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Power Measures and Solutions for Games under Precedence Constraints

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Abstract

In the literature, there exist several models where a cooperative TU-game is enriched with a hierarchical structure on the player set that is represented by a directed graph or digraph. In this paper we consider the *games under precedence constraints* introduced by Faigle and Kern (1992) who also introduce a generalization of the Shapley value for such games. They characterized this solution by efficiency, linearity, the null player property and an axiom called hierarchical strength which states that in a unanimity game the payoffs are allocated among the players in the unanimity coalition proportional to their hierarchical strength in the corresponding coalition. The *hierarchical strength* of a player (node) with respect to a coalition in an acyclic digraph is the number of admissible permutations where this player is the last of that coalition to enter, and where the admissible permutations are those in which successors in the digraph enter before predecessors.

In this paper we introduce and axiomatize a new solution for games under precedence constraints, the so-called hierarchical solution. Unlike the precedence Shapley value, this new solution satisfies the desirable axiom of irrelevant player independence which establishes that the payoffs assigned to relevant players are not affected by the presence of irrelevant players. This hierarchical solution is defined in a similar spirit as the precedence Shapley value but belongs to the class of precedence power solutions being solutions that allocate the dividend of a coalition proportionally to a power measure for acyclic digraphs. The hierarchical solution allocates proportionally to the hierarchical measure. We give an axiomatization of this power measure on the class of acyclic digraphs. In addition we extend the hierarchical measure to regular set systems. Finally we consider the subclasses of acyclic digraphs, consisting of forests and sink forests and consider the normalized version of the hierarchical measure on these subclasses as well as several other power measures.

Keywords: Cooperative TU-game, acyclic digraph, hierarchical strength, irrelevant player, power measure, regular set system, rooted and sink trees.

JEL code: C71

1 Introduction

Faigle and Kern (1992) introduced games under precedence constraints as cooperative TU-games where the player set is endowed with a *precedence relation*. This precedence relation is represented by a partial order (i.e. reflexive, antisymmetric and transitive relation) on the player set. Equivalently it can be represented by an acyclic directed graph. Players enter to form the 'grand coalition' consisting of all players according to some permutation on the player set. Given an acyclic digraph, a permutation of the players is called *admissible*, if players enter after their successors in the digraph. Consequently, a coalition of players is considered *feasible*, if for every player in the coalition all of its successors in the digraph are also present in the coalition. The absolute hierarchical strength of a player i, given a feasible coalition S, is now simply the number of admissible permutations in D where i enters after the players in $S \setminus \{i\}$. The normalized hierarchical strength of a player i, given a feasible coalition S, is obtained by dividing the absolute hierarchical strength by the total number of permutations that are admissible in digraph D. Faigle and Kern (1992) use the hierarchical strength to axiomatically define the so-called *precedence Shapley value*. The axiomatization uses the axioms of efficiency, the null player property (together presented by Faigle and Kern (1992) as the carrier axiom) and linearity, combined with an axiom called hierarchical strength, which replaces the 'standard' symmetry axiom. This axiom states that in a unanimity game under precedence constraints dividend is distributed among the players in the unanimity coalition proportionally to their normalized hierarchical strength in the unanimity coalition.

A player is called *irrelevant* in a game under precedence constraints if it is a null player and all its superiors in the precedence constraint are also null players, where a player is a null player if its marginal contribution in every admissible permutation is zero. Irrelevant player independence is subsequently defined as the property that states that the payoff to relevant players (i.e. players that are not irrelevant) is not affected by the presence of irrelevant players. We consider this a desirable property for solutions for games under precedence constraints. We show that the precedence Shapley value does not satisfy this property. Therefore we introduce the *hierarchical solution* which has the property that the payoffs to relevant players are not affected by the presence of irrelevant players. Like the precedence Shapley value, this new solution for games under precedence constraints satisfies the axioms of efficiency, linearity and the null player property. In addition it uses an alternative for the hierarchical strength axiom. Whereas the precedence Shapley value allocates the dividend of a coalition proportionally to the hierarchical strength applied to the full digraph, the new hierarchical solution allocates this dividend proportionally to the hierarchical strength applied to the subgraph on the unanimity coalition.

We introduce *weight functions* for digraphs which assign to every acyclic digraph and every feasible coalition within that digraph, a weight to the players within that feasible coalition. Both the absolute as well as the normalized hierarchical strength are examples of weight functions. The class of so-called *weighted precedence solutions* consists of solutions that allocate the dividend of a coalition proportionally to some weight function. Both the precedence Shapley value as well as the hierarchical solution are weighted precedence solutions.

We call a weight function *subgraph-invariant* if applied to a feasible coalition it depends only on the subgraph on that coalition. The hierarchical solution is obtained by applying a subgraph-invariant weight function. We show that for all solutions that are obtained by applying a subgraph-invariant weight function, the payoff for relevant players is not affected by the presence of irrelevant players. Moreover, these solutions can be obtained by allocating the dividend of a feasible coalition proportionally to some *power measure* for acyclic digraphs being functions that assign values to the players in an acyclic digraph that can be interpreted as the 'strength' or 'influence' of these players in the digraph.[1] We refer to such solutions as *precedence power solutions*. Our approach of allocating the dividend of a feasible coalition proportionally to some power measure for acyclic digraphs is similar to that of van den Brink, van der Laan and Pruzhansky (2011) for (communication) graph games, generalizing the approach in Borm, Owen and Tijs (1992) to the Myerson value (Myerson (1977)) and position value.

The hierarchical solution is the precedence power solution that allocates dividend proportionally to the *hierarchical measure* being the power measure that assigns to any player in the digraph the number of admissible permutations, where it is preceded in the permutation by all other players. In other words, for any digraph the hierarchical measure is given by the hierarchical strength applied to the grand coalition. We give an axiomatization of the hierarchical measure.

In the literature, the precedence Shapley value has been extended to games associated with combinatorial structures more general than a digraph. For example, Bilbao and Edelman (2000) consider games on convex geometries and Bilbao and Ordoñez (2008) consider games on a class of augmenting systems. Convex geometries have been shown to be contained in the class of so-called regular set systems considered by Lange and Grabisch (2009), who also consider an extension of the precedence Shapley value to games on regular set systems. A set of admissible permutations can be generated from these set systems

[1] Examples of power measures are the ones given by Gould (1987), White and Borgatti (1994), the β-measure of van den Brink and Gilles (2000) and its reflexive version in van den Brink and Borm (2002), the λ-measure of Borm, van den Brink and Slikker (2002), the positional power measure of Herings, van der Laan and Talman (2005) or the centrality measures in del Pozo, Manuel, González-Arangüena and Owen (2011).

similar to how this is done for digraphs. We consider an extension of the hierarchical measure on the class of regular set systems and proceed to give an axiomatization.

The hierarchical measure can be seen to rank players based on a number of permutations of these players. An example where we also encounter ranking of players based on permutations comes from social choice theory. The permutations in this case are the preferences of the voters on a number of alternatives they can choose from and the players are the alternatives. It turns out that the hierarchical measure is similar to the plurality scoring rule for social choice situations. By considering other scoring rules, we can define new solutions for games under precedence constraints.

Finally, we consider the hierarchical measure on the classes of forests and sink forests, which are subclasses of acyclic digraphs. On these classes we consider the normalized version of the hierarchical measure. We consider the application of a number of power measures to river games with multiple springs.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, we consider the irrelevant player property, show that the precedence Shapley value does not satisfy this property and introduce the hierarchical solution that does satisfy this property. We also axiomatize the hierarchical measure as power measure for digraphs which is the power measure on which the hierarchical solution is based. In Section 3 we generalize the hierarchical measure and solution to games on regular set systems and provide an axiomatization. Section 4 considers the normalized hierarchical strength for forests and sink forests. The solutions for sink forest are applied to river games. Finally, Section 5 contains concluding remarks.

2 Preliminaries

2.1 TU-games

A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. We denote the collection of all characteristic functions on player set N by $\mathcal{G}^N$.

A *payoff vector* for game (N, v) is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a function that assigns a payoff vector to every TU-game. One of the most widely used solutions for

TU-games is the *Shapley value* (Shapley (1953a)), given by

$$Sh_i(N,v) = \frac{1}{|N|!} \sum_{\pi \in \Pi(N)} m_i^{\pi}(N,v), \text{ for all } i \in N,$$

where $\Pi(N)$ is the collection of all permutations $\pi\colon N \to N$ on N, and for every permutation $\pi \in \Pi(N)$,

$$m_i^{\pi}(N,v) = v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\}), \tag{2.1}$$

is the marginal contribution of player i to the players that are ranked before him in the order π.

For each $T \subseteq N$, $T \neq \emptyset$, the *unanimity* game (N, u_T) is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that the unanimity games form a basis for $\mathcal{G}^N$. For every $v \in \mathcal{G}^N$ it holds that $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$, where $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ are the *Harsanyi dividends*, see Harsanyi (1959).

2.2 Digraphs, precedence constraints and games under precedence constraints

An *irreflexive directed graph* or *irreflexive digraph* is a pair (N, D) where N is the set of nodes and $D \subseteq \{(i,j) \mid i,j \in N, i \neq j\}$ is an (irreflexive) binary relation on N consisting of ordered pairs called directed links or *arcs*. Since we assume irreflexivity throughout the full paper, we refer to these just as digraphs. Since the nodes will represent players, we often refer to the nodes as players. For $i \in N$, the nodes in $F_D(i) := \{j \in N \mid (i,j) \in D\}$ are called the *followers* or *successors* of i in D, and the nodes in $P_D(i) := \{j \in N \mid (j,i) \in D\}$ are called the *predecessors* of i in D. Further, by $\widehat{F}_D(i)$ we denote the set of successors of i in the *transitive closure* of D i.e., $j \in \widehat{F}_D(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in F_D(h_k)$ for all $1 \leq k \leq t-1$, and $h_t = j$. We refer to the players in $\widehat{F}_D(i)$ as the *subordinates* of i in D, and to the players in the set $\widehat{P}_D(i) = \{j \in N \mid i \in \widehat{F}_D(j)\}$ consisting of all predecessors of i in the transitive closure of D, as i's *superiors*. The digraph (N, D) is called *acyclic* if $i \notin \widehat{F}_D(i)$ for all $i \in N$. We denote the collection of all acyclic digraphs by $\mathcal{D}$. For $S \subseteq N$ and $(N, D) \in \mathcal{D}$, the digraph $(S, D(S))$ is given by $D(S) = \{(i,j) \in D \mid \{i,j\} \subseteq S\}$. By $TOP(N,D) = \{i \in N \mid P_D(i) = \emptyset\}$ we denote the set of 'top players' in (N, D), i.e. the set of players without predecessors.

Faigle and Kern (1992) consider situations where cooperation between players is restricted by a partial order on the player set. They interpret this partial order as a precedence relation. The partial order can be represented by an acyclic digraph. A coalition is feasible, if for any player in the coalition all of its successors in the digraph are also present in the coalition. The set $\Phi^p(N,D)$ of feasible coalitions according to digraph $(N,D) \in \mathcal{D}$ is thus given by

$$\Phi^p(N,D) = \{S \subseteq N \mid F_D(i) \subseteq S \text{ for all } i \in S\}.$$

Faigle and Kern (1992) consider cooperative games, where for acyclic digraph $(N,D) \in \mathcal{D}$ the domain of the characteristic function is given by the set $\Phi^p(N,D)$.[2] A *TU-game under precedence constraints* is a triple (N,v,D), where $N \subseteq \mathbb{N}$ is a finite set of players, $(N,D) \in \mathcal{D}$ is an acyclic digraph, and $v : \Phi^p(N,D) \to \mathbb{R}$ is a characteristic function, that assigns to every set S in $\Phi^p(N,D)$ a worth $v(S)$, where $v(\emptyset) = 0$.

We denote the class of all games under precedence constraints by $\mathcal{G}_{PC}$, and we denote the class of games under precedence constraints on graph $(N,D) \in \mathcal{D}$ by $\mathcal{G}_{PC}^{(N,D)}$. The game under precedence constraints obtained from $(N,v,D) \in \mathcal{G}_{PC}$ by considering only feasible coalition S and its subsets is denoted by $(S, v_S, D(S))$, where $v_S(T) = v(T)$ for all feasible coalitions $T \subseteq S$. We refer to $(S, v_S, D(S))$ as the subgame on S of (N,v,D). For $(N,v,D), (N,w,D) \in \mathcal{G}_{PC}^{(N,D)}$, the sum game $(N, v+w, D)$ is defined by $(v+w)(S) = v(S) + w(S)$, and for $c \in \mathbb{R}$, the game $(N, cv, D) \in \mathcal{G}_{PC}^{(N,D)}$ by $(cv)(S) = cv(S)$ for $S \in \Phi^p(N,D)$.

A permutation $\pi \in \Pi(N)$ is called *admissible* in acyclic digraph (N,D) if $\pi(i) > \pi(j)$ whenever $(i,j) \in D$, i.e. successors enter before their predecessors in the digraph. The set of admissible permutations $\Pi_D(N)$ in D is denoted by

$$\Pi_D(N) = \{\pi \in \Pi(N) \mid \pi(i) > \pi(j) \text{ if } (i,j) \in D\}. \tag{2.2}$$

Note that it holds that the set of admissible permutations in D is the same as that of its transitive closure $tr(D)$: $\Pi_D(N) = \Pi_{tr(D)}(N)$.

The *precedence marginal vector* $m^\pi(N,v,D) \in \mathbb{R}^N$, associated with the game under precedence constraints (N,v,D) and permutation $\pi \in \Pi_D(N)$, is given by

$$m_i^\pi(N,v,D) = m^\pi(N,v) = v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\}),\ i \in N. \tag{2.3}$$

Recall from Section 2.1 that the Shapley value assigns to the players the average over all marginal vectors associated with all permutations of the player set N. The *precedence Shapley value* H of Faigle and Kern (1992) is the solution on $\mathcal{G}_{PC}$ given by

$$H_i(N,v,D) = \frac{1}{|\Pi_D(N)|} \sum_{\pi \in \Pi_D(N)} m_i^\pi(N,v,D), \quad \text{for all } i \in N,$$

[2]This is different from games with a permission structure as in Gilles, Owen and van den Brink (1992) which also is a triple (N,v,D), but where the characteristic function v is still defined on the domain 2^N. Moreover, in the conjunctive approach to these games, a coalition is feasible if for any player in the coalition all of its *predecessors* in the digraph are present.

and assigns to the players in N the average over all precedence marginal vectors of game under precedence constraints (N, v, D). For $(N, v, D) \in \mathcal{G}_{PC}$, all permutations in $\Pi(N)$ are admissible when $D = \emptyset$. In that case the domain of characteristic function v is given by 2^N, and thus is a classical characteristic function of a TU-game. Thus, when $D = \emptyset$, the precedence Shapley value of (N, v, D) yields the Shapley value of (N, v).

Faigle and Kern (1992) give an axiomatization of the precedence Shapley value using the following axioms. The first three axioms are straightforward adaptations of axioms for TU-game solutions.[3] A player $i \in N$ is a *null player* in game under precedence constraints (N, v, D), if for every $\pi \in \Pi_D(N)$ it holds that $m_i^\pi(N, v, D) = 0$.

Efficiency For each game $(N, v, D) \in \mathcal{G}_{PC}$ it holds that $\sum_{i \in N} f_i(N, v, D) = v(N)$.

Linearity For every pair of games (N, v, D) and $(N, w, D) \in \mathcal{G}_{PC}^{(N,D)}$ it holds that $f(N, v + w, D) = f(N, v, D) + f(N, w, D)$, and for $(N, v, D) \in \mathcal{G}_{PC}^{(N,D)}$ and $c \in \mathbb{R}$ it holds that $f(N, cv, D) = cf(N, v, D)$.

