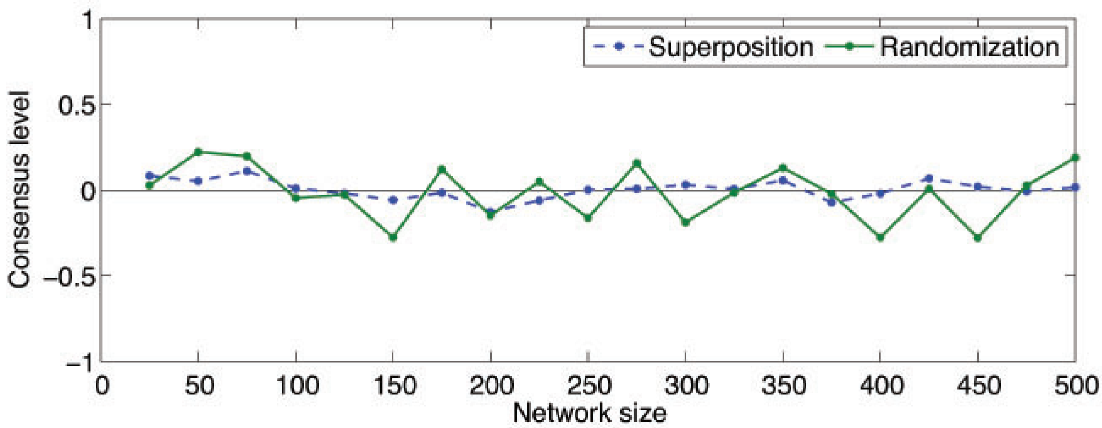


Figure 6: Consensus levels of growing networks.

In order to illustrate that our result is not dependent on the size of the network, we ran the simulation of Example 2 for different sizes $n \in \{25, 50, 75, \ldots, 500\}$ and $\alpha = 0.3$.[12] In Figure 6, consensus levels are plotted against growing network size. As benchmark, consensus outcomes under the corresponding superposed networks are included. The sample paths illustrate that a growing network size does not lead to consensus levels closer to the truth. In fact, we can elaborate this finding by the following simple example.

Example 3. *Consider the scenario in which the learning dynamics starts with the crowd largely influenced by one single agent. For instance, consider an exogenous shock and the expertise of agent* 1 *happens to be the first heard by large media presence. In the setup of our randomization model, assume the outcome of the first random draw is given by the influence matrix*

$$X(n) = \begin{bmatrix} 1 & & & & & & \\ 1-\delta & \delta & 0 & 0 & & & \\ 1-\delta & 0 & \delta & 0 & 0 & & \\ \vdots & & & \ddots & & & \\ 1-\delta & & & & \delta & 0 & 0 \\ 1-\delta & & & & & \delta & 0 \\ 1-\delta & & & & & 0 & \delta \end{bmatrix},$$

with $\delta \in (0,1)$. *For sufficiently small* δ, *convergence to consensus happens almost immediately as every agent follows essentially the belief of agent* 1. *However, it also means that the range of possible beliefs* $I_1(\omega)$ *after one round of updating is a set largely determined by the idiosyncratic error of agent* 1. *For sufficiently small* δ, *this set does not contain the truth* 0, *but will contain only positive or only negative beliefs. Since belief updating implies that* $I_t(\omega) \subseteq I_1(\omega)$ *for all future* t, *distance of consensus to the truth is bounded by* $I_1(\omega)$ *and cannot be improved, regardless of future random draws of updating.*

[12] For each n, we start the belief updating process from $\widehat{\boldsymbol{f}}^{(0)}(n)$ where initial individual beliefs are sampled from the standard normal distribution and then sorted by descending order. Thus the true state of the world is zero. Each simulation run stops at time $\tau(n)$ so that $\left|\max_i[\widehat{f}_i^{(\tau(n))}(n)] - \min_i[\widehat{f}_i^{(\tau(n))}(n)]\right| < \varepsilon = 10^{-3}$.

7 Conclusion

The main topic of this paper concerns a fundamental question in social learning: under what conditions will a society of agents who communicate and naïvely update in a decentralized way reach a consensus that represents the truth? In other words, which conditions ensure wisdom of crowds? We show that consensus is reached almost always, however, the actual level is highly sensitive with respect to the way the social network orchestrates early communication. In contrast to much of the previous literature, this result demonstrates that consensus can by no means be taken to carry any reliable information about the truth. This refutes the idea of the wisdom of crowds even for social networks in which no agent is privileged in terms of influence.

