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# Comonotonic Approximations of Risk Measures for Variable Annuity Guaranteed Benefits with Dynamic Policyholder Behavior

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**Abstract**

The computation of various risk metrics is essential to the quantitative risk management of variable annuity guaranteed benefits. The current market practice of Monte Carlo simulation often requires intensive computations, which can be very costly for insurance companies to implement and take so much time that they cannot obtain information and take actions in a timely manner. In an attempt to find low-cost and efficient alternatives, we explore the techniques of comonotonic bounds to produce closed-form approximation of the risk measures for variable annuity guaranteed benefits. The techniques are further developed in this paper to address in a systematic way risk measures for death benefits with the consideration of dynamic policyholder behavior.

**Key Words.** Variable annuity guaranteed benefit, risk measures, value at risk, conditional tail expectation, geometric Brownian motion, comonotonicity, dynamic policyholder behavior.

## 1 Introduction

Quantitative modeling, pricing and risk management of variable annuities have become an active area of research, driven by rapid market innovation and increasing complexity of guaranteed benefits. Non-traditional quantitative techniques are required for quantifying, assessing and managing embedded option-like investment features. In recent years, regulators in North American markets have set up capital requirement standards for equity-linking insurance products based on Monte Carlo simulations. Among their many great advantages, simulation methods are known for their universal applications to complex systems of product designs and their easy implementation, especially with the rapid improvement of computational power. Bauer et al. (2008) and Bacinello et al. (2011) give comprehensive treatments of major product designs of guaranteed benefits by simulations. However, one should bear in mind that simulation-based techniques are sampling procedures that provide statistical estimation. It is a well-known fact that the sampling error of

Monte Carlo simulation in general goes down by $1/\sqrt{n}$ with $n$ being the sample size. In other words, the sample size has to increase a hundredfold in order for the estimate to improve one significant digit. Many industrial surveys have reported the growing problems of inefficient simulation exercises which make it extremely difficult to obtain useful information and make decisions on pricing and risk management in a timely manner. It is not surprising that practitioners often have to strike a difficult balance between the accuracy of results and the efficiency of their simulations.

There has been growing interest in both the industry and the literature for the improvement of model efficiency by either analytical methods or statistical means. For example, Koursaris (2011) discussed the computation of capital requirement by least squares Monte Carlo simulations. Bauer et al. (2010) compared least squares Monte Carlo simulations and numerical PDEs for valuing surrender options in equity-linking insurance.

The pricing of various types of variable annuity guaranteed benefits is extensively studied in the actuarial literature. Hardy (2003) provides a comprehensive review of option pricing theory and its applications to many investment guarantees. Ulm (2008) and Ulm (2014) derived analytical solutions to guaranteed minimum death benefits (GMDB) with rollup and ratchet options; Chi and Lin (2012) introduced a PDE method for pricing guaranteed minimum maturity benefit (GMMB) and GMDB with continuously paying premiums. As an alternative, a closed-form approximation for the same guarantees with flexible premium payments was derived in Costabile (2013). Marshall et al. (2010) studied the valuation of a guaranteed minimum income benefit (GMIB). Bernard et al. (2014) proposed models for optimal surrender strategy for various guaranteed benefits with surrender options. However, less is known with regard to the risk management of these guaranteed benefits. For many complex product designs, Monte Carlo simulations remain the only available tool for computing risk measures. Nevertheless, efforts have been made in the recent literature to draw analytical techniques non-conventional to actuarial literature to the computation of risk measures. Feng and Volkmer (2012) developed integral solutions to risk measures of GMMB and GMDB net liabilities using Yor's representation of the joint distribution of geometric Brownian motion and its time-integral. An improvement using spectral expansion techniques was made in Feng and Volkmer (2014).

Variable annuities are financial contracts between annuity writers (typically life insurers) and individual policyholders. Policyholders make purchase payments into investment accounts at the inception and expect to reap financial gain on the investment of their accounts. Let us first consider the cash flows of a stand-alone variable annuity contract. The life cycle of a variable annuity contract can be broken down into two phases. The first is known as the *accumulation phase*, in which policyholders' investment accounts grow in proportion to certain equity-indices in which policyholders choose to invest at the inception. Let $\{S_t, 0 \le t \le T\}$ describe the dynamics of the underlying equity-index from the inception of the contract to the maturity $T$ (which is assumed to be an integer) and $\{F_t, 0 \le t \le T\}$ describe the evolution of fund values in a particular policyholder's investment account with $F_0$ being the initial purchase payment. Let us consider the discrete time model with a valuation period of $1/n$ of a time unit, i.e. $t = 1/n, 2/n, \cdots, k/n, \cdots, T$.

The fees and charges by annuity writers are typically taken as a fixed percentage of the-then-current account values on a periodic basis. The equity-linking mechanism for variable annuity dictates that

$$F_{k/n} = F_0 \frac{S_{k/n}}{S_0} \left(1 - \frac{m}{n}\right)^{k-1}, \qquad k = 1, 2, \cdots, nT,$$

where $m$ is the annual rate of total charge compounded $n$ times per year, and charges are made at the beginning of each valuation period. This annual charge $m$ is also referred to in practice as the mortality and expense (M&E) fee. Let $r$ be the continuously compounding yield rate per year on bonds backing up the guaranteed benefits. Observe that the income from the insurer's perspective is generated by a stream of account-value-based payments. The present value of fee incomes, also called margin offset, up to the $k-$th valuation period is given by

$$M_{k/n} = \sum_{j=0}^{k-1} e^{-rj/n} \left(\frac{m_e}{n}\right) F_{j/n}.$$

where $m_e$ is the annual rate of GMMB rider charge compounded $n$ times each year (part of the total charge $m$ allocated to fund the GMMB).

The second phase typically starts at the beginning of payments from guaranteed benefits and is called the *income phase*. The models of the liabilities differ greatly by the designs of investment guarantee. In this paper, we consider the two most common types of benefits.

**Guaranteed Minimum Maturity Benefit (GMMB) - Individual Model**

In the case of a GMMB, the policyholder is guaranteed to receive a minimum balance $G$ in the investment account at maturity $T$. The present value of the gross liability to the insurer is

$$e^{-rT}(G - F_T)_+ I(T_x > T),$$

where $(x)_+ = \max\{x, 0\}$ and $T_x$ is the future lifetime of the policyholder of age $x$ at inception. Consider the net liability of the guaranteed benefits from the insurer's perspective, which is the gross liability of guaranteed benefits in the income phase less the fee incomes in the accumulation phase. The present value of the GMMB net liability is given by

$$L_e^{(n)}(T_x) := e^{-rT}(G - F_T)_+ I(T_x > T) - \sum_{j=0}^{(nT \wedge T_x)-1} e^{-rj/n} \left(\frac{m_e}{n}\right) F_{j/n},$$

where $x \wedge y = \min\{x, y\}$.

We shrink the valuation period to zero by taking $n$ to $\infty$ and observe that $\lim_{n\to\infty}(1 - \frac{m}{n})^n = e^{-m}$ where $m$ in this case should be interpreted as the continuously compounded annual rate of total charges. As a result, for each sample path,

$$F_t = \lim_{n\to\infty} F_{\frac{\lceil nt \rceil}{n}} = \frac{F_0}{S_0} \lim_{n\to\infty} S_{\frac{\lceil nt \rceil}{n}} \left(1 - \frac{m}{n}\right)^{\frac{\lceil nt \rceil - 1}{n} n} = F_0 \frac{S_t}{S_0} e^{-mt}.$$

Similarly, the margin offset can be written as

$$M_t = \lim_{n\to\infty} M_{\frac{\lceil nt \rceil}{n}} = \lim_{n\to\infty} \sum_{j=0}^{\lceil t/n \rceil -1} \frac{1}{n} e^{-rj/n} m_e F_{j/n} = \int_0^t e^{-rs} m_e F_s \,\mathrm{d}s$$

where $m_e$ is interpreted as the continuously compounded annual rate of rider charge allocated to the GMMB rider.

The limit of $L$ leads to a continuous time model. In the case of the GMMB,

$$L_e^{(\infty)}(T_x) = e^{-rT}(G - F_T)_+ I(T_x > T) - \int_0^{T \wedge T_x} e^{-rs} m_e F_s \,\mathrm{d}s. \tag{1.1}$$

**Guaranteed Minimum Death Benefit(GMDB) - Individual Model**

In case of a GMDB, the policyholder is guaranteed to receive a minimum balance $G$ regardless of the performance of the investment account at the end of the $1/n$-th period following his/her death. It is fairly common that the guarantee amount accumulates interest at a fixed rate $\delta > 0$, which is known as a roll-up option. The present value of the gross liability to the insurer is

$$e^{-rK_x}(Ge^{\delta K_x} - F_{K_x})_+ I(K_x < T),$$

where $K_x$ is the curtate future lifetime

$$K_x = \frac{1}{n}\lceil nT_x \rceil,$$

where $\lceil x \rceil$ is the integer ceiling of $x$. The present value of the GMDB net liability is given by

$$L_d^{(n)}(T_x) := e^{-rK_x}(Ge^{\delta K_x} - F_{K_x})_+ I(K_x < T) - \sum_{j=0}^{(nT \wedge K_x)-1} e^{-rj/n} m_e F_{j/n} \left(\frac{1}{n}\right).$$

Similarly, it is easy to use limiting arguments to show that in case of the GMDB,

$$L_d^{(\infty)}(T_x) = e^{-rT_x}(Ge^{\delta T_x} - F_{T_x})_+ I(T_x \le T) - \int_0^{T \wedge T_x} e^{-rs} m_e F_s \,\mathrm{d}s. \tag{1.2}$$

The net liabilities $L$ should be negative with a sufficiently high probability, as the products are designed to be profitable. However, in adverse scenarios, the net liabilities can become positive. The objective of actuarial risk management is to ensure that insurers set aside sufficient capitals to absorb unexpected losses in the adverse scenarios. The amount of minimum capital is often determined by risk measures of insurance liabilities, such as the value-at-risk, also known as quantile risk measure, defined as

$$\mathrm{VaR}_p(L) := \inf\{y : \mathbb{P}[L \le y] \ge p\}.$$

Another risk measure, which incorporates both the likelihood and severity of losses, is the conditional tail expectation

$$\mathrm{CTE}_p(L) := \mathbb{E}[L | L > \mathrm{VaR}_p].$$

The risk measures $\mathrm{VaR}_p(L_e^{(\infty)}), \mathrm{CTE}_p(L_e^{(\infty)}), \mathrm{VaR}_p(L_d^{(\infty)})$, and $\mathrm{CTE}_p(L_d^{(\infty)})$ were studied in Feng and Volkmer (2012) and Feng and Volkmer (2014) by analytical methods.