Null player property For each $(N, v, D) \in \mathcal{G}_{PC}$, if $i \in N$ is a null player in (N, v, D), then $f_i(N, v, D) = 0$.

Besides these three axioms, they introduce an axiom that is based on the hierarchical strength of players. First, for $i \in S \in \Phi^p(N, D)$ the set of permutations $\Pi_D^i(N, S)$ is defined by

$$\Pi_D^i(N, S) = \{\pi \in \Pi_D(N) \mid \pi(i) > \pi(j) \text{ for all } j \in S \setminus \{i\}\}, \tag{2.4}$$

being the collection of those permutations in $\Pi_D(N)$ where i enters after the players in $S \setminus \{i\}$. Note that the collection $\{\Pi_D^i(N, S)\}_{i \in S}$ is a partition of $\Pi_D(N)$.

The *absolute hierarchical strength* is the function h that assigns to every $(N, D) \in \mathcal{D}$ and coalition $S \in \Phi^p(N, D)$ the vector $h(N, D, S) \in \mathbb{R}^S$, where $h_i(N, D, S) = |\Pi_D^i(N, S)|$ is the number of permutations in $\Pi_D(N)$ where $i \in S$ enters after the players in $S \setminus \{i\}$.

The *normalized hierarchical strength* is the function $\overline{h}$ that assigns to every $(N, D) \in \mathcal{D}$ and a coalition $S \in \Phi^p(N, D)$ the vector $\overline{h}(N, D, S) \in \mathbb{R}^S$, where $\overline{h}_i(N, D, S) = \frac{|\Pi_D^i(N,S)|}{|\Pi_D(N)|}$ is the fraction of permutations in $\Pi_D(N)$ where $i \in S$ enters after the players in $S \setminus \{i\}$.[4] Note that $\sum\limits_{i \in S} \overline{h}_i(N, D, S) = 1$ for all $S \in \Phi^p(N, D)$.

[3]We remark that, similar to Shapley (1953a), Faigle and Kern (1992) combine efficiency and the null player property into a carrier axiom.

[4]Both the absolute as well as the normalized hierarchical strength assign a value to a player $i \in N$, given $(N, D) \in \mathcal{D}^N$ and coalition $S \in \Phi^p(N, D)$ and are therefore more correctly denoted by $h_i((N, D), S)$ and $\overline{h}_i((N, D), S)$ respectively. For convenience however we will refer to these functions as $h_i(N, D, S)$ and $\overline{h}_i(N, D, S)$, respectively, throughout this paper.

Unanimity games under precedence constraints are defined similar to classical unanimity TU-games. For each $T \in \Phi^p(N,D)$, $T \neq \emptyset$, the *unanimity* game under precedence constraints $(N, u_T, D) \in \mathcal{G}_{PC}$ is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise, $S \in \Phi^p(N,D)$.[5]

Faigle and Kern (1992) also consider the *dividend* of a coalition $S \in \Phi^p(N,D)$ in game under precedence constraints (N,v,D), $\Delta_v^D(S) = v(S) - \sum_{T \subset S, T \in \Phi^p(N,D), T \neq \emptyset} \Delta_v^D(T)$.

For every $(N,v,D) \in \mathcal{G}_{PC}$, Faigle and Kern (1992) show that the characteristic function in (N,v,D) can be written as a linear combination of the characteristic functions of unanimity game under precedence constraints (N,u_T,D):

$$v = \sum_{\substack{T \in \Phi^p(N,D) \\ T \neq \emptyset}} \Delta_v^D(T) u_T. \tag{2.5}$$

The axiom of hierarchical strength of a solution for games under precedence constraints states that in unanimity games under precedence constraints, the earnings are distributed among the players in the unanimity coalition proportionally to their normalized hierarchical strength in that coalition. Obviously, this is equivalent to distributing the dividends proportionally to the absolute hierarchical strength of the players.

Hierarchical strength For every $(N,D) \in \mathcal{D}$, every $S \in \Phi^p(N,D)$ and every $i,j \in S$, it holds that $\overline{h}_i(N,D,S) f_j(N,u_S,D) = \overline{h}_j(N,D,S) f_i(N,u_S,D)$.

Theorem 2.1 (Faigle and Kern, 1992)
A solution on $\mathcal{G}_{PC}$ is equal to the precedence Shapley value H if and only if it satisfies efficiency, linearity, the null player property and hierarchical strength.

Alternatively, the precedence Shapley value can be defined as the solution that allocates the dividend of a coalition $S \in \Phi^p(N,D)$ proportionally to the hierarchical strength $h(N,D,S)$ of the players in S:

$$H_i(N,v,D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{h_i(N,D,S)}{\sum_{j \in S} h_j(N,D,S)} \Delta_v^D(S) \text{ for all } i \in N.$$

3 Solutions for games under precedence constraints and power measures for acyclic digraphs

3.1 Irrelevant player independence

For a classical TU-game (N,v), player $i \in N$ being a null player in v implies that $\Delta_v(S) = 0$ for all coalitions $S \subseteq N$ with $i \in S$. However, for a game under precedence constraints

[5]Note that, different from classical TU-games, the unanimity game (called simple game by Faigle and Kern) u_T is only defined on the set $\Phi^p(N,D)$.

(N, v, D), player $i \in N$ being a null player does not imply that $\Delta_v(S) = 0$ for all coalitions $S \in \Phi^p(N, D)$ with $i \in S$. We illustrate this with an example.

Example 3.1 *Consider the game under precedence constraints* (N, v, D), *where* $N = \{1, 2\}$, $v = u_{\{1,2\}}$ *is the unanimity game on players 1,2 and* D *is given by* $\{(1, 2)\}$. *The set of feasible coalitions is given by* $\Phi^p(N, D) = \{\emptyset, \{2\}, \{1, 2\}\}$. *The set of admissible permutations is given by* $\Pi_D(N) = \{(2, 1)\}$. *Therefore we only need to consider the precedence marginal vector* $m^{(2,1)}(N, v, D)$, *to decide which players are null players. We obtain* $m^{(2,1)}(N, v, D) = (1, 0)$ *and therefore player 2 is a null player. The dividends of* (N, v, D) *are given by* $\Delta_v^D(\{2\}) = 0, \Delta_v^D(\{1, 2\}) = 1$. *We find that even though player 2 is a null player in* (N, v, D), *not all feasible coalitions that it is contained in have 0 dividend.*

The example shows that the reason why coalitions with null players might have nonzero dividend is that a null player might have predecessors in the digraph that are not null players. Consider the example of a manufacturer of some good, where we can distinguish between agents that perform manual labor at an early stage in the production process, and management that is in charge of distribution and sales of the good. The manual labor produces a good, but this in itself does not generate any worth. The benefit is realized when selling the good.

The following proposition shows that coalitions containing a null player, but not any of this player's predecessors in the digraph, always have zero dividend.

Proposition 3.2 *Consider game under precedence constraints* (N, v, D). *If coalition* $S \in \Phi^p(N, D)$ *contains a null player* i *and* $S \subseteq N \setminus P_D(i)$, *then* $\Delta_v^D(S) = 0$.

Proof

Consider a feasible coalition S containing null player i and no predeccesors of i. Let $H(S) = \{j \mid j \in S \text{ and } j \notin \widehat{F}_D(i)\}$ be the set of players in S that are not subordinates of player i in D. We perform induction on $|H(S)|$.

If $|H(S)| = 0$, then the only feasible subset of S containing player i is S itself. Therefore $v(S) - v(S \setminus \{i\}) = \Delta_v^D(S)$. Since i is a null player, it holds that $v(S) - v(S \setminus \{i\}) = 0 = \Delta_v^D(S)$.

Proceeding by induction, assume that $\Delta_v^D(T) = 0$, when $0 \leq |H(T)| < |H(S)|$. Since $|H(S)| > 0$ it holds that S is no longer the only feasible subset of S containing player i. Let $K(S) = \{T \in \Phi^p(N, D) \mid i \in T \text{ and } T \subset S\}$. It now holds that $v(S) - v(S \setminus \{i\}) = \sum_{T \in K(S)} \Delta_v^D(T) + \Delta_v^D(S)$. Since $|H(T)| < |H(S)|$ for $T \in K(S)$, by induction we have $\Delta_v^D(T) = 0$ for all $T \in K(S)$. Since i is a null player, it holds that $v(S) - v(S \setminus \{i\}) = 0 = \Delta_v^D(S)$.

□

Player $i \in N$ is called an *irrelevant player* in game under precedence constraints (N, v, D) if i is a null player, and any $j \in \widehat{P}_D(i)$ is also a null player (this implies that any $j \in \widehat{P}_D(i)$ is also irrelevant). Call a player $i \in N$ *relevant* if it is not an irrelevant player. We have the following proposition.

Proposition 3.3 *Player $i \in N$ is an irrelevant player in game under precedence constraints (N, v, D) if and only if $\Delta_v^D(S) = 0$ for every coalition $S \in \Phi^p(N, D)$ with $i \in S$.*

Proof

Only if
For $S \subseteq N$, define $P_D(S) = \bigcup_{i \in S} P_D(i)$. For a coalition S, let $P_D^1(S) = P_D(S)$ and $P_D^k(S) = P_D(P_D^{k-1}(S))$, $k = 1, \ldots$. For an irrelevant player $i \in N$, let $\kappa(i)$ be the smallest integer such that $P_D^{\kappa(i)}(\{i\}) = \emptyset$. Let $S \in \Phi^p(N, D)$ such that $i \in S$. We show by induction on $\kappa(i)$ that $\Delta_v^D(S) = 0$.

If $\kappa(i) = 1$, then player i has no predecessors in (N, D). In that case $S \subseteq N = N \setminus P_D(i)$ for any coalition $S \in \Phi^p(N, D)$ such that $i \in S$. Therefore by Proposition 3.2 $\Delta_v^D(S) = 0$ for $S \in \Phi^p(N, D)$ such that $i \in S$.

Proceeding by induction on $\kappa(i)$, assume that for any irrelevant player j such that $\kappa(j) < \kappa(i)$ it holds that $\Delta_v^D(S) = 0$ for all coalitions $S \in \Phi^p(N, D),\ j \in S$. We already know that $\Delta_v^D(S) = 0$ for S such that $S \cap \widehat{P}_D(i) \neq \emptyset$, by the fact that predecessors of irrelevant players are themselves also irrelevant and $\kappa(j) < \kappa(i)$ for any $j \in \widehat{P}_D(i)$. Therefore we only need to consider those feasible coalitions $S \in \Phi^p(N, D)$ such that $S \cap \widehat{P}_D(i) = \emptyset$. For these coalitions it holds that $S \subseteq N \setminus P_D(i)$. Therefore by Proposition 3.2, $\Delta_v^D(S) = 0$.

If
Suppose that i is not an irrelevant player in (N, v, D). If i is not a null player in (N, v, D), then there exists a coalition $S \in \Phi^p(N, D), i \in S$ such that $S \setminus \{i\} \in \Phi^p(N, D)$ and $v(S) - v(S \setminus \{i\}) \neq 0$. We also have $v(S) - v(S \setminus \{i\}) = \sum_{T \subseteq S, i \in T, T \in \Phi^p(N,D)} \Delta_v^D(T)$. It follows that there exists at least one set $S \in \Phi^p(N, D), i \in S$ such that $\Delta_v^D(S) \neq 0$ and we obtain a contradiction. So, i is a null player in (N, v, D).

If $j \in \widehat{P}_D(i)$ is not a null player, we can reason in a similar way to obtain that there exists at least one set $S \in \Phi^p(N, D), j \in S$ such that $\Delta_v^D(S) \neq 0$. Since i is a subordinate of j in D, i must also be in $S \in \Phi^p(N, D)$ and we obtain a contradiction.

□

Let $Irr(N, v, D)$ be the set of irrelevant players in game under precedence constraints (N, v, D). Irrelevant player independence states that removal of irrelevant players from the game, does not affect the payoff to relevant players.

Irrelevant player independence Let $N' = N \backslash Irr(N, v, D)$. For every $(N, v, D) \in \mathcal{G}_{PC}$, it holds that $f_i(N, v, D) = f_i(N', v_{N'}, D(N'))$ for $i \in N'$.

For a collection of sets $\mathcal{F} \subseteq 2^N$ let $\mathcal{F}_S = \{T \in \mathcal{F} \mid T \subseteq S\}$ be the collection of subsets of S in $\mathcal{F}$. It can be seen that, for $N' = N \backslash Irr(N, v, D)$, it holds that $\Phi^p_{N'}(N, D) = \Phi^p(N', D(N'))$, i.e. the collection of feasible subsets of coalition N' obtained from graph (N, D) is equal to the collection of feasible sets obtained from graph $(N', D(N'))$. (Note that this does not have to be the case for all subsets of N). This means that removing irrelevant players from the game does not have an effect on the ability of relevant players to cooperate with each other.

We consider irrelevant player independence a desirable property for a solution for games under precedence constraints to satisfy. Since irrelevant players are null players, they do not make any contribution to their subordinates in the digraph. Moreover, their superiors are also null players, and thus they also do not make a contribution through players that need them to be present in any admissible permutation. Therefore they should not be able to affect the payoffs of those players that do make a contribution in the game. The precedence Shapley value does not satisfy irrelevant player independence, as illustrated by the following example.

Example 3.4 *Consider the* $(N, v, D) \in \mathcal{G}_{PC}$, *where* $N = \{1, 2, 3\}$, $v = u_{\{1,2\}}$ *and* $D = \{(3, 1)\}$. *The set of admissible permutations is given by* $\Pi_D(N) = \{(2, 1, 3), (1, 2, 3), (1, 3, 2)\}$. *Player* 3 *is an irrelevant player in* (N, v, D). *The precedence Shapley value allocates the dividend* $\Delta^D_v(\{1, 2\}) \neq 0$, *among the players 1 and 2 proportionally to the hierarchical strength with player 3 obtaining 0 payoff. Since* $h(N, D, \{1, 2\}) = (1, 2)$, *the payoffs according to the precedence Shapley value are* $H(N, u_{\{1,2\}}, D) = (\frac{1}{3}, \frac{2}{3}, 0)$.

Next consider $(N', u_{\{1,2\}}, D') \in \mathcal{G}_{PC}$, *where* $N' = N \setminus \{3\}$ *and* $D' = \emptyset$. *The set of admissible permutations is given by* $\Pi_{D'}(N') = \{(1, 2), (2, 1)\}$. *Now, the precedence Shapley value of players 1 and 2 is allocated proportionally to the hierarchical strength* $h(N', D', \{1, 2\}) = (1, 1)$, *and therefore* $H(N', u_{\{1,2\}}, D') = (\frac{1}{2}, \frac{1}{2})$.

The presence of irrelevant player 3 changes the payoffs of players 1 and 2 according to the precedence Shapley value from $(\frac{1}{2}, \frac{1}{2})$ *to* $(\frac{1}{3}, \frac{2}{3})$.

It can be shown that for games $(N_m, u_{\{1,2\}}, D_m)$, where $N_m = \{1, \ldots, m\}$ and $D_m = \{(3, 1), (4, 3), \ldots, (m, m-1)\}$, the precedence Shapley value is $H_1(N_m, u_{\{1,2\}}, D_m) = \frac{1}{m}$, $H_2(N_m, u_{\{1,2\}}, D_m) = \frac{m-1}{m}$ and $H_i(N_m, u_{\{1,2\}}, D_m) = 0$ for $i \in N_m \setminus \{1, 2\}$ and so

$\lim_{m\to\infty} H_1(N_m, u_{\{1,2\}}, D_m) = 0$ and $\lim_{m\to\infty} H_2(N_m, u_{\{1,2\}}, D_m) = 1$. We find that the fact that player 1 has many irrelevant players as superiors in the digraph, is detrimental to its payoff, even though, for different values of m, player 1 is present in exactly the same feasible coalitions that contain only relevant players.