We used an extension of the classic DeGroot model to demonstrate this finding. However, our main message does not depend on the precise mechanics of belief updating and is not even challenged by alternative rationality assumptions. Our message is that, yes, social networks influence how smart societies are in the aggregate. Their impact, however, is almost impossible to predict due to the inherent random nature of social interaction. The stylized metaphore that we offer is that society starts with a range of unbiased beliefs around the true value. The likelihood that it *stays* unbiased, however, is practically zero as this would assume that the temporary (finite) neighborhood of agents happens to be unbiased at every date of updating.

From a technical perspective, the DeGroot model with fixed weighted matrices provides a tractable framework for what happens *in expectation* if the social interaction is of more of less stable frequency and recognition of neighbors. Our point is that updating is not a deterministic process such that consensus levels are rather described by a probability distribution. In this interpretation, Golub & Jackson (2010) showed that the large crowd is wise in expectation if and only if the influence of the most influential agent is vanishing as the society grows. Here, influence is measured by a principal eigenvector of the fixed social network matrix. Our point is that, nevertheless, due to the countless ways of representing a fixed matrix as average and the huge variety of possible sample paths, the likelihood of collective beliefs actually ending up at the truth level is zero.

This suggests new insights for several contexts of collective modelling. In situations in which the crowd produces bad judgement such as for example economic bubbles, one aspect might be the mere randomness of endogenous belief updating. This is an entirely different explanation than the classic arguments such as cognitive biases, conformity moves or herding behavior. (For a broad discussion and overview see e.g. Surowiecki (2005); see e.g. Jarrow et al. (2011) for financial markets.)

Our result also suggests to reconsider the measurement of influence in collective dynamics. Measures based on fixed social networks such as centrality measures do not capture the profound impact of early interactions. These early updates, however, serve as an "anchor" followed by decreasing opportunities of belief adjustment. When it comes to setting up a process for collective consensus formation, say in large organizations, it is important to realize that there is an inherent tension in the collective process. On the one hand, there is the need to reveal private information as a valuable trace in search of the truth. On the other hand, it is inevitable that the sequence of expressing beliefs has a significant impact on opinion formation.

The main lesson to be learned from our study is to be careful in drawing conclusions about the truth or best course of action from the opinion or behavior of a crowd. Wisdom of crowds in opinion dynamics seems an illusive concept and it bares the danger of mistaken consensus for truth.

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Appendix A: Proof of Theorem 1

Proof. Recall that for $x \in \mathbb{R}^n$ we let $x^\top = \max(x_i : 1 \leq i \leq n)$, $x^\sharp = \min(x_i : 1 \leq i \leq n)$. We now let

$$\text{span(x)} = x^\top - x^\sharp. \tag{15}$$

Note that span(x) = 0 does not imply $x_i = 0$, for $1 \leq i \leq n$, and span(x) is a pseudo norm.

We have the following properties.

(i) For any $x \in \mathbb{R}^n$ and any square matrix A with equal rows over $\mathbb{R}^n$ it holds that

$$\text{span}(Ax) = 0. \tag{16}$$

(ii) For any row stochastic matrix A, we have

$$\text{span}(Ax) \leq \text{span}(x), \quad (A^k x)^\top \leq x^\top, \quad \text{and} \quad (A^x)^\sharp \geq x^\sharp. \tag{17}$$

Consider any sequence $\{A_\omega^{(t)}\}_{t=1}^\infty$ of networks representing a realisation $\omega \in \Omega$ of the randomized sequence of social networks. Furthermore, let $\boldsymbol{f}^{(0)}$ denote the vector of initial believes and let

$$\boldsymbol{f}^{(t)} = A_\omega^{(t)} A_\omega^{(t-1)} \cdots A_\omega^{(1)} \boldsymbol{f}^{(0)}.$$

Any strongly connected and primitive network A is geometrically ergodic, i.e., there exists $\rho \in (0, 1)$ and $T > 0$ such that for the associated Markov chain we have

$$\|A^k - \Pi_A\| \leq \rho^k, \quad k \geq T.$$

Here, Π_A is ergodic projector of A, that is, Π_A has equal rows that the rows are equal to the unique stationary distribution of A, and $\|\cdot\|$ is some matrix norm. Note that

$$\lim_{k\to\infty} A^k = \Pi_A.$$

Geometric ergodicity implies that there exists a certain N such for $k \geq N$ the rows of A^k are almost equal. In other words, for all $\epsilon > 0$ there exists an N such that for all $k \geq N$

$$\|A^k - \Pi_A\| \leq \epsilon.$$

Let T_1 be the first time that $A(k) = A$ for N consecutive times. Note that since $\text{span}(\Pi_A x) = 0$, see property (i), we have that

$$\text{span}(A_\omega^{(T_1)} A_\omega^{(T_1-1)} \cdots A_\omega^{(1)} \boldsymbol{f}^{(0)}) \leq \epsilon \, \text{span}(\boldsymbol{f}^{(T_1-N)}).$$