**Guaranteed Minimum Maturity Benefit (GMMB) - Average Model**

It is shown in Feng and Shimizu (2014) that if the future lifetimes of all policyholders are mutually independent and all contracts are of equal size, i.e. all policyholders make the same purchase payments and all contracts have the same guarantee level, then as $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N} L_e^{(n)}(T_x^{\langle i\rangle}) \longrightarrow \overline{L}_e^{(n)}, \qquad \text{almost surely,}$$

where $T_x^{\langle i\rangle}$ denotes the future lifetime of the $i$-th individual. The GMMB net liability under the average model is given by

$$\overline{L}_e^{(n)} := e^{-rT}\,{}_Tp_x\,(G-F_T)_+ - \left(\frac{1}{n}\right)\sum_{j=0}^{nT-1} e^{-rj/n}\,{}_{j/n}p_x\, m_e F_{j/n}, \tag{1.3}$$

Observe that the mortality risk is fully diversified in the sense that there is no uncertainty on the timing of cash flows. The continuous time analogue of the GMMB net liability under the average model is given by

$$\overline{L}_e^{(\infty)} := e^{-rT}\,{}_Tp_x(G-F_T)_+ - m_e\int_0^T e^{-rt}\,{}_tp_x F_t\,\mathrm{d}t. \tag{1.4}$$

The risk measures $\mathrm{VaR}_p(\overline{L}_e^{(\infty)})$ and $\mathrm{CTE}_p(\overline{L}_e^{(\infty)})$ were also studied in Feng (2014) using a numerical PDE method. The comparison of risk measures under the two models (1.1) and (1.4) shows in the paper that the financial risk is the dominating factor contributing to positive net liability in comparison with the mortality risk for the GMMB.

**Guaranteed Minimum Death Benefit (GMDB) - Average Model**

Under the same assumption as mentioned above, it is known that as $N \to \infty$,

$$\frac{1}{N}\sum_{i=1}^{N} L_d^{(n)}(T_x^{\langle i\rangle}) \longrightarrow \overline{L}_d^{(n)}, \qquad \text{almost surely.}$$

The GMDB net liability under the average model is given by

$$\overline{L}_d^{(n)} := \sum_{j=1}^{nT} e^{-rj/n}\,{}_{(j-1)/n}p_x\,{}_{1/n}q_{x+(j-1)/n}\,(Ge^{\delta j/n}-F_{j/n})_+ - \left(\frac{1}{n}\right)\sum_{j=0}^{nT-1} e^{-rj/n}\,{}_{j/n}p_x\, m_d F_{j/n}. \tag{1.5}$$

The mortality risk is fully diversified in the sense that there is no uncertainty on the timing of death benefits. The continuous time analogue of the GMDB net liability is determined by letting

$n$ go to infinity

$$\overline{L}_d^{(\infty)} := \int_0^T e^{-rt}\, {}_tp_x\, \mu_{x+t}(Ge^{\delta t} - F_t)_+ \,\mathrm{d}t - m_d \int_0^T e^{-rt}\, {}_tp_x F_t \,\mathrm{d}t,$$

where $\mu_{x+t} = -(\mathrm{d}/\,\mathrm{d}t)\, {}_tp_x /\, {}_tp_x$ is the force of mortality in a continuous mortality model. Unlike the previous models, the numerical PDE method used in Feng (2014) does not apply directly in this case, although other PDE methods may be possible. In this paper, we sort to alternative methods such as comonotonic approximations in this paper.

Although the above formulation of net liabilities can be used for any equity return model, we rely on analytical properties of the underlying processes for the computation of risk measures. Among several commonly used asset return models recommended by the American Academy of Actuaries (c.f. Gorski and Brown (2005)), we use the geometric Brownian motion $\{S_t, t \geq 0\}$, also known as the independent lognormal model by the insurance industry,

$$S_t = S_0 e^{\mu t + \sigma B_t}, \qquad t \geq 0, \tag{1.6}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion. In practice, most equity funds offer a fixed (or relatively stable) make-up of subaccounts by periodic rebalancing. For instance, 30%-high-yield-equity, 30%-low-yield-equity and 40%-bonds. If each of the subaccounts in the equity fund is modeled by a geometric Brownian motion and the proportion attributable to each subaccount is fixed, then the overall equity fund is also driven by a geometric Brownian motion. Continuously rebalanced portfolios were studied with comonotonicity techniques in Dhaene et al. (2006), Marin-Solano et al. (2010) and Dhaene et al. (2005). Even though we only consider the model (1.6) in this paper, most results can be extended to regime-switching geometric Brownian motions, also recommended in the AAA guideline.

The main contributions of the paper can be summarized as follows. (1) While there exists actuarial literature on the pricing of exotic options using comonotonicity (c.f. Simon et al. (2000), Albrecher et al. (2005), Hobson et al. (2005), Chen et al. (2008), Linders et al. (2012) amongst others), this is the first paper to systematically explore this technique for risk measures of variable annuity contracts. (2) In the same technical framework of comonotonicity, we address the computational issue of dynamic policyholder behavior, which was only previously known by simulations in the literature and in practice. (3) Given the complexity of the (time-inhomogeneous) average models, it is quite remarkable that the technique of comonotonicity produces easy-to-implement explicit solutions (Propositions 3.1 and 4.1). The approximation can be viewed as a "back-of-envelope-calculation" alternative to Monte Carlo simulations which require intensive computations. The computational advantage is even more pronounced when policyholder behavior is considered. (4) To the best of our knowledge, this is the first paper in the actuarial literature to provide an analysis of risk metrics with dynamic policyholder behavior using a non-simulation based approach.

In Section 2, we will introduce the concept of comonotonicity, its basic properties and a general form of comonotonic bounds for path dependent equity-link products. Comonotonic bounds for the net liability of GMMB are discussed in detail in Section 3. A more complex development of comonotonic approximations of the net liability of GMDB is introduced in Section 4. In Section 5, we compare the approximations developed in this paper with the benchmark of Monte-Carlo simulation through several numerical examples. In Section 6, we extend the comonotonic approximation approach to address the computation of risk measures for net liabilities taking into account dynamic policyholder behavior.

## 2 Comonotonicity

The theory of comonotonicity was originally studied in the actuarial literature with respect to estimating aggregate claims, which are often sums of dependent random variables representing individual claims. Over the past decades it has seen wider applications ranging from rate-making of property-casualty insurance to pricing of exotic options. Readers can find a comprehensive review on the theory of comonotonicity in Dhaene et al. (2002b), Dhaene et al. (2002a), Dhaene et al. (2006), Deelstra et al. (2011) and the references therein. For the sake of completeness, we briefly review the properties of comonotonic bounds which will be used in our calculations. Although some error analysis is known for comonotonic bounds of option prices (c.f. Vanduffel et al. (2005) and Vanmaele et al. (2006)), there appears to be no error estimation in the previous literature on the TVaR of comonotonic lower bound, which we shall use for approximations of risk measures for variable annuity guaranteed benefits. Hence, we first develop a formula for the error estimation.

### 2.1 Convex order and implication for TVaR

The random variable $X$ is said to be smaller than the random variable $Y$ in convex order, denoted by $X \leqslant_{cx} Y$, if for all $d \in \mathbb{R}$,

$$E(X) = E(Y) \qquad \text{and} \qquad E(X-d)_+ \leqslant E(Y-d)_+.$$

There are several commonly used risk measures for loss random variables. The Value-at-Risk, also known as the quantile risk measure, is defined by

$$\mathrm{VaR}_p(X) := \inf\{x : \mathbb{P}(X > x) < p\}, \qquad p \in [0,1],$$

with $\inf \varnothing = -\infty$. Another risk measure often used for regulatory capital requirements is the Conditional Tail Expectation, defined by

$$\mathrm{CTE}_p(X) := \mathbb{E}[X | X > \mathrm{VaR}_p(X)], \qquad p \in [0,1].$$

Other risk measures include the Left-Tail-Value-at-Risk and the (right-)Tail-Value-at-Risk, defined for $p \in [0,1]$,

$$\mathrm{TVaR}_p(X) := \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\,\mathrm{d}q, \qquad \mathrm{LTVaR}_p(X) := \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\,\mathrm{d}q,$$

where the definitions should be considered as limits of the fractions for $\mathrm{TVaR}_1$ and $\mathrm{LTVaR}_0$. In the applications of this paper, we shall apply these risk measures to continuous random variables such as $L_e^{(n)}$ and $L_d^{(n)}$. It is easy to show that $\mathrm{CTE}_p$ is identical to $\mathrm{TVaR}_p$ for continuous random variables for all $p \in [0,1]$. Hence, we do not distinguish them in this paper.

For insurance applications, we often encounter problems of computing risk measures of random variables arising from complex structure. For example, it may be difficult to directly determine the distribution of some random variable, $X$. Then we may use the convex order relation

$$X^l = \mathbb{E}(X|\Lambda) \leq_{cx} X, \tag{2.1}$$

which implies that

$$\mathrm{TVaR}_p(X^l) \leq \mathrm{TVaR}_p(X), \qquad \forall p \in (0,1). \tag{2.2}$$

The proof of this result can be found, for example, in Dhaene et al. (2006). The TVaR of the comonotonic approximation $X^l$ is sometimes much easier to compute than that of the original variable $X$ and serves as a lower bound. This is in particular the case where $X$ is the sum of the components of a multivariate lognormal random vector. For numerical implementation, we want to know the magnitude of the errors of the lower bound.

**Proposition 2.1.** *For all* $p \in [0,1)$,

$$\boxed{\mathrm{TVaR}_p(X) - \mathrm{TVaR}_p(X^l) \leq \frac{1}{2(1-p)}\mathbb{E}(|X - X^l|).} \tag{2.3}$$

*Proof.* It is easy to prove that for two real-valued functions $f$ and $g$ bounded from below with the same domain,

$$\inf f + \inf g \leq \inf\{f + g\},$$

which implies

$$\inf f - \inf g \leq -\inf\{g - f\}.$$

Furthermore, if $g - f$ is bounded, then

$$\inf f - \inf g \leq \sup\{f - g\}.$$

We know from (Denuit et al., 2005, p.75) that

$$\mathrm{TVaR}_p(X) = \inf_{a\in\mathbb{R}} \left\{a + \frac{1}{1-p}\mathbb{E}(X-a)_+\right\}.$$

Let us denote the expression inside the brackets by $f_X(a)$. Then it is clear that

$$f'_X(a) = 1 - \frac{1}{1-p}\mathbb{P}(X > a).$$

Therefore, there exists some number $a_0$ such that $f$ is non-increasing on $(\infty, a_0)$ and non-decreasing on $(a_0, \infty)$. Since

$$\lim_{a\to-\infty} a + \frac{1}{1-p}\mathbb{E}(X-a)_+ \geq \frac{1}{1-p}\lim_{a\to-\infty} a + \mathbb{E}(X-a)_+ = \frac{1}{1-p}\mathbb{E}(X) > -\infty,$$

we find that $f_X$ is indeed bounded from below. Note that by Jensen's inequality,

$$f_S(a) - f_{S^l}(a) = \frac{1}{1-p}\{\mathbb{E}(S-a)_+ - \mathbb{E}(S^l-a)_+\} \geq 0, \qquad \forall a \in \mathbb{R}.$$

Moreover, $f_S - f_{S^l}$ is differentiable and

$$\lim_{a\to+\infty} f_S(a) - f_{S^l}(a) = 0.$$

Thus $f_S - f_{S^l}$ is bounded. Therefore,

$$\mathrm{TVaR}_p(S) - \mathrm{TVaR}_p(S^l) \leq \frac{1}{1-p}\sup_{a\in\mathbb{R}}\Big\{\mathbb{E}(S-a)_+ - \mathbb{E}(S^l-a)_+\Big\}.$$

It follows from (Rogers and Shi, 1995, (3.5)) that

$$\mathbb{E}(Y_+) - \mathbb{E}(\mathbb{E}(Y|\Lambda)_+) \leq \frac{1}{2}\mathbb{E}(|Y - \mathbb{E}(Y|\Lambda)|).$$

Let $Y = S - a$. Therefore, we find the error bound (2.3). □

Hereafter we provide some examples to demonstrate that (2.1) is a tight upper bound of the difference in the sense that the upper bound can be reached for a particular choice of $p \in (0,1)$.

