TI 2014-131/VII
Tinbergen Institute Discussion Paper

Price Competition on Graphs

Pim Heijnen
Adriaan Soetevent

University of Groningen, the Netherlands.

Price competition on graphs

Pim Heijnen* Adriaan R. Soetevent†

September 26, 2014

Abstract

This paper extends Hotelling's model of price competition with quadratic transportation costs from a line to graphs. We derive an algorithm to calculate firm-level demand for any given graph, conditional on prices and firm locations. These graph models of price competition may lead to spatial discontinuities in firm-level demand. We show that the existence result of D'Aspremont *et al.* (1979) does not extend to simple star graphs and conjecture that this non-existence result holds more generally for all graph models with two or more firms that cannot be reduced to a line or circle.

JEL classification: D43, L10, R12
Keywords: spatial competition, Hotelling, graphs.

*Corresponding author: Faculty of Economics and Business, University of Groningen, P.O. Box 800, 9700AV, Groningen, e-mail: p.heijnen@rug.nl

†Faculty of Economics and Business, University of Groningen, and Tinbergen Institute. e-mail: a.r.soetevent@rug.nl. Soetevent's research is supported by the Netherlands Organisation for Scientific Research under grant 451-07-010. Most of this work was done while Soetevent was affiliated to the University of Amsterdam. The study benefited from comments by Robert Adams, Matteo Alvisi, Daniele Condorelli, Nicholas Economides, Paolo Garella, Jeroen Hinloopen, Stephen Martin, Nicolas de Roos and participants at the IIOC 2011, CeNDEF at 15 workshop, ASSA 2014 EEA 2014 and EARIE 2014 meetings, and seminar participants at the Vrije Universiteit Amsterdam. The usual disclaimer applies.

1 Introduction

Firms face two opposing incentives in the decision where to locate relative to competitors. A location close to one's competitors maximizes the opportunities to capture one's competitors' consumers, but at the same time, little spatial or product differentiation increases price competition among firms. Hotelling (1929) introduced a stylized linear model of spatial (product) differentiation to analyze which of these is the dominant force.[1]

The current paper generalizes Hotelling's line model of spatial (product) differentiation to graphs. For markets with two competing firms and transportation cost quadratic in distance, we outline an algorithm that calculates firm-specific demand as a function of the firms' prices and conditional on their position in the graph. In other words, for any structure of lines and intersections that one can draw on a piece of paper and the position of the firms on this structure, the algorithm will give firm-level demand. The constructed graphs may be as arbitrary as the patterns of released sticks in the game of Pick-up sticks, after the isolated sticks have been removed. Consumers are assumed to be uniformly distributed on the graph's edges. The line model with quadratic transportation cost as studied by D'Aspremont *et al.* (1979) arises as a special case.

The prime motivation for this extension is that in reality, firms cannot locate just anywhere on a plane but are constrained by zoning, geography and roads. As a result, observations of clustering by firms in physical space are not the exclusive result from firm conduct but may as well reflect the structure of the product space. Recent empirical studies acknowledge this and use techniques from spatial statistics to develop measures of spatial clustering that correct for this (Picone *et al.*, 2009).[2] To the best of our knowledge, no theoretical models exist that evaluate what price profit-maximizing firms would choose on a graph, conditional on their own location and those of competitors. Throughout this

[1]D'Aspremont *et al.* (1979) show the invalidity of Hotelling's original claim that with transportation cost linear with respect to distance, firms tend to minimally differentiate. They demonstrate that, in a model with transportation cost quadratic in distance, a price equilibrium solution exists for any pair of locations and firms maximally differentiate. Irmen and Thisse (1998) conclude that, despite differences in modeling assumptions, the outcome of most theoretical models is that firms seek to differentiate in order to avoid price competition. They however show that when the analysis is extended from one-dimensional to multi-dimensional characteristics space while upholding the quadratic transportation costs, firms only maximally differentiate in a single dimension and thus Hotelling was "almost right".

[2]The presented graphs are best interpreted as models of differentiation in physical space but interpretations as models of differentiation in product space may be possible.

paper, firm's location will be taken exogenous. That is, we focus on the second stage of the two-stage game with firms competing in prices in the second stage after having chosen their location in the first stage.

Section 2 introduces the model. In Section 3, we analyze how consumer demand behaves on each edge of the graph. We subsequently aggregate over all edges and construct firm-level demand as a function of prices. We show that these demand functions are piecewise linear with at most one discontinuity, at the point where both firms set equal prices. This allows us to derive in Section 4 sufficient conditions for the non-existence of pure-strategy Nash equilibria in any given graph with a given firm location configuration. With some examples, we illustrate for which graphs and firm locations one is less likely to find pure-strategy price equilibria. To complete, we prove in a straightforward extension of Dasgupta and Maskin (1986b, Th. 3) that equilibria in mixed strategies always exist.

