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Abstract

We introduce heterogeneous expectations in a standard housing market model linking housing rental levels to fundamental buying prices. Using quarterly data we estimate the model parameters for eight different countries, US, UK, NL, JP, CH, ES, SE and BE. We find that the data support heterogeneity in expectations, with temporary endogenous switching between fundamental mean-reverting and trend-following chartists beliefs based on their relative performance. For all countries we identify temporary house price bubbles, amplified by trend extrapolation, and crashes reinforced by fundamentalists. The qualitative predictions of such non-linear models are very different from standard linear benchmarks, with important policy implications. The fundamental price becomes unstable, e.g. when the interest rate is set too low or mortgage tax deductions too high, giving rise to multiple non-fundamental equilibria and/or global instability.

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1 Introduction

Do house prices exhibit expectations-driven temporary bubbles and crashes? The aim of this paper is to obtain insights into the expectations feedback and large price fluctuations in housing markets. This is important, since housing market bubbles are considered to be leading indicators of financial instability and crises (Leamer, 2007). Financial crises and recessions are often preceded by a decline in housing investments (Reinhart and Rogoff, 2009). For this reason, a good understanding of house price dynamics and the booms and busts they can generate are crucial for central banks. Unfortunately, a good understanding of the housing market and the business cycle are still lacking. Even state-of-the art dynamic stochastic general equilibrium (DSGE) models with housing consumption and production (Davis and Heathcote, 2005) are unable to match house price fluctuations and do not capture the phenomenon that house investment leads GDP. Since the 1980s, e.g., Shiller has repeatedly warned that changes in fundamentals cannot account for the large swings in home prices observed empirically; see e.g. Case and Shiller (2003).

Traditional housing market models enable one to derive fundamental relations between housing prices and rents. But fundamental factors apparently are insufficient to explain the observed large booms and busts in house prices. What could be a reasonable alternative model for house price dynamics? Glaeser and Nathanson (2014) argue that many non-rational explanations for housing bubbles exist, but the most promising theories emphasize some form of trend-chasing, which in turn reflects boundedly rational learning. In this paper we construct a model where house price fluctuations are partly driven by almost self-fulfilling expectations (animal spirits) of boundedly rational heterogeneous agents. During a housing boom expectations regarding future housing prices are typically optimistic, while they are pessimistic during a bust. There is both anecdotal as well as empirical evidence that house price expectations were unrealistically high in the US during the housing boom (Case *et al.*, 2012; Cheng *et al.*, 2014). This is in line with the observation by Frankel and Froot (1987) that expectations in asset prices are heterogeneous and probably boundedly rational, which ignited the emergence of behavioral finance and heterogeneous agent models (Hommes, 2013). Surveys of stock market forecasts, e.g. in (Shiller, 1987; Shiller, 2000) and (Vissing-Jorgensen, 2004), show that heterogeneity in price expectations changes over time. Shiller (2007) and Piazzesi and Schneider (2009) obtained similar results in surveys among home owners and showed that the data are characterized by heterogeneity and mutual feedback between house price expectations and realised house prices.

Blanchard (2014) recently stressed that "*The main lesson of the crisis is that we were much closer to "dark corners" –situations in which the economy could badly malfunction – than we thought. Now that we are more aware of nonlinearities and the dangers they pose, we should explore them further theoretically and empirically.*" Blanchard argued for the coexistence of non-linear models along the standard, general equilibrium models: "*If macroeconomic policy and financial regulation are set in such a way as to maintain a healthy distance from dark corners, then our models that portray normal times may still be largely appropriate. Another class of economic models, aimed at measuring systemic risk, can be used to give warning signals that we are getting too close to dark corners, and that steps must be taken to reduce risk and increase distance. Trying to create a model that integrates normal times and systemic risks may be beyond the profession's conceptual and technical reach at this stage.*"

The purpose of this paper is to develop a nonlinear heterogeneous agent model for the housing market with *endogenous switching* between optimistic and pessimistic expectations and estimate the model using empirical housing price data from different countries. Heterogeneous expectations feedback relations may drive the price-rent ratio temporarily away from its long-run fundamental value and at other times expectations may reinforce mean-reversion back to fundamentals. The fundamental price-rent ratio of the more traditional housing market literature is nested as a special case within our nonlinear model where expectations coincide with or mean-revert to the fundamental values. Our heterogeneous expectations housing market model thus provides an empirical test whether behavioral heterogeneity and booms and busts deviating from fundamental value are economically and statistically significant. The structural nonlinear switching model also provides an early warning signal for identifying housing bubbles, when the average market sentiment –i.e. the average extrapolation factor– exceeds 1, indicating that house price dynamics and deviations from fundamentals become temporarily explosive.

Heterogeneous agents models (HAMs) provide a recently developed tool, which can be used to study temporary deviations from market equilibria in economics and finance. Rather than a single, representative, fully rational, agent, HAMs allow for bounded rationality and heterogeneity of expectations among agents. HAMs were originally introduced by Brock and Hommes (1997, 1998) to describe financial asset price fluctuations. They showed that heterogeneous beliefs, together with switching between beliefs based on recent past performance, can lead to situations where the fundamental price is locally unstable and asset prices show multiple equilibria or chaotic fluctuations, with irregular bubbles and crashes, around the fundamental value; see Hommes (2006) for an overview. More recently HAMs have also been applied to macroeconomics, particularly extending the New Keynesian framework, e.g., Gali and Gertler (1999), Branch and Evans (2006), DeGrauwe (2011), Branch and McGough (2009), Kurz (2011), Massaro (2013) and Cornea *et al.* (2012).

Endogenous switching, the ability of agents to change views about the future at regular points in time, is a pivotal feature of HAMs. Empirical evidence shows that switching based on past performance is relevant for a number of real financial markets. For example, Ippolito (1989), Chevalier and Ellison (1997), Sirri and Tufano (1998) and Karceski (2002) found that money flows out of past poor performers into good performers in mutual funds data. Pension funds also switch away from bad performers DelGuercio and Tkac (2002). Investors in the stock market display similar switching behaviour when choosing between different strategies.

HAMs with performance-based endogenous switching have been successfully estimated for different financial markets. Boswijk *et al.* (2007) estimated the Brock and Hommes (1997, 1998) HAM using annual S&P 500 stock market data from 1871 to 2003. They found evidence for the presence of heterogeneity and endogenous switching for a fundamental price based on both the price/earnings ratio and the price/dividend ratio. Their model explains the dot com bubble as being triggered by economic fundamentals (good news about the economy, because of a new internet technology), strongly amplified by investors' switching to trend-following behaviour. Hommes and in't Veld (2014) follow a similar approach, using both the dynamic Gordon present-discounted-value and the Campbell-Cochrane consumption habit fundamental benchmarks, using quarterly S&P500 data 1950-2013 and conclude that the financial crises has been amplified by switching between fundamentalists and trend-following strategies. Lof (2014) estimates a HAM with different VAR-model specifications to the S&P500 index and finds temporary switching between fundamentalists and rational and contrarian speculators. Lux (2009) estimated the parameters of a dynamic opinion formation process with social interactions based on survey data on business expectations (sentiment index data). Franke and Westerhoff (2011, 2012) estimate HAMs with structural stochastic volatility using S&P 500 index data. De Jong *et al.* (2009) estimated a HAM for the EMS exchange rate dynamics and for Asian stock markets during the Asian crisis (De Jong *et al.*, 2010).

Our paper focuses on the empirical relevance of performance based switching in the housing market. We develop a stylised 2-type heterogeneous expectations feedback model, with a trend-extrapolation versus a mean-reversion forecasting rule, and estimate the model for eight different countries, the United States (US), Japan (JP), United Kingdom (UK), The Netherlands (NL), Switzerland (CH), Spain (ES), Sweden (SE) and Belgium (BE) for the period 1970–2013. We arrived at this set of countries by starting with the US and adding countries until we had a fairly mixed list of countries that have recently seen a housing bubble (US, CH, JP), are currently near the peak of a bubble (UK, SE and BE) and are in a price corrective regime (NL, ES). For all countries, the estimated parameters measuring the heterogeneity and switching behaviour turn out to be significant and lie in – or close to – the region where the fundamental equilibrium is unstable (and hence it does not prevail). For all eight countries we identify long-lasting periods of temporary housing bubbles, amplified by an explosive market

sentiment. Our simple model can be used as an early warning indicator to identify housing bubbles. As the bubble bursts, a crash in house prices is reinforced by switching back to the mean-reverting fundamental forecast rule. However, while the model can tell us whether we are in a period of instability (boom), it cannot necessarily predict the timing of market switches/corrections. Nevertheless, the conditions of instability derived are linked explicitly to policy variables, so we are able to discuss which policies — e.g. interest rate policies or reduction of mortgage tax deduction — can stabilise booms and busts or may destabilise housing prices. In this respect our model can inform policy makers about the possibility of the economy approaching, or even being, at what Blanchard described as 'dark corners', places where variables react in very non-linear ways, such that even small, and otherwise innocuous, shocks may produce very large and unpredictable effects (Blanchard, 2014). Our nonlinear heterogeneous agents switching model thus can provide important insights for policy makers to avoid the "dark corners" of the economy.

House price fluctuations have been studied extensively in the literature. Ambrose *et al.* (2013) examined a long time series of house price data of Amsterdam from 1650 to 2005, and found that substantial deviations from fundamentals persisted for decades and are corrected mainly through price adjustments and to a lesser extent through rent adjustments. Based on the same dataset, Eichholtz *et al.* (2013) found that there is evidence for switching in expectation formation between fundamental and trend following beliefs. Theoretical models for house price dynamics involving HAMs have been considered, for instance by Dieci and Westerhoff (2012, 2013). Kouwenberg and Zwinkels (2014) estimated a HAM model specifically for the US housing market using quarterly data from 1960 until 2012. An important difference with our approach is that their model (following Dieci and Westerhoff, 2012, 2013) uses a price adjustment rule based on excess demand, while we use a temporary equilibrium pricing model (as well as that we estimate the model for a number of different countries). Geanakoplos *et al.* (2012) develop an agent-based model to explain the housing boom and crash, 1997-2009 in the Washington DC area. Adam *et al.* (2011) consider a housing market model with Bayesian learning of an "internally rational" representative agent; Ascari *et al.* (2014) extend this model to the case of heterogeneous expectations with fundamentalists versus chartists. Another related theoretical paper is Burnside *et al.* (2011). Their approach differs in two important ways from ours. Firstly, their agents disagree about the fundamental value of housing, whereas we assume that agents agree on the fundamental value of houses but disagree on how prices return to it. Secondly, their model is epidemiological in nature, in that agents infect each other, while in our approach strategy switching is based upon relative performance.

The paper is organised as follows. Section 2 extends a standard housing model with heterogeneous expectations. In Section 3 we estimate the heterogeneous expectations model using

data from eight countries. We find evidence of temporary bubbles in all countries and discuss in-sample and out-of-sample forecasting. Section 4 derives conditions under which the 'classical' fundamental equilibrium is stable, and shows that for an unstable fundamental equilibrium multiple equilibria and global instability may occur. We also discuss policy implications and which policies might prevent a critical transition or tipping point to global instability. Finally, Section 5 concludes.

2   A Housing Market Model with Heterogeneous beliefs

In this section we develop a standard housing asset pricing model with heterogeneous beliefs following Brock and Hommes (1997, 1998). Agents are boundedly rational and have different views about the future values of asset prices. At the same time, agents are allowed to switch from one period to the next between a number of available strategies, based on how well they have performed in the recent past. Since our goal is to estimate a simple heterogeneous expectations model, we restrict attention to two types of agents $h \in \{1, 2\}$, but the case with H types is straightforward.

2.1   The model

Our point of departure is an asset pricing model where agents choose between either buying or renting a house. In equilibrium the annual cost of home ownership –in the literature known as the "imputed rent" (e.g. Himmelberg *et al.* (2005))– must equal the housing rent. In the model agents base their decisions at time t on the expected excess return R_{t+1} on investing in housing relative to renting during the period between time t and $t+1$. Let P_t denote the price of one unit of housing at time t. Let the price for renting one unit of housing in the period between times t and $t+1$ be given by Q_t. Since rents are typically payed up-front (at time t), to express the rent at time t in terms of currency at time $t+1$, it should be inflated by a factor $(1 + r^{\mathrm{rf}})$, where r^{rf} denotes the risk free mortgage rate. Therefore, the cost of renting in the period between time t and $t+1$, expressed in terms of currency at time $t+1$, is given by $(1+r^{\mathrm{rf}})Q_t$ rather than Q_t. The ex post excess return R_{t+1} on investing in housing during the period between time t and $t+1$ then is given by the sum of the capital gain minus mortgage/maintenance costs and the saving on rent (see also Campbell *et al.*, 2009; Ambrose *et al.*, 2013)

$$R_{t+1} = \frac{(P_{t+1} - (1 + r_t)P_t) + (1 + r^{\mathrm{rf}})Q_t}{P_t} = \frac{P_{t+1} + (1 + r_t^{\mathrm{rf}})Q_t}{P_t} - (1 + r_t),$$

where $r_t = r_t^{\text{rf}} + \omega_t$, with r_t^{rf} the risk-free mortgage rate and ω_t the maintenance costs/tax rate.