For a proof, use that $\Pi_A \, \boldsymbol{f}^{(T_1-N)}$ has span zero. Since $\text{span}(\boldsymbol{f}^{(t)})$ is bounded by $\text{span}(\boldsymbol{f}^{(0)})$ for all t, we obtain

$$\text{span}(\boldsymbol{f}^{(T_l)}) \leq \epsilon^l \, \text{span}(\boldsymbol{f}^{(0)}),$$

where T_l denotes the l-th time that $A_\omega^{(t)} = A$ for N consecutive times. Hence, provided that T_l tends to infinity for l to infinity, we have

$$\lim_{t\to\infty} \text{span}(\boldsymbol{f}^{(t)}) = 0.$$

Since (ii) the values of $\boldsymbol{f}^{(t)}$ are bounded, i.e.,

$$(\boldsymbol{f}^{(t-1)})^\sharp \leq (\boldsymbol{f}^{(t)})^\sharp \leq (\boldsymbol{f}^{(t)})^\top \leq (\boldsymbol{f}^{(t-1)})^\top,$$

for all t. This implies

$$\lim_{t\to\infty} \boldsymbol{f}^{(t)} = \lim_{t\to\infty} (\boldsymbol{f}^{(t)})^\sharp.$$

To summarize, if T_l tends to infinity for l to infinity, then the believe vector converges to a vector with all components equal. In other words we reach consensus.

We now turn to proving (7). Recall that under superposition, the process (1) becomes deterministic after initial beliefs $f_i^{(0)}$'s being drawn. However, in (3) each $\widehat{\boldsymbol{f}}^{(t)}$, $t = 1, 2, \ldots,$ is a random variable with bounded support provided that $\widehat{\boldsymbol{f}}^{(0)}$ is given. The following Lemma follows from Lebesgue's dominated convergence theorem, see e.g. Doob (1994).

Lemma 1 (Dominated Convergence)**.** *If there exists a random variable* $\widehat{\boldsymbol{f}} : \Omega \to \mathbb{R}^n$ *such that*

$\widehat{\boldsymbol{f}}^{(t)}$ converges to $\widehat{\boldsymbol{f}}$ almost surely as $t \to \infty$, then it holds that

$$\mathrm{E}[\lim_{t\to\infty} \widehat{\boldsymbol{f}}^{(t)} \mid \widehat{\boldsymbol{f}}^{(0)}] = \lim_{t\to\infty} P^t \widehat{\boldsymbol{f}}^{(0)} = \mathbf{1} \cdot \boldsymbol{\pi}^\top \cdot \widehat{\boldsymbol{f}}^{(0)}.$$

We have proven in the first part of this proof that $\widehat{\boldsymbol{f}}$ converges almost surely if $\mathcal{A}$ consists of primitive matrices which confirms (7).

□

Appendix B: Perturbation Analysis under Superposition

Here we consider a non-wise sequence $\{X(n)\}_{n=n_0}^{\infty}$ and a wise sequence $\{Y(n)\}_{n=n_0}^{\infty}$. Let $\{P(n)\}_{n=n_0}^{\infty}$ be the blended sequence of $\{X(n)\}$ and $\{Y(n)\}$ with respect to $\theta(n) \in (0, 1)$, i.e. $P(n) = \theta(n)X(n)+(1 - \theta(n))Y(n)$ for all $n \geq n_0$. We assume both $X(n)$ and $Y(n)$ are primitive for $n \geq n_0$. It holds that

$$P(n) = \theta(n)X(n) + (1 - \theta(n))Y(n) = Y(n) + \theta(n)(X(n) - Y(n)). \tag{18}$$

Of interest here is under what conditions $\{P(n)\}_{n=n_0}^{\infty}$ will be wise. In the following discussion we drop variable n for convenience.

The deviation matrix associated with a Markov chain characterized by transition matrix $\tilde{P}$ is defined by

$$\begin{aligned} D_{\tilde{P}} &= \sum_{n=0}^{\infty}(\tilde{P}^n - \Pi_{\tilde{P}}) \\ &= \sum_{n=0}^{\infty}(\tilde{P} - \Pi_{\tilde{P}})^n - \Pi_{\tilde{P}} \\ &= (I - \tilde{P} + \Pi_{\tilde{P}})^{-1} - \Pi_{\tilde{P}} \\ &= F_{\tilde{P}} - \Pi_{\tilde{P}}, \end{aligned}$$

where $F_{\tilde{P}} := (I - \tilde{P} + \Pi_{\tilde{P}})^{-1}$ is the fundamental matrix of $\tilde{P}$. For any two primitive A and B, one has $A = B + (A - B)$ and