Importantly, a number of standard results do not carry over from the unit interval to graph models of price competition. First, when transportation costs are quadratic – as we will assume throughout – spatial discontinuities in firm-level demand may occur. That is, consumer's with a preference for firm i's product are surrounded by consumers with a preference for its competitor's product. Second, in contrast to D'Aspremont *et al.* (1979), the assumption of quadratic transportation cost no longer is a sufficient condition for the existence of a Nash equilibrium in pure strategies.[3]

We show that for arguably the simplest extension of the line model, the "Hotelling line with a junction" (a $K_{1,3}$ graph), there always exist firm location configurations for which the price competition game does not possess a noncooperative equilibrium in pure strategies. We compare this result with the non-existence result in Varian (1980) and argue that the market context and line of proof is different here. This non-existence result is the main reason for not endogenizing firm locations.[4] We conjecture that for

[3]The assumption of quadratic transportation cost, where disutility rises more than proportional with distance is often thought to be more appropriate in models where "distance" is not interpreted as a physical distance but proxies for the difference between the characteristics of the product bought and the most preferred variety. Within the current model, non-linear transportation costs may however reflect increased search cost: the greater the distance between the consumer and the firm, at the more crossroads the consumer has to take the right turn to reach the firm.

[4]See Osborne and Pitchik (1987) for the complexity of characterizing the mixed strategy equilibria even in the original Hotelling model with travel cost proportional to distance. Implementing the first-stage is also computationally difficult because in each step one has to evaluate the profits associated with

every graph with at least one node with degree 3, firms can always be located such that no equilibrium price solution exists. We derive conditions that suggest a method for finding such locations (Proposition 6) and we illustrate this method by means of an example.

The paper is related to studies on pricing on networks that have appeared, like Bloch and Querou (2013). These studies however locate firms and consumers at nodes, following the modeling methodology common in social network analysis. In particular, the edges in these models are "void": they are not inhabited by a density of consumers but only serve the purpose of connecting two nodes.[5] The graphs presented here are fundamentally different because consumers are assumed uniformly distributed along the edges of the graph.[6] This is in the spirit of Hotelling's line model, Salop's (1979) circular model and Von Ungern-Sternberg's (1991) pyramid model. In a recent contribution, Fournier and Scarsini (2014) also assume uniformly distributed consumers. With firms competing on location but not on price, they show the existence of a pure Nash equilibrium when the number of firms is sufficiently large. Firms can only choose location in their model, whereas in our model location is given, but firms set prices.

A number of papers (Mills and Lav, 1964; Eaton and Lipsey, 1976; Greenhut, Hwang and Ohta, 1976; Holahan and Schuler, 1981) have studied location choice and price competition on two-dimensional spatial markets with constant transport cost per unit distance and free entry. The starting point of this literature is the well-known result first asserted by Lösch (1954) and formally proven by Bollobás and Stern (1972) that, conditional on every consumer in the plane being served and constant transport cost per unit distance, a division of market demand into hexagons is socially optimal. Subsequent contributions have questioned whether the hexagonal configuration is the unique equilibrium when the number of firms is given and the extent to which this configuration results under free entry.[7] These studies have in common that consumers are assumed uniformly distributed

infinite number of possible locations for firm i, conditional on the position of its competitors.

[5]Buechel and Röhl (2015) characterize the set of equilibria in a firm location game on such a network. They establish that, in robust equilibria, firms cluster, thereby vindicating Hotelling's intuition.

[6]Commuting behavior of consumers is not considered, see Claycombe and Mahan (1991); Raith (1996) for theoretical contributions and Houde (2012) for a state-of-the-art empirical study.

[7]Eaton and Lipsey (1976) demonstrate that without entry and/or exit, next to the hexagonal configuration, equilibrium configurations of squares and rectangles can occur but that the rectangular lattice

over a plane. Instead, the current paper is concerned with studying the generic properties of spatial models of product competition with consumers uniformly distributed along the edges of a given graph. We do not study which firm configurations result under entry for any particular (class of) graphs.

We limit attention to the situation with two firms and quadratic transportation cost but conceptually, the analytical approach can be extended in a straightforward manner to cover situations with non-quadratic cost and multi-firm competition. In fact, as in the pyramid model in Von Ungern-Sternberg (1991), graphs (or network structures) easily allow for multi-firm competition. The difference with the model by Von Ungern-Sternberg is that no analytical solutions are available for less stylized graphs.

2 Model and preliminary results

There are two firms, A and B, who have zero cost of production and who simultaneously set prices; p_A and p_B, respectively. Consumers are uniformly distributed along a graph (with a total mass of one) on which the firms have a fixed location.