The demand, $z_{h,t}$, of agents of belief type h is determined by maximizing one-period ahead expected excess returns adjusted for risk:

$$\mathbb{E}_{h,t}\left(R_{t+1}z_{h,t}\right) - \frac{a}{2}\text{Var}_{h,t}\left(R_{t+1}z_{h,t}\right), \tag{1}$$

where a is a measure of risk aversion. For simplicity we assume r_t^{rf} and ω_t to be constant over time: $r_t^{\text{rf}} = r^{\text{rf}}$, $\omega_t = \omega$ (and hence $r_t = r$). Agents are assumed to be homogeneous with respect to their expectations regarding the conditional variance of the excess return, that is, $\text{Var}_{h,t}\left((P_{t+1} + (1+r^{\text{rf}})Q_t)/P_t - (1+r)\right) = V_t$, while they are heterogeneous concerning their expectations of excess return $\mathbb{E}_{h,t}\left((P_{t+1} + (1+r^{\text{rf}})Q_t)/P_t - (1+r)\right)$. Maximizing Eq. (1) leads to the demand for housing:

$$z_{h,t} = \frac{\left(\mathbb{E}_{h,t}\left(P_{t+1} + (1+r^{\text{rf}})Q_t\right)/P_t - (1+r)\right)}{aV_t} = \frac{\mathbb{E}_{h,t}(R_{t+1})}{aV_t}, \tag{2}$$

of agents of type $h \in \{1,2\}$.

Upon aggregation of the demand across these two types of agents, the market clearing condition is:

$$\sum_{h=1}^{2} \frac{n_{h,t}\left(\mathbb{E}_{h,t}\left(P_{t+1} + (1+r^{\text{rf}})Q_t\right)/P_t - (1+r)\right)}{aV_t} = S_t, \tag{3}$$

where S_t is the stock of housing, and $n_{h,t}$ is the fraction of agents in period t that hold expectations of type h.[1] Solving the market clearing condition for the price P_t leads to the following price equation:

$$P_t = \frac{1}{1+r+\alpha}\sum_{h=1}^{2} n_{h,t}\mathbb{E}_{h,t}\left(P_{t+1} + (1+r^{\text{rf}})Q_t\right), \tag{4}$$

where $\alpha \equiv aV_t \times S_t$ is assumed to be constant in order to keep the model tractable. It is beyond the scope of this paper to include a model of the supply side S_t of the market, which not only would require a model and data concerning the supply of new houses, but also concerning demographic and cultural changes over time, such as the tendency for families to decrease in size. Our goal here is to develop a simple stylized heterogeneous expectations housing model that can be estimated using date from different countries. Agents require a rate of return on housing equal to $r + \alpha$ rather than $r = r^{\text{rf}} + \omega$. Therefore the parameter α can be interpreted

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[1]We assume that S_t is large enough, such that neither type has an incentive to sell short on houses.

as a risk premium of buying a house over renting a house. Treating α as a constant in the model allows for estimating this extra required rate of return.

We next turn to the fundamental housing price. Following Boswijk *et al.* (2007), we assume that the fundamental process underlying the model, i.e. the rent Q_t, follows a geometric Brownian motion with drift, i.e.

$$\log Q_{t+1} = \mu + \log Q_t + v_{t+1}, \quad \{v_t\} \overset{\text{i.i.d.}}{\sim} N(0, \sigma_v^2),$$

with commonly known parameters μ and σ_v^2, from which one obtains

$$\frac{Q_{t+1}}{Q_t} = (1+g)\varepsilon_{t+1},$$

with $g = e^{\mu + \frac{1}{2}\sigma_v^2} - 1$ and $\varepsilon_{t+1} = e^{v_{t+1} - \frac{1}{2}\sigma_v^2}$, such that $\mathbb{E}_t(\varepsilon_{t+1}) = 1$.

We define the fundamental price as the price that would prevail under homogeneous rational expectations $\mathbb{E}_t(R_{t+1})$ about the conditional mean of R_t, while taking into account the risk premium α. Taking into account the risk premium in the fundamental price is convenient, as it will provide an equilibrium fundamental price from which the market price will deviate by an amount which averages out to zero in long time series.

Under rational expectations on the first conditional moment, we can re-write the price Eq. (4) as

$$(1 + r + \alpha)P_t = \mathbb{E}_t\left(P_{t+1} + (1 + r^{\text{rf}})Q_t\right).$$

By applying the law of iterated expectations and imposing the transversality condition, we obtain the fundamental price at time t, denoted by P_t^*.

$$P_t^* = \mathbb{E}_t\left[\sum_{i=0}^{\infty} \frac{(1 + r^{\text{rf}})Q_{t+i}}{(1 + r + \alpha)^{i+1}}\right] = \sum_{i=0}^{\infty} \frac{(1+g)^i(1 + r^{\text{rf}})Q_t}{(1 + r + \alpha)^{i+1}} = \frac{1 + r^{\text{rf}}}{r + \alpha - g}Q_t, \qquad r + \alpha > g. \quad (5)$$

This shows that the fundamental price of housing is directly proportional to the actual rent level. In practice we often only observe indices for housing price levels and rents, with possibly different base years, so that the proportionality factor between prices and rents is not known and needs to be estimated from the data.

Define $X_t = \frac{P_t}{P_t^*} - 1$ as the relative deviation of the price from the fundamental price. A straightforward computation (see Appendix A.1) shows that, for the model with two belief

types, the price equation (4) simplifies to:

$$X_t = \frac{1}{R + \bar{\alpha}} \sum_{h=1}^{2} n_{h,t} \mathbb{E}_{h,t} \left(X_{t+1} \right),$$

where $R = \frac{1+r}{1+g}$ and $\bar{\alpha} = \frac{\alpha}{1+g}$, and where we have assumed that $\mathbb{E}_{h,t}(Q_{t+1}X_{t+1}) = (1 + g)Q_t\mathbb{E}_{h,t}(X_{t+1})$, which, for instance, is the case if agents believe that X_{t+1} and Q_{t+1} are conditionally independent given the information available at time t. Notice that in the special case where both belief types are fundamentalists, i.e. $\mathbb{E}_{h,t} \equiv 0$, $h = 1, 2$, the housing price always equals its fundamental value. Hence, the special homogeneous RE benchmark is nested as a special case. Fig. 1 shows the fundamental housing price together with the realized housing prices and deviations from the fundamental for eight countries, as discussed in more detail in the data Section 3. These plots suggest excess volatility, that is, house prices fluctuate much more than underlying fundamentals. The nest subsection introduces a two-type heterogeneous expectations switching model with boundedly rational agents to capture the excess volatility in housing price dynamics.

2.2   Two types of agents

Following Boswijk *et al.* (2007), henceforth BHM, we assume that each of the two types of agents have simple linear beliefs about X_{t+1}, but with different values of the coefficient ϕ:

$$\begin{aligned}
\mathbb{E}_{1,t} \left(X_{t+1} \right) &= \phi_1 X_{t-1}, \\
\mathbb{E}_{2,t} \left(X_{t+1} \right) &= \phi_2 X_{t-1}.
\end{aligned}$$

Hence, the two types disagree about the speed of convergence to or divergence from the fundamental benchmark. In particular, $\phi_1 < 1$ corresponds to fundamentalists believing in mean-reversion towards the fundamental, while $\phi_2 > 1$ corresponds to trend-followers believing that prices further divert from the fundamental.

We have assumed the presence of two belief types here, but for the sake of argument, consider homogeneous beliefs: $\phi_1 = \phi_2$. The homogeneous case $\phi_1 = \phi_2 < R + \bar{\alpha}$ would lead to the price converging to the fundamental price, whereas homogeneous beliefs $\phi_1 = \phi_2 > R + \bar{\alpha}$ would imply a bubble, where prices would deviate more and more from the fundamental price.[2] Next consider the heterogeneous case $\phi_1 \neq \phi_2$. If one of the belief parameters, ϕ_1 say, is smaller than $R + \bar{\alpha}$, and the other, ϕ_2, larger than $R + \bar{\alpha}$, the fractions $n_{1,t}$ and $n_{2,t} = 1 - n_{1,t}$ of agents of belief type 1 and 2, determine whether prices are temporarily converging to the

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[2]A house price bubble occurs when agents have unreasonably high expectations about future capital gains, leading them to perceive their user cost to be lower than it actually is and thus pay "too much" to purchase a house today.
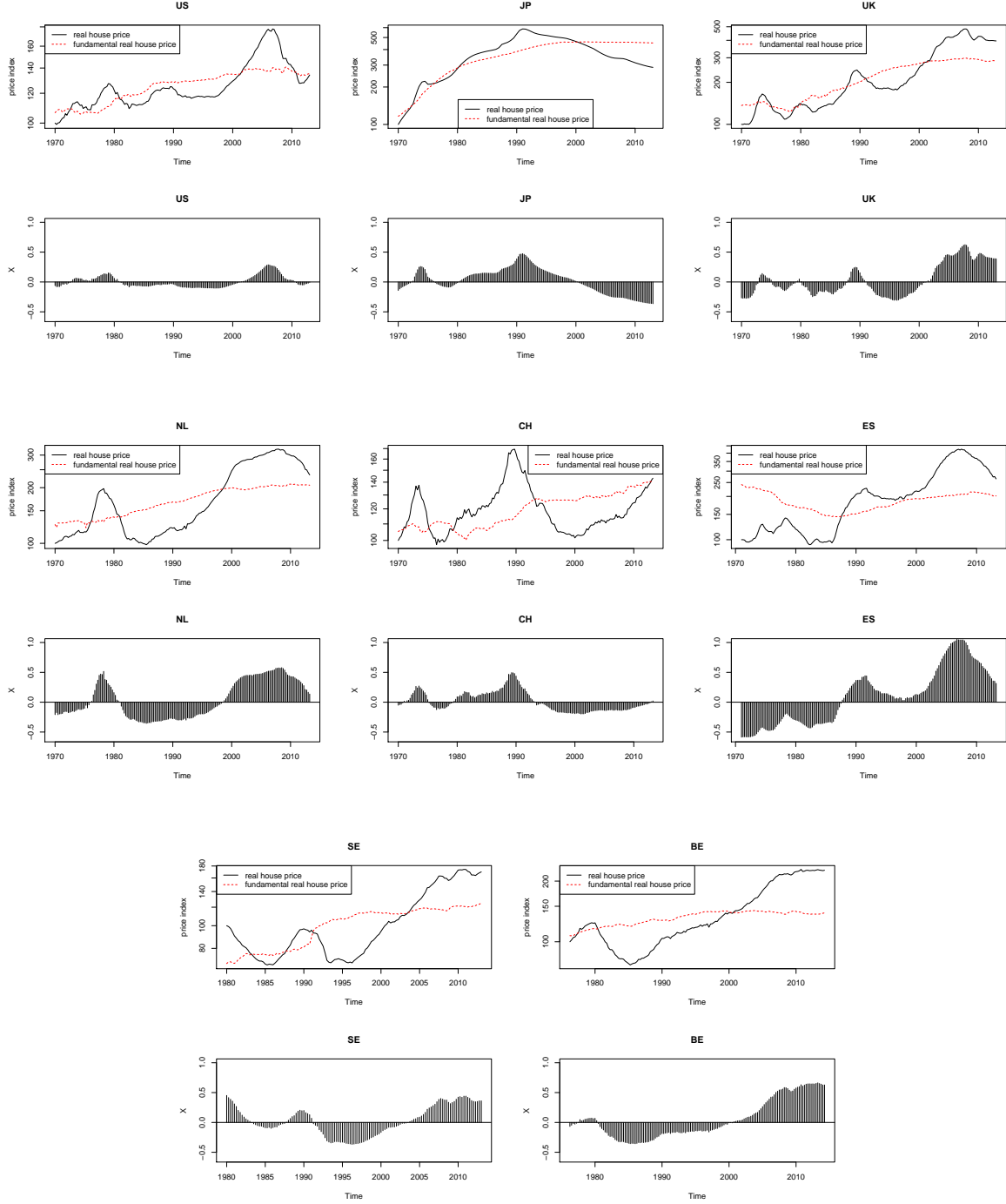
Figure 1: House price indices (top sub-panels, solid lines, 1970Q1=100), estimated fundamental real house prices (left, dashed lines) and corresponding relative over-valuation X_t (bottom sub-panels).

fundamental price or diverging from it. Since agents are allowed to switch between the two different types of beliefs, the fractions themselves are changing over time. This in turn implies that the system may temporarily be in a bubble regime, where prices deviate further from

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fundamentals, or in a correction or mean reversion regime, with prices converging back to the fundamental.