3.2 The hierarchical solution for games under precedence constraints

Next, we consider the solution for games under precedence constraints that is obtained by replacing hierarchical strength in Theorem 2.1 by the axioms of irrelevant player independence and weak hierarchical strength. Weak hierarchical strength is a weaker version of the hierarchical strength axiom in that it only requires the equality for unanimity games of the grand coalition. This axiom can be interpreted as follows. If unanimity among all players must be reached before any non-zero worth can be generated, we might consider the players equals with respect to the game. Therefore, worth allocation should depend only on the strength of the players in the digraph. The strength of each player in the digraph is measured by the hierarchical strength.

Weak hierarchical strength For every $(N, D) \in \mathcal{D}$ and every $i, j \in N$, it holds that $\overline{h}_i(N, D, N) f_j(N, u_N, D) = \overline{h}_j(N, D, N) f_i(N, u_N, D)$.

We further note that the null player property can be replaced by the following weaker property on irrelevant players.[6]

Irrelevant player property For each $(N, v, D) \in \mathcal{G}_{PC}$, if $i \in N$ is an irrelevant player in (N, v, D), then $f_i(N, v, D) = 0$.

We show that there is a unique solution for games under precedence constraints that satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence, and weak hierarchical strength.

Theorem 3.5 *There is a unique solution f on $\mathcal{G}_{PC}$ that satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence, and weak hierarchical strength.*

PROOF
Since f satisfies linearity it is sufficient to consider uniqueness of f on unanimity games under precedence constraints. For the unanimity game under precedence constraints

[6] It is straightforward to show that the null player property can also be replaced by the irrelevant player property in the axiomatization of the precedence Shapley value.

(N, u_S, D) on some coalition $S \in \Phi^p(N, D)$, the set of irrelevant players is given by $N \setminus S$ (since $S \in \Phi^p(N, D)$ implies that $\widehat{F}_D(S) \subseteq S$). By the irrelevant player property these players are assigned a 0 payoff by f. By irrelevant player independence for the players in S, it holds that $f_i(N, u_S, D) = f_i(S, u_S, D)$ for all $i \in S$.[7]
From efficiency it follows that

$$\sum_{k \in S} f_k(S, u_S, D) = u_S(S) = 1. \tag{3.6}$$

Now consider any player $i \in S$. If $|S| = 1$, then efficiency determines $f_i(S, u_S, D)$. Therefore, suppose that $|S| \geq 2$. Since (S, u_S, D) is a unanimity game on the grand coalition S, we can apply weak hierarchical strength to player i and any player $k \in S \setminus \{i\}$ to obtain that

$$\overline{h}_i(S, D, S) f_k(S, u_S, D) = \overline{h}_k(S, D, S) f_i(S, u_S, D). \tag{3.7}$$

We distinguish the following two cases:

(i) Suppose that $\overline{h}_i(S, D, S) = 0$.

Since $\sum_{j \in S} \overline{h}_j(S, D, S) = 1$, there exists at least one $l \in S \backslash \{i\}$ such that $h_l(S, D, S) \neq 0$. It follows from Equation (3.7) applied to players i and l that $f_i(S, u_S, D) = 0$.

(ii) Suppose that $\overline{h}_i(S, D, S) > 0$.

For $k \in S \setminus \{i\}$ it follows from Equation (3.7) that

$$f_k(S, u_S, D) = \frac{\overline{h}_k(S, D, S)}{\overline{h}_i(S, D, S)} f_i(S, u_S, D).$$

By substituting this expression in Equation (3.6) we obtain

$$\sum_{k \in S} \frac{\overline{h}_k(S, D, S)}{\overline{h}_i(S, D, S)} f_i(S, u_S, D) = 1$$

Since $\overline{h}_k(S, D, S)$ is known for $k \in S$ we find that $f_i(S, u_S, D)$ is uniquely determined.

□

Next we define the hierarchical solution $\widetilde{H}$ for games under precedence constraints.

[7] For convenience we write the subgame on S by (S, u_S, D) instead of $(S, u_S|_S, D(S))$.

Definition 3.6 *The hierarchical solution* $\widetilde{H}$ *is the solution on* $\mathcal{G}_{PC}$ *given by*

$$\widetilde{H}_i(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{h_i(S, D(S), S)}{\sum_{j \in S} h_j(S, D(S), S)} \Delta_v^D(S), \; i \in N.$$

This hierarchical solution allocates the dividend of every feasible coalition over the players in that coalition proportional to the hierarchical strength in the subgraph on that coalition. Next, we provide an example which calculates the hierarchical solution and the precedence Shapley value highlighting that in general both solutions are different.

Example 3.7 *Consider the game under precedence constraints* (N, v, D) *with* $N = \{1, 2, 3, 4\}$, $v = u_{\{1,2,4\}}$ *and* $D = \{(3, 1), (3, 2), (4, 2)\}$. *The dividends of* v *are given by* $\Delta_v^D(\{1, 2, 4\}) = 1$ *and* $\Delta_v^D(S) = 0$, *otherwise. The set of admissible permutations is*

$$\Pi_D(N) = \{(1, 2, 3, 4), (1, 2, 4, 3), (2, 1, 3, 4), (2, 1, 4, 3), (2, 4, 1, 3)\}.$$

In this case, for $S = \{1, 2, 4\} \in \Phi^p(N, D)$, *we have*

$$h_1(N, D, S) = 1, \; h_2(N, D, S) = 0, \; h_4(N, D, S) = 4,$$

yielding the precedence Shapley value $H(N, v, D) = \left(\frac{1}{5}, 0, 0, \frac{4}{5}\right)$.

The set of admissible permutations on subgraph $(S, D(S))$ *is given by*

$$\Pi_{D(S)}(S) = \{(1, 2, 4), (2, 1, 4), (2, 4, 1)\}.$$

Therefore, $h_1(S, D(S), S) = 1$, $h_2(S, D(S), S) = 0$, $h_4(S, D(S), S) = 2$, *yielding the hierarchical solution* $\widetilde{H}(N, v, D) = \left(\frac{1}{3}, 0, 0, \frac{2}{3}\right)$.

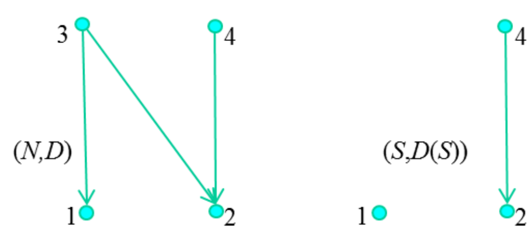

Figure 1: Digraphs (N, D) and $(S, D(S))$

It is straightforward to show that the hierarchical solution satisfies the axioms of Theorem 3.5 and thus is characterized by these axioms. Therefore, we obtain the following characterization.

Theorem 3.8 *A solution on* $\mathcal{G}_{PC}$ *is equal to the hierarchical solution* $\widetilde{H}$ *if and only if it satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence, and weak hierarchical strength.*

3.3 Weighted precedence solutions for games under precedence constraints

Consider a game under precedence constraints (N, v, D). Both the precedence Shapley value H as well as the hierarchical solution $\widetilde{H}$ allocate the dividend $\Delta_v^D(S)$ of a coalition $S \in \Phi^p(N, D)$ among the players in S. The precedence Shapley value allocates proportionally to the hierarchical strength $h(N, D, S)$, while the hierarchical solution allocates proportionally to the hierarchical strength $h(S, D(S), S)$. To make clear the difference, we introduce the following notion.

Definition 3.9 *A weight function is a function w that assigns to every digraph $(N, D) \in \mathcal{D}$ and $S \in \Phi^p(N, D)$ a vector $w(N, D, S) \in \mathbb{R}^S$, where $\sum_{i \in S} w_i(N, D, S) > 0$.*

We note that both the absolute as well as the normalized hierarchical strength are weight functions. A weight function w is called *subgraph-invariant* if $w(N, D, S) = w(S, D(S), S)$ for all $(N, D) \in \mathcal{D}$ and $S \in \Phi^p(N, D)$. The weight vector assigned to a digraph $(N, D) \in \mathcal{D}$ and $S \in \Phi^p(N, D)$ by a subgraph-invariant weight function depends only on the subgraph on S. Let the collection of all subgraph-invariant weight functions be denoted by W^I.

Next we consider the class of solutions that for any game (N, v, D) allocate the dividend of a coalition $S \in \Phi^p(N, D)$ according to $w(N, D, S)$ for some weight function w. We will refer to solutions in this class as *weighted precedence solutions*.

Definition 3.10 *For a game under precedence constraints (N, v, D) and weight function w, the weighted precedence solution is the solution f^w given by*

$$f_i^w(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{w_i(N, D, S)}{\sum_{j \in S} w_j(N, D, S)} \Delta_v^D(S) \text{ for all } i \in N.$$

The precedence Shapley value H is the weighted precedence solution f^h, where h is the absolute (or normalized) hierarchical strength. The hierarchical solution $\widetilde{H}$ is the weighted precedence solution $f^{\widetilde{h}}$, where $\widetilde{h}$ is the weight function given by

$$\widetilde{h}(N, D, S) = h(S, D(S), S) \text{ for } (N, D) \in \mathcal{D} \text{ and } S \in \Phi^p(N, D),$$

being the weight function that assigns to every coalition S in a digraph its hierarchical strength in the subgraph on coalition S. For these solutions, dividend allocation of a feasible coalition depends only on the subgraph on that coalition.

Proposition 3.11 *Every weighted precedence solution obtained from some subgraph-invariant weight function satisfies irrelevant player independence.*

Proof

Consider a game under precedence constraints (N, v, D) and solution f^w obtained from a subgraph-invariant weight function w. Let $N' = N \setminus Irr(N, v, D)$. By Proposition 3.3, it holds that $\Delta_v^D(S) = 0$ if $S \cap Irr(N, v, D) \neq \emptyset$ and from the expression of dividends given by Faigle and Kern (1992) it follows that $\Delta_v^D(S) = \Delta_{v_{N'}}^D(S)$ for $S \in \Phi_{N'}^p(N, D)$.

For a player $i \in N'$, it holds that $f_i^w(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{w_i(S,D(S),S)}{\sum_{j \in S} w_j(S,D(S),S)} \Delta_v^D(S) = \sum_{\substack{S \in \Phi_{N'}^p(N,D) \\ i \in S}} \frac{w_i(S,D(S),S)}{\sum_{j \in S} w_j(S,D(S),S)} \Delta_{v_{N'}^D}(S)$, and $f_i^w(N', v_{N'}, D(N')) = \sum_{\substack{S \in \Phi^p(N',D(N')) \\ i \in S}} \frac{w_i(S,D(N')(S),S)}{\sum_{j \in S} w_j(S,D(N')(S),S)} \Delta_{v_{N'}}^{D(N')}(S)$. For all coalitions $S \in \Phi^p(N', D(N')) = \Phi_{N'}^p(N, D)$ it holds that $D(S) = D(N')(S)$, and therefore $w_i(S, D(S), S) = w_i(S, D(N')(S), S)$ for all $i \in S$. Hence, we conclude that $f_i^w(N, v, D) = f_i^w(N', v_{N'}, D(N'))$.

□

3.4 Power measures for digraphs and precedence power solutions for games under precedence constraints

A *power measure* for acyclic digraphs is a function p, that to every acyclic digraph $(N, D) \in \mathcal{D}$ assigns a vector $p(N, D) \in \mathbb{R}^N$ to the players in N. For a player $i \in N$, $p_i(N, D)$ is a measure of its 'power' or 'influence' in (N, D). We call a power measure p positive if $\sum_{j \in N} p_j(N, D) > 0$ for all $(N, D) \in \mathcal{D}$. In this paper we consider only positive power measures. Let the collection of all positive power measures be denoted by P.

Let $t : W^I \to P$ be the function that assigns to every subgraph-invariant weight function $w \in W^I$ the power measure $p \in P$ that for every acyclic digraph $(N, D) \in \mathcal{D}$ is given by $p(N, D) = w(N, D, N)$.

Proposition 3.12 *The function t is a bijection.*

Proof

We show that t is both injective and surjective.

(i) t is injective.

Let $p = t(w_1) = t(w_2)$. Since both w_1 and w_2 are subgraph-invariant, it holds that $w_1(N, D, S) = w_1(S, D(S), S)$ and $w_2(N, D, S) = w_2(S, D(S), S)$. Since $p(S, D(S)) = w_1(S, D(S), S)$ and $p(S, D(S)) = w_2(S, D(S), S)$, we have $w_1(N, D, S) = w_1(S, D(S), S) = p(S, D(S)) = w_2(S, D(S), S) = w_2(N, D, S)$.

(ii) t is surjective.

For any power measure $p \in P$, consider the weight function $w \in W^I$ given by $w(N, D, S) = p(S, D(S))$. Clearly it holds that $t(w) = p$.

□

For positive power measure p, we define the *p-hierarchical solution* as the solution that allocates the dividend of a coalition $S \in \Phi^p(N, D)$ among the players in S proportionally to $p(S, D(S))$ according to some positive power measure p.

Definition 3.13 *For positive power measure p, the p-hierarchical solution is the solution on $\mathcal{G}_{PC}$ given by*

$$H_i^p(N, v, D) = \sum_{\substack{S \in \Phi^p(N,D) \\ i \in S}} \frac{p_i(S, D(S))}{\sum_{j \in S} p_j(S, D(S))} \Delta_v^D(S) \text{ for all } i \in N.$$

We will refer to the class consisting of all p-hierarchical solutions as the class of precedence power solutions. It turns out that these are exactly the weighted precedence solutions obtained from a subgraph-invariant weight function.

Proposition 3.14 *The collection of weighted precedence solutions obtained from a subgraph-invariant weight function is equivalent to the collection of precedence power solutions.*

Proof

Consider function t from Proposition 3.12. From Definition 3.10 and Definition 3.13 we have that $f^w = H^{t(w)}$. The proposition then follows from bijectivity of t.

□

From Proposition 3.11 and 3.14 we obtain the following corollary.

Corollary 3.15 *Every precedence power solution for games under precedence constraints satisfies irrelevant player independence.*

In order to axiomatize the p-hierarchical solution we introduce the p-strength axiom. This axiom has an interpretation similar to that of weak hierarchical strength from Theorem 3.5. If unanimity among all players must be reached to generate any non-zero worth, we might consider the players equals with respect to the game. Therefore, worth allocation should only depend on the strength of players in the digraph. The p-hierarchical solution uses the power measure p to measure the strength of each player in the digraph.

p-strength For every $(N, D) \in \mathcal{D}$ and every $i, j \in N$, it holds that
$p_i(N, D) f_j(N, u_N, D) = p_j(N, D) f_i(N, u_N, D)$.

The p-hierarchical solution is axiomatized by replacing in Theorem 3.5 weak hierarchical strength by p-strength.

Theorem 3.16 *A solution for games under precedence constraints is equal to the p-hierarchical solution H^p if and only if it satisfies efficiency, linearity, the irrelevant player property, irrelevant player independence and p-strength.*

PROOF
It is straightforward to show that H^p satisfies efficiency, linearity, the irrelevant player property and p-strength. H^p satisfying irrelevant player independence follows from Corollary 3.15
The proof of uniqueness follows similar as in Theorem 3.5.