A graph $\mathcal{G}$ is described by a finite set of nodes $N = \{1, \ldots, n\}$ and a set of edges $E \subset N \times N$. Generic elements of N are denoted by i and j. If i and j share an edge, then $(i, j) \in E$. We abbreviate (i, j) by ij. Furthermore, node i has a physical location $x_i \in \mathbb{R}^2$. Consequently, we think of an edge ij as a straight line connecting x_i and x_j. The length of edge ij is then $\ell_{ij} = \|x_i - x_j\|$. We assume that $\mathcal{G}$ is connected.

Let firm A be located at node 1 and firm B at node 2. This is without loss of generality in the sense that we can always create an extra node on the edge where the firm is located and relabeling the nodes accordingly. Mostly this will generate redundant nodes that lie on a straight line between two other nodes. If the firm happens to be located on a node, then it will only lead to relabeling of the nodes. The firms' location does not change the underlying graph. However, when locations are fixed, the analysis is considerably easier if we are allowed to exclude the possibility that the firms are located in the interior of an edge.

A location on an edge is denoted by $z \in ij$. Note that $z = tx_i + (1 - t)x_j$ for some

seems most robust to entry.

$t \in (0,1)$. It is useful to be able to talk about small perturbations along edges: let $z_\varepsilon = z + \varepsilon(x_j - x_i)$, where ε is sufficiently close to zero such that $z_\varepsilon \in ij$.

We assume that the market is covered and that consumers minimize transportation cost plus price. Transportation cost is the square of the distance of the consumer to the firm. On a graph, distance from one point to another is less trivial than in Euclidean spaces. Shortest paths are usually unique, but there can be locations for which the direction of the shortest path changes. For the moment we postpone the discussion of how to calculate these shortest paths and assume that for each node $i > 2$, the distance to node 1 and node 2 are known.[8] We denote the corresponding distance to firm A and B by d_{Ai} and d_{Bi}, respectively. This information is sufficient to calculate the minimal distance for each point on the graph, for $z \in ij$:

$$d_A(z) = \min\{d_{Ai} + \|z - x_i\|, d_{Aj} + \|z - x_j\|\}, \tag{1}$$

and $d_B(z)$ is defined analogously. From this formula, we see that the shortest path goes either via node i or via node j. On each edge, there can be at most one point where the shortest path is non-unique because distance via i or j changes linearly along the edge, increasing in one direction and decreasing in the other. We refer to these points as *change points* (for firm A or B) as these are the points where the direction of the shortest path (to firm A or B) changes. The distance of a perturbation z_ε of z to firm A is related to the distance of z to firm A, provided ε is sufficiently small:

$$d_A(z_\varepsilon) = d_A(z) + \varepsilon\ell_{ij} \text{ (if the shortest path goes via } i\text{)}, \tag{2}$$

$$d_A(z_\varepsilon) = d_A(z) - \varepsilon\ell_{ij} \text{ (if the shortest path goes via } j\text{)}, \tag{3}$$

$$d_A(z_\varepsilon) = d_A(z) - |\varepsilon|\ell_{ij} \text{ (if } z \text{ is a change point)}. \tag{4}$$

Note that the last expression states that change points for firm A are located further from firm A than points in its neighborhood, which is a direct consequence of the fact that the minimum of two affine functions is a concave function (cf. (1)).

[8] Dijkstra's algorithm (Bertsekas, 1991, p. 68-75), for example, calculates the shortest path from a node to all other nodes.

3 Demand

A consumer located at z buys from firm A if $d_A(z)^2 + p_A < d_B(z)^2 + p_B$ or if

$$f(z) \equiv d_A(z)^2 - d_B(z)^2 < \mu \equiv p_B - p_A,$$

where μ is the price difference. Hence the consumer buys from A if $f(z) < \mu$, buys from B is $f(z) > \mu$ and is indifferent when $f(z) = \mu$. Assume that demand from indifferent consumers is split equally between the firms. As in Hotelling's model, the location of the indifferent consumers allow us to calculate demand. On Hotelling's line, all consumers on the left of the indifferent consumer will buy from one firm, those on the right from the other. On more complicated graphs, the picture is less straightforward. To illustrate all different possibilities, we first focus on a single edge and determine demand on this edge.

3.1 Demand on a single edge

Consider edge ij and suppose that the price difference is μ. In most of the proofs below, we compare the value of $f(z_\varepsilon)$ to the value of $f(z)$. By combining (2–4) with the definition of $f(\cdot)$, we get the following cases:

$$f(z_\varepsilon) = f(z) - 2\ell_{ij} d_A(z)\Box_A - 2\ell_{ij} d_B(z)\Box_B, \tag{5}$$

where $\Box_A \in \{-\varepsilon, \varepsilon, |\varepsilon|\}$ and $\Box_B \in \{\varepsilon, -\varepsilon, -|\varepsilon|\}$, depending on whether the shortest path to firm A (B) goes via i, via j or whether z is a change point, respectively.