The switching between the two types of beliefs is based on the recent past performance of the strategies, measured in terms of realised profits, $\pi_{h,t-1}$, as in Brock and Hommes (1997, 1998). We derive the realised profits $\pi_{h,t-1}$ at time $t-1$ along the lines of Boswijk *et al.* (2007), starting from

$$\pi_{h,t-1} = R_{t-1}z_{h,t-2} = R_{t-1}\frac{\mathbb{E}_{h,t-2}(R_{t-1})}{aV_{t-2}(R_{t-1})}.$$

We can express R_{t-1} as (see Appendix A.2)

$$R_{t-1} \approx (1+g)Y_{t-2}\left(X_{t-1} + \bar{\alpha} - RX_{t-2}\right),$$

where $Y_{t-2} = \frac{P_{t-2}^*}{P_{t-2}}$.

Note that $\bar{\alpha}$ represents an endogenously determined risk premium for home owners. To see this, suppose X_{t-1} and X_{t-2} are zero, that is, house prices are at fundamental value). If the stock of housing S, the risk aversion parameter a and the perceived variance V are positive, $\bar{\alpha}$ is positive, and the excess return on housing R_{t-1} is positive even if prices evolve according to the fundamental price.

The performance measure is the product of the excess return R_{t-1} and the demand $z_{h,t-2}$, which by (2) is proportional to the expected excess return

$$z_{h,t-2} = \frac{\mathbb{E}_{h,t-2}(R_{t-1})}{aV_{t-2}},$$

where

$$\mathbb{E}_{h,t-2}(R_{t-1}) = (1+g)Y_{t-2}\left(\mathbb{E}_{h,t}\left(X_{t-1}\right) + \bar{\alpha} - RX_{t-2}\right),$$

and

$$V_{t-2} = \text{Var}_{t-2}\left(\frac{P_{t-1} + (1+r^{\text{rf}})Q_{t-2} - (1+r)P_{t-2}}{P_{t-2}}\right).$$

Following BHM we assume that agents have fundamental beliefs regarding the conditional variance. This leads to

$$V_{t-2} = Y_{t-2}^2\eta^2,$$

where $\eta^2 = \text{Var}\left((1+g)\varepsilon_t\right)$ (see Appendix A.3).

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Taking together the expectations on returns and conditional variance gives

$$\pi_{h,t-1} = z_{h,t-2}R_{t-1} = \frac{(1+g)^2}{a\eta^2}\left(X_{t-1} + \bar{\alpha} - RX_{t-2}\right)\left(\mathbb{E}_{h,t-2}\left(X_{t-1}\right) + \bar{\alpha} - RX_{t-2}\right),$$

i.e. a constant involving the risk aversion times the realised excess return on housing at time $t-1$, times the expected (at time $t-2$) one-step-ahead excess return. We define the latter product as the fitness measure at time $t-1$ for type h:

$$U_{h,t-1} = \left(X_{t-1} + \bar{\alpha} - RX_{t-2}\right)\left(\mathbb{E}_{h,t-2}\left(X_{t-1}\right) + \bar{\alpha} - RX_{t-2}\right). \tag{6}$$

The fractions are determined by a logistic switching model with a-synchronous updating:

$$
\begin{aligned}
n_{1,t} &= \delta n_{1,t-1} + (1-\delta)\frac{e^{\beta U_{1,t-1}}}{e^{\beta U_{1,t-1}} + e^{\beta U_{2,t-1}}}\\
&= \delta n_{1,t-1} + (1-\delta)\frac{1}{1 + e^{-\beta(X_{t-1}+\bar{\alpha}-RX_{t-2})(\phi_1-\phi_2)X_{t-3}}}\\
n_{2,t} &= 1 - n_{1,t}.
\end{aligned}
\tag{7}
$$

The term a-synchronous updating refers to the fact that only a fraction $(1-\delta)$ of agents re-evaluates and updates beliefs according to the logit model in each given period. Parameter β, referred to as the intensity of choice, represents the sensitivity of agents' to small changes in past performance $\pi_{h,t-1}$[3].

The full dynamical system for the housing market model with heterogeneous beliefs is given by

$$
\begin{aligned}
n_{1,t} &= \delta n_{1,t} + (1-\delta)\frac{1}{1+e^{-\beta(x_{t-1}+\bar{\alpha}-Ry_{t-1})(\phi_1-\phi_2)z_{t-1}}}\\
x_t &= \frac{n_{1,t}\phi_1+(1-n_{1,t})\phi_2}{R+\bar{\alpha}}x_{t-1}\\
y_t &= x_{t-1}\\
z_t &= y_{t-1},
\end{aligned}
$$

where $x_t = X_t$. At the fundamental equilibrium $n_{1,t} = \frac{1}{2}$ and $x_t = y_t = z_t = 0$. By linearising the dynamics around this equilibrium one finds that the fundamental equilibrium is locally stable if

$$\left|\frac{\phi_1+\phi_2}{2(R+\bar{\alpha})}\right| = \left|\frac{(1+g)(\phi_1+\phi_2)}{2(1+r+\alpha)}\right| < 1. \tag{8}$$

We estimate the model in two steps. First we estimate the fundamental parameters and

[3]Note that by working with βU_{h_t-1} rather than $\beta\pi_{h,t-1}$ in Eq. (7), a factor $(1+g)/(a\eta^2)$ has been absorbed in the definition of β. This has the advantage that we do not need to estimate this term, but it should be kept in mind that it makes direct comparisons between β-estimates for different countries difficult.

the corresponding deviations $\{X_t\}$, of prices from their fundamental value, using housing price and rent indices. Second we use the deviations to estimate the behavioural parameters of the agent-based model. Assuming white noise disturbances u_t on the price equation, the agent-based model is in fact a nonlinear time-varying AR(1) model

$$X_t = \frac{n_{1,t}\phi_1 + n_{2,t}\phi_2}{R + \bar{\alpha}}X_{t-1} + u_t,$$

the parameters of which we estimate using nonlinear least squares (NLS), since $n_{1,t}$ and $n_{2,t}$ depend nonlinearly on the model parameters. We will refer to

$$\frac{n_{1,t}\phi_1 + n_{2,t}\phi_2}{R + \bar{\alpha}}, \tag{9}$$

as the (time-varying) *market sentiment*, representing the average mean-reverting or mean-diverting beliefs in the housing market. When the market sentiment exceeds 1, the market is explosive and in a temporary bubble state.

3 Data and Empirical Results

We use an OECD housing dataset similar to that described in Rousová and Van den Noord (2011), but extended to include more recent observations. This dataset contains quarterly data for nominal and real house prices for 20 countries, starting from 1970Q1 for most countries (see Rousová and Van den Noord (2011), Appendix 1-2, for the list of countries and corresponding data sources). We use data downloaded in autumn 2013, with data until 2013Q1 or 2013Q2, with the exception of Belgium, which was added recently to our database as a 'live' case study, and for which we have data until 2014Q2 (see Table 1 for the exact start and end quarters of our sample per country). The nominal house price is indexed using 2005 as base year. The real house price index is derived by deflating with the private final consumption expenditure deflator, available from the OECD Economic Outlook 89 database. The price-to-rent ratio is defined as the nominal house price index divided by the rent component of the consumer price index, made available by the OECD. Long term interest rates are also retrieved from the OECD Economic Outlook 89 database. For each country we also have the consumer price index over the same sample period, so that all nominal rates can be converted to real rates[4]

Note that since we are using indices we are only able to calculate the ratio Q_t/P_t from the data up to an unknown factor from the indices of Q_t and P_t. To overcome this, we calibrated the series Q_t/P_t by using observed rent-to-price ratios at particular, country-specific, reference

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[4]Whether nominal or real rates are used has no consequences for the estimation of the fundamental model parameters.

Country	start	end	rent/price	reference quarter
US	1970Q1	2013Q1	0.0429	2013Q1
JP	1970Q1	2013Q1	0.0553	2013Q1
UK	1970Q1	2013Q1	0.0209	2013Q1
NL	1970Q1	2013Q2	0.0568	2013Q2
CH	1970Q1	2013Q2	0.0294	2013Q2
ES	1971Q1	2013Q2	0.0392	2013Q2
SE	1980Q1	2013Q1	0.0700	2005Q3
BE	1976Q2	2014Q2	0.0436	2013Q2

Table 1: Start and end of sample period for each country, as well as the rental yield (rent-to-price ratio's) used to calibrate the price-to-rent ratio time series, and the corresponding reference quarter.

dates for each of the countries [source: GlobalPropertyGuide.com]. The rent-to-price ratio's used for calibration are given in Table 1.

In what follows we estimate the two-type heterogenous beliefs BHM model and present the results for the housing markets of the US, JP, UK, NL, CH, ES, SE and BE. We arrived at this subset of countries by starting our analysis with the US and adding countries until we had a fairly mixed list of countries that have recently seen a housing bubble (US, CH, JP), are currently near the peak of a bubble (UK, SE, BE) or are in a price corrective regime (NL, ES).

3.1 Estimation of the fundamental model parameters

Before estimating the behavioural model we calibrate the fundamental model parameters, namely $R(= \frac{1+r}{1+g})$ and $\bar{\alpha}(= \frac{\alpha}{1+g})$. The fundamental relation between quarterly prices and rents in Eq. (5) can be re-written as

$$R = 1 + \frac{(1 + r^{\mathrm{rf}})Q_t}{(1 + g)P_t^*} - \bar{\alpha} \approx 1 + \frac{Q_t}{P_t^*} - \bar{\alpha}, \tag{10}$$

where we used the approximation $(1 + r^{\mathrm{rf}})/(1 + g) \approx 1$, which is reasonable since quarterly interest and growth rates are small relative to 1. Note that even with this simplification the fundamental parameters R and $\bar{\alpha}$ cannot be estimated independently, since we have one fundamental equation and two unknown fundamental parameters[5]. We address this problem by fixing $\bar{\alpha}$, which is hard to obtain empirically for individual countries, at a plausible value.

Himmelberg *et al.* (2005) estimated the risk premium of house owning relative to renting to be about 2% per year for the US, which would correspond to $\bar{\alpha} \approx 0.005$. However, they

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[5]Recall that R and α appear independently in the fitness measure for strategy switching in Eq. (6).

also noted that this is probably an over-estimation, since it ignores relevant factors such as the insurance value of owning a house in hedging risk associated with future changes in rent. Based on this, we consider 0 to be a lower limit and 0.005 an upper limit for $\bar{\alpha}$. It turns out that the exact choice of $\bar{\alpha}$ within this range has relatively small effects on the estimated behavioural parameters.

For a given value of $\bar{\alpha}$ we estimate R as

$$R = 1 + \bar{y} - \bar{\alpha}$$

where $\bar{y}$ is the mean quarterly rental yield (average of Q_t/P_t) over the sample period. Since the data involved yearly rental yields, $\bar{y}$ was converted to quarterly rental yields prior to estimating R. The estimates of $\bar{y}$ and $R + \bar{\alpha}$ are given in Table 2. For completeness we also give the estimated inflation rate π, and the nominal and real growth rates g and r.

| Country | π | $\bar{y}$ | nominal | | real | | $R + \bar{\alpha}$ |
			g	$r + \alpha$	g	$r + \alpha$	
US	1.050	1.053	1.188	2.253	0.138	1.192	1.0105
JP	0.658	0.889	0.788	1.684	0.130	1.021	1.0089
UK	1.424	0.739	1.852	2.604	0.428	1.170	1.0074
NL	0.854	1.646	1.138	2.803	0.283	1.934	1.0165
CH	0.643	0.751	0.809	1.566	0.166	0.918	1.0075
ES	1.766	1.403	1.659	3.085	-0.107	1.294	1.0140
SE	0.892	1.988	1.342	3.357	0.450	2.447	1.0199
BE	0.754	1.837	0.925	2.779	0.171	2.011	1.0184

Table 2: Empirically observed mean quarterly inflation rate π, mean quarterly rental yield $\bar{y}$, nominal and real growth rate g, nominal and real value of $r + \alpha$ and corresponding values of $R + \bar{\alpha} = (1 + r + \alpha)/(1 + g)$. All quarterly rates are multiplied by 100, except $R + \bar{\alpha}$, which is not.

Fig. 1 presents the house price indices P_t with the corresponding estimated fundamental values P_t^* (left panel) and the log-difference between the two, $X_t = \log P_t - \log P_t^*$ (right panel). All eight countries exhibit longlasting periods of persistent under- or overvaluation of house prices compared to fundamentals, ranging from over-valuations of 25% for the US, around 50% for JP, UK, NL, CH, SE and BE, to 100% for ES. House prices in US, NL and ES have been increasing rapidly since the mid-1990s and have peaked around 2008. Since then house prices have dropped considerably for those countries, but are still in decline and above the fundamental value for NL and ES by the end of the sample. The JP and CH housing prices peaked earlier, around 1990, and subsequently declined to levels below the fundamental values. While the decline continues to the end of the sample for JP, CH price levels have recovered to the fundamental level at the end of the sample. House prices in the UK and SE are still

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around 35% above their fundamental values at the end of the sample, while those in Belgium are still around 50% above the fundamental value at the end of the sample.