□

3.5 The hierarchical measure for digraphs

Above we found that the hierarchical solution $\widetilde{H}$ is equivalent to the weighted precedence solution $f^{\tilde{h}}$, where $\tilde{h}$ is the subgraph-invariant weight function given by $\widetilde{h}(N, D, S) = h(S, D(S), S)$ for $(N, D) \in \mathcal{D}$ and $S \in \Phi^p(N, D)$. Proposition 3.14 implies that there exists a power measure p such that $H^p = f^{\tilde{h}} = \widetilde{H}$.[8] This is the following power measure.

Definition 3.17 *The hierarchical measure is the power measure on $\mathcal{D}$ given by*

$$\eta_i(N, D) = \widetilde{h}_i(N, D, N) = h_i(N, D, N) = |\Pi^i_D(N)| \text{ for all } i \in N.$$

Since $\widetilde{H} = H^\eta$, from here on we will denote the hierarchical solution by H^η. Note that η is defined for any $N \subset \mathbb{N}$, and thus also for all $S \subset N$.

Faigle and Kern (1992) only use the hierarchical strength as a tool to axiomatize the precedence Shapley value H. No motivation is given about the reason why the hierarchical strength is used, instead of any other measure. Although other motivations for the precedence Shapley value are given, here we motivate the use of power measure η by giving an axiomatization of this power measure on the class of acyclic digraphs. Therefore, we introduce several axioms that can be satisfied by a generic power measure p for acyclic digraphs.

The first axiom, 1-normalization, states that if digraph (N, D) contains only one player then this player has power one. This property is satisfied by many power measures in the literature.

1-Normalization For every $(N, D) \in \mathcal{D}$ with $N = \{i\}$, it holds that $p_i(N, D) = 1$.

[8] For the hierarchical strength h itself, we cannot find such a corresponding power measure. This follows from the precedence Shapley value not satisfying irrelevant player independence.

The second axiom, the non-top property, states that players that are not top players in the graph have zero power. The digraph is interpreted as a hierarchical structure, where the only players that can enter as last player, and therefore are not depending on players that always enter after them, are players without predecessors. This property is also satisfied by, for example the λ-measure of Borm, van den Brink and Slikker (2002).

Non-top property For every $(N, D) \in \mathcal{D}$ and $i \in N$ such that $P_D(i) \neq \emptyset$, it holds that $p_i(N, D) = 0$.

The third axiom is independence of successors and states that the power of a player does not depend on its successors. This property reflects that the power of a player does not depend on players it dominates, but more on the players that it is dominated by.

For a player $i \in N$, let $out_D(i) := \{(k, l) \in D \mid k = i\}$ be the set of outgoing arcs from i in digraph D.

Independence of successors For every $(N, D) \in \mathcal{D}$ it holds that $p_i(N, D) = p_i(N, D \setminus out_D(i))$.

Finally, the isolated player property states that the power of an isolated player (i.e. a player having no successors nor predecessors) is equal to the sum of the powers of all other players in the subgraph without this isolated player. Since isolated players do not have any predecessors, these players might be considered to interact freely with any of the other players in the graph. Since isolated players also do not have any successors, it might be said that their power in the graph comes only from this interaction with other players. The isolated player property reflects that the power of an isolated player depends on the combined strength of the relations it is able to have with any of the other players, where the strength of each relation depends on the powers of the other players in the subgraph without the isolated player.

Isolated player property For every $(N, D) \in \mathcal{D}$ and $i \in N$ such that $F_D(i) \cup P_D(i) = \emptyset$, it holds that $p_i(N, D) = \sum_{j \in N \setminus \{i\}} p_j(N \setminus \{i\}, D_{-i})$.

The four previous axioms characterize the hierarchical measure on acyclic digraphs.[9]

Now consider $(N, D) \in \mathcal{D}$. For any admissible permutation $\pi \in \Pi_D(N)$ and $S \subseteq N$ let $\pi_S \in \Pi(S)$ be such that $\pi_S(i) < \pi_S(j)$ if $\pi(i) < \pi(j), i, j \in S$, i.e. it is the permutation on players in coalition S obtained by considering the relative order of these players in π.

[9]Note that for $(N, D) \in \mathcal{D}$ and $i \in N$ such that $F_D(i) \cup P_D(i) = \emptyset$ (and so i is an isolated player), it holds that $D_{-i} = D$. Therefore in the definition of the isolated player property it does not matter whether we use D_{-i} or D.

Furthermore, for $1 \leq i \leq |N|$, let π_i be such that $|\{k \in N \mid \pi(k) \leq \pi_i\}| = i$, i.e. it is the player preceded by $i-1$ players in permutation π.

In order to characterize the hierarchical measure by these four axioms, we use the following proposition.

Proposition 3.18 *Consider an acyclic digraph $(N, D) \in \mathcal{D}$. Let $P = \{P_1, \ldots, P_m\}$ be a partition of N such that for every pair $k, l \in \{1, \ldots, m\}, k \neq l$, there do not exist $i \in P_k, j \in P_l$ such that either $(i, j) \in D$ or $(j, i) \in D$. Then $\pi \in \Pi_D(N)$, if and only if for every $k \in \{1, \ldots, m\}$ it holds that $\pi_{P_k} \in \Pi_{D(P_k)}(P_k)$.*

This follows from the fact that the set of admissible permutations of an acyclic digraph (N, D) is determined only by successors entering before predecessors. Successors are guaranteed to enter before predecessors for those permutations π of N, where for any partition of (N, D) into subgraphs that have no arcs between them, the relative orders in π of players in those subgraphs are admissible permutations of those subgraphs.

Theorem 3.19 *A power measure for acyclic digraphs is equal to the hierarchical measure η if and only if it satisfies 1-normalization, the non-top property, independence of successors and the isolated player property.*

Proof

Consider acyclic digraph (N, D). It is straightforward to show that the hierarchical measure η satisfies 1-normalization since the only permutation on $N = \{i\}$ is (i).

Since $j \in P_D(i)$ implies that $\pi(i) < \pi(j)$ for all $\pi \in \Pi_D(N)$, there is no $\pi \in \Pi_D(N)$ such that $\pi(i) = n$, and therefore $\eta_i(N, D) = 0$, showing that η satisfies the non-top property.

Suppose that $j \in F_D(i)$. Since $\Pi_D(N) \subset \Pi_{D \setminus \{(i,j)\}}(N)$ and $\pi \in \Pi_{D \setminus \{(i,j)\}}(N) \setminus \Pi_D(N)$ implies that $\pi(j) > \pi(i)$, it holds that $\eta_i(N, D) = \eta_i(N, D \setminus \{(i, j)\})$. Repeated application for all arcs in $out_D(i)$ shows that η satisfies independence of successors.

For an isolated player $i \in N$ there does not exist $j \in N \setminus \{i\}$ such that $(i, j) \in D$ or $(j, i) \in D$. By Proposition 3.18 it therefore holds that $\pi \in \Pi_D(N)$ if and only if $\pi_{N \setminus \{i\}} \in \Pi_{D_{-i}}(N \setminus \{i\})$. The number of admissible permutations in $\Pi^i_D(N)$ is therefore equal to the number of possible relative orders $\pi_{N \setminus \{i\}}$ of the players in $N \setminus \{i\}$. It follows that $|\Pi^i_D(N)| = |\Pi_{D_{-i}}(N \setminus \{i\})|$. Furthermore by definition of the hierarchical measure it holds that $|\Pi_{D_{-i}}(N \setminus \{i\})| = \sum_{j \in N \setminus \{i\}} \eta_j(N \setminus \{i\}, D_{-i})$. Therefore $\eta_i(N, D) = |\Pi^i_D(N)| = |\Pi_{D_{-i}}(N \setminus \{i\})| = \sum_{j \in N \setminus \{i\}} \eta_j(N \setminus \{i\}, D_{-i})$ showing that η satisfies the isolated player property.

The proof of uniqueness is given as follows. Let p be a positive power measure satisfying the axioms. We perform induction on $|N|$. If $|N| = 1$ then $p_i(\{i\}, D) = 1$ by

1-normalization. Proceeding by induction, assume that $p(N', D')$ is uniquely determined whenever $|N'| < |N|$, and consider $(N, D) \in \mathcal{D}$. If $P_D(i) \neq \emptyset$ then $f_i(N, D) = 0$ by the non-top property. Therefore, suppose that $P_D(i) = \emptyset$. Then, $p_i(N, D) = p_i(N, D \setminus out_D(i)) = p_i(N, D_{-i}) = \sum_{j \in N \setminus \{i\}} p_j(N \setminus \{i\}, D_{-i})$, where the first equality follows from independence of successors and the last equality follows from the isolated player property. By the induction hypothesis, $p_j(N \setminus \{i\}, D_{-i})$, $j \in N \setminus \{i\}$, are known, and thus $p_i(N, D)$ is uniquely determined.

□

We show logical independence by the following alternative power measures.

1. The power measure that is given by $p_i(N, D) = 0$ for all $(N, D) \in \mathcal{D}$ and $i \in N$ satisfies the non-top property, independence of successors and the isolated player property. It does not satisfy 1-normalization.

2. If $D = \emptyset$ then $|\Pi_D(N)| = |N|!$. In $\frac{1}{|N|}$ of these permutations player $i \in N$ is the last player. Ignoring the digraph D and assigning to every player the value equal to the number of permutations of N where it is last yields the power measure that is given by $p_i(N, D) = (|N| - 1)!$ for all $(N, D) \in \mathcal{D}$ and $i \in N$. This power measure satisfies 1-normalization, independence of successors and the isolated player property. It does not satisfy the non-top property.

3. The power measure given by $p_i(N, D) = (|TOP(N, D)| - 1)!$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies 1-normalization, the non-top property and the isolated player property. It does not satisfy independence of successors.

4. The power measure given by $p_i(N, D) = 1$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies 1-normalization, the non-top property and independence of successors. It does not satisfy the isolated player property.

4 Regular set systems and the hierarchical measure

4.1 Chains and regular set systems

In the literature, the precedence Shapley value has been extended to games associated with combinatorial structures more general than a digraph. For example, Bilbao and Edelman (2000) consider games on convex geometries, Algaba, Bilbao, van den Brink, and Jiménez-Losada (2003, 2004) consider games on antimatroids, and Bilbao and Ordoñez (2009) consider games on augmenting systems where the grand coalition is feasible. Convex geometries have been shown to be contained in the class of so-called regular set systems considered by Lange and Grabisch (2009), who also consider an extension of the precedence Shapley value to games on regular set systems. A set of admissible permutations is generated by considering the so-called chains of feasible sets.

Consider a feasible set system $\mathcal{F} \subseteq 2^N$ for which $\emptyset$, $N \in \mathcal{F}$. A *chain* in set system $\mathcal{F}$ from $S \in \mathcal{F}$ to $T \in \mathcal{F}$, $S \subset T$, is an ordered collection $C = (C_0, ..., C_k)$ of feasible sets, such that $C_0 = S$, $C_k = T$ and $C_i \subset C_{i+1}$ for $i \in \{0, \ldots, k-1\}$. A chain $C = (C_0, ..., C_k)$ is maximal, when there exists no set $S \in \mathcal{F}$ such that for some $i \in \{0, \ldots, k-1\}$ it holds that $C_i \subset S \subset C_{i+1}$. The length $l(C)$ of a chain C is defined as the number of sets that it contains. Let the collection of all maximal chains from $\emptyset$ to N be denoted by $C^N_{\mathcal{F}}$. For a chain $C = (C_0, ..., C_k)$, when $S = C_i$ for some $i \in \{0, \ldots, k\}$, we will say that set S is on chain C. From here on, when we refer to maximal chains, we will mean the maximal chains from $\emptyset$ to N.

Example 4.1 *Let* $N = \{1, 2, 3, 4, 5\}$ *and let*
$\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3, 4\}, \{2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4, 5\}\}$.
The collection of maximal chains $C^N_{\mathcal{F}}$ *is given by* $\{C_1, C_2, C_3\}$*, where*
$C_1 = (\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}), C_2 = (\emptyset, \{2\}, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\})$,
and $C_3 = (\emptyset, \{2\}, \{1, 2\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\})$. *Note that* $l(C_1) = l(C_3) = 5$*, while* $l(C_2) = 6$.

In this section we consider the hierarchical measure on regular set systems which are introduced by Honda and Grabisch (2006).

Definition 4.2 *A set system* $\mathcal{F} \subseteq 2^N$ *is a regular set system if it satisfies the following axioms:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(feasible grand coalition) $N \in \mathcal{F}$,

(regularity property) for every maximal chain $C \in C^N_{\mathcal{F}}$ *it holds that* $l(C) = |N| + 1$.

Regularity of $\mathcal{F}$ implies that $|C_k \setminus C_{k-1}| = 1$ for all $k \in \{1, \ldots, n\}$ and $C = (C_0, C_1, \ldots, C_n) \in C^N_{\mathcal{F}}$. Note that the set system $\mathcal{F}$ in Example 4.1 is not regular since $l(c_1) = l(c_3) < |N| + 1$. We obtain the following proposition with respect to regular set systems.

Proposition 4.3 *Let* $\mathcal{F} \subseteq 2^N$ *be such that* $\emptyset, N \in \mathcal{F}$. *Then* $\mathcal{F}$ *is a regular set system if and only if for all* $S, T \in \mathcal{F}$ *with* $S \subset T$ *there exists at least one maximal chain* $C = (C_0, \ldots, C_n)$ *containing coalitions* S *and* T.

Proof

Only if: It follows directly from the regularity property.

If: Suppose that $\mathcal{F}$ is not a regular set system. Then there exists a maximal chain $C \in C_{\mathcal{F}}^N$ such that $l(C) < |N| + 1$, i.e., there exists $C = (C_0, \ldots, C_k) \in C_{\mathcal{F}}^N$ with $k < n$. Therefore, there must exist C_h and C_{h+1}, $h \in \{0, \ldots, k-1\}$, on C such that $C_h \subset C_{h+1}$, $|C_{h+1}| \geq |C_h| + 2$, and there is no $S \in \mathcal{F}$, $C_h \subset S \subset C_{h+1}$. Hence, we conclude that any chain containing C_h, $C_{h+1} \in \mathcal{F}$ has at most n elements which is a contradiction with the hypothesis. □

Honda and Grabisch (2006) show that all convex geometries are regular set systems, while Grabisch (2013) shows that augmenting systems $\mathcal{F}$ satisfying $N \in \mathcal{F}$ are regular set systems.

Definition 4.4 *A set system $\mathcal{F} \subseteq 2^N$ is an augmenting system if it satisfies the following axioms:*

(feasible empty set) $\emptyset \in \mathcal{F}$,

(union stability) $S, T \in \mathcal{F}$ *with* $S \cap T \neq \emptyset$, *implies that* $S \cup T \in \mathcal{F}$,

(augmentation) for $S, T \in \mathcal{F}$ *with* $S \subset T$, *there exists* $i \in T \setminus S$ *such that* $S \cup \{i\} \in \mathcal{F}$.

Note that in general, augmenting systems are not regular set systems, since N need not be feasible, see also Bilbao (2003). Moreover, in general, convex geometries and augmenting systems containing the grand coalition are proper subsets of the class of regular set systems. Normalized antimatroids are particular cases of augmenting systems containing the grand coalition, and therefore, normalized antimatroids are also particular cases of regular set systems.