Proposition 1. *An edge ij contains an interval of indifferent consumers if and only if $\mu = 0$ and there exists $z \in ij$ such that $d_A(z) = d_B(z)$ and both shortest paths have the same direction.*

Proof. The "only if"-part is proved as follows. Suppose there is an interval of indifferent consumers and let z be a location on this interval. Then for ε sufficiently close to zero, we have

$$f(z) = f(z_\varepsilon),$$
$$0 = 2\ell_{ij}\varepsilon[\pm d_A(z) \pm d_B(z)].$$

Note that the RHS can only be equal to zero when $d_A(z) = d_B(z)$ and the shortest paths have the same direction. Since $d_A(z) = d_B(z)$ and z is the location of an indifferent consumer, it follows that $\mu = 0$.

The proof of the if-part is as follows. Suppose there exists $z \in ij$ such that $d_A(z) = d_B(z)$, then $f(z) = 0$. Since $\mu = 0$, the consumer located at z is indifferent. Since the shortest paths have the same direction, from (5) we get,

$$f(z_\varepsilon) = f(z) \pm 2\ell_{ij}\varepsilon[d_A(z) - d_B(z)] = 0.$$

Hence all consumers in the neighborhood of z are also indifferent. ■

This establishes that intervals of indifferent consumers can only appear when both firms set the same price (i.e. $\mu = 0$) and gives necessary and sufficient conditions for such intervals of indifference to occur in terms of the properties of the graph and the position of the firms therein. In Figure 1, the simplest example is illustrated. In this case, $d_A(z) = d_B(z)$ for all $z \in ij$: the entire edge is indifferent. Figure 2 adds one edge to the simplest example, which leads to a change point for firm A on the edge ij, labeled z_A. The conditions of Proposition 1 no longer apply between z_A and x_j since the direction of the shortest path to firm A now points in another direction as the shortest path to B. Only consumer between x_i and z_A are indifferent.

In both examples, if one of the firms lowers its price, then all these consumers will buy from the firm with the lowest price. Therefore, demand for the firm may not be continuous at $\mu = 0$. The next proposition gives a sharper characterization of edges where an interval of the consumers is indifferent.

Proposition 2. *Suppose $\mu = 0$ and there exists $z \in ij$ such that $d_A(z) = d_B(z)$ and both shortest paths have the same direction. Then either the entire edge is indifferent between*

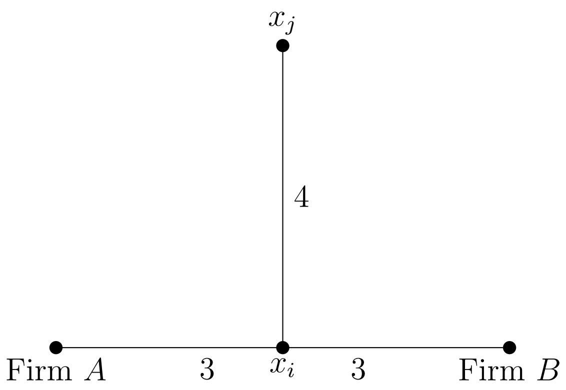


Figure 1: At $\mu = 0$, all consumers on edge ij are indifferent (numbers indicate distances).

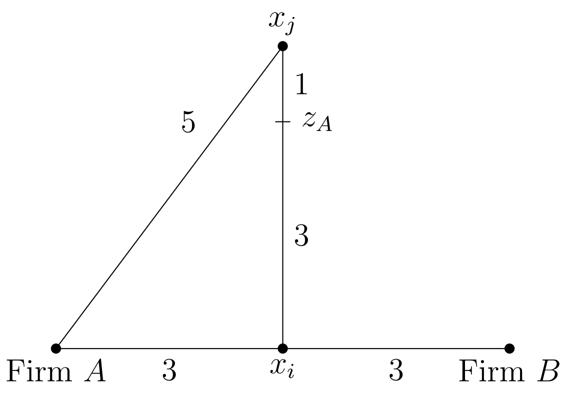


Figure 2: If $\mu = 0$, all consumers between x_i and z_A are indifferent (numbers indicate distances).

A and B or all indifferent consumers are located between one of the nodes and a change point.

Proof. Since change points are precisely those points where the direction of the shortest path changes, it is clear that if there exists $z \in ij$ such that $d_A(z) = d_B(z)$ in between a node and a change point, then all consumers on that part of the edge are indifferent. Moreover, on an edge without change points, the entire edge will be indifferent. We have to show that it is not possible for consumers located between two change points to be indifferent. Let z_A denote the change point for the shortest path to A, and z_B the change point for the shortest path to B. Assume without loss of generality that z_A is closer to x_i than z_B. Points between z_A and z_B are closer to x_j than z_A and closer to x_i than z_B. Since the direction of the shortest path changes at z_A and z_B, we conclude that for points in between z_A and z_B, the direction of the shortest path to A goes via x_j and the direction of the shortest path to B goes via x_i. Hence, the shortest paths have different direction and consumers located between z_A and z_B are not indifferent. ■

Note that this implies that if, for a given edge, one of the end nodes is equidistant from both firms and the shortest paths have the same direction at this node, then part of the edge will be indifferent when $\mu = 0$. If this is true for both nodes (such as in Figure 1), then the entire edge is indifferent.