3.2   Estimation of the heterogenous agents model

The behavioural model parameters are estimated based on the time series $X_t = \frac{P_t}{P_t^*} - 1 \approx \log P_t - \log P_t^*$, that is, the relative deviation of the house price from the estimated long-run fundamental ratio between P_t and Q_t. When presenting the empirical results we focus on the case $\bar{\alpha} = 0$, unless stated otherwise explicitly[6].

The estimated behavioural model parameters ϕ_1, $\Delta\phi$, β and δ are given in Table 3. the estimated values of $\Delta\phi = \phi_2 - \phi_1$ are significant for all countries. The data therefore confirm the presence of time-varying heterogeneity in the way agents form expectations. We also find that β is not significantly different from zero. Note, however, that we cannot put $\beta = 0$, firstly because β is restricted to be strictly larger than zero in the model since otherwise the fractions $n_{1,t}$ and $n_{2,t}$ will converge to the constant 0.5. Secondly, setting $\beta = 0$ leads to the problem that $\Delta\phi$ is not identified in that case. To avoid such identification problems and to be able to identify the significance of the differences in the forecast rules, β should be nonzero. From this perspective, the fact that β is found to be insignificant is merely an indication that the model's forecast accuracy is not very sensitive to the exact value of β, and the other parameters can to a large extent compensate for changes in β[7]. For BE, β would converge to any (large) upper value we would specify, so we had to provide an upper limit which was not unrealistically large, and settled for an upper limit of $10,000$.

The asynchronous updating parameter δ is significant for the US, NL, CH and ES, with roughly half of the agents re-evaluating their strategies per period. For JP, UK and SE no evidence for asynchronous updating is found, suggesting that in those countries more than half of the agents re-evaluate their strategies each period.

The bottom part of Table 3 provides a number of coefficients expressed as linear functions of the estimated parameters, and hence their standard errors could be easily calculated based on the variance-covariance matrix of the estimated parameters. These coefficients are important for the underlying dynamics of the nonlinear model. The first of these functions is $\phi_2 = \phi_1 + \Delta\phi$, which appears to be significantly > 1 for all countries. The second is $\phi_2 - (R + \bar{\alpha})$.

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[6]As discussed below, the results are rather robust when fixing the risk premium to the estimate $\bar{\alpha} = 0.005$ of Himmelberg *et al.* (2005); see Appendix A4.

[7]Hommes and in't Veld (2014) estimate a similar 2-type switching model on the relative deviations of the S&P500 stock market index from two benchmark fundamentals, the Gordon growth model and the Campbell-Cochrane consumption-habit model and show that the likelihood function is very flat w.r.t. the intensity of choice parameter β. They use quarterly data 1950-2013 and show by Monte-Carlo simulations that the test to reject the null of a switching model with estimated parameter values has essentially zero power for small samples of 250 observations. This explains the large standard deviations in the estimates for β and its non-significance.

Country	US	JP	UK	NL	CH	ES	SE	BE
				fundamental parameters				
$\bar{\alpha}$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$R + \bar{\alpha}$	1.0105	1.0089	1.0074	1.0165	1.0075	1.0140	1.0199	1.0184
				estimated behavioural parameters				
ϕ_1	0.8808***	0.9296***	0.9023***	0.9557***	0.8592***	0.9673***	0.9544***	0.9913***
	(0.0245)	(0.0105)	(0.0287)	(0.0248)	(0.0281)	(0.0060)	(0.0158)	(0.0134)
$\Delta\phi$	0.2158***	0.1589***	0.2007***	0.1358***	0.2710***	0.1034***	0.1537***	0.0826***
	(0.0340)	(0.0190)	(0.0568)	(0.0489)	(0.0464)	(0.0105)	(0.0325)	(0.0233)
$\beta\ (\times 10^3)$	26.33	5.565*	0.795	2.195	3.872	3.792*	3.579	10.00
	(24.04)	(2.377)	(0.594)	(2.271)	(2.684)	(1.955)	(2.755)	(14.23)
δ	0.6237***	0.1227	0.0000	0.4607*	0.6090***	0.3972*	0.1456	0.7208***
	(0.1013)	(0.1801)	(0.2465)	(0.2267)	(0.1027)	(0.1419)	(0.1668)	(0.1179)
BIC	-1011.043	-959.1851	-725.0588	-718.4524	-816.0738	-766.2773	-610.2841	-786.4019
			implied behavioural parameters/local stability					
ϕ_2	1.0967***	1.0886***	1.1030***	1.0915***	1.1302***	1.0707***	1.1081***	1.0739***
	(0.0157)	(0.0120)	(0.0303)	(0.0262)	(0.0228)	(0.0073)	(0.0205)	(0.0117)
$\phi_2 - (R + \bar{\alpha})$	0.0861***	0.0797***	0.0956***	0.0751**	0.1227***	0.0567***	0.0882***	0.0555***
	(0.0157)	(0.0120)	(0.0303)	(0.0262)	(0.0228)	(0.0073)	(0.0205)	(0.0117)
$\frac{\phi_1+\phi_2}{2(R+\bar{\alpha})}$	0.9784***	1.0002***	0.9953***	1.0070***	0.9873***	1.0050***	1.0111***	1.0140***
	(0.0115)	(0.0060)	(0.0081)	(0.0072)	(0.0107)	(0.0041)	(0.0082)	(0.0046)
loc. stability	stable	unstable	stable	unstable	stable	unstable	unstable	unstable
$R_{\text{crit.}}$	0.9887***	1.0091***	1.0027***	1.0236***	0.9947***	1.0190***	1.0312***	1.0326***
	(0.0116)	(0.0060)	(0.0082)	(0.0073)	(0.0107)	(0.0041)	(0.0084)	(0.0047)

Table 3: Fundamental parameters (fixed during estimation of the behavioural parameters), estimated behavioural model parameters for $\bar{\alpha} = 0$, and corresponding implied behavioural parameters. Signif. codes: 0.00 '***' 0.001 '**' 0.01 '*' 0.05 '∘' 0.10.

If this is positive, the dynamics allow for temporary bubbles or explosive behaviour around the fundamental price, provided that a sufficiently large fraction of agents is of type 2. We find this parameter to be significantly larger than zero for all countries considered. The third function is $(\phi_1 + \phi_2)/(2(R + \bar{\alpha}))$, the left-hand-side of the stability condition (8). This is the implied value of the AR-coefficient at equilibrium, determining whether the estimated model has a stable or an unstable fundamental equilibrium. Whether the equilibrium was found to be stable or not is indicated in the row immediately below. The implied ratios $(\phi_1 + \phi_2)/(2(R + \bar{\alpha}))$ on the left-hand-side of the stability condition (8) can be seen to be surprisingly close to 1 for all countries, which implies that the fundamental equilibrium is very close to the border of (in)stability. This means that in all cases the dynamics are very close to a unit root process (a random walk) near the fundamental equilibrium (i.e. for X_t small). The fourth and final implied parameter is $R_{\text{crit.}} = (\phi_1 + \phi_2)/2 - \bar{\alpha}$, the critical value of R below which the fundamental equilibrium would become unstable if the behavioural model parameters were held fixed at their estimated values. The role of $R_{\text{crit.}}$ will be discussed in more detail below, when we discuss multiple equilibria and policy implications.

As a robustness check we have repeated the estimation for $\bar{\alpha} = 0.005$, which, as discussed above, we consider to be the upper bound of the risk premium $\bar{\alpha}$. The results, shown in Table 4 in Appendix are qualitatively similar to those obtained for $\bar{\alpha} = 0$ for most countries. Only one country (NL) moves from being just stable to just unstable. For all countries except SE, $\Delta\phi$ and $\phi_2 - (R + \bar{\alpha})$ remain significantly different from zero, although the significance has become less pronounced for the US, NL and ES. The asynchronous updating parameter δ is no longer significant for the US, NL and ES when we take $\bar{\alpha} = 0.005$ instead of 0, and less significant for CH. Note that the BIC-values indicate that the model with $\bar{\alpha} = 0$, presented in the main text, is more suitable than the model with $\bar{\alpha} = 0.05$. This is the case for all countries, except the UK, for which both models have practically equal BIC-values.

3.3 Temporary bubbles in house prices

Fig. 2 shows for all eight countries the log-difference between house prices and fundamentals (upper panels), the estimated proportion n_1 of agents forming expectations of type 1 associated with $\phi_1 < 1$ (middle panels), in other words, those agents who expect mean-reversion towards the fundamental value, and finally the estimated time-varying AR(1) coefficients in (9) showing the time variation of the market sentiment (lower panels).

There is considerable time-variation in the distribution of the fundamental-reverting and fundamental-diverting rule across the population and in the average market sentiment. An immediate observation is that in all countries temporary housing market bubbles occur, that is, there are prolonged periods of several years during which the market sentiment is explosive
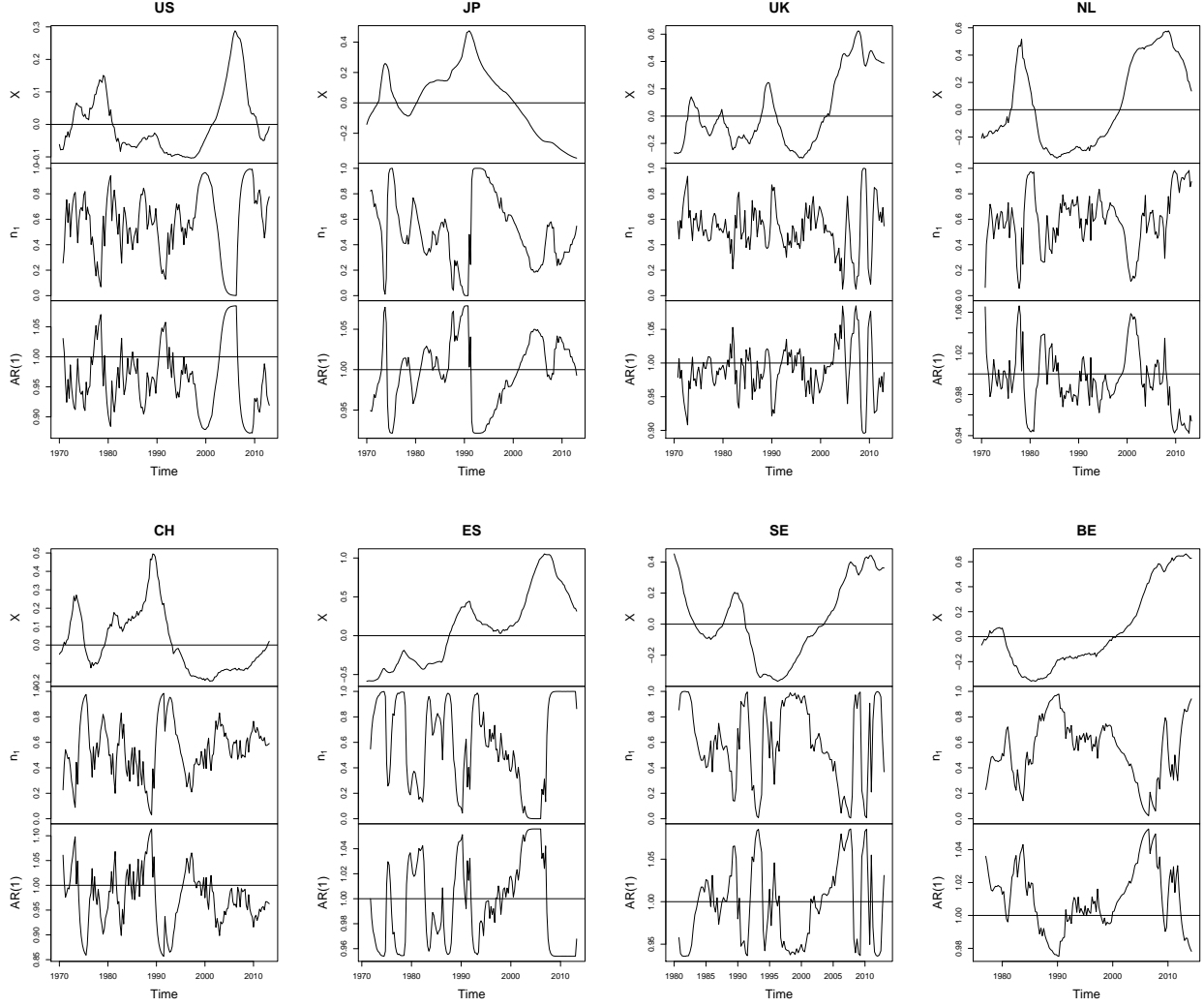
Figure 2: Relative price deviations X_t from fundamentals (top panels), estimated fractions of agents of type 1, i.e. fundamental-reverting agents (middle panels) and implied AR(1) coefficients, i.e. market sentiment (bottom panels).