**Example 2.1.** *Let $\Lambda$ be independent of $X$, then (2.1) is reduced to the special form*

$$\mathrm{TVaR}_p(X) - \mathbb{E}(X) \leqslant \frac{1}{2(1-p)}\mathbb{E}|X - \mathbb{E}X|$$

1. *Consider $X$ to be a standard normal random variable with cdf $\Phi$. It follows immediately that*

$$\mathrm{VaR}_p(x) = \Phi^{-1}(p) \qquad \text{and} \qquad \mathrm{TVaR}_p(X) = \frac{1}{(1-p)\sqrt{2\pi}}e^{-\frac{1}{2}(\Phi^{-1}(p))^2}$$

   *Since $\mathbb{E}X = 0$ and $\mathbb{E}|X| = \sqrt{2/\pi}$, the upper bound is attained if and only if $p = \frac{1}{2}$.*

2. *Consider $X$ to be an exponential random variable with mean $1/\lambda$. Then*

$$\mathrm{VaR}_p(x) = -\frac{1}{\lambda}\ln(1-p) \qquad \text{and} \qquad \mathrm{TVaR}_p(X) = \frac{1-\ln(1-p)}{\lambda}$$

   *and $\mathbb{E}|X - \mathbb{E}X| = 2/(\lambda e)$ where $e$ is the Euler's constant. It turns out that the upper bound is attained at the constant $p = (e-1)/e \approx 0.6321205588$.*

3. *Let $X$ be a gamma random variable with mean $\alpha/\beta$ and variance $\alpha^2/\beta$. Then* $\mathrm{VaR}_p(x)$ *is the inverse function of* $F(x) = \frac{1}{\Gamma(\alpha)}\gamma(\alpha, \beta x)$ *and $\gamma$ is the lower incomplete gamma function. And*

$$\mathrm{TVaR}_p(X) = \frac{1}{(1-p)\beta\Gamma(\alpha)}\Gamma(\alpha+1, \beta\mathrm{VaR}_p(x))$$

$$\mathbb{E}|X - \mathbb{E}X| = \frac{2\alpha^\alpha e^{-\alpha}}{\beta\Gamma(\alpha)}$$

*where $\Gamma(\cdot)$ and $\Gamma(\cdot,\cdot)$ are the gamma function and the upper incomplete gamma function. It can be shown that the upper bound is attained at the constant $p = \gamma(\alpha,\alpha)/\Gamma(\alpha)$, which is independent of $\beta$.*

**Remark 2.1.** *In applications, it is often difficult to determine a closed-form expression for $\mathbb{E}(|X - X^l|)$. Therefore, using the fact that $\mathbb{E}|X - X^l| \leq \mathrm{Var}(X|\Lambda)$, we find the following weaker upper bound:*

$$\mathrm{TVaR}_p(X) - \mathrm{TVaR}_p(X^l) \leq \frac{1}{2(1-p)}\mathbb{E}(\mathrm{Var}(X|\Lambda)^{1/2}), \qquad \textit{for } p \in [0,1). \tag{2.4}$$

*Here,* Var *is to used to denote the variance, not to be confused with the value-at-risk* $\mathrm{VaR}_p$*. Note, however, the upper bound in* (2.4) *may not be attained for any $p$ in* $[0,1)$*.*

Numerical experiments indicate that the bounds in (2.3) and (2.4) are generally very conservative for large $p$, and actual errors are orders of magnitude smaller.

## 2.2 Comonotonic bounds for sums of random variables

We are often interested in the aggregate sum of random variables such as $S = X_1 + X_2 + \cdots + X_n$, where the marginal distributions of random variables $X_1, X_2, \cdots, X_n$ are known but their joint distribution is either unknown or too complex to be useful for computations. In such cases, one can exploit the theory of comonotonic bounds to find closed-form approximations that can be implemented efficiently.

By definition, the random vector $(X_1, X_2, \cdots, X_n)$ is comonotonic if

$$(X_1, X_2, \cdots, X_n) \sim \left(F_1^{-1}(U), F_2^{-1}(U), \cdots, F_n^{-1}(U)\right),$$

where $\sim$ means equality in distribution and $F_k^{-1}$ is the (generalized) inverse distribution function of $X_k$ for $k = 1, \cdots, n$. For any random vector $(X_1, X_2, \cdots, X_n)$ and any random variable $\Lambda$,

$$S^l := \sum_{i=1}^n \mathbb{E}(X_i|\Lambda) \leqslant_{cx} \sum_{i=1}^n X_i \leqslant_{cx} \sum_{i=1}^n F_{X_i}^{-1}(U)$$

where $U$ has a uniform distribution on $[0,1]$. We call $S^l$ the comonotonic lower bound of $S$ based on $\Lambda$. The right-hand side of the second inequality is called the comonotonic upper bound.

It is known in the literature (c.f. Vanduffel et al. (2005)) that in a multivariate lognormal setup with appropriate choices of $\Lambda$, the comonotonic lower bound $S^l$ provides a better approximation of

$S$ than the comonotonic upper bound. Since (2.1) implies (2.2), we have that for any conditioning random variable $\Lambda$,

$$\mathrm{TVaR}_p(S^l) \le \mathrm{TVaR}_p(S).$$

Then we can try to obtain the maximum value of the lower bound $\mathrm{TVaR}_p(S^l)$,

$$\max_{\Lambda \in \Theta} \mathrm{TVaR}_p(S^l), \tag{2.5}$$

as the approximation of $\mathrm{TVaR}(S)$, where $\Lambda$ is taken from a family $\Theta$ of normal random variables.

# 3 Guaranteed minimum maturity benefit

For brevity, we suppress the superscripts $(n)$ and subscripts $e$ in the notation introduced earlier, as the frequency of charges and the type of benefit are clear from the context of this section. Given that $L > 0$, we consider the net liability in the average model (1.3):

$$L = e^{-rT} \,{}_T p_x G - \left( \frac{1}{n} m_e F_0 + S \right),$$

where

$$S = \sum_{i=1}^{nT-1} \alpha_i e^{Z_i}, \qquad Z_i = (\mu - r - m)\frac{i}{n} + \sigma B_{i/n},$$

and the $\alpha_i$'s are positive constants defined by

$$\alpha_i = \begin{cases} \frac{1}{n} \,{}_{i/n} p_x m_e F_0, & i = 1, \cdots, nT-1, \\ {}_T p_x F_0, & i = nT. \end{cases}$$

Consider the comonotonic lower bound of $S$ given by

$$S^l = \mathbb{E}[S|\Lambda] = \sum_{i=1}^{nT-1} \alpha_i \mathbb{E}[e^{Z_i}|\Lambda],$$

where the conditioning random variable $\Lambda$ is a linear combination of $M$ appropriately chosen normal random variables, $\{N_1, \cdots, N_M\}$, derived from $\{B_t, t \ge 0\}$, i.e.

$$\Lambda = \sum_{k=1}^{M} \lambda_k N_k.$$

In the work of Vanduffel et al. (2008b) and Vanduffel et al. (2008a), a total of $M = nT$ normal random variables was used with $N_k = Z_k$. Under this choice of random variables, the weights $\{\lambda_k, k = 1, \cdots, M\}$ were derived to approximately maximize $\mathrm{TVaR}_p(S^l)$ or $\mathrm{Var}(S^l)$, see Appendix B. However, as we shall demonstrate in the numerical examples, it is possible to achieve roughly the same maximum with a fewer number of random variables, which requires less computational efforts. For example, we can let $N_k = Z_{nk}$ and $M = T$.

Using properties of conditional distributions, we obtain

$$S^l = \sum_{i=1}^{nT} \alpha_i e^{\mathbb{E}[Z_i]+\frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i}(\Lambda-\mathbb{E}[\Lambda])/\sigma_\Lambda},$$

where $\mathbb{E}[Z_i]$ and $\sigma_{Z_i}^2$ are the expectation and variance of $Z_i$, i.e.

$$\mathbb{E}[Z_i] = (\mu - r - m)\frac{i}{n}, \qquad \sigma_{Z_i} = \sigma\sqrt{\frac{i}{n}},$$

while $r_i$ is the correlation coefficient of $Z_i$ and $\Lambda$, and $\sigma_\Lambda^2$ is the variance of $\Lambda$. Owing to the structure of the sum $S^l$, we can find explicit expressions for the risk measures

$$\mathrm{VaR}_p[S^l] = \sum_{i=1}^{nT-1} \alpha_i e^{\mathbb{E}[Z_i]+\frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i}\Phi^{-1}(p)}$$

and

$$\mathrm{TVaR}_p[S] \geqslant \mathrm{TVaR}_p[S^l] = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_p[S^l]\,\mathrm{d}p = \frac{1}{1-p}\sum_{i=1}^{nT-1} \alpha_i \mathbb{E}[e^{Z_i}]\Phi(r_i\sigma_{Z_i} - \Phi^{-1}(p)).$$

The derivation can be found in Vanduffel et al. (2008b).