Example 4.5 *Consider the regular set system* $(N, \mathcal{F})$ *given by* $N = \{1, 2, 3, 4, 5\}$ *and*

$$\mathcal{F} = \{\emptyset, \{1\}, \{1,2\}, \{1,2,5\}, \{1,2,5,3\}, \{4\}, \{4,2\}, \{4,2,3\}, \{4,2,3,5\}, N\}.$$

There are two maximal chains with 6 elements:

$$\begin{aligned} C_1 &: \emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,5\} \subset \{1,2,5,3\} \subset N, \\ C_2 &: \emptyset \subset \{4\} \subset \{4,2\} \subset \{4,2,3\} \subset \{4,2,3,5\} \subset N. \end{aligned}$$

However, this regular set system is not a convex geometry since $\{1,2\}, \{4,2\} \in \mathcal{F}$ *and* $\{1,2\} \cap \{4,2\} = \{2\} \notin \mathcal{F}$. *Moreover, this set system is also not an augmenting system since it does not satisfy union stability because* $\{1,2\}, \{4,2\} \in \mathcal{F}$ *with* $\{1,2\} \cap \{4,2\} \neq \emptyset$, *but* $\{1,2\} \cup \{4,2\} = \{1,2,4\} \notin \mathcal{F}$.

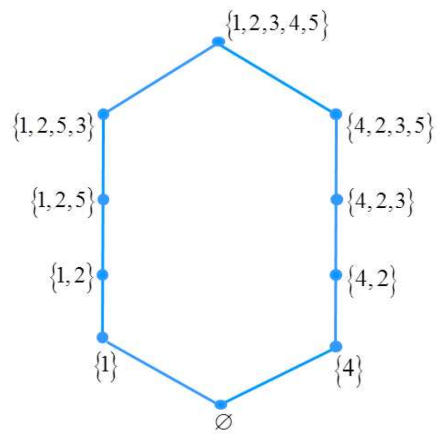

Figure 2: Regular Set System $(N, \mathcal{F})$

For a regular set system $\mathcal{F} \subseteq 2^N$ we can generate admissible permutations of the players in N by considering the maximal chains in $C^N_{\mathcal{F}}$. The set of *admissible permutations* $\Pi_{\mathcal{F}}(N)$ associated with a regular set system $\mathcal{F}$ is given by

$$\Pi_{\mathcal{F}} = \left\{ \pi \in \Pi(N) \,\middle|\, (\emptyset, \{\pi_1\}, \bigcup_{i=1}^{2}\{\pi_i\}, ..., \bigcup_{i=1}^{|N|-1}\{\pi_i\}, N) \in C^N_{\mathcal{F}} \right\}, \tag{4.8}$$

where π_i is the player that is at position i in permutation π.

Note that for any feasible set system $\mathcal{F} \subseteq 2^N$ the set of admissible permutations $\Pi_{\mathcal{F}}$ maps $\mathcal{F}$ to the set of permutations on N. We find that this mapping is injective, but not surjective.

Proposition 4.6 *For every two regular set systems $\mathcal{E}$ and $\mathcal{F}$ with $\Pi_{\mathcal{E}} = \Pi_{\mathcal{F}}$, we have $\mathcal{E} = \mathcal{F}$.*

Proof

Suppose $\Pi_{\mathcal{E}} = \Pi_{\mathcal{F}}$. Suppose without loss of generality that $S \in \mathcal{E}$. We show that then also $S \in \mathcal{F}$. Since for every regular set system it holds that every feasible set is on a maximal chain, coalition S is on some maximal chain C in $C^N_{\mathcal{E}}$. Let $\pi \in \Pi_{\mathcal{E}}$ be the permutation corresponding to this chain. Since also $\pi \in \Pi_{\mathcal{F}}$ the chain C must also be a chain in $\mathcal{F}$ and therefore $S \in \mathcal{F}$.

□

Not every collection of permutations can be the set of admissible permutations in some regular set system. In other words, there exist subsets $\Pi \subset \Pi(N)$ for which there is no regular set system $\mathcal{F}$ such that $\Pi_{\mathcal{F}} = \Pi$. We illustrate this with the following example.

Example 4.7 *Consider $N = \{1,2,3\}$ and $\Pi = \{(1,2,3),(2,3,1),(3,1,2)\}$. If Π is the set of admissible permutations of some regular set system $\mathcal{F}$ then it must hold that the set of maximal chains $C^N_{\mathcal{F}}$ from $\emptyset$ to N is given by $(\emptyset,\{1\},\{1,2\},\{1,2,3\})$, $(\emptyset,\{2\},\{2,3\},\{1,2,3\})$, $(\emptyset,\{3\},\{1,3\},\{1,2,3\})$. The set system $\mathcal{F}$ is then given by the union of the sets on these chains and therefore $\mathcal{F} = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. However in that case the ordered collection $(\emptyset,\{3\},\{2,3\},\{1,2,3\})$ is also a chain in $\mathcal{F}$ and it must hold that $(3,2,1) \in \Pi_{\mathcal{F}}$. Since $(3,2,1)$ is not in Π it holds that Π cannot be generated as the set of admissible permutations from the chains of some regular set system.*

4.2 The hierarchical measure for regular set systems

In this subsection we will study the hierarchical measure in the context of regular set systems. We will denote the class of all regular set systems by $\mathcal{R}$. For any coalition T in a regular set system $\mathcal{F}$, the set of coalitions in $\mathcal{F}$ that are a subset of T is a regular set system on T.

Proposition 4.8 *If $\mathcal{F}$ is a regular set system on N and $T \in \mathcal{F}$, then $\mathcal{F}_T = \{S \in \mathcal{F} \mid S \subseteq T\}$ is a regular set system on T.*

Proof

Obviously, $\emptyset, T \in \mathcal{F}_T$. Suppose that $\mathcal{F}_T$ is not a regular set system on T. Then there exists a maximal chain $C_T \in C^T_{\mathcal{F}_T}$ such that $l(C_T) < t+1$, with $t = |T|$. Since $T \in \mathcal{F}_T$, there exists $C_T = (C_0, \ldots, C_r)$ with $r < t$ and $C_r = T$. Since $T, N \in \mathcal{F}$, $T \subset N$ and as $\mathcal{F}$ is a regular set system, there exists a maximal chain $C^* = \left(C^*_0, \ldots, C^*_t, C^*_{t+1} \ldots, C^*_n\right) \in C^N_{\mathcal{F}}$,with $C^*_t = T$. Consider the chain $C' = \left(C_0, \ldots, C_r, C^*_{t+1} \ldots, C^*_n\right)$. Notice that $C' \in C^N_{\mathcal{F}}$ but $l(C') < n+1$ which is a contradiction with $\mathcal{F}$ being a regular set system on N.

□

A *power measure* p for regular set systems is a function that assigns to every regular set system $(N,\mathcal{F}) \in \mathcal{R}$ a vector $p(N,\mathcal{F}) \in \mathbb{R}^N$, where $p_i(N,\mathcal{F})$ measures the 'power' or 'influence' of player i in $(N,\mathcal{F})$. We call a power measure p positive if $\sum_{j\in N} p_j(N,\mathcal{F}) > 0$ for all $(N,\mathcal{F}) \in \mathcal{R}$. In this paper we consider only positive power measures.

For $i \in N$, the set of permutations $\Pi^i_{\mathcal{F}}(N)$ is defined by

$$\Pi^i_{\mathcal{F}}(N) = \{\pi \in \Pi_{\mathcal{F}}(N) \mid \pi(i) > \pi(j) \text{ for all } j \in N \setminus \{i\}\}. \tag{4.9}$$

being the collection of permutations in $\Pi_{\mathcal{F}}(N)$ where i is preceded by the players in $N \setminus \{i\}$.

The *hierarchical measure* η assigns to a player i the number of admissible permutations of regular set system $(N,\mathcal{F})$ such that player i is preceded by the players in $N \setminus \{i\}$.

Definition 4.9 *The hierarchical measure is the power measure η on $\mathcal{R}$ given by*

$$\eta_i(N, \mathcal{F}) = \left|\Pi^i_{\mathcal{F}}(N)\right| \text{ for all } i \in N.$$

We use the following axioms to characterize the hierarchical measure on regular set systems. First, chain efficiency states that the sum of the power scores of the players is exactly equal to the number of maximal chains.

Chain efficiency For every $(N, \mathcal{F}) \in \mathcal{R}$ it holds that $\sum_{i \in N} p_i(N, \mathcal{F}) = |C^N_{\mathcal{F}}|$.

The second axiom is similar to the non-top property for acyclic digraphs.

Non-tail property For $(N, \mathcal{F}) \in \mathcal{R}$, if for every maximal chain $C = (C_0, ..., C_{|N|-1}, C_{|N|}) \in C^N_{\mathcal{F}}$, it holds that $C_{|N|} \setminus C_{|N|-1} \neq \{i\}$, then $p_i(N, \mathcal{F}) = 0$.

The third axiom states that the only chains that affect the power of a player are the maximal chains where it is the last player.

Independence of irrelevant chains For $(N, \mathcal{F}), (N, \mathcal{F}') \in \mathcal{R}$ such that $\Pi^i_{\mathcal{F}}(N) = \Pi^i_{\mathcal{F}'}(N)$, we have $p_i(N, \mathcal{F}) = p_i(N, \mathcal{F}')$.

Now let $\mathcal{F}^i = \{S \in \mathcal{F} \mid \text{there is a chain } C = (C_0, ..., C_{|N|-1}, C_{|N|}) \text{ in } C^N_{\mathcal{F}}, \text{ such that } S = C_k \text{ for some } k \in \{1, \dots, |N|\}, \text{ and } C_{|N|} \setminus C_{|N|-1} = \{i\}\}$ being the set system that is obtained from $(N, \mathcal{F})$ by removing exactly the sets that are not on any maximal chain where i only occurs in set N.

Proposition 4.10 *If $\mathcal{F} \subseteq 2^N$ is a regular set system with $\Pi^i_{\mathcal{F}}(N) \neq \emptyset$, then $\mathcal{F}^i$ is also a regular set system.*

Proof

Since $\emptyset, N \in \mathcal{F}^i$ we only have to consider whether $\mathcal{F}^i$ satisfies the regularity property. Suppose that there exists a maximal chain $C \in C^N_{\mathcal{F}^i}$ such that $l(C) < |N| + 1$. Let $(C_0, ..., C_k)$ be this chain. It holds that C is also a chain from $\emptyset$ to N in set system $\mathcal{F}$. In $\mathcal{F}$ every maximal chain has length $|N| + 1$. Therefore chain C is not maximal in $\mathcal{F}$. This means that for some $m \in \{0, \dots, k-1\}$ there exists $S \in \mathcal{F}$ such that $C_m \subset S \subset C_{m+1}$. We distinguish two cases:

(i) Suppose that $C_{m+1} = N$.
In that case it holds that $|C_m| = l$ for some $l < |N| - 1$. From the fact that $C_m \in \mathcal{F}^i$ it follows that there exists a maximal chain $C' = (C'_0, C'_1, \dots, C'_n) \in C^N_{\mathcal{F}}$ such that $C_m = C'_l$ and $C'_{|N|} \setminus C'_{|N|-1} = \{i\}$. Now consider the set C'_{l+1}. It holds that $C'_{l+1} \in \mathcal{F}^i$ and $C'_{l+1} \neq N$ by $l < |N| - 1$. Since $C_m = C'_l \subset C'_{l+1} \subset N$ it must hold that $C \notin C^N_{\mathcal{F}^i}$ and we obtain a contradiction.

(ii) Suppose that $C_{m+1} \neq N$.
Since $i \notin C_{m+1}$, we also have $i \notin S$. Now let $|C_{m+1}| = p$. By Proposition 4.3 there exists a chain $C' \in C^N_{\mathcal{F}}$ such that both S and C_{m+1} are on C'. From the fact that $C_{m+1} \in \mathcal{F}^i$ it follows that there exists a chain $C'' \in C^N_{\mathcal{F}}$ such that $C_{m+1} = C''_p$ and $C''_{|N|} \setminus C''_{|N|-1} = \{i\}$. From C' and C'' we construct a chain $C''' \in C^N_{\mathcal{F}}$ such that $C'''_{|N|} \setminus C'''_{|N|-1} = \{i\}$ and S is on C''' as follows: let $C'''_j = C'_j$ for $j < p$, $C'''_p = C''_p = C'_p = C_{m+1}$ and $C'''_j = C''_j$ for $j > p$. It follows that $S \in \mathcal{F}^i$ and we obtain a contradiction with $C \in C^N_{\mathcal{F}}$.

□

Let $C^N_{\mathcal{F}}|_i \subseteq C^N_{\mathcal{F}}$ be the collection of those maximal chains $C \in C^N_{\mathcal{F}}$ such that $i \notin C_k$ for $k \in \{1, \ldots, |N|-1\}$, i.e. it is the collection of those maximal chains generating admissible permutations where player i is preceded by the other players in N.

Proposition 4.11 *If $\mathcal{F}$ is a regular set system on N and $i \in N$ then $C^N_{\mathcal{F}}|_i = C^N_{\mathcal{F}^i}$.*

The proof follows straightforwardly from $\mathcal{F}^i$ containing exactly the sets in $\mathcal{F}$ that are on chains in $C^N_{\mathcal{F}}|_i$.

Theorem 4.12 *A power measure on $\mathcal{R}$ is equal to the hierarchical measure η if and only if it satisfies chain efficiency, the non-tail property and independence of irrelevant chains.*

Proof
It is straightforward to show that the hierarchical measure satisfies these axioms.
The proof of uniqueness is given as follows.

Let p be a positive power measure satisfying the axioms. Let $(N, \mathcal{F}) \in \mathcal{R}$ and $i \in N$. If for every maximal chain $C = (C_0, ..., C_{n-1}, C_{|N|}) \in C^N_{\mathcal{F}}$, it holds that $C_{|N|} \setminus C_{|N|-1} \neq \{i\}$, then by the non-tail property $p_i(N, \mathcal{F}) = 0$. Now assume that $C_{|N|} \setminus C_{|N|-1} = \{i\}$ for at least one maximal chain in $C^N_{\mathcal{F}}$. We consider the regular set system $\mathcal{F}^i$. The non-tail property implies that $p_j(N, \mathcal{F}^i) = 0$ for all $j \in N \setminus \{i\}$. Then chain efficiency implies that $p_i(N, \mathcal{F}^i) = |\mathcal{F}^i|$. Since $C^N_{\mathcal{F}}|_i = C^N_{\mathcal{F}^i}$ we can apply independence of irrelevant chains to determine that $p_i(N, \mathcal{F}) = p_i(N, \mathcal{F}^i) = |C^N_{\mathcal{F}^i}|$ is uniquely determined. □

We show logical independence by the following alternative power measures.

1. The power measure that is given by $p_i(N, \mathcal{F}) = 0$, $i \in N$, satisfies the non-tail property and independence of irrelevant chains. It does not satisfy chain efficiency.

2. Consider the following power measure p. If $|N| = 1$, then $p(N, \mathcal{F}) = \eta(N, \mathcal{F})$. If $|N| > 1$ and players $1, 2 \in N$, then $p_1(N, \mathcal{F}) = \eta_1(N, \mathcal{F}) - 1$, $p_2(N, \mathcal{F}) = \eta_2(N, \mathcal{F}) + 1$ and $p_i(N, \mathcal{F}) = \eta_i(N, \mathcal{F})$ for $i \in N \setminus \{1, 2\}$. This power measure satisfies chain efficiency and independence of irrelevant chains. It does not satisfy the non-tail property.