(i.e. exceeds 1). For the US, for example, bubbles are identified in the late 1970s and from 2004-2007. The UK, NL, SE and ES also exhibit housing bubbles with explosive market sentiment in the years 2004-2007 or even for a longer period (BE). Exceptions are CH and JP for which house bubbles, with exploding market sentiment, arose much earlier between 1980-1990. JP experienced a strong negative bubble with explosive market sentiment during most of 2000-2010, during which housing prices continuously declined. Our simple stylized model thus provides an easy to use tool for identifying housing market bubbles and the market sentiment may serve as an early warning signal of housing bubbles signalling when the market becomes explosive. When a bubble bursts after a few years, a majority of agents typically switches to

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the type 1 fundamental forecasting rule, and strong mean-reverting market sentiment brings house prices back closer to fundamentals. In particular, after 2007 in the US, and somewhat later also in the UK, NL, ES and SE, the housing market is dominated by fundamentalists and strongly mean-reverting market sentiment bringing prices back closer to fundamentals.

Similar heterogeneous expectations models have been estimated on various data in recent years, including stock market data, exchange rates, survey data and macroeconomic data (see the references in the introduction). An interesting and characteristic feature of the housing market data are the long-lasting bubbles, over many years, detected in all eight countries. For example, ES has a long-lasting bubble characterized by temporary exploding market sentiment of about 10 years from the late 1990s to the financial crisis, while Japan has a declining bubble over roughly the same period. For SE and BE, the two countries in the sample that have not gone through a correction, the duration is arguably longer. In other data sets such long-lasting bubbles are rare, with the only exception being the dot-com bubble in the stock market. Using yearly data of the S&P500, Boswijk et al. (2007) estimate the dot-com bubble to last for 6 years, 1995-2000, while Hommes and in't Veld (2014) find the same for quarterly data.

3.4   Fancharts

We next examine how the forecasting performance of our heterogenous agents model forecasts compared to a linear AR benchmark model.

Figs. 3–10 provide quantiles of density forecasts constructed for both the nonlinear BHM switching model and AR models, for each country. The dotted/dashed lines correspond to estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts, based on the BHM model (top panels) and an AR(5) model (lower panels). An AR order of 5 was used to ensure that both models have the same number of parameters. The variable of interest, the deviation from the fundamental X_t, is represented by the black solid lines. The left hand side panels shows in-sample forecasts for the last 10 years of the sample. The right-hand-side panels correspond to 5-year ahead out-of sample forecasts based on information available up to and including the end of the data-set. In all cases the model parameters were estimated based on the data prior to the base of the forecast (the last observation conditioned upon).

We observe that the density forecasts from the BHM model and the linear model are qualitatively very different. For instance, the in-sample median forecasts for the UK are strikingly different, forecasting opposite directions of change. Again focusing on the median forecast, we see that the BHM model in some cases (NL and ES in-sample forecasts) predicts the over-valuation to resume, even after the peak of a bubble, while the AR(5) model consistently predicts a progressive convergence of prices back to the corresponding fundamental value.

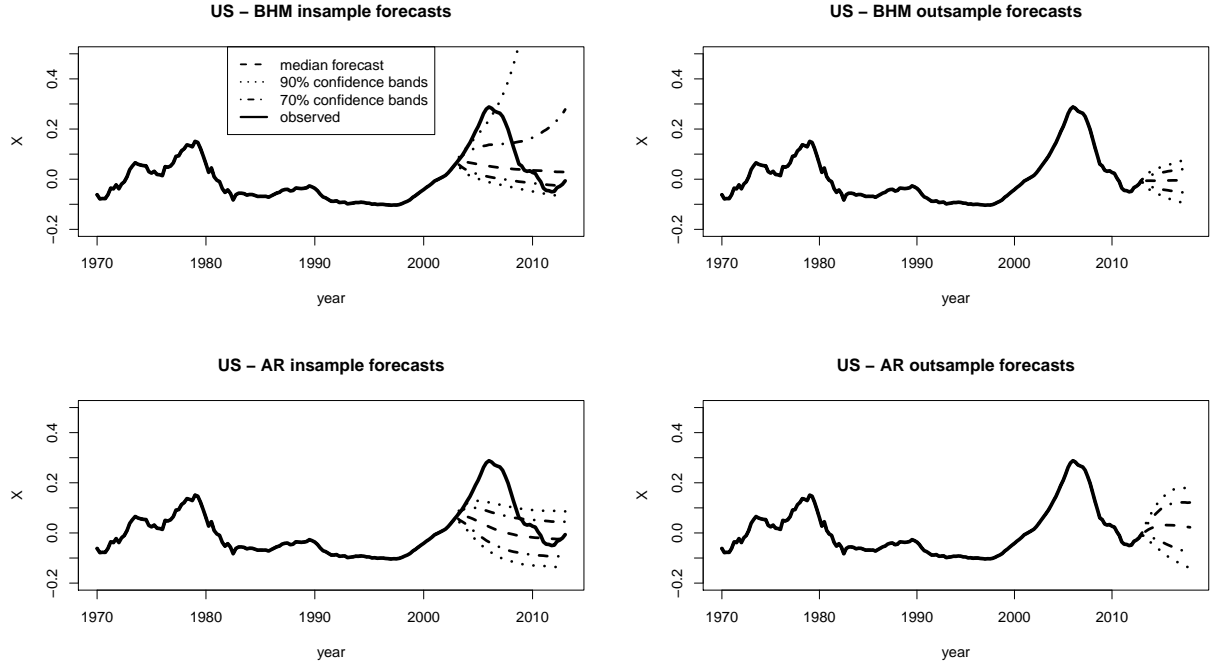Also the width of the predictive intervals differs substantially between the linear and the

Figure 3: Fancharts for US data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.
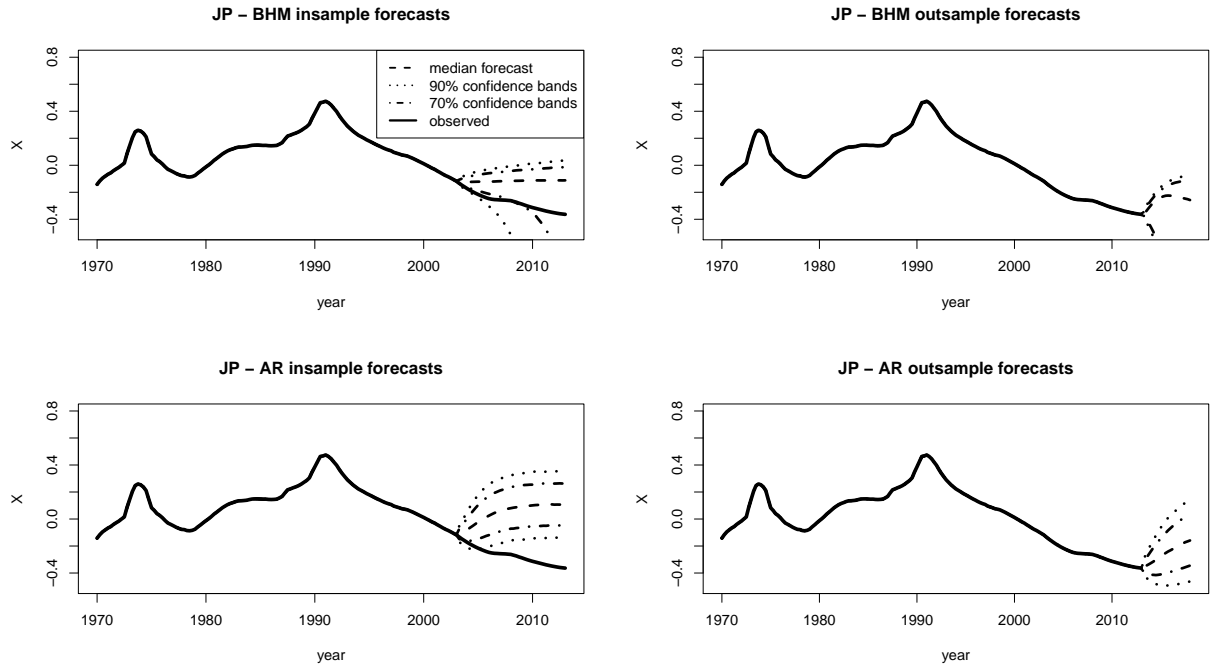

Figure 4: Fancharts for JP data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.
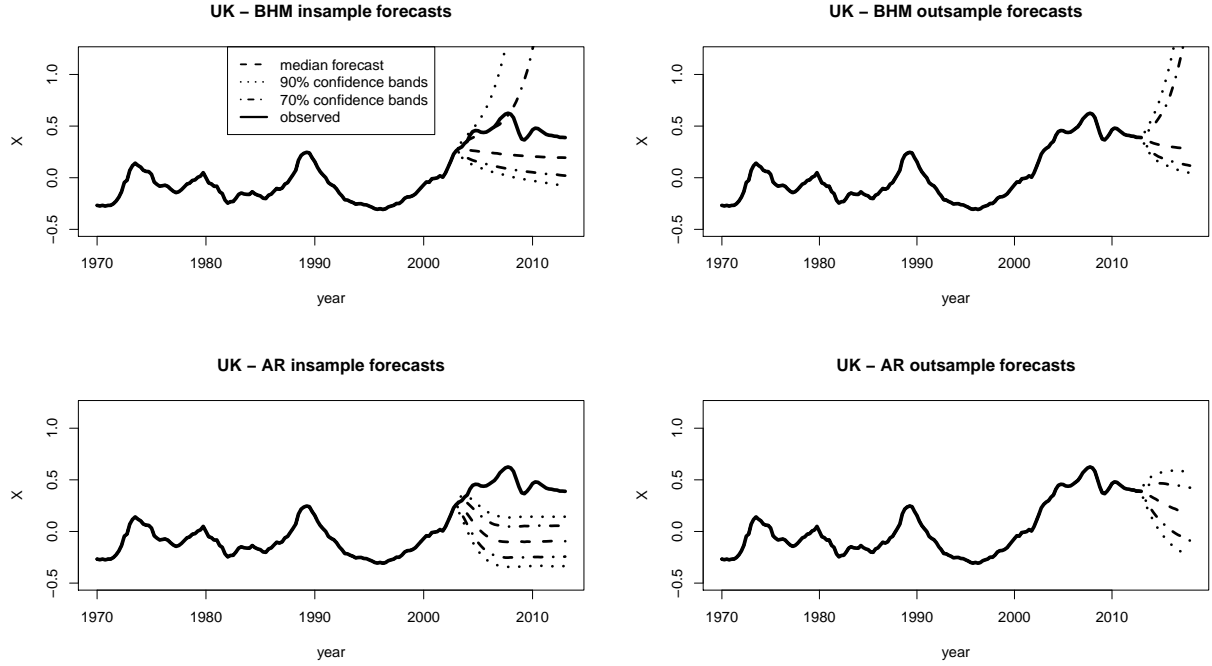
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Figure 5: Fancharts for UK data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.

Figure 6: Fancharts for NL data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.
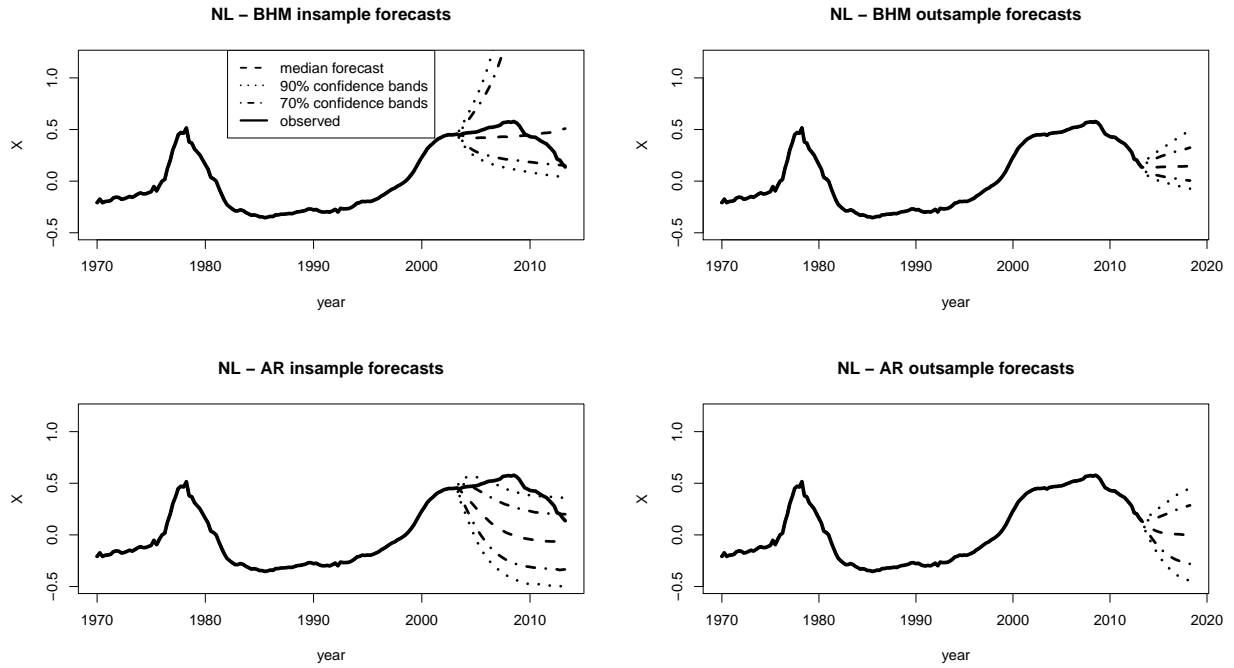
Figure 7: Fancharts for CH data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.