**Proposition 3.1.** *Consider the net liability for the GMMB in the average model* (1.3)*. The value-at-risk and the conditional tail expectation of the comonotonic lower bound* $L^l = E[L|\Lambda]$ *are given by*

$$\boxed{\mathrm{VaR}_p(L^l) = e^{-rT}\,{}_Tp_x G - \left(\frac{1}{n}m_e F_0 + \mathrm{VaR}_{1-p}(S^l)\right)} \tag{3.1}$$

*and*

$$\boxed{\mathrm{CTE}_p(L^l) = e^{-rT}\,{}_Tp_x G - \frac{1}{n}m_e F_0 - \mathrm{LTVaR}_{1-p}(S^l)} \tag{3.2}$$

*where*

$$\mathrm{VaR}_p(S^l) = \sum_{i=1}^{nT-1} \alpha_i e^{\mathbb{E}[Z_i]+\frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i}\Phi^{-1}(p)},$$

*and*

$$\mathrm{LTVaR}_{1-p}(S^l) = \frac{1}{1-p}\sum_{i=1}^{nT-1} \alpha_i \mathbb{E}[e^{Z_i}](1 - \Phi(r_i\sigma_{Z_i} - \Phi^{-1}(1-p))).$$

**Remark 3.1.** *Note that the risk measures* $\mathrm{VaR}_p(S^l)$ *and* $\mathrm{LTVaR}_p(S^l)$ *only depend on the unspecified vector* $(\lambda_1, \cdots, \lambda_M)$ *through the vector* $(r_1, \cdots, r_M)$*. Rather than searching for* $\lambda_i$*'s, Vanduffel et al. (2008a) proposed two methods for selecting optimal* $r_i$*'s. Their first approach is to maximize the first-order approximation of the variance of* $S^l$ *(globally optimal choice), in which case* $N_k = Z_k$ *for* $k = 1, \cdots, nT$*. The second approach maximizes the first-order approximation of* $\mathrm{TVaR}_p(S^l)$ *(locally optimal choice). The exact formulas for the optimal choice of* $r_i$*'s can be found in the Appendix. The first numerical example in Section 5 provides a testimony to the remarkable effectiveness of the approximations. However, even the first-order approximations can be difficult to find for the GMDB. Hence, we propose to use numerical optimization algorithms to find* $\lambda_i$*'s that achieve* (2.5)*.*

# 4 Guaranteed minimum death benefit

For the net liability of the GMDB in the average model (1.5),

$$L = \sum_{i=1}^{nT} u_i (G e^{\delta i/n} - F_{i/n})_+ - \sum_{i=0}^{nT-1} v_i F_{i/n}$$

where

$$u_i := e^{-ri/n} \; {}_{(i-1)/n}p_x \; {}_{1/n}q_{x+(i-1)/n}, \qquad v_i := \frac{1}{n} e^{-ri/n} m_d \; {}_{i/n}p_x.$$

We use random variable $N_k$'s with mean zero,

$$\Lambda = \sum_{k=1}^{M} \lambda_k N_k.$$

Since $F_{i/n} = \exp(Z_i)$ where $Z_i = (\mu - m)\frac{i}{n} + \sigma B_{i/n}$, we must have

$$Z_i | \Lambda = \lambda \sim \text{Norm}\left( \mu_i \left( \frac{\lambda}{\sigma_\Lambda} \right), \sigma_i^2 \right),$$

where

$$\mu_i(y) := (\mu - m)\frac{i}{n} + r_i \sigma \sqrt{\frac{i}{n}} y, \qquad \sigma_i^2 := \sigma^2 \frac{i}{n}(1 - r_i^2).$$

Recall that

$$\mathbb{E}\left[ (G - F_0 e^{\mu + \sigma \Phi^{-1}(U)})_+ \right] = G\Phi\left( \frac{\ln(G/F_0) - \mu}{\sigma} \right) - F_0 e^{\mu + \sigma^2/2} \Phi\left( \frac{\ln(G/F_0) - \sigma^2 - \mu}{\sigma} \right).$$

Let $\mu_i^*(y) = \mu_i(y) - \delta i/n$. Consider the conditional expectation

$$\begin{aligned} \mathbb{E}[L|\Lambda = \lambda] = & \sum_{i=1}^{nT} u_i \left[ G e^{\delta i/n} \Phi\left( \frac{\ln(G/F_0) - \mu_i^*\left(\frac{\lambda}{\sigma_\Lambda}\right)}{\sigma_i} \right) - F_0 e^{\mu_i^*\left(\frac{\lambda}{\sigma_\Lambda}\right) + \sigma_i^2/2} \Phi\left( \frac{\ln(G/F_0) - \sigma_i^2 - \mu_i^*\left(\frac{\lambda}{\sigma_\Lambda}\right)}{\sigma_i} \right) \right] \\ & - \sum_{i=0}^{nT-1} v_i \, F_0 \exp\left\{ \mu_i \left( \frac{\lambda}{\sigma_\Lambda} \right) + \sigma_i^2/2 \right\}. \end{aligned} \tag{4.1}$$

It is easy to show that $\mathbb{E}\left[ (G - F_0 e^{\mu + \sigma \Phi^{-1}(U)})_+ \right]$ is a decreasing function of $\mu$. Therefore, each term in (4.1) is a decreasing function of $\lambda$. This implies that we have closed-form formulas for both risk measures of $L^l := \mathbb{E}[L|\Lambda]$.

**Proposition 4.1.** *Consider the net liability for the GMDB in the average model* (1.5). *The value-at-risk and the conditional tail expectation of the comonotonic lower bound* $L^l$ *are given by*

$$\begin{aligned} \text{VaR}_p(L^l) = & \sum_{i=1}^{nT} u_i \left[ G e^{\delta i/n} \Phi\left( \frac{\ln(G/F_0) - \mu_i^*\left(\Phi^{-1}(1-p)\right)}{\sigma_i} \right) \right. \\ & \left. - F_0 e^{\mu_i^*\left(\Phi^{-1}(1-p)\right) + \sigma_i^2/2} \Phi\left( \frac{\ln(G/F_0) - \sigma_i^2 - \mu_i^*\left(\Phi^{-1}(1-p)\right)}{\sigma_i} \right) \right] \\ & - \sum_{i=0}^{nT-1} v_i \, F_0 \exp\left\{ \mu_i\left(\Phi^{-1}(1-p)\right) + \sigma_i^2/2 \right\} - \frac{1}{n} m_d \, F_0, \end{aligned} \tag{4.2}$$

*and*

$$\boxed{\begin{aligned} \mathrm{CTE}_p(L^l) &= \frac{1}{1-p}\sum_{i=1}^{nT} u_i \Bigg[ G e^{\delta i/n} H\big(\Phi^{-1}(1-p); a_i, b_i\big) \\ &- F_0 \exp\left\{\mu_i^*(0) + \frac{1}{2}\sigma^2 \frac{i}{n}\right\} H\left(\Phi^{-1}(1-p) - r_i\sigma\sqrt{\frac{i}{n}}; a_i, b_i - \frac{i\sigma^2}{n\sigma_i}\right)\Bigg] \\ &- \frac{1}{1-p}\sum_{i=0}^{nT-1} v_i\, F_0 \exp\left\{\mu_i(0) + \frac{1}{2}\sigma^2\frac{i}{n}\right\}\Phi\left(\Phi^{-1}(1-p) - r_i\sigma\sqrt{\frac{i}{n}}\right) - \frac{1}{n} m_d\, F_0, \end{aligned}} \tag{4.3}$$

*where*

$$a_i = \frac{r_i}{\sqrt{1-r_i^2}}, \qquad b_i = \frac{\ln(G/F_0) - (\mu - m - \delta)i/n}{\sigma_i}.$$

*and $H$ is a special function whose definition and computation are discussed in the Appendix.*

The derivation is largely based on the simple identity (A.1). It should be pointed out that it is in general difficult to find explicit formulas for optimal choices of $r_i$ because of the complex structure of $H$ functions. Nevertheless, numerical methods for nonlinear optimization problems are widely available in computational software packages such as Matlab. In Section 5, we shall provide an example in which the optimization procedure is implemented.

# 5 Numerical example

We illustrate the computation of risk measures for variable annuity guaranteed benefits by two examples, which are based on the following assumptions. The policyholder is 65-year-old at policy issue, and the term of the variable annuity is 10 years, i.e. $T = 10$. The mean and standard deviation of log-returns per annum in the Black-Scholes model (1.6) are set as $\mu = 0.09$ and $\sigma = 0.3$ respectively. The yield rate per annum of the assets backing up the guarantee liabilities is $r = 0.04$. The M&E fee per annum is $m = 0.01$, and rider charge $m_e$ or $m_d$ is assumed to be 35 basis points per annum of the separate account. The initial guarantee amount is set to be the initial purchase payment $G = F_0$. To model the future lifetime of policyholders, we use the life tables published in the actuarial study by the U.S. Social Security Administration in 2005.

All computations in Sections 5 and 6 are carried out on a personal computer with Intel Core i7-4700MQ CPU at 2.40GHz and an RAM of 8.00 GB.

## 5.1 Guaranteed minimum maturity benefit

The purpose of the first example is to test the accuracy and efficiency of the comonotonic approximations proposed in Proposition 3.1. The computation of risk measures for the GMMB under the continuous time average model (1.4) was investigated in (Feng, 2014, Tables 5 and 6) through

| $x$ | $q_x$ | $k$ | ${}_kp_x$ | $x$ | $q_x$ | $k$ | ${}_kp_x$ |
|---|---|---|---|---|---|---|---|
| 65 | 0.01753 | 0 | 1 | 71 | 0.03059 | 6 | 0.87275 |
| 66 | 0.01932 | 1 | 0.98246 | 72 | 0.03343 | 7 | 0.84606 |
| 67 | 0.02122 | 2 | 0.96348 | 73 | 0.03633 | 8 | 0.81778 |
| 68 | 0.02323 | 3 | 0.94304 | 74 | 0.03942 | 9 | 0.78807 |
| 69 | 0.02538 | 4 | 0.92113 | 75 | 0.04299 | 10 | 0.75700 |
| 70 | 0.02785 | 5 | 0.89775 | | | | |

Table 1: Life Table

a numerical PDE method. As we shall demonstrate, the comonotonic approximations appear to be very efficient with only small compromise of accuracy, which is likely negligible for practical purposes. Hence, the comonotonic approximation is arguably superior to the PDE method for this example.

| Method | $\mathrm{VaR}_{0.9}$ | $\mathrm{CTE}_{0.9}$ | Time (secs) |
|---|---|---|---|
| Global optimization $\Lambda^{\mathrm{MV}}$ | 0.14900 | 0.25944 | 0.04 |
| Local optimization $\Lambda^{(p)}$ | 0.14901 | 0.25948 | 0.04 |
| Nonlinear optimization | 0.14902 | 0.25948 | 33.95 |
| Monte Carlo (1 million) | 0.14914 (0.00043) | 0.25966 (0.00034) | 179.84 |
| Monte Carlo (100 millions) | 0.14902 (0.00007) | 0.25949 (0.00004) | 18042.09 |

Table 2: GMMB

We first run Monte Carlo simulations for the average model (1.3) with $n = 4$, i.e. fees are collected on a quarterly basis. The probability of the policyholder surviving a non-integer period is calculated under the assumption of constant force of mortality in each year. For each scenario of investment accounts generated by the geometric Brownian motion, we calculate the net liability based on the formulation (1.3). After repeating the simulation 1 million or 100 million times, the net liability values form an empirical distribution, from which we use order statistics to obtain one estimate of the value-at-risk and conditional tail expectation. Then we repeat the whole procedure 20 times to obtain a sample of risk measure estimates. In Table 2, we show the mean and standard deviation (in brackets) of the estimated risk measures.