For the remainder of this paragraph, we exclude the possibility of intervals of indifferent consumers and focus on isolated indifference points by assuming that either $\mu \neq 0$, or there does not exist a $z \in ij$ such that $d_A(z) = d_B(z)$ and both shortest paths have the same direction. We start with an investigation of the location of indifferent consumers. It is useful to distinguish between weak and strong indifference points.

Definition 1 (Weak and strong indifference points)**.** *A location z is an indifference point for a given value of μ if $f(z) = \mu$. A* weak *indifference point is an indifference point with the property that consumers on both sides of z strictly prefer to buy from the same firm. A* strong *indifference point separates consumers who buy from firm A from those who buy from firm B.*

The importance of weak indifference points is that we expect a bifurcation to occur at weak indifference points, i.e. a structural change from an edge where everybody buys A (or alternatively B) to an edge where in the middle of the edge a group of consumers prefer B over A (with two strong indifference points on either side). Weak indifference points are linked to change points, as the following proposition establishes.

Proposition 3. *A weak indifference point is a change point either for firm A or for firm B (or both). Moreover, if z is a change point for firm A and $f(z) > 0$ (or for firm B and $f(z) < 0$), then it is a weak indifference point for $\mu = f(z)$.*

Proof. The proof of the first part is by contradiction. Suppose z is a weak indifference point (for some value of μ) but not a change point. There are four different cases (depending on the direction of the shortest paths). We focus on the case where, at z, the shortest path to both firm A and B leaves the edge via node i, the proof for the other three cases is similar. Then, since z is assumed not to be a change point,

$$d_A(z_\varepsilon) = d_A(z) + \varepsilon \ell_{ij}$$
$$d_B(z_\varepsilon) = d_B(z) + \varepsilon \ell_{ij}$$

and, consequently,

$$f(z_\varepsilon) = f(z) + 2\ell_{ij}\varepsilon[d_A(z) - d_B(z)] = \mu + 2\ell_{ij}\varepsilon[d_A(z) - d_B(z)].$$

Note that $d_A(z) \neq d_B(z)$ by assumption. If $d_A(z) > d_B(z)$, then $f(z_\varepsilon) < \mu$ for $\varepsilon < 0$ and $f(z_\varepsilon) > \mu$ for $\varepsilon > 0$. If $d_A(z) < d_B(z)$, then $f(z_\varepsilon) < \mu$ for $\varepsilon > 0$ and $f(z_\varepsilon) > \mu$ for $\varepsilon < 0$. In both cases, z separates consumers who buy from firm A from those who buy from firm B. It follows that z is a strong indifference point and therefore not weak.

Suppose z is a change point for firm A and $f(z) > 0$. In that case (1) $d_A(z_\varepsilon) = d_A(z) - |\varepsilon|\ell_{ij}$ and (2) $d_A(z) > d_B(z)$. We focus on the case where $d_B(z_\varepsilon) = d_B(z) + \varepsilon\ell_{ij}$. The other case is similar. Then

$$f(z_\varepsilon) = f(z) - 2\ell_{ij}[d_A(z)|\varepsilon| + d_B(z)\varepsilon]$$

It follows that $f(z_\varepsilon) < f(z)$ for both $\varepsilon > 0$ and $\varepsilon < 0$. Hence for $\mu = f(z)$, z is a weak indifference point. ■