Figure 8: Fancharts for ES data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.
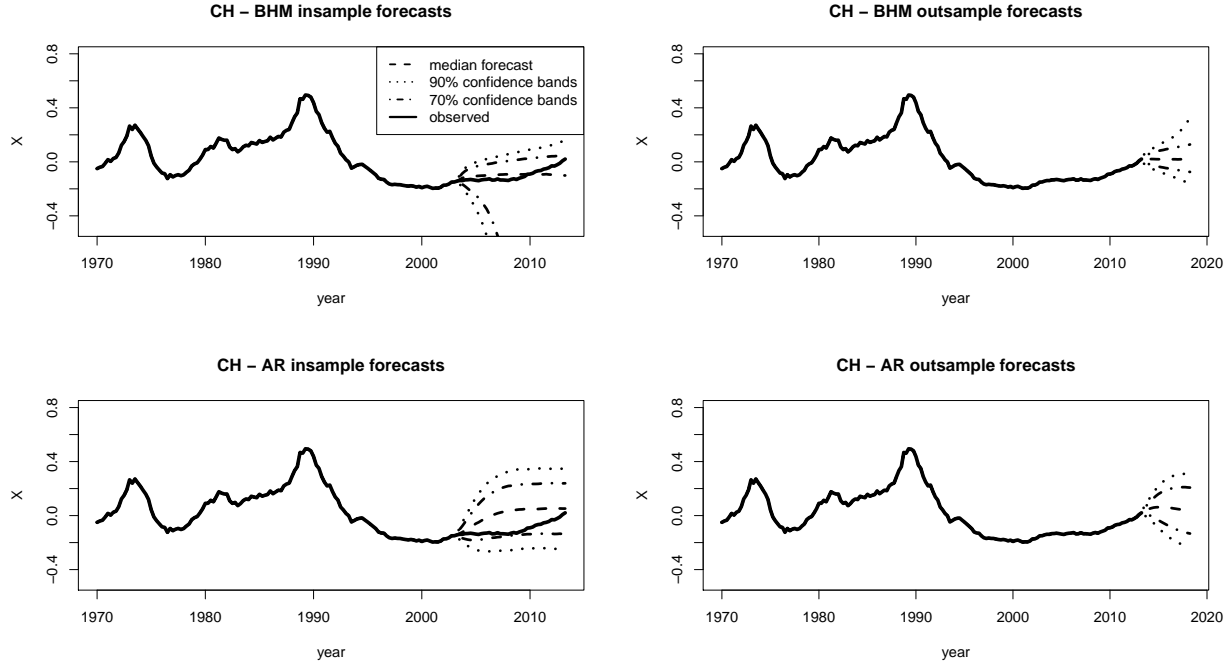
Figure 9: Fancharts for SE data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.
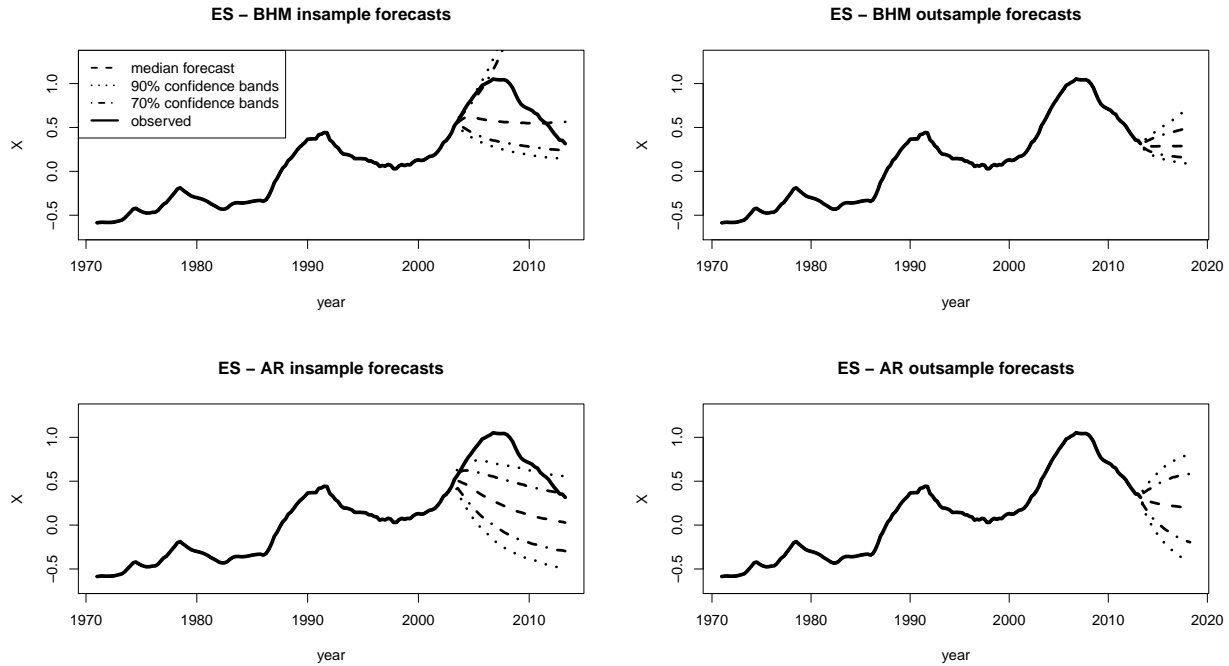

Figure 10: Fancharts for BE data: estimates of the 5, 15, 50, 85 and 95% quantiles of the density forecasts (dashed lines) based on the BHM model (top panels) and an AR(5) model (lower panels). The deviations from the fundamental are represented by the solid lines.
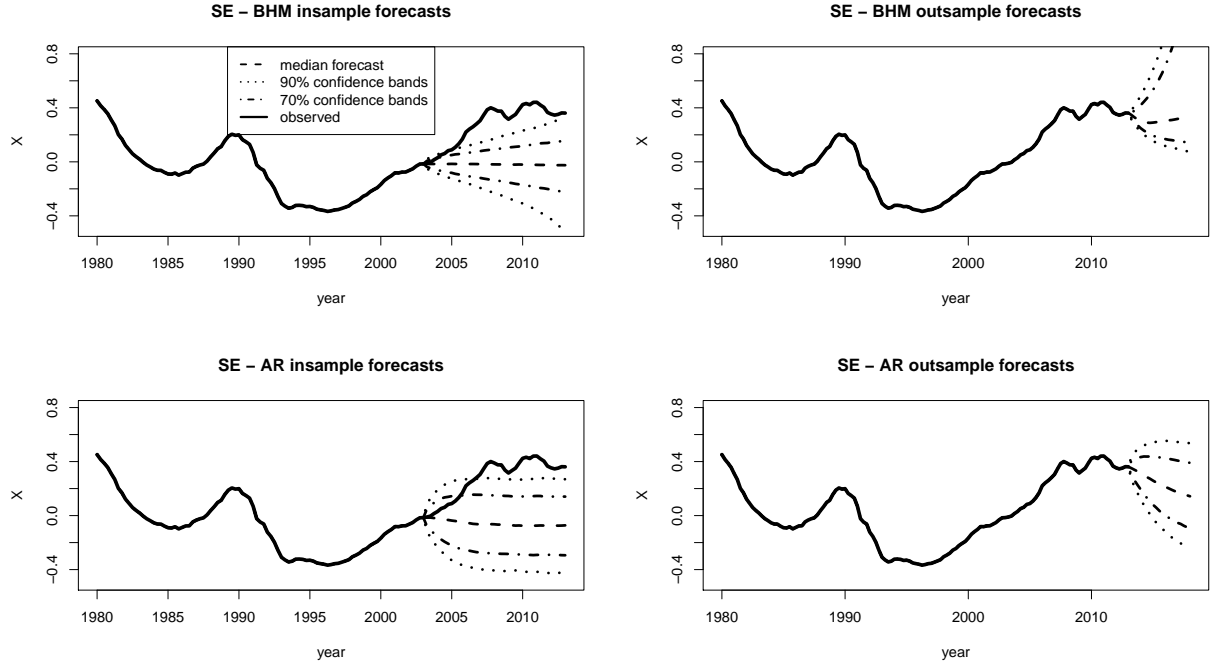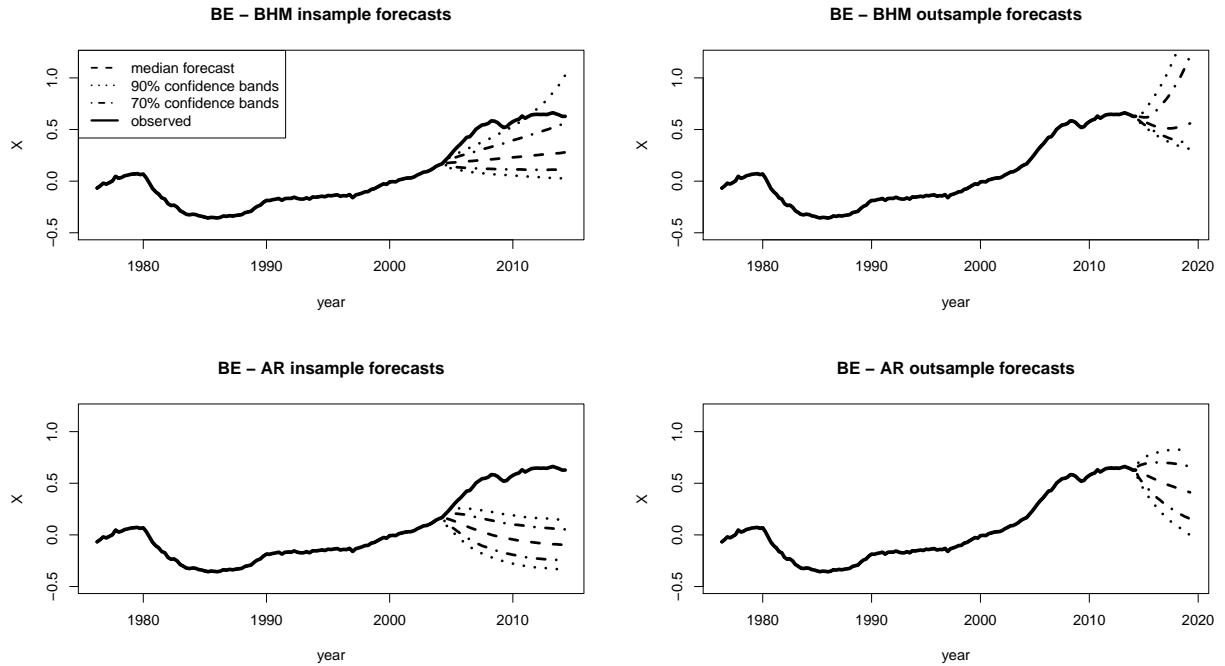
non-linear forecasts. For instance, we observe that for the US the out-of-sample forecasts for the agent-based model have much smaller predictive intervals than for the AR model. This can be explained as follows. The price at the start of the out-of-sample forecast is close to the fundamental price, in which case both models produce similar one-step ahead mean (and median) forecasts. The width of the predictive density is then proportional to the residual standard deviation, which is smaller for the agent-based model than for the AR model (i.e. the agent-based model gave a better fit). However, it can be observed in the in-sample forecasts for the US that the agent-based model gives rise to wider forecast intervals for longer horizons. This is related to the fact that for the AR model the width of the forecast interval is independent of the state (it only depends on the forecast horizon, in which the forecast variance grows linearly) while in the non-linear agent-based model the width of the forecast interval also depends on the state itself. In fact, for all in-sample forecasts during a housing bubble, e.g. in the US, UK, NL, BE and ES, the forecast intervals of the BHM model include large bubbles, while the AR model predicts mean-reversion towards the fundamental. Moreover, for JP the in-sample forecast of the BHM model (correctly) predicted a further decline, while the AR model predicted a recovery back to fundamentals.

For some of the agent-based model forecasts (for in-sample US, UK, NL, BE and out-of-sample SE) it can be observed that the top (95%) forecast quantile diverges as the forecast horizon increases. This suggests that not only the fundamental equilibrium is unstable, but the system is globally unstable. However, note that our one-step-ahead model is used here to construct forecasts over multiple step forecast horizons, while it really is constructed and calibrated as a one-step-ahead model. Therefore, these medium-term forecasts should be interpreted with care. To obtain better medium-term forecasts, a medium-term model should be constructed and estimated instead.