We test the comonotonic approximations (3.1) and (3.2) with various choices of $\Lambda$. In the first case, we set $N_k = Z_k$ for $k = 1, \cdots, nT$ and use the *globally optimal choice* of $\Lambda$, proposed by Vanduffel et al. (2008b), which is the optimization of the linear approximation of $\mathrm{Var}(S^l)$ as a function of the vector $(r_1, \cdots, r_M)$. The exact expressions for $r_i$'s are given in (B.1). In the second case, we use the *locally optimal choice* of $\Lambda$, proposed in the same paper, which is the optimization of the linear approximation of $\text{TVaR}_p(S^l)$ as a function of the vector $(r_1, \cdots, r_M)$. The exact expressions for $r_i$'s are given in (B.2). In the third case, we treat the risk measures as nonlinear functions of the vector $(\lambda_1, \cdots, \lambda_M)$ and use Matlab's `fminsearch` to find the optimal value of the vector that minimizes $-\text{TVaR}_p(S^l)$. This algorithm uses a version of the Nelder-Mead simplex search method to obtain a local minimum value of the function. It works well in unconstrained nonlinear optimization system, which suits our situation. Based on empirical data, we observe that $\text{TVaR}_p(S^l)$ is in fact unimodal and hence the numerical algorithm is very stable and efficient.

It is not surprising that the approximations based on the global and local optimal choices of $r_i$'s are more efficient than the nonlinear optimization, as the former pins down the values of $r_i$'s by closed-form formulas where as the latter invokes a search algorithm for $\lambda_i$'s. It is also worth to note that the nonlinear optimization brings $\text{TVaR}_p(S^l)$ closer to the true value of $\text{TVaR}(S)$.

### 5.2 Guaranteed minimum death benefit

Consider the net liability of the GMDB rider in (1.5). For simplicity, the net liabilities are evaluated under the same valuation basis as in the GMMB case. Keep in mind that we no longer have closed-form solutions to $(\lambda_1, \cdots, \lambda_M)$. In the case of $N_k = Z_k$ for $k = 1, \cdots, 4T$ (quaterly valuation), a 10-year contract with quarterly fee payments require 40-dimensional optimization (the first row in Table 3). Therefore, we intend to reduce computational efforts by restricting the space of normal random variables $\Theta$ in (2.5). We use the results from Monte Carlo simulations as the bench mark for accuracy and efficiency. In the second row of Table 3, the normal random variables are sampled every half-year, i.e. $N_k = Z_{2k}$ for $k = 1, \cdots 2T$. (The number of random variables is reduced by half from the quarterly case.) In the third row of Table 3, the normal random variables are chosen on yearly basis, i.e. $N_k = Z_{4k}$ for $k = 1, \cdots, T$. (The number of random variables is reduced by 75% from the quarterly case.) It appears that the running time can be reduced drastically with small compromises of accuracy.

| Method | $\text{VaR}_{0.9}$ | $\text{CTE}_{0.9}$ | Time (secs) |
|---|---|---|---|
| Nonlinear optimization | 0.03035 | 0.06126 | 69.97 |
| Nonlinear optimization ( 50% reduced) | 0.03031 | 0.06123 | 30.50 |
| Nonlinear optimization ( 75% reduced) | 0.03018 | 0.06111 | 7.83 |
| Monte Carlo (1 million) | 0.03059 (0.00013) | 0.06137 (0.00010) | 226.16 |
| Monte Carlo (100 millions) | 0.03035 (0.00002) | 0.06128 (0.00002) | 22602.80 |

Table 3: Risk measures for the GMDB net liability with $\delta$=0

In the next example, we intend to consider the impact of offering a roll-up bonus on the GMDB net liability. In this case, the guarantee base accumulates interests at the rate of $\delta = 0.06$ per annum. In comparison with the tail behavior in Table 3 with no roll-up, the 90% risk measures show that the tail of the net liability is heavier owing to the richer benefit payments. We have also experimented with the reduction of normal random variables in $\Lambda$ as was done in the previous case.

| Method | $\text{VaR}_{0.9}$ | $\text{CTE}_{0.9}$ | Time (secs) |
|---|---|---|---|
| Nonlinear optimization | 0.10318 | 0.13681 | 63.75 |
| Nonlinear optimization ( 50% reduced) | 0.10315 | 0.13677 | 31.08 |
| Nonlinear optimization ( 75% reduced) | 0.10301 | 0.13664 | 7.23 |
| Monte Carlo (1 million) | 0.10346 (0.00016) | 0.13710 (0.00009) | 223.51 |
| Monte Carlo (100 millions) | 0.10335 (0.00001) | 0.13706 (0.00002) | 21955.19 |

Table 4: Risk measures for the GMDB net liability with $\delta$=0.06

## 6 GMDB with dynamic policyholder behavior

We now incorporate into the average model a feature of dynamic policyholder behavior (DPB) commonly used in practice. In many product designs, variable annuity policyholders have the option to surrender their contracts subject to a certain surrender charge. It is common to see that policies tend to lapse at a higher rate when the guarantees are out-of-money than when they are in-the-money. In the past few years, the Society of Actuaries has been publishing annual reports on its industrial surveys on dynamic policyholder behavior. Interested readers are recommended to consult IAA (2010) and Campbell et al. (2014) for the practitioner's approach to modeling policyholder behavior. For illustration purpose, we construct a model based on the concepts described in (IAA, 2010, II-50,IV-9). Practitioners typically break down lapse rates as

$$\text{dynamic lapse rate} = \text{base lapse rate} \times \text{dynamic lapse factor},$$

where the base rate reflects the average experience that varies with the duration of the contract and the dynamic factor is modeled by a decreasing function of the in-the-moneyness ratio. According to the SOA 2012 survey (c.f. PBITT working group (2013)), "a majority of insurers now use dynamic lapse functions for GMDBs. The percentage increased from 25% in 2008 to over 55% in 2011 and 2012." The majority of participating companies vary their basic lapse rates by applying dynamic factors once the in-the-moneyness exceeds 10%. The definition of the in-the-moneyness ratio (ITM) and the dynamic factor function vary company by company. Nevertheless, based on the most common practice described in IAA (2010), we define the ITM as either

$$\text{ITM} = \frac{\text{present value of the guaranteed benefit}}{\text{account value}}, \tag{6.1}$$

or alternatively,

$$\text{ITM} = \frac{\text{present value of the guaranteed benefit}}{\text{surrender value}}. \tag{6.2}$$

Hence, we consider the lapse factor determined by

$$f(x) = \begin{cases} \gamma_1, & x \geq b_1 \\ \gamma_2, & b_2 \leq x < b_1, \\ \vdots & \\ \gamma_w, & b_w \leq x < b_{w-1}, \end{cases}$$

where $x$ represents the ITM, the thresholds $b_0 = \infty > b_1 > b_2 > \cdots > b_w = 0$ and dynamic factors $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_w \leq 1$. An example can be found in Table 6 in the numerical example. Yearly base lapse rates are typically estimated from experience data. In other words, the base rate is considered as a decreasing piecewise function of time, which we denote by $q^b_{x+t-1}$.

The surrender charges are typically designed to decline over time so as to discourage early policy surrender. We denote the surrender charges by a decreasing function of time, $c_t$. An example of base lapse rate and surrender charge is given in Table 5. If the ITM is defined as a percentage of the current account value as in (6.1), then the dynamic lapse rate is determined by $q^l_{x+t-1}(F_t)$, with $q^l$ given by

$$q^l_{x+t-1}(y) = q^b_{x+t-1} \cdot f\left(\frac{(Ge^{\delta t} - y)_+}{y}\right) = q^b_{x+t-1} \sum_{k=1}^{m} \gamma_k I\left(\frac{Ge^{\delta t}}{b_{k-1}+1} < y \leq \frac{Ge^{\delta t}}{b_k + 1}\right).$$

If the ITM is defined as a percentage of the surrender value as in (6.2), then the dynamic lapse rate is given by $q^l_{x+t-1}(F_t)$ where $q^l$ is determined by

$$q^l_{x+t-1}(y) = q^b_{x+t-1} \cdot f\left(\frac{(Ge^{\delta t} - y)_+}{(1-c_t)y}\right) = q^b_{x+t-1} \sum_{k=1}^{m} \gamma_k I\left(\frac{Ge^{\delta t}}{(1-c_t)b_{k-1}+1} < y \leq \frac{Ge^{\delta t}}{(1-c_t)b_k + 1}\right).$$

The lapse survival rate is path-dependent, making the computation very difficult. Therefore, we only consider the current state for the survival rate and then exponentially interpolate the path of account values between the initial purchase payment $F_0$ and the current value:

$${}_{k/n}p^l_x(y) \approx \prod_{i=1}^{k} \Big(1 - {}_{1/n}q^l_{x+(i-1)/n}\big(y^{i/k} F_0^{1-i/k}\big)\Big). \tag{6.3}$$

The net liability of the GMDB with the dynamic policyholder behavior is then given by

$$\begin{aligned} L^* := &\sum_{k=1}^{nT} e^{-rk/n} \,{}_{(k-1)/n}p_x \,{}_{1/n}q_{x+(k-1)/n} \,{}_{(k-1)/n}p^l_x(F_{k/n}) \,(Ge^{\delta t} - F_{k/n})_+ \\ &- \sum_{k=1}^{nT} e^{-rk/n} \,{}_{(k-1)/n}p_x \,{}_{(k-1)/n}p^l_x(F_{k/n}) {}_{1/n}q^l_{x+(k-1)/n}(F_{k/n}) \,c_{k/n} F_{k/n} \\ &- \sum_{k=0}^{nT-1} \frac{1}{n} e^{-rk/n} m_d \,{}_{(k-1)/n}p_x \,{}_{(k-1)/n}p^l_x(F_{k/n}) \big[1 - {}_{1/n}q^l_{x+(k-1)/n}(F_{k/n}) - {}_{1/n}q_{x+(k-1)/n}\big] F_{k/n}. \end{aligned} \tag{6.4}$$

The first term of the formula represents the present value of outgoing payments of death benefits to in-force policies up to $k-1$ periods for which death occurs in the $k$-th period. The second term shows the present value of incoming payments of surrender charges from in-force policies that lapse during the $k$-th period. If the policy remains in force after the $k$-th period, the fees are collected as a percentage of the then-current account value, as is shown in the third term of the formula.