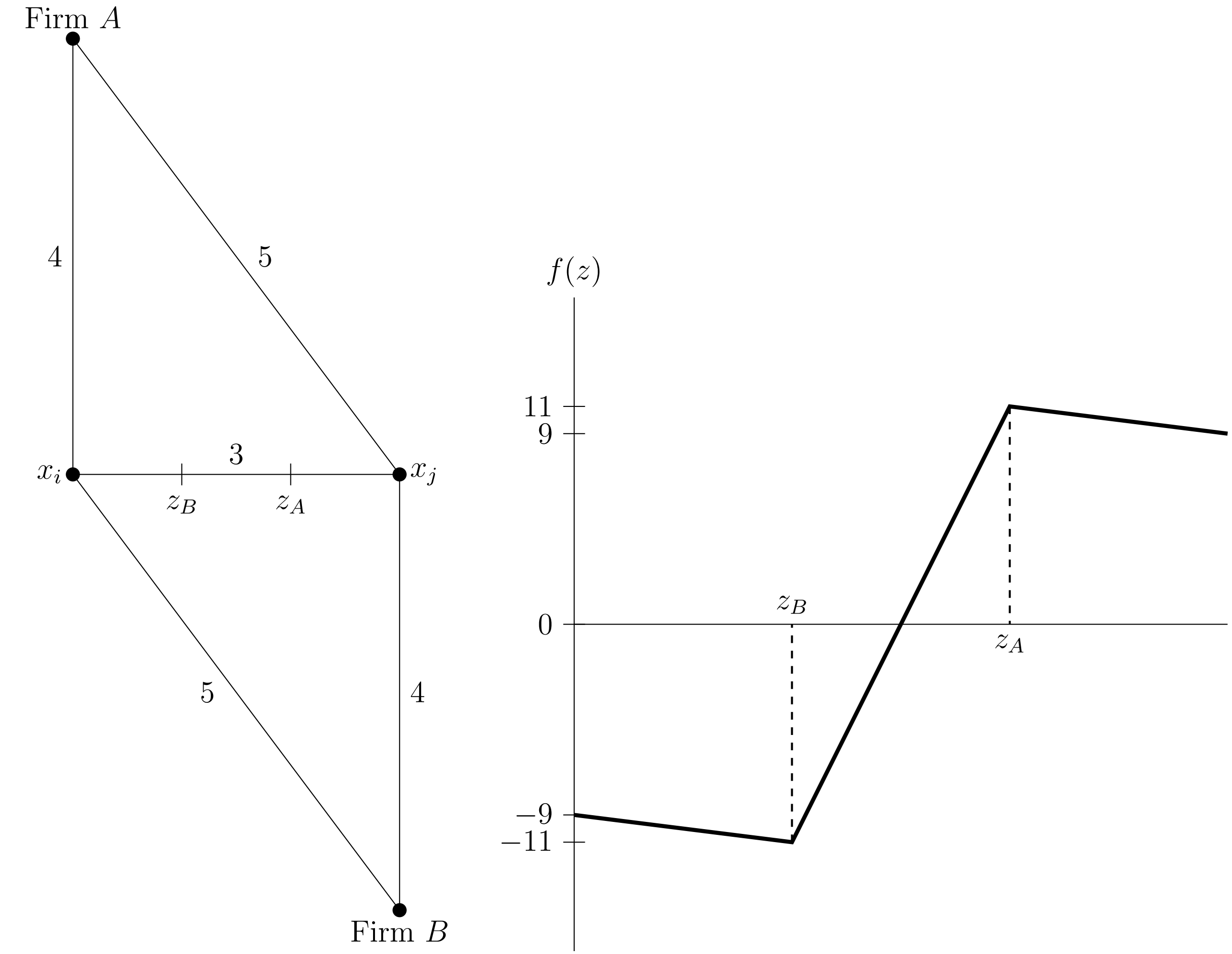
With the aid of this Proposition, we can immediately establish whether a change point is a weak indifference point. Figure 3(a) shows an example of a graph where one of the edges has both a change point for firm A, located at $z_A = \frac{1}{3}x_i + \frac{2}{3}x_j$, and a change point for firm B, located at $z_B = \frac{2}{3}x_i + \frac{1}{3}x_j$. Observe that $f(z_A) = 11 > 0$ and $f(z_B) = -11 < 0$. Hence both change points are weak indifference points for $\mu = 11$ and $\mu = -11$ respectively.[9] Figure 3(b) shows how the value of f changes along the edge. We see that if $\mu < -11$, then $f(z) > \mu$ for the entire edge and everyone buys from firm B. Similarly, if $\mu > 11$, then the entire edge buys from A. When $\mu \in (-9, 9)$, there is a strong indifference point on this edge and everyone to the left of the indifference point buys from firm B, the rest from firm A. When $\mu \in (-11, -9)$, everyone buys from firm B with the exception of the consumers located around z_B who buy firm A's product. The opposite happens for $\mu \in (9, 11)$.

There is nothing special about strong indifference points, in the sense that any location on the graph will be a strong indifference point for some value of μ. At a strong indifference point, the difference in transportation cost is either increasing or decreasing. At weak indifference points, the difference in transportation cost instead reaches a minimum or a maximum because these can only occur at change points. Therefore, in the absence of weak indifference points, we know that the entire edge buys from firm A (or B) if the consumers located at node i and j buy from A (or B). When the consumers located at these nodes buy from different firms, there has to be a strong indifference point, separating those who buy from A from those who buy from B.

By combining these observations, one can determine the demand for firm A on any given edge as follows.

Proposition 4. *Suppose that for a given edge ij we know the following: the distance to firm A and B from node i and j, the location of the change points (and the distance to firm A and B from these points). Then demand for firm A, $\nu_{ij}(\mu)$, along the edge can be*

[9] At this point, it is important to remember that not all change points necessarily are weak indifference points for some value of μ. Suppose for example that we change the graph in Figure 3(a) such that the length of the edges x_jB and x_iB becomes 8 and 9, respectively, then $f(z_A) = d_A(z_A)^2 - d_B(z_A)^2 = 36 - 64 = -28 < 0$ and there does not exist any price difference $\mu \equiv p_A - p_B$ that would make the consumer at point z_A indifferent while all consumers in her direct neighborhood would have a strict preference to buy from the same firm.