4 Dynamics and Policy Implications

What are the policy implications of our nonlinear model? In order to draw some policy conclusions an analysis of the key features of the nonlinear dynamics is necessary. We therefore first investigate the housing price dynamics of the nonlinear heterogeneous expectations switching model for the range of parameter values containing the estimated coefficients for all countries. In particular, we discuss *multiple steady state equilibria* and *global instability* of the system and then discuss how policy can avoid these "dark corners" of the economy (Blanchard, 2014). We focus on the role of R as the policy parameter, as it can be influenced by policy makers.

As a first step in the analysis of nonlinear systems, one needs to study the local stability and bifurcations of the fundamental steady state. A *bifurcation* is a qualitative change in the dynamics of a nonlinear model, for example a change of stability of a steady state, when a

parameter changes. Recall from the stability condition (8) that the fundamental equilibrium $X_t = 0$ is locally stable when

$$\frac{\phi_1 + \phi_2}{2(R + \bar{\alpha})} = \frac{(1+g)(\phi_1 + \phi_2)}{2(1 + r + \alpha)} < 1.$$

where we have dropped the absolute value, since all parameters are positive.

Note that the fundamental equilibrium is locally *unstable* when the average of ϕ_1 and ϕ_2 is larger than $R + \bar{\alpha}$, which means that the local stability is affected directly by behavioural parameters as well as institutional parameters.

For all eight countries, the estimated parameter values of the model were found to be close to the border of (in)stability of the fundamental steady state. The bifurcation diagrams in Fig. 11 show numerically the transitions that occur for each country as R decreases, with all other parameters fixed at the country specific estimates. The fundamental steady state destabilises when R decreases and hits the critical threshold

$$R_{crit} = \frac{\phi_1 + \phi_2}{2} - \bar{\alpha},$$

the estimated values of which are shown in the last row of Table 3. Notice that lowering R is equivalent to lowering the risk-free mortgage interest rate r^{rf}, lowering the tax rate ω or increasing the growth rate g of the rent.

The bifurcation diagrams in Fig. 11 are constructed by plotting for each parameter value R (from a finite grid) 100 subsequent states visited by the model after a transient of 100 iterations. A small amount of dynamic noise was added, $\text{NID}(0, \sigma^2)$ with $\sigma = 10^{-4}$, to enable the system to move away from the equilibrium once it becomes unstable.

There are two different bifurcation scenarios for different countries. For UK, NL, ES, BE and SE *two co-existing stable non-fundamental steady states* arise, one above and one below the unstable fundamental steady state, for values of R below the critical threshold. Hence, for these countries the nonlinear model exhibits *multiple stable steady state equilibria* for low values of R. The diagram also illustrates that as R decreases further, the model becomes globally unstable with exploding house prices. In contrast, the primary bifurcations to instability for US, UK and CH are different: the fundamental steady state becomes *globally unstable*, with exploding house prices, for R values immediately below the critical threshold. A careful reader may also observe that the critical thresholds R_{crit} for the US and CH are below 1 and therefore might think that a bifurcation towards instability can not arise. However, as will become clear below, even when the fundamental steady state is locally stable for this nonlinear system, global instability may arise due to small noise shocks.
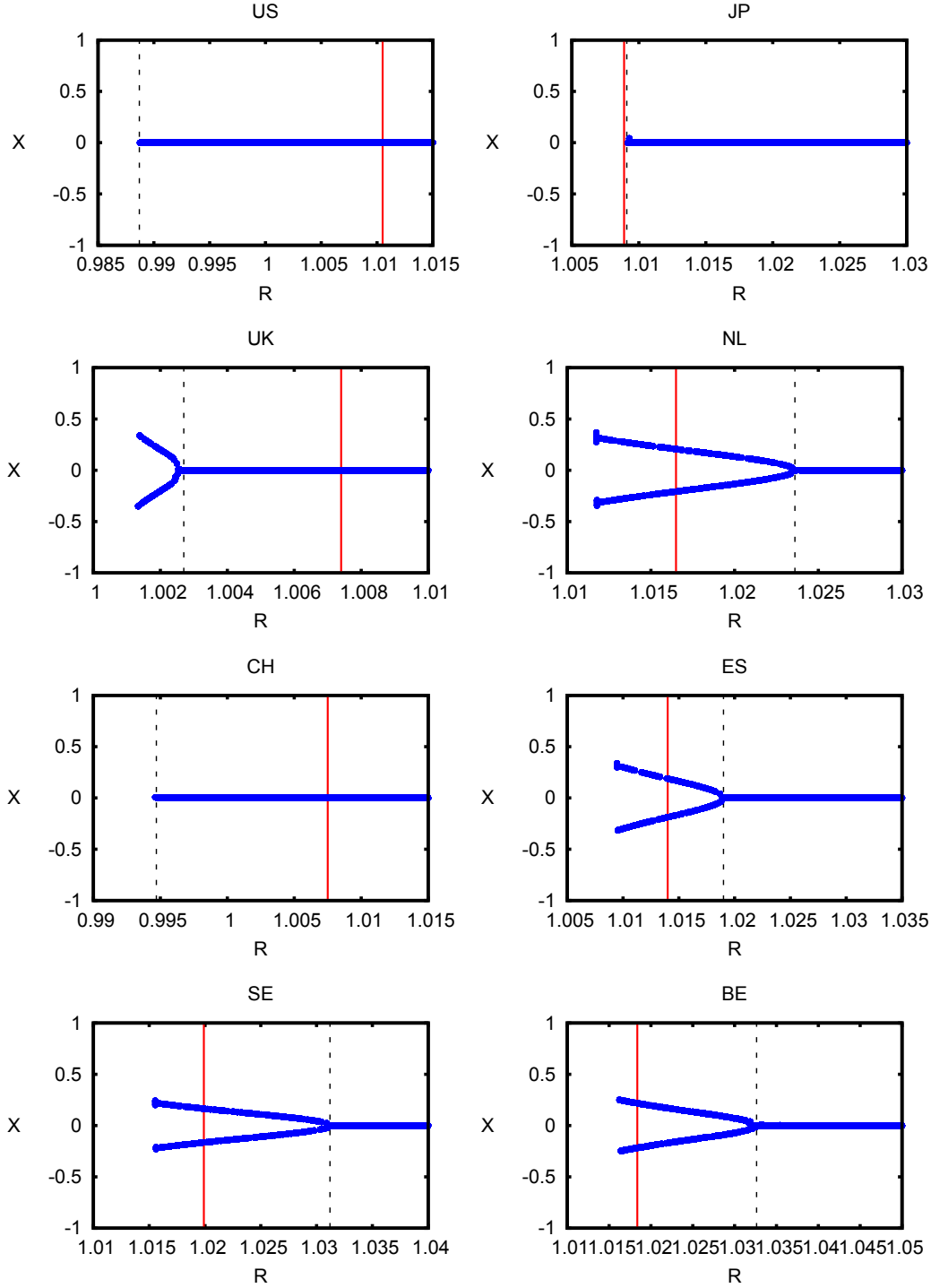
Figure 11: Bifurcation diagrams showing the long run behaviour for each country as a function of the bifurcation parameter R, with other parameters fixed at the estimated values. The solid vertical lines indicate the estimated values of R and the dashed vertical lines the implied critical value R_{crit} of R at which the primary bifurcation occurs (transition from local stability to (local) instability of the fundamental steady state).

Pitchfork bifurcation

The bifurcations observed in the housing model for the eight countries in Fig. 11 are so-called *pitchfork bifurcations*. There are two different cases for the pitchfork bifurcation, a supercritical and a subcritical, and both theoretical cases, together with the estimates for the eight countries, are illustrated in Figure 12.[8] In both cases, below the critical value, R_{crit}, the fundamental steady state is *unstable*. In the case of a supercritical pitchfork (left panel), when the fundamental steady state becomes unstable, two additional non-fundamental steady states are created, one above and one below the fundamental, both of which are *stable*. These non-fundamental steady states lie on a "parabola" emanating from the stable fundamental steady state to the left.
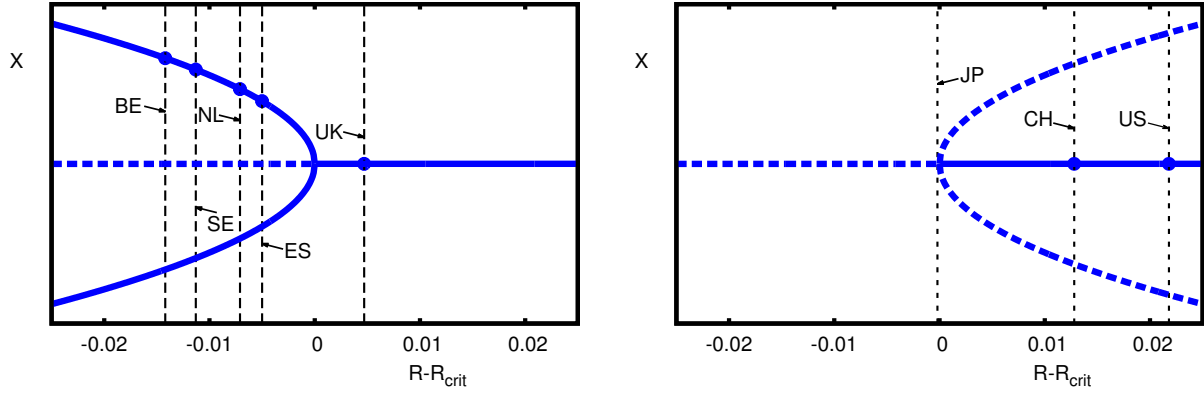

Figure 12: Pitchfork bifurcation w.r.t. the policy parameter R. Bold curves denote stable steady states; dotted curves are unstable steady states. **Left Panel:** supercritical pitchfork bifurcation with stable fundamental steady state for $R > R_{\text{crit}}$ and unstable fundamental steady state surrounded by two stable non-fundamental steady states for $R < R_{\text{crit}}$. **Right Panel:** subcritical pitchfork bifurcation with stable fundamental steady state surrounded by a corridor of stability bounded by two unstable non-fundamental steady states for $R > R_{\text{crit}}$ and (globally) unstable fundamental steady state with exploding dynamics for $R < R_{\text{crit}}$. The dots represent the estimated values $R - R_{crit}$ for the eight countries.

The second case of a subcritical pitchfork bifurcation is illustrated in Figure 12 (right panel). As before, the fundamental steady state becomes unstable as the parameter R decreases below its critical value R_{crit}. In the subcritical case, however, the two non-fundamental steady states exist for $R > R_{\text{crit}}$, lying on a "parabola" emanating from the fundamental steady state to the right. Both non-fundamental steady states are *unstable* and they form a *corridor of stability* around the stable fundamental steady state. Initial states between the two non-

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[8]The first order condition for a pitchfork bifurcation is that the linearized system has an eigenvalue $+1$; whether the pitchfork is super- or subcritical depends on higher order derivatives of the system at bifurcation. See Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory.

fundamental steady states converge to the fundamental steady state; initial states outside this corridor of stability diverge (possibly to infinity, depending on higher order nonlinearities). A decrease of the policy parameter R may now have dramatic consequences: if R decreases below R_{crit}, the fundamental steady state becomes globally unstable, as the corridor of stability shrinks to zero and disappears at the critical value R_{crit} and the system becomes explosive. Notice that, when the system is buffeted with small stochastic shocks, even for $R > R_{\text{crit}}$ close to the bifurcation point, that is, before the critical threshold is reached, the price dynamics may easily escape and diverge from fundamentals along exploding bubbles.

It should be stressed that our simple nonlinear housing model and the pitchfork bifurcation are by no means artificial or exceptional. On the contrary, the pitchfork bifurcation, supercritical as well as subcritical, is a *generic* phenomenon that can arise in many (higher dimensional) nonlinear systems for example in a more detailed model of the housing market.[9]

Policy Implications

Estimation of the nonlinear heterogeneous expectations switching model shows the occurrence of long-lasting housing bubbles in all eight countries. What can a policy maker do to stabilise house price bubbles and prevent market instability? This is a partial equilibrium approach that aims to classify house price changes into categories of acceptable and unacceptable (i.e. dangerous) movements. To understand the welfare implications of such instability one would need to incorporate them into general equilibrium models.

Nevertheless, what this model can do is provide early warnings when the system may be approaching what (Blanchard, 2014) called the 'dark corners'[10]. The general lesson for policy makers to be drawn from our analysis of the nonlinear housing model is that structural knowledge of the system may yield important insights in policies that can prevent local or even global instability. In general terms, the policy maker should prevent the system from getting too close to bifurcation points that may destabilise the system. An interesting new methodological contribution of our analysis of the stylized housing model is that the policy maker should in particular be aware of preventing the so-called "hard bifurcations" of the system, such as the subcritical pitchfork, which may cause a sudden, *critical transition* of the system leading to an exploding bubble or market collapse.