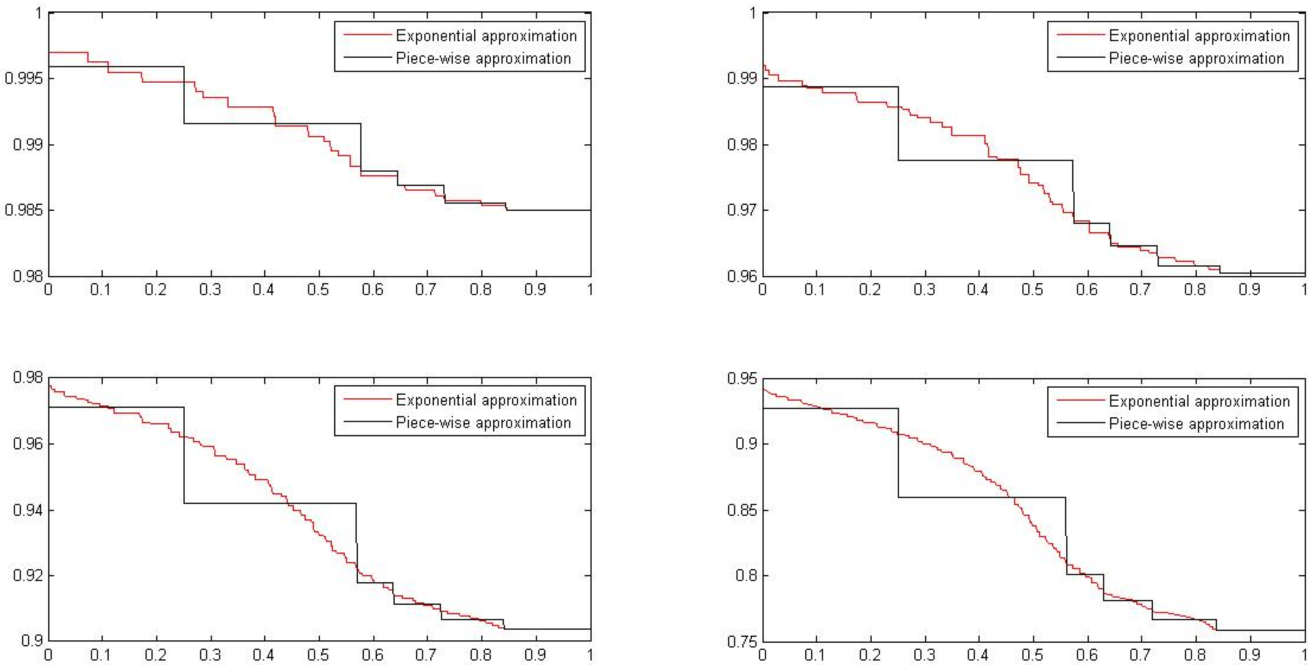


Figure 1: Approximations of dynamic lapse survival rates

To take advantage of the method developed in earlier sections, the random variables should only be a linear combination of $(Ge^{\delta/n} - F_{k/n})_+$ and $F_{k/n}$. For the CTE calculation, each of $(Ge^{\delta/n} - F_{k/n})_+$ terms in the net liability corresponds an $H$ function. As mentioned in the previous section, the computation of the H function is the most time-consuming part in the whole procedure. To make the algorithm efficient, we approximate $p_t^l$ by a piecewise constant function with the same partition of the domain $(0, \infty)$ as $q_t^l$. The partition points $\eta_{ki}$ are determined by $\eta_{k0} = 0$, $\eta_{kw} = +\infty$ and

$$\eta_{ki} = \frac{Ge^{\delta k/n}}{(1 - c_{k/n})b_i + 1}, \qquad k = 1, \cdots, nT; i = 1, \cdots, w-1.$$

Then we set the constants of $q_{k/n}^l$ and $p_{k/n}^l$ for the interval $(\eta_{ki}, \eta_{k(i+1)})$:

$$\alpha_{ki} := {}_{1/n}q^b_{x+(k-1)/n}\gamma_i, \qquad \beta_{ki} := \frac{1}{2}\big[{}_{k/n}p_x^l(\eta_{ki}) + {}_{k/n}p_x^l(\eta_{k(i+1)})\big].$$

In other words, we approximate the lapse survival rate $p^l$ by

$$ {}_{k/n}\tilde{p}_x^l(y) = \sum_{k=0}^{w-1} \beta_{ki} I(\eta_{ki} \le y < \eta_{k(i+1)}). \tag{6.5} $$

Examples of the survival rates based on exponential interpolation and piecewise constant approximation are given in Figure 2, for which the parameters can be found in the numerical example below.

If we replace the lapse survival rate $p^l$ by its approximation $\tilde{p}^l$ in (6.4), then the approximated net liability of the GMDB with the dynamic policyholder behavior can be written as

$$ \begin{aligned} L^* \approx L := & \sum_{k=1}^{nT} \sum_{i=0}^{w-2} u_k \, {}_{1/n}q_{x+(k-1)/n} \beta_{ki} \, (Ge^{\delta k/n} - F_{k/n})_+ I_{ki} - \sum_{k=1}^{nT} \sum_{i=0}^{w-1} c_{k/n} u_k \, \beta_{ki} \alpha_{ki} \, F_{k/n} I_{ki} \\ & - \sum_{k=0}^{nT-1} \sum_{i=0}^{w-1} \frac{m_d}{n} u_k \, \beta_{ki} (1 - \alpha_{ki} - {}_{1/n} \, q_{x+(k-1)/n}) F_{k/n} I_{ki}, \end{aligned} \tag{6.6} $$

where

$$ u_k := e^{-rk/n} \, {}_{(k-1)/n}p_x, \qquad I_{ki} := I\left(\eta_{ki} < F_{k/n} \le \eta_{k(i+1)}\right). $$

We are now ready to apply the technique of comonotonicity to determine closed-form expressions for the risk measures of $L^l = \mathbb{E}[L|\Lambda]$, which we propose to use as approximations of the risk measures of $L^*$.

**Remark 6.1.** *Consider the net liability for the GMDB in the average model with DPB* (6.6)*. The value-at-risk and the conditional tail expectation of $L^l$ can be calculated by*

$$ \begin{aligned} \mathrm{VaR}_p(L^l) = & \sum_{k=1}^{nT} u_k \, {}_{\frac{1}{n}}q_{x+\frac{k-1}{n}} Ge^{\delta k/n} \sum_{i=0}^{w-2} \beta_{ki} \left[ \Phi\left(\frac{g_{k(i+1)}}{\sigma_k}\right) - \Phi\left(\frac{g_{ki}}{\sigma_k}\right) \right] \\ & - \sum_{k=1}^{nT} u_k \, {}_{\frac{1}{n}}q_{x+\frac{k-1}{n}} F_0 e^{\mu_k(\phi)+\sigma_k^2/2} \sum_{i=0}^{w-2} \beta_{ki} \left[ \Phi\left(\frac{g_{k(i+1)} - \sigma_k^2}{\sigma_k}\right) - \Phi\left(\frac{g_{ki} - \sigma_k^2}{\sigma_k}\right) \right] \\ & - \sum_{k=1}^{nT} u_k \, F_0 e^{\mu_k(\phi)+\sigma_k^2/2} \sum_{i=0}^{w-1} \beta_{ki} c_{k/n} \alpha_{ki} \left[ \Phi\left(\frac{g_{k(i+1)} - \sigma_k^2}{\sigma_k}\right) - \Phi\left(\frac{g_{ki} - \sigma_k^2}{\sigma_k}\right) \right] \\ & - \sum_{k=0}^{nT-1} u_k \, F_0 e^{\mu_k(\phi)+\sigma_k^2/2} \sum_{i=0}^{w-1} \beta_{ki} \frac{m_d}{n} (1 - \alpha_{ki} - {}_{\frac{1}{n}}q_{x+\frac{k-1}{n}}) \left[ \Phi\left(\frac{g_{k(i+1)} - \sigma_k^2}{\sigma_k}\right) - \Phi\left(\frac{g_{ki} - \sigma_k^2}{\sigma_k}\right) \right], \end{aligned} \tag{6.7} $$

*where* $\phi = \Phi^{-1}(1-p), g_{ki} = \ln(\eta_{ki}/F_0) - \mu_k(\phi)$, *and*

$$
\begin{aligned}
\mathrm{CTE}_p(L^l) = & \frac{1}{1-p}\sum_{k=1}^{nT} u_{k\ 1/n}q_{x+(k-1)/n} G e^{\delta k/n} \sum_{i=0}^{w-2} \beta_{ki} \Big[ H\big(\phi; a_k, b_{k(i+1)}\big) - H\big(\phi; a_k, b_{ki}\big) \Big] \\
& - \frac{1}{1-p}\sum_{k=1}^{nT} u_{k 1/n}q_{x+(k-1)/n} F_0 e^{\mu_k(0)+\frac{k\sigma^2}{2n}} \sum_{i=0}^{w-2} \beta_{(k,i-1)} \\
& \times \left[ H\left(\phi - r_k\sigma\sqrt{\frac{k}{n}}; a_k, b_{k(i+1)} - \frac{k\sigma^2}{n\sigma_k}\right) - H\left(\phi - r_k\sigma\sqrt{\frac{k}{n}}; a_k, b_{ki} - \frac{k\sigma^2}{n\sigma_k}\right) \right] \\
& - \frac{1}{1-p}\sum_{k=1}^{nT} u_k F_0 e^{\mu_k(0)+\frac{k\sigma^2}{2n}} \sum_{i=0}^{w-1} \beta_{ki} c_{k/n} \alpha_{ki} \\
& \times \left[ H\left(\phi - r_k\sigma\sqrt{\frac{k}{n}}; a_k, b_{k(i+1)} - \frac{k\sigma^2}{n\sigma_k}\right) - H\left(\phi - r_k\sigma\sqrt{\frac{k}{n}}; a_k, b_{ki} - \frac{k\sigma^2}{n\sigma_k}\right) \right] \\
& - \frac{1}{1-p}\sum_{k=0}^{nT-1} u_k F_0 e^{\mu_k(0)+\frac{k\sigma^2}{2n}} \sum_{i=0}^{w-1} \beta_{ki} \frac{m_d}{n}(1 - \alpha_{ki} - {}_{1/n}q_{x+(k-1)/n}) \\
& \times \left[ H\left(\phi - r_k\sigma\sqrt{\frac{k}{n}}; a_k, b_{k(i+1)} - \frac{k\sigma^2}{n\sigma_k}\right) - H\left(\phi - r_k\sigma\sqrt{\frac{k}{n}}; a_k, b_{ki} - \frac{k\sigma^2}{n\sigma_k}\right) \right]
\end{aligned}
\tag{6.8}
$$

*where*

$$
a_k = \frac{r_k}{\sqrt{1-r_k^2}}, \qquad b_{ki} = \frac{\ln(\eta_{ki}/F_0) - (\mu - m)k/n}{\sigma_k}.
$$

In the derivation of the expressions above, we divided (6.6) into three parts: the death benefit part A, the surrender charge part B, and the rider charge part C:

$$
\begin{aligned}
A(\lambda) &:= \mathbb{E}\left( \sum_{k=1}^{nT}\sum_{i=0}^{w-2} \left[ u_{k\ 1/n}q_{x+(k-1)/n}\beta_{ki}\,(Ge^{\delta k/n} - F_{k/n})_+ \right] I_{ki} \,\middle|\, \Lambda = \lambda \right); \\
B(\lambda) &:= \mathbb{E}\left( -\sum_{k=1}^{nT}\sum_{i=0}^{w-1} u_k\,\beta_{ki}\,c_{k/n}\alpha_{ki}\,F_{k/n} I_{ki} \,\middle|\, \Lambda = \lambda \right); \\
C(\lambda) &:= \mathbb{E}\left( -\sum_{k=1}^{nT}\sum_{i=0}^{w-1} u_k\,\beta_{ki}\,\frac{m_d}{n}(1 - \alpha_{ki} -_{1/n} q_{x+(k-1)/n})\,F_{k/n} I_{ki} \,\middle|\, \Lambda = \lambda \right).
\end{aligned}
$$

Although it is difficult to prove monotonicity of $A, B, C$, numerical experiments show that they appear to be decreasing functions in the example under consideration. Figure 2 shows the pattern of $\mathbb{E}[L|\Lambda = \lambda]$ as a decreasing function of $\lambda$ in the following numerical example. Using the monotonicity, we obtain the approximations of risk measures in Remark 6.1. Each of the three sums correspond to the risk measures of $A, B, C$ respectively.