(a) Graph (numbers indicate length of edges)

(b) Difference in transportation cost

Figure 3: An example with multiple weak indifference points

determined by the following algorithm.

1. IF *there are no change points:*

   (a) IF $\max\{f(x_i), f(x_j)\} = 0$ *and* $\mu = 0$, *then* $\nu_{ij}(\mu) = \frac{1}{2}\ell_{ij}$

   (b) ELSE*:*

      - *If* $\min\{f(x_i), f(x_j)\} \geq \mu$, *then* $\nu_{ij}(\mu) = 0$
      - *If* $\max\{f(x_i), f(x_j)\} \leq \mu$, *then* $\nu_{ij}(\mu) = \ell_{ij}$
      - *Else find the value of* t *such that* $f(tx_i + (1-t)x_j) = \mu$
        - *If* $f(x_i) \leq \mu$, *then* $\nu_{ij}(\mu) = t\ell_{ij}$.
        - *Else* $\nu_{ij}(\mu) = (1-t)\ell_{ij}$

2. ELSE: *Suppose the change points are located at* $z_0, z_1, \cdots, z_m$. *Then create auxiliary edges* ik_0, k_0k_1, $\cdots$, k_mj *from* x_i *to* z_0, z_0 *to* z_1, $\cdots$ *and* z_k *to* x_j. *Then* $\nu_{ij}(\mu) = \nu_{ik_0}(\mu) + \nu_{k_0k_1}(\mu) + \cdots \nu_{k_mj}(\mu)$.

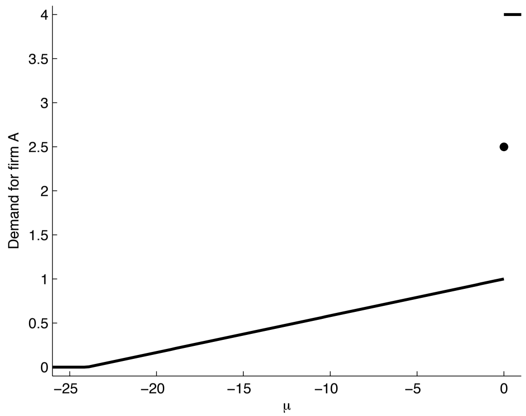
The reason why this algorithm will properly calculate the demand on a single edge is that it breaks up edges at change points if these change points are located in their interior edge. In doing so, we circumvent all special cases discussed above.

Figure 4 shows demand for the edges in the examples discussed earlier in this section.[10] Note that demand is piecewise linear. This follows from the fact that if z is a strong indifference point, but not a change point and $\mu \neq 0$, then from the implicit function theorem we get

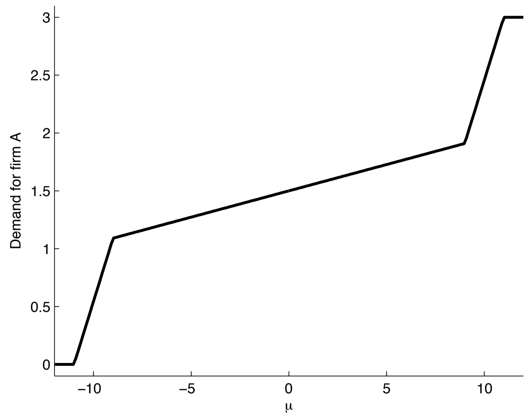
$$\left.\frac{dz_\varepsilon}{d\mu}\right|_{\varepsilon=0} = 2\ell_{ij}(\pm d_A(z) \pm d_B z), \tag{6}$$

which does not depend on μ. The points in Figure 4, where the slope changes, are at values of μ for which either one of the change points on the edge is a weak indifference point or one of the end nodes of the edge is an indifference point.

[10] The demand functions are computed numerically: the Matlab-files are available on request.



(a) Example in Figure 2: demand jumps at $\mu = 0$.



(b) Example in Figure 3(a): kinks because of the weak indifference points.