$$\Pi_A = \Pi_B + \Pi_A(A - B)D_B, \tag{19}$$

which is shown in **?**. Inserting (19) into Π_A on the right-hand side of (19) recursively, one has

$$\Pi_A = \Pi_B \sum_{k=0}^{t} \{(A-B)D_B\}^k + \Pi_A \{(A-B)D_B\}^{t+1}$$

for all $t \geq 0$. By defining $H(t)$ and $R(t)$ as

$$H(t) = \Pi_B \sum_{k=0}^{t} \{(A-B)D_B\}^k,$$
$$R(t) = \Pi_A \{(A-B)D_B\}^{t+1},$$

one can approximate Π_A by $H(t)$ as $t \to \infty$ where the remainder term $R(t)$ is shown to be convergent with an upper bound that decays to zero at a geometric rate, provided the following technical condition holds.

(**C**): *There exists a finite number T such that we can find $\delta_T \in (0,1)$ that satisfies*

$$\left\| ((A-B)D_B)^T \right\|_v < \delta_T,$$

where $\|\cdot\|_v$ is the v-norm of matrices on $\mathbb{R}^{S\times S}$, such that with function $v : S \to [1,\infty)$,

$$\|A\|_v = \sup_{s\in S} \frac{1}{v(s)} \sum_{s'\in S} |A_{ss'}|\, v(s').$$

Note that by letting $v(s) = 1$ for $s \in S$, the v-norm recovers the supremum norm.

Similarly, with Equation (18) it can be obtained that

$$\Pi_P = \Pi_Y + \Pi_X \theta (X-Y) D_Y, \tag{20}$$

and correspondingly

$$\Pi_P = \Pi_Y \sum_{k=0}^{t} \{\theta(X-Y)D_Y\}^k + R(t,\theta)$$

where $R(t,\theta) = \Pi_X \{\theta(X-Y)D_Y\}^{t+1}$. Here $R(t,\theta)$ can also be shown to be convergent with an upper bound that decays to zero at a geometric rate, under a slightly modified version of condition (**C**). Define $\delta(n)$ by

$$\delta(n) := \frac{1}{\left\|(X(n) - Y(n))D_Y(n)\right\|_1}.$$

Theorem 2. *Given* $\{X(n)\}_{n=n_0}^{\infty}$ *is non-wise and* $\{Y(n)\}_{n=n_0}^{\infty}$ *is wise,* $\{P(n)\}_{n=n_0}^{\infty}$ *is wise if*

$$\lim_{n\to\infty} \frac{\theta(n)}{\delta(n)} = 0.$$

Proof. Equation (20) can be rewritten as

$$\Pi_P(\theta(n), n) = \Pi_Y(n) + \theta(n)\Pi_X(n)(X(n) - Y(n))D_Y(n).$$

Since every element of the second term on the right-hand side is bounded by $\|\theta(n)\Pi_X(n)(X(n) - Y(n))D_Y(n)\|_1$, taking the (j, i) element of the above equation yields that

$$\begin{aligned} \pi_{P,i}(\theta(n), n) &\leq \pi_{Y,i}(n) + \theta(n)\, \|\Pi_X(n)\|_1\, \|(X(n) - Y(n))D_{Y(n)}\|_1 \\ &= \pi_{Y,i}(n) + \theta(n)\, \|(X(n) - Y(n))D_{Y(n)}\|_1, \end{aligned} \tag{21}$$

where the first inequality is due to the sub-multiplicativity of the operator norm $\|\cdot\|_v$, that is $\|AB\|_v \leq \|A\|_v \cdot \|B\|_v$, and the last equality follows from the fact that $\|\Pi_X(n)\|_1 = 1$. By definition,

$$\|(X(n) - Y(n))D_{Y(n)}\|_1 = \frac{1}{\delta(n)}.$$

Letting n tend to infinity in (21) yields

$$0 \leq \lim_{n\to\infty} \pi_{P,i}(\theta(n), n) \leq \lim_{n\to\infty} \pi_{Y,i}(n) + \lim_{n\to\infty} \frac{\theta(n)}{\delta(n)}.$$

The first term on the right-hand side of the above inequality tends to zero since we have assumed that $\{Y(n)\}_{n=n_0}^{\infty}$ is wise, and the second term tends to zero by assumption. We may thus conclude that

$$\lim_{n\to\infty} \pi_{P,i}(\theta(n), n) = 0.$$

The above argument can be applied to any agent i and we arrive at

$$\lim_{n\to\infty} \max_i \pi_{P,i}(\theta(n), n) = 0,$$

which then proves the claim. Note that taking the maximum in the above limit takes care of the effect that the maximizing index may depend on $\theta(n)$. $\square$