In terms of our stylized housing model and the bifurcation diagrams in Figures 11 and 12 the policy maker should keep the parameter R sufficiently large, so that the system stays away

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[9]To be mathematically precise, one should say that pitchfork bifurcations are generic in systems that are symmetric w.r.t. to a coordinate axis. When symmetry breaks down, the pitchfork bifurcation "breaks up" into the generic non-symmetric case with two curves, an equilibrium curve and a saddle-node bifurcation curve, with very similar dynamics as for the pitchfork. We refer once more to Kuznetsov (1995) for a detailed mathematical treatment of bifurcation theory and its importance in applications.

[10]See the quotes from Blanchard (2014) in our Introduction.

from the locally or even globally unstable fundamental steady state (the dark corners). Recall that $R = \frac{1+r}{1+g}$, where g is the growth rate of housing rents and $r = r^{rf} + \omega$ is the sum of the risk-free mortgage rate and the tax/maintenance cost rate. Hence, stabilizing policies include an increase of the mortgage interest rate, a decrease of mortgage tax deduction rates, an increase of tax rates for home owners, and/or a decrease of (the growth of) housing rents. Notice also an important difference in policy implications compared to the RE fundamental benchmark model. Under RE these policies *only* affect the level to fundamental house prices. In our behavioural 2-type switching model these policies affect both the level of the fundamental steady state and the market volatility, that is, the *stability of the dynamics* around the steady state is directly affected by these policies.

5 Concluding Remarks

We have developed a nonlinear empirical housing market model with heterogeneous beliefs. The standard model with housing price fundamentals based on imputed rents is nested as a special case within the general heterogeneous setup. Agents however disagree in their expectations about future house prices and switch between a mean-reverting, fundamentalists and a mean-diverting, trend-following forecasting rule, based upon their relative performance. The heterogeneous beliefs housing model with endogenous switching displays nonlinear aggregate price fluctuations, with booms and busts, around the fundamental price triggered by stochastic shocks and strongly amplified by self-fulfilling expectations.

Using quarterly data on rents and house prices, we estimate the model parameters for eight different countries, US, UK, NL, JP, CH, ES, SE and BE. In all countries the data support heterogeneity in expectations, with temporary switching between fundamental-reverting and trend-following beliefs. For all countries we identify long-lasting temporary house price bubbles, driven or amplified by trend extrapolation. For four countries, US, UK, NL and ES, we identify strong housing bubbles in the period 2004-2007, while for JP and CH housing bubbles in the 1980-1990s are identified, in all cases strongly amplified by trend-following behaviour. When these bubbles burst, the majority of agents switches to a mean-reverting fundamental strategy reinforcing a strong correction of housing prices. As indicated by the fancharts, the qualitative in-sample and out-of-sample predictions of the non-linear switching model are very different from those of standard linear benchmark models with a rational representative agent. For two countries, BE and SE the results identified high overvaluations (bubble) but where the correction has not happened. The instability shown by the AR coeffcient as well as the convergence of opinions in favour of prices returning to their fundamental level, suggest that this correction is still to happen The speed and extent of this correction is difficult to predict however, especially in light of respective authorities already taking action to stabilize

the system (as is also the case for the UK).

These results have important policy implications. The underlying nonlinear switching model exhibits multiple equilibria and/or global instability for parameter-values close to the estimated values for all countries. We have argued that a decrease of the (mortgage) interest rate, a decrease of the tax rate for home owners, an increase of mortgage tax deduction rates and/or an increase of housing rents all shift the nonlinear system closer to multiple equilibria and global instability. Policy should prevent the system getting too close to bifurcation in order to avoid critical transitions to global instability. The market sentiment, that is the average extrapolation factor of our model, may serve as an early warning indicator when the system is approaching the border of instability.

Our housing model with heterogeneous beliefs is very stylized and the results should be viewed as a 'proof of principle' to show that a nonlinear model with switching can lead to very different behaviour than a benchmark linear model. Nonlinear systems may easily turn unstable, but structural nonlinear modeling can also provide new insights for policy makers on how to prevent critical transitions towards instability and collapse. Our stylised nonlinear housing market model is just a simple, but empirically relevant, example, illustrating potential policy tools for taming instability. Building more realistic nonlinear economic models based upon country specific institutional details can give important new insights for policy makers, particularly in extreme times of crises.

In order to study realistic policy scenarios in more detail future research should focus on the following extensions. First of all, our benchmark fundamental value is very stylized and general, and for policy analysis it would be important to include more country specific institutional details of the housing market, such as tax rules and mortgage requirements, in the fundamental value. One could then easily re-estimate the model and study the amplification mechanism of endogenous belief switching around an improved and more realistic fundamental benchmark. Second, our two forecasting rules are in fact the simplest linear examples with only one lag. It would be of interest to have further guidance on which types of forecasting rules to use in these models, for example through laboratory experiments on expectations with human subjects and/or using surveys of forecasts. In general, building more realistic behavioural models for policy analysis should be high on the research agenda of academics and policy makers.

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A Appendix

A.1 Derivation of dynamics of X_t

Substituting $X_t = \frac{P_t}{P_t^*} - 1$ in terms of X_t and P_t^* into the price equation (4) gives

$$(1 + r + \alpha)(P_t^* X_t + P_t^*) = \sum_{h=1}^{2} n_{h,t} \mathbb{E}_{h,t} \left(P_{t+1}^* X_{t+1} + P_{t+1}^* + (1 + r^{\text{rf}})Q_t \right),$$

or, using $P_t^* = \frac{1 + r^{\text{rf}}}{r + \alpha - g} Q_t$

$$(1 + r + \alpha)Q_t(X_t + 1) = \sum_{h=1}^{2} n_{h,t} \mathbb{E}_{h,t} \left(Q_{t+1}(X_{t+1} + 1) + (r + \alpha - g)Q_t \right).$$

Since the fundamental process satisfies

$$(1 + r + \alpha)Q_t = \sum_{h=1}^{2} n_{h,t} \mathbb{E}_{h,t} \left(Q_{t+1} + (r + \alpha - g)Q_t \right),$$

what remains is

$$(1 + r + \alpha)Q_t X_t = \sum_{h=1}^{2} n_{h,t} \mathbb{E}_{h,t} \left(Q_{t+1} X_{t+1} \right).$$

The right hand side equals $(1 + g)Q_t \mathbb{E}_{h,t}(X_{t+1})$ if $\mathbb{E}_{h,t}(Q_{t+1}X_{t+1}) = \mathbb{E}_{h,t}(Q_{t+1})\mathbb{E}_{h,t}(X_{t+1}) = (1 + g)Q_t \mathbb{E}_{h,t}(X_{t+1})$, $h = 1, 2$, resulting in

$$X_t = \frac{1 + g}{1 + r + \alpha} \sum_{h=1}^{2} n_{h,t} \mathbb{E}_{h,t} \left(X_{t+1} \right).$$

A.2 Expressing R_{t-1} in terms of X_{t-1} and X_{t-2}

Since

$$X_t = \frac{P_t}{P_t^*} - 1 = \frac{P_t}{\left(\frac{1 + r^{\text{rf}}}{r + \alpha - g} Q_{t-1} \right)} - 1,$$

we can write $P_t = (X_t + 1)\frac{1+r^{\text{rf}}}{r+\alpha-g}Q_t$, and express R_{t-1} as

$$
\begin{aligned}
R_{t-1} &= \frac{P_{t-1} + (1+r^{\text{rf}})Q_{t-2} - (1+r)P_{t-2}}{P_{t-2}} \\
&= (X_{t-1}+1)\frac{1+r^{\text{rf}}}{r+\alpha-g}\frac{Q_{t-1}}{P_{t-2}} + \left[(1+r^{\text{rf}}) - (1+r)\frac{1+r^{\text{rf}}}{r+\alpha-g}(X_{t-2}+1)\right]\frac{Q_{t-2}}{P_{t-2}} \\
&\approx (X_{t-1}+1)\frac{1+r^{\text{rf}}}{r+\alpha-g}(1+g)\frac{Q_{t-2}}{P_{t-2}} + \left[(1+r^{\text{rf}}) - (1+r)\frac{1+r^{\text{rf}}}{r+\alpha-g}(X_{t-2}+1)\right]\frac{Q_{t-2}}{P_{t-2}} \\
&= \frac{1+r^{\text{rf}}}{r+\alpha-g}\frac{Q_{t-2}}{P_{t-2}}\left((X_{t-1}+1)(1+g) + \left[(r+\alpha-g) - (1+r)(X_{t-2}+1)\right]\right) \\
&= (1+g)Y_{t-2}\left(X_{t-1} + \bar{\alpha} - RX_{t-2}\right),
\end{aligned}
$$

where $Y_{t-2} = \frac{1+r^{\text{rf}}}{r+\alpha-g}\frac{Q_{t-2}}{P_{t-2}} = \frac{P^*_{t-2}}{P_{t-2}}$.

A.3 Conditional variance

Assuming that agents have fundamental beliefs regarding the conditional variance leads to

$$
\begin{aligned}
V_{t-2} &= \text{Var}_{t-2}\left(\frac{P_{t-1} + (1+r^{\text{rf}})Q_{t-2} - (1+r)P_{t-2}}{P_{t-2}}\right) \\
&= \frac{Q^2_{t-2}}{P^2_{t-2}}\text{Var}_{t-2}\left(\frac{P_{t-1}}{Q_{t-2}}\right) \\
&= \frac{Q^2_{t-2}}{P^2_{t-2}}\text{Var}_{t-2}\left(\frac{P^*_{t-1}}{Q_{t-2}}\right) \\
&= \frac{Q^2_{t-2}}{P^2_{t-2}}\text{Var}_{t-2}\left(\frac{1+r^{\text{rf}}}{r+\alpha-g}\frac{Q_{t-1}}{Q_{t-2}}\right) \\
&= Y^2_{t-2}\text{Var}\left((1+g)\varepsilon_t\right) \\
&= Y^2_{t-2}\eta^2,
\end{aligned}
$$

where $\eta^2 = \text{Var}\left((1+g)\varepsilon_t\right)$.

A.4 Estimated behavioural parameters for $\bar{\alpha} = 0.005$

Country	US	JP	UK	NL	CH	ES	SE	BE
	fundamental parameters							
$\bar{\alpha}$	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050
$R+\bar{\alpha}$	1.0105	1.0089	1.0074	1.0165	1.0075	1.0140	1.0199	1.0184
	estimated behavioural parameters							
ϕ_1	0.8360*** (0.0705)	0.9364*** (0.0140)	0.9093*** (0.0222)	0.9552*** (0.0256)	0.8782*** (0.0267)	0.9587*** (0.0088)	0.8854*** (0.1129)	0.9909*** (0.0128)
$\Delta\phi$	0.3097* (0.1384)	0.1558*** (0.0252)	0.1674*** (0.0380)	0.1130* (0.0498)	0.2292*** (0.0430)	0.1119*** (0.0151)	0.2826 (0.2257)	0.0701*** (0.0201)
$\beta\ (\times 10^3)$	1.285 (1.508)	3.098° (1.623)	1.457 (1.025)	2.875 (3.515)	2.680 (1.957)	1.375* (0.653)	0.434 (0.789)	10.00 (12.95)
δ	0.0000 (0.2545)	0.0000 (0.2369)	0.0000 (0.2460)	0.3996 (0.2727)	0.4624** (0.1525)	0.1761 (0.1878)	0.0699 (0.1683)	0.5726*** (0.1572)
BIC	-1007.543	-938.6408	-725.8254	-718.0138	-808.7961	-763.0292	-606.0777	-784.6270
	implied parameters/local stability							
ϕ_2	1.1457*** (0.0693)	1.0922*** (0.0134)	1.0767*** (0.0191)	1.0681*** (0.0260)	1.1075*** (0.0211)	1.0706*** (0.0082)	1.1680*** (0.1134)	1.0610*** (0.0095)
$\phi_2-(R+\bar{\alpha})$	0.1351* (0.0693)	0.0833*** (0.0134)	0.0693*** (0.0191)	0.0517* (0.0260)	0.0999*** (0.0211)	0.0566° (0.0082)	0.0958 (0.1134)	0.0427 (0.0095)
$\frac{\phi_1+\phi_2}{2(R+\bar{\alpha})}$	0.9804*** (0.0097)	1.0053*** (0.0054)	0.9857*** (0.0082)	0.9952*** (0.0068)	0.9854*** (0.0106)	1.0006*** (0.0039)	1.0067*** (0.0081)	1.0075*** (0.0046)
loc. stability	stable	unstable	stable	stable	stable	unstable	unstable	unstable
$R_{\text{crit.}}$	0.9858*** (0.0098)	1.0093*** (0.0054)	0.9880*** (0.0083)	1.0067*** (0.0069)	0.9879*** (0.0107)	1.0097*** (0.0040)	1.0217*** (0.0082)	1.0210*** (0.0047)

Table 4: Fundamental parameters (fixed during estimation of the behavioural parameters), estimated behavioural model parameters for $\bar{\alpha} = 0.005$, and corresponding implied behavioural parameters. Signif. codes: 0.00 '***' 0.001 '**' 0.01 '*' 0.05 '∘' 0.10.