**Remark 6.2.** *The formulas for the risk measures* (6.7) *and* (6.8) *in the model with DPB are more general than their counterparts* (4.2) *and* (4.3) *in the model without DPB. When setting* $\gamma_k = 1$

*for* $k = 1, \cdots, w$, *and* $q^b_{x+n} = c_n = 0$ *for all* $n$, *we observe that* (6.7) *and* (6.8) *reduce to* (4.2) *and* (4.3), *respectively, after cancellations in the telescoping series.*

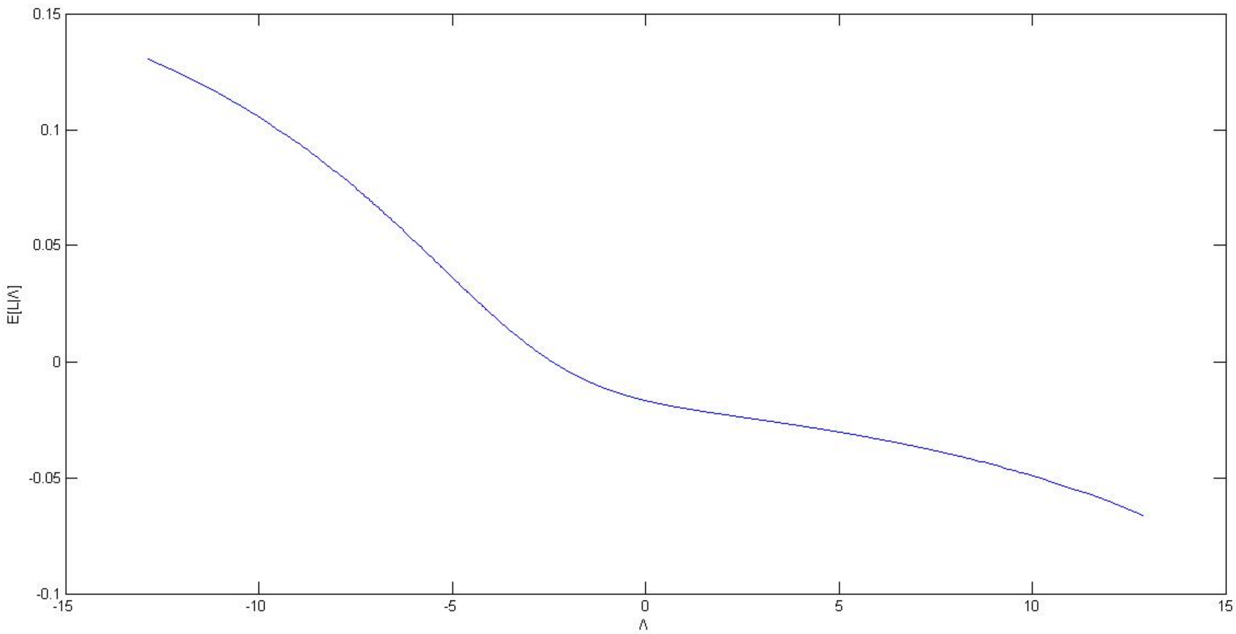


Figure 2: Numerical illustration of the monotonicity of $L^l$

We provide a numerical example to demonstrate the effectiveness of the comonotonic approximation for the net liability under the average model with DPB. As widely acknowledged in the insurance industry, Monte Carlo simulations with models of policyholder behavior are very time-consuming. With all computations performed on a personal computer, we restrict the policy term to be $T = 5$ with half-years fee collections ($n = 2$) in order to save computational efforts. There is no roll-up of the guarantee base in this example, i.e. $\delta = 0$. The rest of the valuation assumptions are the same as in Section 5, while additional assumptions on policyholder behavior are shown in Tables 5 and 6. The surrender charges decline with time, which is to discourage policyholders from early lapse. Accordingly, the basic lapse rates are relatively small in early years and then rise drastically immediately after the lapse rates decrease to 0%. The first few columns of Table 6 show the assumption that the contracts tend to persist ($\gamma \downarrow$) when the guarantees are deep-in-the-money (ITM $\uparrow$). The last few columns show the opposite: the lapse rates are more or less around base rates when the guarantees are at-the-money or out-of-the-money.

We first test the accuracy of approximation formulas developed in Proposition 6.1 for the model with DPB by showing its convergence to the model without DPB, when damping down the lapse rates. The convergence statements in Remark 6.2 are numerically verified in Table 7. The Matlab algorithm `Fminsearch` is used in each of the calculations.

| Policy year ($t$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | beyond |
|---|---|---|---|---|---|---|---|---|---|
| Surrender charge ($c_t$) | 8% | 7% | 6% | 5% | 4% | 3% | 2% | 0% | 0% |
| Base lapse rate ($q_t^b$) | 1.5% | 2.5% | 3.0% | 3.0% | 4.0% | 5.0% | 8.0% | 25.0% | 10.0% |

Table 5: Assumptions on surrender charges and base lapse rates

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $b_k$ | 100% | 80% | 60% | 40% | 20% | 0% |
| $\gamma_k$ | 20% | 40% | 60% | 80% | 90% | 100% |

Table 6: Assumptions on dynamic factors

Table 8 summarizes the risk measures under various models and levels of approximations. The first row shows the risk measures in the model (1.5) without DPB using Monte Carlo simulations. The second row shows the risk measures in the model (6.6) where the dynamic factor is set to constant 1. Note that in this case the lapse rates are deterministic and the model (6.6) can be incorporated into the model (1.5) by adding the lapse rates to the mortality rates.

Important is to bear in mind that we made three steps of approximations in order to achieve a linear structure of the net liability. The first approximation is the exponential interpolation for turning the path-dependent survival rate into a non-path-dependent one in (6.3), while the second approximation in (6.5) produces a piece-wise constant approximation function. The last step is to approximate $\mathrm{CTE}_p(L)$ by the comonotonic bound $\mathrm{CTE}_p(L^l)$. We demonstrate the loss of accuracy in each step of the approximation in Table 8. In all Monte Carlo procedures, we simulate 10-million sample paths of account values for each estimate of the risk measure. The sample mean and variance of 20 estimates are reported in Table 8. Despite the accumulation of approximation errors in three steps, the relative errors appear to be under 5% for both risk measures, the nonlinear optimizations of $\mathrm{VaR}_{0.9}(L^l)$ and $\mathrm{CTE}_{0.9}(L^l)$ reduce the time consumption by at least hundred times.

## 7 Conclusion and Extension

This paper proposes a general framework for computing risk measures of variable annuity liabilities using the techniques of comonotonicity. The framework allows us to analyze the tail events of net liabilities under various guaranteed benefits and provides closed-form approximations of risk measures, which are easy to compute with special functions such as the normal distribution function and Owen's T function. The paper also proposes an extension to the average model of the GMDB, which incorporates the analysis of dynamic policyholder behavior. To the authors' best knowledge, no alternative method other than Monte Carlo simulation has ever been attempted in the previous literature on models of DPB. Despite the model complexity, the same analytic framework allows us to propose closed-form approximations, which have been numerically tested to be very efficient.

| Lapse rates | Formula | $\text{VaR}_{0.9}$ | $\text{CTE}_{0.9}$ |
|---|---|---|---|
| $q^b$ | (6.7)&(6.8) | 0.0453 | 0.0749 |
| 0.2 $q^b$ | (6.7)&(6.8) | 0.05019 | 0.07937 |
| 0.1 $q^b$ | (6.7)&(6.8) | 0.05083 | 0.07995 |
| 0 | (6.7)&(6.8) | 0.05150 | 0.08054 |
| 0 | (4.2)&(4.3) | 0.05150 | 0.08054 |

Table 7: GMDB Comparison

| Method | $\text{VaR}_{0.9}$ | $\text{CTE}_{0.9}$ | Time (secs) |
|---|---|---|---|
| MC with no lapse rate | 0.05133 (0.00010) | 0.08057 (0.00012) | 1162.02 |
| MC with constant dynamic factor 1 | 0.04317 (0.00009) | 0.07002 (0.00009) | 26645.13 |
| MC with dynamic factor | 0.04473 (0.00016) | 0.07402 (0.00014) | 23420.11 |
| MC with exponential interpolation | 0.04460 (0.00021) | 0.07376 (0.00018) | 51358.37 |
| MC with piecewise approximation | 0.04541 (0.00009) | 0.07564 (0.00011) | 21782.75 |
| Nonlinear optimization ($L^l$) | 0.04529 | 0.07490 | 237.32 |

Table 8: GMDB with dynamic policyholder behavior

It should be pointed out that the framework can be easily extended to consider risk measures of flexible premium variable annuity, where purchase payments are allowed throughout the accumulation phrase. A related work on the pricing of flexible premium variable annuity is done in Bernard et al. (2015). In the recent literature, there have been proposals in the actuarial literature to introduce state-dependent fee rates to replace the constant fee rates in the classical cases, see Delong (2014), Bernard et al. (2013). The techniques in Section 6 can also be used to compute risk measures of net liabilities under the models with state-dependent fees.

# A Appendix: Special function $H$

A key element in the computation of the conditional tail expectation is the double integral

$$H(z) = \int_{-\infty}^{z} \int_{-\infty}^{b-ay} \frac{1}{2\pi} e^{-(x^2+y^2)/2} \, \mathrm{d}x \, \mathrm{d}y.$$

This function was not previously studied in the literature. Although the integral can be evaluated numerically, our application requires efficient computation as the integrals appear repeatedly for multiple time points. Hence, we take advantage of the Owen's T-function, for which fast and accurate algorithms have been developed in the statistics literature. Owen's T function was introduced in Owen (1956). For $a, h \in \mathbb{R} \cup \pm\infty$, $T(h, a)$ is defined by

$$T(h, a) = \frac{1}{2\pi} \int_{0}^{a} \frac{\exp\left\{-\frac{1}{2}h^2(1+x^2)\right\}}{1+x^2} \, \mathrm{d}x.$$

This special function was implemented in Mathematica and can be computed very efficiently. The probabilistic interpretation of the function is as follows: $T(h, a)$ stands for the probability mass of two independent standard normal random variables falling in the domain on a plane between $y = 0$ and $y = ax$ and to the right of $x = h$, which is referred to as a polygon in Owen (1956).