3. Let $E = \{i \in N \mid C_{\mathcal{F}}^N|_i = \emptyset\}$ be the set of those players that are never the last player in a maximal chain. The power measure that is given by $p_i(N, \mathcal{F}) = 0$ for $i \in E$ and $p_i(N, \mathcal{F}) = \frac{|C_{\mathcal{F}}^N|}{|N \setminus E|}$ otherwise satisfies chain efficiency and the non-tail property. It does not satisfy independence of irrelevant chains.

A different axiomatization of the hierarchical measure for regular set systems is obtained by replacing the non-tail property by non-negativity.

Non-negativity For every $(N, \mathcal{F}) \in \mathcal{R}$ it holds that $p_i(N, \mathcal{F}) \geq 0$ for all $i \in N$.

A power measure on $\mathcal{R}$ that satisfies chain efficiency, non-negativity and independence of irrelevant chains also satisfies the non-tail property.

Theorem 4.13 *Let p be a power measure on $\mathcal{R}$ that satisfies chain efficiency, non-negativity and independence of irrelevant chains. Then p also satisfies the non-tail property.*

PROOF
For regular set system $(N, \mathcal{F})$ let $\mu_{\mathcal{F}} = |C_{\mathcal{F}}^N|$ be the total number of maximal chains. By independence of irrelevant chains, for every $i \in N$ there exists a number $c_i \in \mathbb{R}$ such that for every regular set system $(N, \mathcal{F})$ in $\mathcal{R}$ with $C_{\mathcal{F}}^N|_i = \emptyset$, it holds that $p_i(N, \mathcal{F}) = c_i$. For arbitrary $i \in N$ it holds that $C_{\mathcal{F}^i}^N|_j = \emptyset$ for $j \in N \setminus \{i\}$, and therefore $p_j(N, \mathcal{F}^i) = c_j$. By chain efficiency we therefore have $p_i(N, \mathcal{F}^i) = \mu_{\mathcal{F}^i} - \sum_{j \in N \setminus \{i\}} c_j$. Let $E = \{i \in N \mid C_{\mathcal{F}}^N|_i = \emptyset\}$ be the set of those players that are never the last player in a maximal chain. By independence of irrelevant chains, we also have $p_i(N, \mathcal{F}) = p_i(N, \mathcal{F}^i)$ for $i \in N \setminus E$, and thus $p_i(N, \mathcal{F}) = \mu_{\mathcal{F}^i} - \sum_{j \in N \setminus \{i\}} c_j$. By chain efficiency we have $\mu_{\mathcal{F}} = \sum_{j \in N} p_j(N, \mathcal{F}) = \sum_{j \in E} c_j + \sum_{j \in N \setminus E} [\mu_{\mathcal{F}^j} - \sum_{k \in N \setminus \{j\}} c_k] = \sum_{j \in N \setminus E} \mu_{\mathcal{F}^j} - (|N \setminus E| - 1) \sum_{j \in N} c_j$. Since $\mu_{\mathcal{F}} = \sum_{j \in N \setminus E} \mu_{\mathcal{F}^j}$ it holds that $(|N \setminus E| - 1) \sum_{j \in N} c_j = 0$. By non-negativity we therefore have that $c_j = 0$ for $j \in N$. It follows that p satisfies the non-tail property.

□

From Theorems 4.12 and 4.13 we obtain the following corollary.

Corollary 4.14 *A power measure on $\mathcal{R}$ is equal to the hierarchical measure η if and only if it satisfies chain efficiency, non-negativity and independence of irrelevant chains.*

Finally, we remark that the hierarchical measure or any other positive power measure for regular set systems can be used to define a solution for games on a regular set system or *regular games*, introduced by Lange and Grabisch (2009). These regular games are triples $(N, v, \mathcal{F})$ where $v : \mathcal{F} \to \mathbb{R}$ with $v(\emptyset) = 0$, and $\mathcal{F} \subseteq 2^N$ a regular set system. New solutions for these regular games, based on power measures, can be obtained similar to the p-hierarchical solutions for games under precedence constraints.

4.3 The Plurality and Borda measure

Note that in order to calculate the hierarchical measure for regular set systems, it is sufficient to know the set of admissible permutations. Instead of generating admissible permutations from some feasible set system, any set of permutations on the set of players can be used as the set of admissible permutations. The hierarchical measure can be seen to rank players based on these permutations. An example where we encounter the ranking of players from permutations comes from social choice theory.[10] The 'permutations of the players' in this case are the preferences of the voters on the alternatives they can choose from if they all have a different preference ordering. Here the absolute hierarchical measure can be seen as the plurality scoring rule that assigns an alternative a score of 1 for any permutation where it enters last (i.e. where it is the most preferred alternative).

The other way around, we can apply social choice theory to define new power measures for acyclic digraphs. A *social choice situation* is described by a triple (V, N, p) where V is the set of voters, N is the set of alternatives and $p = (p_k)_{k \in V}$ is a preference profile. A preference profile $p = (p_k)_{k \in V}$ consists of a preference relation p_k for every voter $k \in V$, being a weak order on the set of alternatives N. We denote the collection of all social choice situations by $\mathcal{S}$.[11] Two main questions social choice theory tries to answer for each social choice situation are (i) what can be considered the 'socially best' alternatives, and (ii) how to aggregate the individual preferences into one 'social preference relation'. The first question is dealt with by considering so-called social choice functions which assign to every social choice situation a subset of the alternatives that can be considered the 'social choice'. The second is dealt with by considering social welfare functions which assign to every social choice situation a preference relation on the set of alternatives. A specific class of social choice functions and social welfare functions are those based on a *scoring method* being a function $\sigma \colon \mathcal{S} \to \bigcup_{K \subseteq \mathbb{N}} \mathbb{R}^K$ such that $\sigma(V, N, p) \in \mathbb{R}^N$ for every $(V, N, p) \in \mathcal{S}$, which assigns a real number, its score, to every alternative in a social choice situation. As the social choice one can simply take the alternatives with the highest score, and one can define a social preference profile simply by ordering the alternatives in non-increasing

[10] For a survey of various scoring methods we refer to Laslier (1997).

[11] For the application we describe in this section the set of voters must be variable.

order of their score. Given scoring method σ, in this paper we consider the social welfare function that assigns to social choice situation (V, N, p) the weak order $\succeq^\sigma$ given by $i \succeq^\sigma j$ if and only if $\sigma_i(V, N, p) \geq \sigma_j(V, N, P)$, where $i \succeq^\sigma j$ can be interpreted as 'i is at least as good as j'.[12]

Famous ranking methods are based on the plurality score and the Borda score. The *plurality score* of an alternative is simply the number of preference relations in the preference profile where it is ranked highest, i.e. $\sigma_i^{plur}(V, N, p) = \#\{k \in V \mid i \; p_k \; j \text{ for all } j \in N\}$.

Now, given a regular set system, we can consider the set of all admissible permutations (derived from the chains) as a preference profile with the nodes (players) as the alternatives. It does not matter what the set of voters is in this application, as long as there as many voters as maximal chains. Then we can define the *plurality measure* of player $i \in N$ in regular set system $\mathcal{F}$ as $\widehat{\sigma}_i^{plur}(N, \mathcal{F}) = \sigma_i^{plur}(V, N, p^{\mathcal{F}})$, where $(V, N, p^{\mathcal{F}})$ is the social choice situation derived from $\mathcal{F}$ with $V = C_{\mathcal{F}}^N$ and for every $k = (\emptyset, C_1, C_2, \ldots, C_n) \in C_{\mathcal{F}}^N$, the preference profile p_k is given by $i \; p_k \; j$ if and only if $i \in C_l \Rightarrow j \in C_l$. Obviously, since the plurality score just counts the number of preference profiles in which an alternative is ranked highest, applied in this way the plurality score of a player in a regular set system is the number of chains in which it enters last, i.e. its score according to the hierarchical measure. This gives an alternative definition of the hierarchical measure.

Proposition 4.15 *If $\mathcal{F} \subseteq 2^N$ is a regular set system then $\eta(N, \mathcal{F}) = \widehat{\sigma}^{plur}(N, \mathcal{F})$.*

A main disadvantage of the plurality ranking method is that it only looks at the best alternative for every voter, but does not take into account the rest of the preference profile. For example, an alternative that is second best for every voter might be a 'good' social choice, but it will have the lowest plurality score (zero). Alternatively, one can use the *Borda score* which assigns for every voter $|N|-1$ points to the best alternative, $|N|-2$ points to the second best alternative, and so on to zero points for the worst alternative, i.e. $\sigma_i^{Borda}(V, N, p) = \sum_{k \in V} (\#\{j \in N \mid i \; p_k \; j\} - 1)$.[13] The *Borda measure* of player $i \in N$ in regular set system $\mathcal{F} \subseteq 2^N$ is defined as $\widehat{\sigma}^{Borda}(N, \mathcal{F}) = \sigma_i^{Borda}(V, N, p^{\mathcal{F}})$.

Example 4.16 *Consider the digraph (N, D) on $N = \{1, 2, 3, 4, 5, 6\}$ given by $D = \{(1, 2), (2, 3), (3, 4), (4, 5), (6, 5)\}$.*

[12] Although $\succeq^\sigma$ is a preference relation, just as p_k, $k \in V$, we use a different notation to stress that $\succeq^\sigma$ represents the social preference.

[13] Both the plurality score as well as the Borda score are special cases of the class of scoring methods where for a fixed set of r alternatives scoring numbers s_r, $r \in \{1, \ldots, n\}$ with $s_r \geq s_l$ if $r < l$, are given, and for every voter the best alternative gets s_1 points, the second best alternative gets s_2 points, and so on.

The admissible permutations are $(5,4,3,2,1,6), (5,4,3,2,6,1), (5,4,3,6,2,1), (5,4,6,3,2,1)$ *and* $(5,6,4,3,2,1)$. *So, the plurality score of the corresponing regular set system* $\mathcal{F}$ *is* $\widehat{\sigma}^{plur}(N,\mathcal{F}) = (4,0,0,0,0,1)$ *being the absolute hierarchical strength. The Borda score gives* $\widehat{\sigma}^{Borda}(N,\mathcal{F}) = (4,3,2,1,0,5) + (5,3,2,1,0,4) + (5,4,2,1,0,3) + (5,4,3,1,0,2) + (5,4,3,2,0,1) = (24,18,12,6,0,15)$. *So, according to hierarchical strength player 6 is the second ranked player (and in fact is, besides player 1, the only player who has a positive score). But according to the Borda strength, player 2 (who gets zero in the hierarchical strength) has a higher Borda score than player 6.*

Other scoring methods from social choice theory (or multiple criteria decision making) can also be applied.

5 The normalized hierarchical measure for forests and sink forests

The *normalized hierarchical measure* $\overline{\eta}$ assigns to a player i the fraction of permutations in $\Pi_D(N)$ where i enters after the players in $N \setminus \{i\}$.

Definition 5.1 *The normalized hierarchical measure* $\overline{\eta}$ *is the power measure on* $\mathcal{D}$ *given by*

$$\overline{\eta}_i(N,D) = \frac{\eta_i(N,D)}{\sum_{j \in N} \eta_j(N,D)} \text{ for all } i \in N.$$

In this section we consider the normalized hierarchical measure $\overline{\eta}$ for two special classes of digraphs, namely rooted trees and sink trees. A digraph (N,D) is a *rooted tree* if and only if there is an $i_0 \in N$ such that (i) $P_D(i_0) = \emptyset$, (ii) $\widehat{F}_D(i_0) = N \setminus \{i_0\}$, and (iii) $|P_D(i)| = 1$ for all $i \in N \setminus \{i_0\}$. Player i_0 is called the *root* of the tree. A digraph (N,D) is a *sink tree* if the digraph (N,D^-) with $D^- = \{(i,j) \in N \times N : (j,i) \in D\}$ is a rooted tree. In other words, a digraph (N,D) is a sink tree if and only if there is an $i_s \in N$ such that (i) $F_D(i_s) = \emptyset$, (ii) $\widehat{P}_D(i_s) = N \setminus \{i_s\}$, and (iii) $|F_D(i)| = 1$ for all $i \in N \setminus \{i_s\}$. Player i_s is called the *sink* of the tree. A digraph (N,D) is a line graph, when it is both a rooted tree and a sink tree. Rooted trees and sink trees are often encountered in the economic and OR literature, for example, in the literature on cooperative river water allocation games initiated by Ambec and Sprumont (2002) and in polluted river problems in Dong, Ni and Wang (2012).

A *component* in a directed graph is a set of players that is maximally connected in the underlying undirected graph. Two players $i,j \in N$ are connected in $(N,D) \in \mathcal{D}$ if there exists a sequence of players $(i_1,\ldots,i_m)$ such that $i_1 = i,\ i_m = j$ and $\{(i_k,i_{k+1}),(i_{k+1},i_k)\} \cap$

$D \neq \emptyset$ for all $k = 1, \ldots, m-1$. A set of players $S \subseteq N$ is connected in $(N, D) \in \mathcal{D}$ if every two players $i, j \in S$ are connected in $(S, D(S))$. A subset K of N is a *component* in (N, D) if the digraph $(K, D(K))$ is maximally connected, i.e., $(K, D(K))$ is connected and, for every $j \in N \setminus K$, the digraph $(K \cup \{j\}, D(K \cup \{j\}))$ is not connected. We denote the set of all components in (N, D) by $C_D(N)$.

5.1 An axiomatization of the normalized hierarchical measure for forests

Next, we consider the classes of digraphs where every component is a rooted tree, also known as a forest. We denote the set of forest digraphs by $\mathcal{D}_R$. From the axioms discussed in subsection 3.5, the normalized hierarchical measure satisfies the non-top property and 1-normalization. It satisfies an even stronger version of 1-normalization that normalizes the total power of all players to one for every digraph. Note that this property also holds on the class of all acyclic digraphs, although we will use it only on forests and sink forests.

Normalization For every $(N, D) \in \mathcal{D}_R$, it holds that $\sum_{i \in N} p_i(N, D) = 1$.

The normalized hierarchical measure satisfies the even stronger property that the cumulative power of the players in any one component is equal to the fraction of players in that component, i.e. when a component contains $|K|$ players this cumulative power is equal to $\frac{|K|}{|N|}$.

Component normalization For every $(N, D) \in \mathcal{D}_R$, if $K \in C_D(N)$, we have

$\sum_{i \in K} p_i(N, D) = \frac{|K|}{|N|}$.

Note that component normalization implies normalization. In the axiomatization of the normalized hierarchical measure that follows, we will use component normalization.

Before we continue with the axiomatization we state the following propositions.

Proposition 5.2 *Consider a digraph $(N, D) \in \mathcal{D}$ and let $K \in C_D(N)$. Now let permutation $\pi' \in \Pi_{D(K)}(K)$ and permutation $\pi'' \in \Pi_{D(N \setminus K)}(N \setminus K)$. The total number of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N \setminus K} = \pi''$ is given by $\binom{|N|}{|K|}$.*

PROOF

Consider the vector $x = (x_1, \ldots, x_{|N \setminus K|+1}) \in \mathbb{N}^{|N \setminus K|+1}$, where x_1 is the number of players in K that precede player π''_1 in permutation π, $x_{|N \setminus K|+1}$ is the number of players in K that are preceded by player $\pi''_{|N \setminus K|}$ in permutation π and finally (if $|N \setminus K| > 1$) for $1 < i < |N \setminus K| + 1$, x_i is the number of players in K that are preceded by player π''_{i-1} in permutation π, but precede player π''_i. The number of permutations π satisfying $\pi_K = \pi'$ and $\pi_{N\setminus K} = \pi''$ is now equal to the number of solutions to $x_1 + \ldots + x_{|N\setminus K|+1} = |K|$, where $0 \leq x_i \leq |K|$ for $i \in \{1, \ldots, |N \setminus K| + 1\}$. From combinatorics we obtain that this number is equal to $\binom{|N|}{|K|}$.

□

Next we consider the fraction of these permutations, where a player from K enters last.

Proposition 5.3 *Consider a digraph $(N, D) \in \mathcal{D}$ and let $K \in C_D(N)$. Now let permutation $\pi' \in \Pi_{D(K)}(K)$ and permutation $\pi'' \in \Pi_{D(N\setminus K)}(N \setminus K)$. The fraction of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N\setminus K} = \pi''$, and where $\pi_{|N|} \in K$ is given by $\frac{|K|}{|N|}$.*

Proof

From Proposition 5.2 we obtain that the total number of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N\setminus K} = \pi''$ is given by $\binom{|N|}{|K|}$. To calculate the number of permutations with a player from K at the end, we can use a combinatorial argument similar to that from the proof of Proposition 5.2. Since the position of one of the players in K is now fixed (after any of the players in $N \setminus K$), we only have to know the number of ways to position the other $|K| - 1$ players in K relative to those in $N \setminus K$ to fully determine $\pi(i)$ for any player $i \in N$. Following the proof of Proposition 5.2, the total number of permutations such that a player from K is at the end is therefore given by the number of solutions to $x_1 + \ldots + x_{|N\setminus K|+1} = |K| - 1$, where $0 \leq x_i \leq |K| - 1$ for $i \in \{1, \ldots, |N \setminus K| + 1\}$. This number is equal to $\binom{|N|-1}{|K|-1}$. Therefore the fraction of permutations $\pi \in \Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N\setminus K} = \pi''$, and where a player from K enters last is given by $\frac{\binom{|N|-1}{|K|-1}}{\binom{|N|}{|K|}} = \frac{|K|}{|N|}$

□

For any specific pair of permutations $\pi' \in \Pi_{D(K)}(K)$ and $\pi'' \in \Pi_{D(N\setminus K)}(N \setminus K)$, by Proposition 5.3 we know that the fraction of permutations in $\Pi_D(N)$ such that $\pi_K = \pi'$ and $\pi_{N\setminus K} = \pi''$ and where a player from component K enters last is given by $\frac{|K|}{|N|}$. At the same time by Proposition 3.18, we know that for any permutation $\pi \in \Pi_D(N)$, there exist $\pi' \in \Pi_{D(K)}(K)$ and $\pi'' \in \Pi_{D(N\setminus K)}(N \setminus K)$ such that $\pi_K = \pi'$ and $\pi_{N\setminus K} = \pi''$. The following proposition therefore follows immediately from these Propositions.

Proposition 5.4 *Consider a digraph $(N, D) \in \mathcal{D}$ and let $K \in C_D(N)$. The number of permutations $\pi \in \Pi_D(N)$, such that $\pi \in \Pi^i_D(N)$ for some $i \in K$, is given by $\frac{|K|}{|N|}$.*

Next we provide an axiomatization of the normalized hierarchical measure for forest digraphs. The non-top property used in this axiomatization is adapted in a straightforward way from the class of all acyclic digraphs to the class of forests $\mathcal{D}_R$, meaning that players with predecessors are assigned 0 power.

Theorem 5.5 *A power measure on $\mathcal{D}_R$ is equal to the normalized hierarchical measure $\overline{\eta}$ if and only if it satisfies component normalization and the non-top property.*

Proof

The normalized hierarchical measure $\overline{\eta}$ satisfying component normalization follows straightforwardly from Proposition 5.4. The non-top property follows immediately from the absolute hierarchical measure η satisfying the non-top property.

The proof of uniqueness is given as follows. Let p be a power measure satisfying the axioms. In a forest the top players in $TOP(N, D)$ are the roots of the components. The non-top property implies that $p_i(N, D) = 0$ for all $i \in N \setminus TOP(N, D)$. Since every component has a unique root, component normalization then uniquely determines the values for the players in $TOP(N, D)$. □

We show logical independence by the following alternative power measures.

1. The absolute hierarchical measure η satisfies the non-top property on $\mathcal{D}_R$. It does not satisfy component normalization on $\mathcal{D}_R$.
2. The power measure that is given by $p_i(N, D) = \frac{1}{|N|}$ for all $(N, D) \in \mathcal{D}$ and $i \in N$ satisfies component normalization on $\mathcal{D}_R$. It does not satisfy the non-top property on $\mathcal{D}_R$.

In the above axiomatization we used component normalization instead of the weaker normalization. Another weak version of component normalization requires only that the cumulative power of any one component is assigned proportionally to the number of players in that component.

Component comparability For every pair of components $K, K' \in C_D(N)$ of $(N, D) \in \mathcal{D}_R$, we have $\frac{\sum_{i \in K} p_i(N,D)}{\sum_{i \in K'} p_i(N,D)} = \frac{|K|}{|K'|}$.

It is straightforward to see that normalization and component comparability together are equivalent to component normalization.

Proposition 5.6 *A power measure satisfies component normalization if and only if it satisfies normalization and component comparability.*

Therefore, in the above axiomatization (and also in the one from the next subsection) it is possible to replace component normalization by normalization and component comparability.

5.2 An axiomatization of the normalized hierarchical measure for sink forests

Next, we consider the normalized hierarchical measure on the class of sink forests, being digraphs where every component is a sink tree, also known as a sink forest. We denote the set of sink forests by $\mathcal{D}_S$.

The axioms of component normalization and the non-top property from Theorem 5.5 are adapted in a straightforward way to the class of sink forests $\mathcal{D}_S$, meaning that the cumulative power of the players in any one component is equal to the fraction of players in that component, respectively, that players with predecessors are assigned zero power. Together these axioms are not sufficient to axiomatize the normalized hierarchical measure on the class of sink forests. Therefore, in addition we consider equal dependence on bottom players which states that if within any component deleting the player who has no successors (and thus is a subordinate of all other players in this component) from the digraph does not change the power ratios of the top players in that component (who now might belong to different components). We will refer to a player without sucessors as a bottom player.

Equal dependence on bottom players For every $(N, D) \in \mathcal{D}_S$, $K \in C_D(N)$ such that $|TOP(K, D(K))| \geq 2$, $i, j \in TOP(K, D(K))$, $i \neq j$, and $h \in K$ such that $F_D(h) = \emptyset$ (and therefore $\widehat{P}_D(h) = K \setminus \{h\}$), it holds that $\frac{p_i(N,D)}{p_j(N,D)} = \frac{p_i(N\setminus\{h\},D_{-h})}{p_j(N\setminus\{h\},D_{-h})}$.

Note that $(N\setminus\{h\}, D_{-h})$ is a sink forest if (N, D) is a sink forest and $\widehat{P}_D(h) = K\setminus\{h\}$ for some $K \in C_D(N)$.

Theorem 5.7 *A power measure on $\mathcal{D}_S$ is equal to the normalized hierarchical measure $\overline{\eta}$ if and only if it satisfies component normalization, the non-top property and equal dependence on bottom players.*

Proof

The normalized hierarchical measure $\overline{\eta}$ satisfying component normalization follows from Proposition 5.4. The normalized hierarchical measure $\overline{\eta}$ satisfying the non-top property follows immediately from the absolute hierarchical measure η satisfying the non-top property.

To show that the normalized hierarchical measure $\overline{\eta}$ satisfies equal dependence on bottom players. Let $K \in C_D(N)$, $i, j \in TOP(K, D(K))$, $i \neq j$, and $h \in K$ such that $F_D(h) = \emptyset$ (and so $\widehat{P}_D(h) = K \setminus \{h\}$). Equal dependence on bottom players is satisfied if the ratio of the admissible permutations in $(N \setminus \{h\}, D_{-h})$ where i is the last to enter, and those where j is the last to enter, is the same as in (N, D).

Obviously, if K is a component in (N, D), $i, j \in TOP(K, D(K))$ and $h \in K$ such that $F_D(h) = \emptyset$ and $\widehat{P}_D(h) = K \setminus \{h\}$, then every permutation admissible in D has h entering before any other player in K. But then, the permutations admissible in $(N \setminus \{h\}, D_{-h})$ are exactly those admissible in (N, D) but without player h (and all players entering after h move one position to the front). Hence, the ratios of the normalized hierarchical strength within each component does not change, i.e. $\frac{f_i(N,D)}{f_j(N,D)} = \frac{f_i(N \setminus \{h\}, D_{-h})}{f_j(N \setminus \{h\}, D_{-h})}$, showing that $\overline{h}$ satisfies equal dependence of bottom players.

The proof of uniqueness is given as follows. Let p be a positive power measure satisfying the axioms, and let K be a component in (N, D). We perform induction on $|K|$. If $|K| = 1$ then $p_i(\{i\}, D) = \frac{1}{|N|}$ by component normalization. Proceeding by induction, assume that $p_i(N, D')$ is uniquely determined for all $i \in K'$ whenever $|K'| < |K|$, with $p_i(N, D) > 0$ if $P_D(i) = \emptyset$. By component normalization, it holds that $\sum_{i \in K} p_i(N, D) = \frac{|K|}{|N|}$. If $P_D(i) \neq \emptyset$ then $p_i(N, D) = 0$ follows from the non-top property. Therefore, $\sum_{i \in TOP(K,D(K))} p_i(N, D) = \frac{|K|}{|N|}$. By equal dependence on bottom players, $\frac{p_i(N,D)}{p_j(N,D)} = \frac{p_i(N \setminus \{h\}, D_{-h})}{p_j(N \setminus \{h\}, D_{-h})}$, for all $i, j \in TOP(K, D(K))$ and $h \in K$ such that $F_D(h) = \emptyset$ (and therefore $\widehat{P}_D(h) = K \setminus \{h\}$). (Note that every component in a sink forest has exactly one such a player h.) Since $p_i(N \setminus \{h\}, D_{-h}) p_j(N \setminus \{h\}, D_{-h}) > 0$ by the induction hypothesis, this implies that the values $p_i(N, D)$, $i \in TOP(K, D(K))$, are uniquely determined. $\square$

We show logical independence by the following alternative power measures.

1. The absolute hierarchical measure η satisfies the non-top property and equal dependence on bottom players on $\mathcal{D}_S$. It does not satisfy component normalization on $\mathcal{D}_S$.

2. The power measure that is given by $p_i(N, D) = \frac{1}{|N|}$ for all $(N, D) \in \mathcal{D}$ and $i \in N$ satisfies component normalization and equal dependence on bottom players on $\mathcal{D}_S$. It does not satisfy the non-top property on $\mathcal{D}_S$.

3. Let K_i be the component in (N, D) containing player i. Let ω assign to every digraph the exogenously given vector of weights $\omega(N, D) \in \mathbb{R}^N_{++}$, where $\omega_i(N, D) > 0$, $i \in N$. The weighted hierarchical measure h^ω given by $\eta_i^\omega(N, D) = \frac{\omega_i(N,D)}{\sum_{j \in TOP(K_i, D(K_i))} \omega_j(N,D)} \sum_{j \in TOP(K_i, D(K_i))} \overline{\eta}_j(N, D)$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies component normalization and the non-top property on $\mathcal{D}_S$. It does not satisfy equal dependence on bottom players on $\mathcal{D}_S$.

5.2.1 Alternative normalizations and power measures for sink forests

In the literature on power measures it has been shown that just applying a different normalization can have an important impact.[14] In this subsection we consider two alternative versions of component normalization, and show that together with the non-top property and equal dependence on bottom players, these characterize other power measures for sink forests. The first alternative to component normalization requires that the cumulative power of the players in a component is the same for each component.

Component normalization 2 Let $(N, D) \in \mathcal{D}_S$. For $K \in C_D(N)$, it holds that $\sum_{i \in K} p_i(N, D) = \frac{1}{|C_D(N)|}$.

The second alternative requires that the cumulative power of the players in a component is equal to the share of top players in that component.

Component normalization 3 Let $(N, D) \in \mathcal{D}_S$. For $K \in C_D(N)$, it holds that $\sum_{i \in K} p_i(N, D) = \frac{|TOP(K, D(K))|}{|TOP(N, D)|}$.

Next, we show what power measures for sink forests are characterized by replacing component normalization in Theorem 5.7 by one of the above two alternatives.[15]

Theorem 5.8 *(i) A power measure for sink forests satisfies component normalization 2, the non-top property and equal dependence on bottom players if and only if it is the power measure p^2 given by*

$$p^2(N, D) = \begin{cases} \frac{1}{|C_D(N)| \prod_{j \in \hat{F}_D(i)} |P_D(j)|} & \text{if } P_D(i) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

(ii) A power measure for sink forests satisfies component normalization 3, the non-top property and equal dependence on bottom players if and only if it is the power measure p^3 given by

$$p^3(N, D) = \begin{cases} \frac{1}{|TOP(N, D)|} & \text{if } P_D(i) = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

[14] For example, van den Brink and Gilles (2000) provide axiomatizations of the outdegree (score) and β-measures that differ only in the normalization that is used, implying that a different normalization can lead to a different ranking of, for example, teams in a sports competition. In van den Brink, Rusinowska and Steffen (2012) axiomatizations of a power and satisfaction score in a (voting) model with opinion leaders are given differing only in the normalization that is applied.

[15] We can also define the axioms of component normalization 2 and component normalization 3 on the class of forest digraphs. It is straightforward to show that together with the non-top property both axioms characterize the solution that assigns a power of $p_i(N, D) = \frac{1}{|C_D(N)|} = \frac{|TOP(K, D(K))|}{|TOP(N, D)|} = \frac{1}{|TOP(N, D)|}$ if $i \in N$ is a top player and $p_i(N, D) = 0$ otherwise.

Proof

It is straightforward to verify that the power measures p^2 and p^3 satisfy the corresponding axioms. Uniqueness follows as in the proof of Theorem 5.7, replacing component normalization by component normalization 2 and component normalization 3, respectively. □

Together with the non-top property and equal dependence on bottom players, component normalization 2 yields the power measure p^2 where the power ratio between two top players equals the product of the number of predecessors of each of their subordinates. Considering a 'flow' argument, this power measure can be described as follows. Suppose you start a random walk at the sink and walk through the network along the arcs to one of the top players. At every non-top player you select one of the arcs to its predecessors with equal probability and continue your walk along this arc. In that case p^2 describes the probability of ending up at each top player.[16]

Together with the non-top property and equal dependence on bottom players, component normalization 3 yields the power measure p^3 that distributes the power equally over the top players in the digraph.