**Proposition A.1.** *For $a, b > 0$ and $z \neq 0$, one has that*

$$\begin{aligned} H(z; a, b) = &\frac{1}{2}\operatorname{sgn}(z)\Phi(|z|) + T\left(z, \frac{az-b}{z}\right) \\ &+ \frac{1}{2}\Phi\left(\frac{b}{\sqrt{1+a^2}}\right) - T\left(\frac{b}{\sqrt{1+a^2}}, \frac{(1+a^2)z - ab}{b}\right). \end{aligned}$$

*For $a > 0, b < 0$ and $z \neq 0$, one has that*

$$\begin{aligned} H(z; a, b) = &-\frac{1}{2}\operatorname{sgn}(z)\Phi(-|z|) + T\left(z, \frac{az-b}{z}\right) \\ &+ \frac{1}{2}\Phi\left(\frac{b}{\sqrt{1+a^2}}\right) - T\left(\frac{b}{\sqrt{1+a^2}}, \frac{(1+a^2)z - ab}{b}\right). \end{aligned}$$

*When $z = 0$, the expressions are given by their limits:*

$$H(0; a, b) = \frac{1}{2}\Phi\left(\frac{b}{\sqrt{1+a^2}}\right) + T\left(\frac{b}{\sqrt{1+a^2}}, a\right).$$

*Similarly, when $b = 0$, then one finds*

$$H(z; a, 0) = \frac{1}{2}\Phi(z) + T(z, a).$$

*Proof.* Consider the case where $z > 0$. Even though Figure 1 only illustrates the case where $0 < z < b/a$, all the decompositions and expressions in the case where $z > b/a$ are exactly the same. Hence, without loss of generality, we derive the expressions based on Figure 1. Note that the line segments connecting $A, F, G, E, I, J$ extend to infinity. Since the value $H(z)$ is the total probability mass of two independent standard normal random variables lying in the area below $JBI$, we intend to decompose the total mass over polygons on which the probabilities can be represented as Owen's T functions. Note that the desired domain can be viewed as the area below $JBG$ less the sum of polygons $GBDE$ and $EDI$. We indicate the probability mass over the latter two polygons by (1) and (2), respectively.

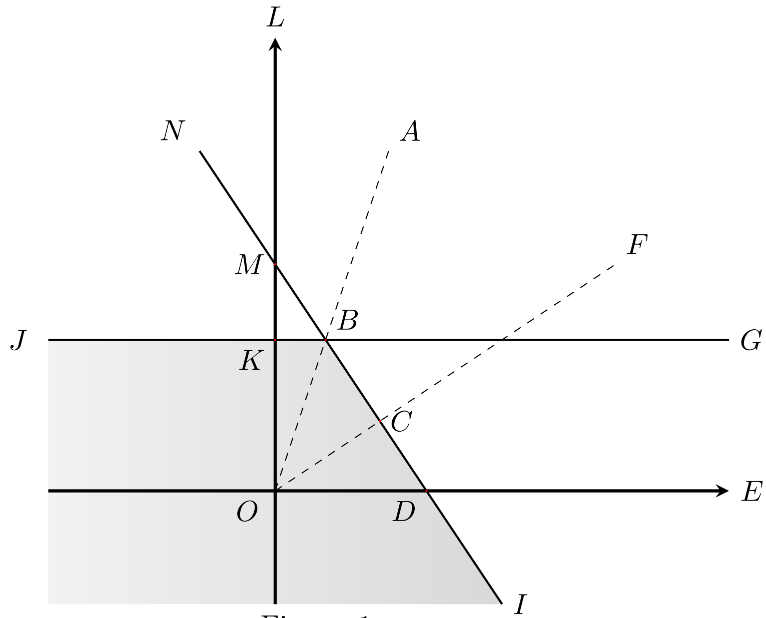

Figure 1

The probability mass (1) can be determined by the probability mass over $ABCDE$, denoted by (3), less that over $ABG$, denoted by (4). On one hand, note that $ABCDE$ is the union of $ABCF$ and $FCDE$. Hence,

$$(3) = T\left(\frac{b}{\sqrt{1+a^2}}, a\right) + T\left(\frac{b}{\sqrt{1+a^2}}, \frac{(1+a^2)z - ab}{b}\right).$$

On the other hand, we see that $ABG$ is equal to the area above $LKG$ less the area bounded by the polygon $LKBA$, which means that

$$(4) = \frac{1}{2}(1 - \Phi(z)) - T\left(z, \frac{b - az}{z}\right).$$

The probability mass (2) is simply $FCI$ less $FCDE$, i.e.

$$(2) = T\left(\frac{b}{\sqrt{1+a^2}}, \infty\right) - T\left(\frac{b}{\sqrt{1+a^2}}, a\right) = \frac{1}{2}\Phi\left(-\frac{|b|}{\sqrt{1+a^2}}\right) - T\left(\frac{b}{\sqrt{1+a^2}}, a\right).$$

Putting all pieces together, we obtain

$$H(z) = \Phi(z) - (1) - (2) = \Phi(z) - (3) + (4) - (2),$$

which yields the desired expression for $H$ after rearrangement.

When $z < 0$, the domain of integration of $H$ is shown in Figure 2 as the area below $JBI$. We shall use a slightly different decomposition to derive the expression for $H$.

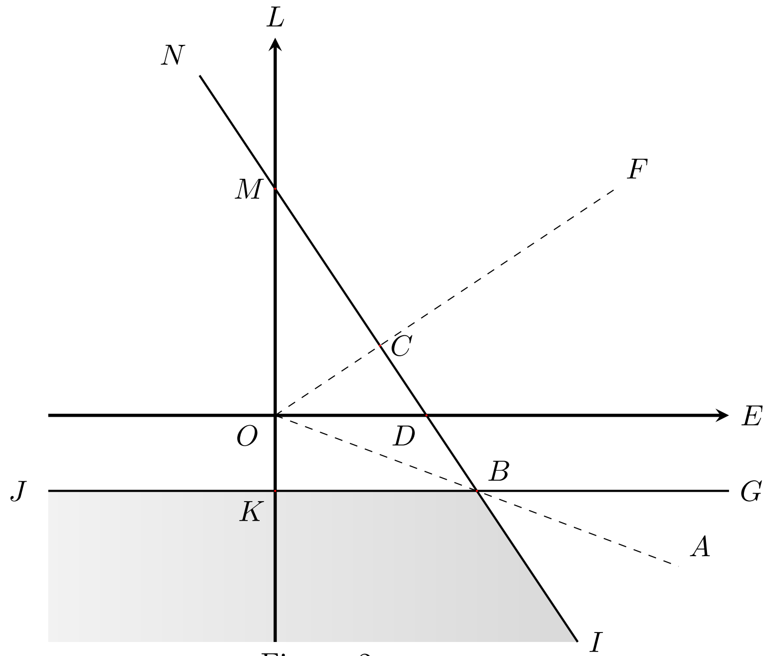


Figure 2

Think of the area below $JBI$ as the area below $JBG$ less the area bounded by the polygon $GBI$. We denote the probability masses over $GBI$, $FCBG$ and $GBA$ by (5), (6), (7) respectively. Note that $GBA$ is same as the area below $GKL$ less $ABKL$, i.e.

$$(7) = \frac{1}{2}\Phi(z) - T\left(-z, \frac{b-az}{-z}\right) = \frac{1}{2}\Phi(z) + T\left(z, \frac{b-az}{z}\right).$$

Clearly, $FCBG$ is equal to $FCBA$ less $GBA$. Thus

$$(6) = T\left(\frac{b}{\sqrt{1+a^2}}, \frac{ab-(1+a^2)z}{b}\right) - (7).$$

Then $GBI$ is equal to $FCI$ less $FCBG$, which determines

$$(5) = T\left(\frac{b}{\sqrt{1+a^2}}, \infty\right) - (6).$$

Finally, the total probability over the area below $JBI$ is given by

$$\begin{aligned} H(z) &= \Phi(z) - (5) \\ &= \frac{1}{2}\Phi(z) - \frac{1}{2}\Phi\left(-\frac{b}{\sqrt{1+a^2}}\right) + T\left(\frac{b}{\sqrt{1+a^2}}, \frac{ab-(1+a^2)z}{b}\right) - T\left(z, \frac{b-az}{z}\right). \end{aligned}$$

The proof for the expression of $H$ in the cases where $a > 0, b < 0$ is very similar to the previous cases and hence omitted. □

**Proposition A.2.**

$$\int_p^1 \exp\{C\Phi^{-1}(1-p)\}\Phi\big(B - A\Phi^{-1}(1-q)\big)\,\mathrm{d}q = \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{C^2}{2}\right\} H\left(\Phi^{-1}(1-q) - C; A; B - AC\right). \tag{A.1}$$

*Proof.* Let $y = \Phi^{-1}(1-q)$. Then the left-hand side of (A.1) is equal to

$$\int_{-\infty}^{\Phi^{-1}(1-q)} e^{Cy}\Phi(B-Ay)\phi(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}}e^{c^2/2}\int_{-\infty}^{\Phi^{-1}(1-q)-C} \Phi(B-AC-Ay)\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\,\mathrm{d}u,$$

which is the right-hand side of (A.1) by definition. □

# B Appendix: Choices of the conditioning random variable $\Lambda$

As discussed in Vanduffel et al. (2008a), the globally optimal choice of $\Lambda$ refers to the set of $r_i$'s that maximizes the linear approximation of the variance of $S^l$:

$$\mathrm{Var}[S^l] \approx (\mathrm{Corr}[\sum_{j=1}^{n}\alpha_j\mathbb{E}[e^{Z_j}],\Lambda])^2\mathrm{Var}[\sum_{j=1}^{n}\alpha_j E(e^{Z_j})Z_j],$$

which attains its maximum value when

$$\Lambda^{MV} = \sum_{j=1}^{n}\alpha_j\mathbb{E}[e^{Z_j}]Z_j,$$

and

$$r_k^{MV} = \frac{1}{\sigma_{Z_k}\sigma_\Lambda}\sum_{j=1}^{n}\alpha_j\mathbb{E}[e^{Z_j}]\mathrm{Cov}[Z_k,Z_j]. \tag{B.1}$$

The covariances of the $Z_i$'s and the variance of $\Lambda$ can be calculated from the basic properties of Brownian motion:

$$\mathrm{Cov}[Z_k,Z_j] = \frac{\sigma^2}{n}\min\{k,j\} \qquad \sigma_\Lambda^2 = \frac{\sigma^2}{n}\sum_{i=1}^{nT}(\sum_{j=i}^{nT}\lambda_j)^2.$$

For a locally optimal choice, Vanduffel et al. (2008a) proposed a linear approximation of $\mathrm{CTE}_p$:

$$\mathrm{CTE}_p[S^l] \approx \frac{1}{1-p}\sum_{j=1}^{n}\alpha_j\mathbb{E}[e^{Z_j}]\Phi'[r_j^{MV}\sigma_{Z_j}-\Phi^{-1}(p)]r_j\sigma_{Z_j} + \text{constant}.$$

Its value is maximized when

$$\Lambda^{(p)} = \sum_{j=1}^{n}\alpha_j\mathbb{E}[e^{Z_j}]\Phi'[r_j^{MV}\sigma_{Z_j}-\Phi^{-1}(p)]Z_j,$$

and

$$r_k^{(p)} = \frac{1}{\sigma_{Z_k}\sigma_\Lambda}\sum_{j=1}^{n}\alpha_j\mathbb{E}[e^{Z_j}]\Phi'[r_j^{MV}\sigma_{Z_j}-\Phi^{-1}(p)]\mathrm{Cov}[Z_k,Z_j]. \tag{B.2}$$

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