Example 5.9 *Consider the sink tree* (N, D) *with* $N = \{1, 2, 3, 4, 5\}$ *and* $D = \{(1, 3), (2, 3), (3, 5), (4, 5)\}$. *Then the normalized hierarchical measure* $\overline{\eta}(N, D)$, $p^2(N, D)$ *and* $p^3(N, D)$, *respectively, are given by*

$$\overline{\eta}(N, D) = (\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{4}, 0),$$

$$p^2(N, D) = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}, 0), \textit{and}$$

$$p^3(N, D) = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0).$$

5.2.2 An application of power measures for sink forests to river games

We end this section by applying the hierarchical strength and the other power measures discussed in this section to the river games mentioned before. These games are introduced by Ambec and Sprumont (2002) for rivers with a single spring and a single source. They consider river water allocation problems (N, e, b) where agents are located along a single-stream river from upstream to downstream. There is a nonnegative water inflow $e_i \geq 0$ at the territory of every agent $i \in N$. Every agent is assumed to have quasi-linear preferences

[16] Alternatively, suppose that a unit of water flows from the sink to the top players in such a way that at each non-top player the water stream splits into multiple streams of equal amounts, for every arc to a predecessor. Then p^2 describes the expected amount of water that arrives at the top players.

over river water and money, where the benefit of consuming an amount of water is given by a differentiable, strictly increasing, and strictly concave benefit function $b_i \colon \mathbb{R}_+ \to \mathbb{R}$, where $b_i(x_i)$ is the benefit of agent i of consuming an amount x_i of water. An allocation of the river water among the agents is efficient when it maximizes the total sum of benefits. Every water allocation and transfer schedule yields a welfare distribution, where the utility of an agent is equal to its benefit from water consumption plus its monetary transfer, which can be negative. Ambec and Sprumont (2002) derive a cooperative river game where the worth of every connected coalition equals the welfare to these agents when they optimally allocate (i.e. maximizing the sum of their individual benefits) the water inflow in their own territories among each other (under the condition that water can be sent from upstream to downstream agents but not the other way around). As solution they suggest and axiomatize their *downstream incremental solution* which requires the (unique) optimal allocation of water over the agents and monetary transfers such that the resulting welfare distribution is given by the marginal vector of the river game where agents enter from upstream to downstream. As noted by van den Brink, van der Laan and Vasil'ev (2007) this means that all the surplus of cooperation of a connected coalition is allocated to the downstream agent in the coalition. As alternative, van den Brink, van der Laan and Vasil'ev (2007) and Ambec and Ehlers (2008) proposed the *upstream incremental solution* which requires the (unique) optimal allocation of water over the agents and monetary transfers such that the resulting welfare distribution is given by the marginal vector of the river game where agents enter from downstream to upstream, and thus the surplus of cooperation is allocated to the upstream agents.

Khmelnitskaya (2010) and van den Brink, van der Laan and Moes (2012) consider the more general river structures where there can be mutliple springs, but still a single source.[17] Whereas it is straightforward to generalize the downstream incremental solution to multiple spring rivers (since the marginal vector corresponding to any permutation where upstream agents enter before their downstream neighbors is the same), it is less obvious how to generalize the upstream incremental solution to multiple spring rivers. This section offers three possibilities by considering the weighted hierarchical solution corresponding to the hierarchical strength and the power measures p^2 and p^3.

In van den Brink, van der Laan and Moes (2012) a class of solutions for mutliple spring rivers that contains the downstream and the three upstream incremental solutions considered here is axiomatized. This is the class of weighted hierarchical solutions, being convex combinations of the so-called hierarchical outcomes introduced by Demange (2004), see also Béal, Rémila and Solal (2010). To every player $i \in N$ is assigned a *hierarchical*

[17]Khmelnitskaya (2010) also considers rivers with a single spring but multiple sinks, whereas van den Brink, van der Laan and Moes (2012), like Ambec and Ehlers (2008), allow for more general benefit functions where the agents can be satiable.

outcome being the marginal vector where agent h enters before agent j if j is on the (unique) path from i to h. All marginal vectors corresponding to a permutation that satisfies this condition with respect to player i are the same, and it is called the hierarchical outcome corresponding to player i. A solution is a weighted hierarchical solution if there exist weights $\alpha_i \geq 0,\ i \in N$, satisfying $\sum_{i\in N} \alpha_i = 1$, such that for every river problem (N, e, b), it assigns the corresponding convex combination of the hierarchical outcomes. Clearly, we obtain the downstream incremental solution by giving weight one to the (unique) most downstream agent and weight zero to all other agents. By taking any of the weight systems given by the hierarchical strength or power measures p^2 or p^3 of this section where, by the non-top property only the springs can have a nonzero weight, we obtain generalizations of the upstream incremental solution. Although a theoretical analysis is beyond the scope of this paper, we illustrate these three upstream incremental type solutions with an example.[18]

Example 5.10 *Consider the river problem (N, e, b, D) with $N = \{1, 2, 3, 4, 5\}$ digraph D as given in Example 5.9, $e_1 = e_2 = e_4 = 1,\ e_3 = e_5 = 0,\ b_5(x_5) = \sqrt{x_5}$ and $b_i(x_i) = 0$ for all $i \in \{1, 2, 3, 4\}$.[19] The associated river game is the game (N, v) given by $v(\{4, 5\}) = v(\{1, 3, 5\}) = v(\{2, 3, 5\}) = v(\{1, 4, 5\}) = v(\{2, 4, 5\}) = v(\{3, 4, 5\}) = v(\{1, 2, 4, 5\}) = 1,\ v(\{1, 2, 3, 5\}) = v(\{1, 3, 4, 5\}) = v(\{2, 3, 4, 5\}) = \sqrt{2},\ v(\{1, 2, 3, 4, 5\}) = \sqrt{3}$ and $v(S) = 0$ otherwise. The three hierarchical outcomes t^i, where $i \in TOP(N, D)$ is a top player, are given by*

$$t^1(N, v, D) = \left(\sqrt{3} - \sqrt{2}, 0, \sqrt{2} - 1, 0, 1\right),$$

$$t^2(N, v, D) = \left(0, \sqrt{3} - \sqrt{2}, \sqrt{2} - 1, 0, 1\right) \text{ and}$$

$$t^4(N, v, D) = \left(0, 0, 0, \sqrt{3} - \sqrt{2}, \sqrt{2}\right).$$

By Example 5.9, we have $\overline{\eta}(N, D) = (\frac{3}{8}, \frac{3}{8}, 0, \frac{1}{4}, 0)$, $p^2(N, D) = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}, 0)$ and $p^3(N, D) = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0)$. This yields the following weighted hierarchical outcomes according to $\overline{\eta}, p^2$ and p^3:

$$\begin{aligned} w^{\overline{\eta}}(N, v, D) &= \frac{3}{8}t^1(N, v, D) + \frac{3}{8}t^2(N, v, D) + \frac{1}{4}t^4(N, v, D) \\ &= \left(\frac{3\left(\sqrt{3} - \sqrt{2}\right)}{8}, \frac{3\left(\sqrt{3} - \sqrt{2}\right)}{8}, \frac{3\left(\sqrt{2} - 1\right)}{4}, \frac{\left(\sqrt{3} - \sqrt{2}\right)}{4}, \frac{\left(3 + \sqrt{2}\right)}{4}\right), \end{aligned}$$

[18] We refer to the literature for formal definitions of hierarchical outcomes and river games. In the example we give the associated games to illustrate the use of power measures to define solutions for river problems.

[19] Although these benefit functions do not satisfy the assumptions of Ambec and Sprumont (2002), we use them for illustration.

$$\begin{aligned} w^{p^2}(N,v,D) &= \frac{1}{4}t^1(N,v,D)+\frac{1}{4}t^2(N,v,D)+\frac{1}{2}t^4(N,v,D) \\ &= \left(\frac{(\sqrt{3}-\sqrt{2})}{4},\frac{(\sqrt{3}-\sqrt{2})}{4},\frac{(\sqrt{2}-1)}{2},\frac{(\sqrt{3}-\sqrt{2})}{2},\frac{(1+\sqrt{2})}{2}\right) \text{ and} \end{aligned}$$

$$\begin{aligned} w^{p^3}(N,v,D) &= \frac{1}{3}(t^1(N,v,D)+t^2(N,v,D)+t^4(N,v,D)) \\ &= \left(\frac{(\sqrt{3}-\sqrt{2})}{3},\frac{(\sqrt{3}-\sqrt{2})}{3},\frac{2\,(\sqrt{2}-1)}{3},\frac{(\sqrt{3}-\sqrt{2})}{3},\frac{(2+\sqrt{2})}{3}\right). \end{aligned}$$

□

Instead of applying the weights to define such upstream incremental type solutions, we could also apply the power measures as weights to define Harsanyi solutions (see Vasil'ev and van der Laan (2001), also called sharing values by Derks, Haller and Peters (2000)), including the weighted Shapley values (see Shapley (1953b) and Kalai and Samet (1987)) for river games.[20]

Example 5.11 *Consider the river problem from Example 5.10. The dividends of the river game are given by:* $\Delta_v(\{4,5\}) = \Delta_v(\{1,3,5\}) = \Delta_v(\{2,3,5\}) = 1$, $\Delta_v(\{1,2,3,5\}) = \Delta_v(\{1,3,4,5\}) = \Delta_v(\{2,3,4,5\}) = \sqrt{2}-2$, $\Delta_v(\{1,2,3,4,5\}) = \sqrt{3}-3\sqrt{2}+3$ *and* $\Delta_v(S) = 0$ *otherwise.*

The Harsanyi solution (or sharing value) associated to a sharing system $\omega = [\omega^T]^{T\in 2^N\setminus\{\emptyset\}}$ *such that for every* $T \in 2^N \setminus \{\emptyset\}$, $\omega^T \in \mathbb{R}^T_+$, $\sum_{i\in T}\omega^T_i = 1$, *allocates the dividend of any coalition proportional to the weights of the players in the coalition. Applying the three power measures discussed in this section, we obtain the following welfare distributions.*

Allocating dividends according to the normalized hierarchical measure yields welfare distribution $\widetilde{H}^{\overline{\eta}}$ *given by*

$$\begin{aligned} \widetilde{H}^{\overline{\eta}}_1(N,v,D) = \widetilde{H}^{\overline{\eta}}_2(N,v,D) &= 1+\frac{1}{2}\left(\sqrt{2}-2\right)+\frac{2}{3}\left(\sqrt{2}-2\right)+\frac{3}{8}\left(\sqrt{3}-3\sqrt{2}+3\right) \\ &= \frac{3}{8}\sqrt{3}+\frac{1}{24}\sqrt{2}-\frac{5}{24} \end{aligned}$$

and

$$\begin{aligned} \widetilde{H}^{\overline{\eta}}_4(N,v,D) &= 1+\frac{1}{3}\left(\sqrt{2}-2\right)+\frac{1}{3}\left(\sqrt{2}-2\right)+\frac{1}{4}\left(\sqrt{3}-3\sqrt{2}+3\right) \\ &= \frac{1}{4}\sqrt{3}-\frac{1}{12}\sqrt{2}+\frac{5}{12}. \end{aligned}$$

[20] Similar as with the previous example, we refer to the literature for the definitions of weighted Shapley values and in the example we just illustrate the application of power measures to define weighted Shapley values for river problems.

For those coalitions S, such that player 3 and/or player 5 is a top player of $(S, D(S))$, it holds that $\Delta_v(S) = 0$. For coalitions S such that $\Delta_v(S) \neq 0$, it holds that players 3 and 5 are not top players of $(S, D(S))$. Therefore their power is zero and these players obtain zero payoff. We have $\widetilde{H}_3^{\overline{\eta}}(N, v, D) = \widetilde{H}_5^{\overline{\eta}}(N, v, D) = 0$ (the same holds for power measures p^2 and p^3).

For power measure p^2 we obtain welfare distribution

$$\begin{aligned}\widetilde{H}_1^{p^2}(N, v, D) = \widetilde{H}_2^{p^2}(N, v, D) &= 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{4}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\ &= \frac{1}{4}\sqrt{3} + \frac{1}{4}\sqrt{2} - \frac{1}{4}\end{aligned}$$

and

$$\begin{aligned}\widetilde{H}_4^{p^2}(N, v, D) &= 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{3} - 3\sqrt{2} + 3\right) \\ &= \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2} + \frac{1}{2}.\end{aligned}$$

with $\widetilde{H}_3^{p^2}(N, v, D) = \widetilde{H}_5^{p^2}(N, v, D) = 0$.

Finally, power measure p^3 always assigns equal power to the springs, so we obtain the following welfare distribution

$$\widetilde{H}_1^{p^3}(N, v, D) = \widetilde{H}_2^{p^3}(N, v, D) = \widetilde{H}_4^{p^3}(N, v, D) =$$

$$= 1 + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{2}\left(\sqrt{2} - 2\right) + \frac{1}{3}\left(\sqrt{3} - 3\sqrt{2} + 3\right) = \frac{1}{3}\sqrt{3}$$

and $\widetilde{H}_3^{p^3}(N, v, D) = \widetilde{H}_5^{p^3}(N, v, D) = 0$.

6 Concluding remarks

We showed that the precedence Shapley value for games under precedence constraints of Faigle and Kern (1992) does not satisfy irrelevant player independence. We introduced a class of solutions for games under precedence constraints that do satisfy irrelevant player independence. The solutions in this class allocate dividend according to power measures for acyclic digraphs. We introduced the hierarchical measure as a power measure for acyclic digraphs inspired by the hierarchical strength. We analyzed this measure from an axiomatic point of view. We also generalized the hierarchical measure to regular set systems. This wider class of structures allows for more applications. Score rankings are considered in defining new solutions for games under precedence constraints. We also defined the normalized hierarchical measure and axiomatized it on the class of forests of rooted trees and forests of sink trees. On forests of sink trees we obtained different power measures, by using alternative versions of component normalization. Finally, we considered the application of these measures on forests of sink trees to water allocation problems.

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Appendix: Normalization and component comparability

In this appendix we show that in the axiomatizations in Section 4 of the normalized hierarchical measure on forests and sink forests it is not possible to weaken component normalization to either normalization or component comparability.

First, we show logical independence of the axioms of Theorem 5.5 if we replace component normalization by normalization and component comparability.

1. The hierarchical measure η satisfies the non-top property and component comparability on $\mathcal{D}_R$. It does not satisfy normalization on $\mathcal{D}_R$.

2. The power measure that is given by $p_i(N, D) = \frac{1}{|C_D(N)|}$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies normalization and the non-top property on $\mathcal{D}_R$. It does not satisfy component comparability on $\mathcal{D}_R$.

3. The power measure that is given by $p_i(N, D) = \frac{1}{|N|}$ for all $(N, D) \in \mathcal{D}$ and $i \in N$ satisfies normalization and component comparability on $\mathcal{D}_R$. It does not satisfy the non-top property on $\mathcal{D}_R$.

Second, we show logical independence of the axioms of Theorem 5.7 if we replace component normalization by normalization and component comparability.

1. The hierarchical measure η satisfies component comparability, the non-top property and equal dependence on bottom players on $\mathcal{D}_S$. It does not satisfy normalization on $\mathcal{D}_S$.

2. The power measure that is given by $p_i(N, D) = \frac{1}{|TOP(K,D(K))| \cdot |C_D(N)|}$ if $P_D(i) = \emptyset$, and $p_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies normalization, the non-top property and equal dependence on bottom players on $\mathcal{D}_S$. It does not satisfy component comparability on $\mathcal{D}_S$.

3. The power measure that is given by $p_i(N, D) = \frac{1}{|N|}$ for all $(N, D) \in \mathcal{D}_S$ and $i \in N$ satisfies normalization, component comparability and equal dependence on bottom playerson $\mathcal{D}_S$. It does not satisfy the non-top property on $\mathcal{D}_S$.

4. Let K_i be the component in (N, D) containing player i. Let ω assign to every digraph the exogenously given vector of weights $\omega(N, D) \in \mathbb{R}^N_{++}$, where $\omega_i(N, D) > 0$, $i \in N$. The weighted hierarchical measure η^ω given by

   $\eta^\omega_i(N, D) = \frac{\omega_i(N,D)}{\sum_{j \in TOP(K_i, D(K_i))} \omega_j} \sum_{j \in TOP(K_i, D(K_i))} \overline{\eta}_j(N, D)$ if $P_D(i) = \emptyset$, and $\eta^\omega_i(N, D) = 0$ if $P_D(i) \neq \emptyset$, satisfies normalization, component comparability and the non-top property on $\mathcal{D}_S$. It does not satisfy equal dependence on bottom players on $\mathcal{D}_S$.