Figure 4: Demand on a single edge

3.2 Firm-level demand and profit

Demand for firm A's product is simply the sum over demand on each single edge:

$$D(\mu) = \frac{\sum_{ij\in E} \nu_{ij}(\mu)}{\sum_{ij\in E} \ell_{ij}}$$

where we have normalized demand to 1. Demand for firm B is $1 - D(\mu)$. The profits are given by:

$$\pi_A(p_A, p_B) = D(p_B - p_A)p_A$$
$$\pi_B(p_A, p_B) = [1 - D(p_B - p_A)]p_B$$

Firms simultaneously set prices. The properties of $D(\cdot)$ will determine whether equilibria in pure strategies exist. We close this section by observing that $D(\mu)$ inherits most of its properties from ν_{ij}, namely:

Proposition 5. *Demand for firm A as a function of the price differential μ is a non-decreasing, piecewise linear, function that is continuous everywhere, except possibly at $\mu = 0$. Furthermore*

1. $\check{\mu} = \sup\{\mu \mid D(\mu) = 0\} \in (-\infty, 0]$
2. $\hat{\mu} = \inf\{\mu \mid D(\mu) = 1\} \in [0, \infty)$
3. *The set of points where $D(\mu)$ is non-differentiable contains finitely many elements.*

Proof. The first properties follow directly from the properties of demand on a single edge.

Ad (1) and (2): Note that there exists a location z^* which minimizes f on $\mathcal{G}$. Set $\check{\mu} = f(z^*)$. Consequently $f(z) \geq \check{\mu}$ for all $z \in \mathcal{G}$. Hence $D(\check{\mu}^*) = 0$. Note that $\check{\mu}$ is finite. To show that is non-positive, observe that $f(z^*) \leq f(x_1) \leq 0$ since x_1 is the location of firm A. The proof for $\hat{\mu}$ goes along similar.

Ad (3): From (6) we derive that the slope changes only if μ is such that the position of the indifferent consumer coincides with a change point, or when a small change in μ leads the indifferent consumer to move to another edge (with possibly a different length). Since there are at most two change points per edge and at most two points where the indifferent consumer can drift off the edge, the number of non-differentiable points is at most 4 times the number of edges, which is finite. ■

Observe that $\check{\mu} = \hat{\mu} = 0$ corresponds to the pathological case where firm A and firm B are located at the same node. Since the equilibrium is trivial in this point ($p_A = p_B = 0$), we will exclude this possibility.

A natural question to ask is whether it is possible to abstract from the graph-theoretic foundation and to reduce the consumers' characteristics that are relevant for firm-level demand to a single dimension. This is possible. Suppose consumers are indexed by $\theta \in \mathbb{R}$ and that the distribution of θ is given by the cumulative distribution function F. Consumer θ receives utility $-p_A$ if he buys from firm A and utility $\theta - p_B$ if he buys from B. Therefore, he buys from A if $-p_A > \theta - p_B \Rightarrow \theta < \mu$. Firm-level demand for firm A is thus given by $F(\mu)$. However, without the graph-theoretic foundation, there is no justification for the result in Proposition 5 that this demand function is piecewise linear and continuous, except possibly at $\mu = 0$.

4 Equilibrium existence

We want to establish conditions under which pure strategy Nash-equilibria exist. As Vives (1999, Ch. 6) shows, in games with differentiated products and simultaneous price-setting, the existence of pure-strategy equilibria is not guaranteed.[11] As a first step, let us examine the properties of the reaction function.[12] The reaction function for firm A is

$$R_A(p_B) = \arg\max_{p_A \geq 0} D(p_B - p_A)p_A.$$

We have:

1. The best-response to $p_B = 0$ is to set $p_A > p_B$: Suppose $p_B = 0$. Note that the profit of firm A is zero if it sets $p_A = 0$. However demand, and therefore profit, is strictly positive for any $p_A \in (0, -\check{\mu})$. It follows that $R_A(0) > 0$.

[11] Note that the existence-result by Anderson *et al.* (1997) does not apply in our case because the graph-theoretic foundation does not imply log-concavity of $D(\mu)$.

[12] We present the proofs for firm A, the proofs for firm B are similar.

2. The best-response to $p_B = \frac{\hat{\mu}D(0)+\check{\mu}}{1-D(0)}$ is to set $p_A < p_B$. Note that $p_A = p_B - \hat{\mu} = \frac{D(0)[\hat{\mu}-\check{\mu}]}{1-D(0)} > 0$. At this price, firm A captures the entire market and profit is therefore equal to $p_B - \hat{\mu}$. This is a lower bound on profit when firm A undercuts firm B. We show that profit for any $p_A \geq p_B$ is beneath this lower bound. Note that

$$\max_{p_A \geq p_B} D(p_B - p_A)p_A = \max_{p_A \in [p_B, p_B - \check{\mu}]} D(p_B - p_A)p_A < D(0)(p_B - \check{\mu}) = p_B - \hat{\mu},$$

where the first equality follows from the fact that demand is zero for prices above $p_B - \check{\mu}$ and the inequality follows from the fact that demand is strictly decreasing. We conclude that for p_B sufficiently high, $R_A(p_B) < p_B$.

3. The best-response curve is upward-sloping whenever the maximizer is at a differentiable point of the profit function: The first-order condition for profit-maximization is

$$-D'(p_B - p_A)p_A + D(p_B - p_A) = 0.$$

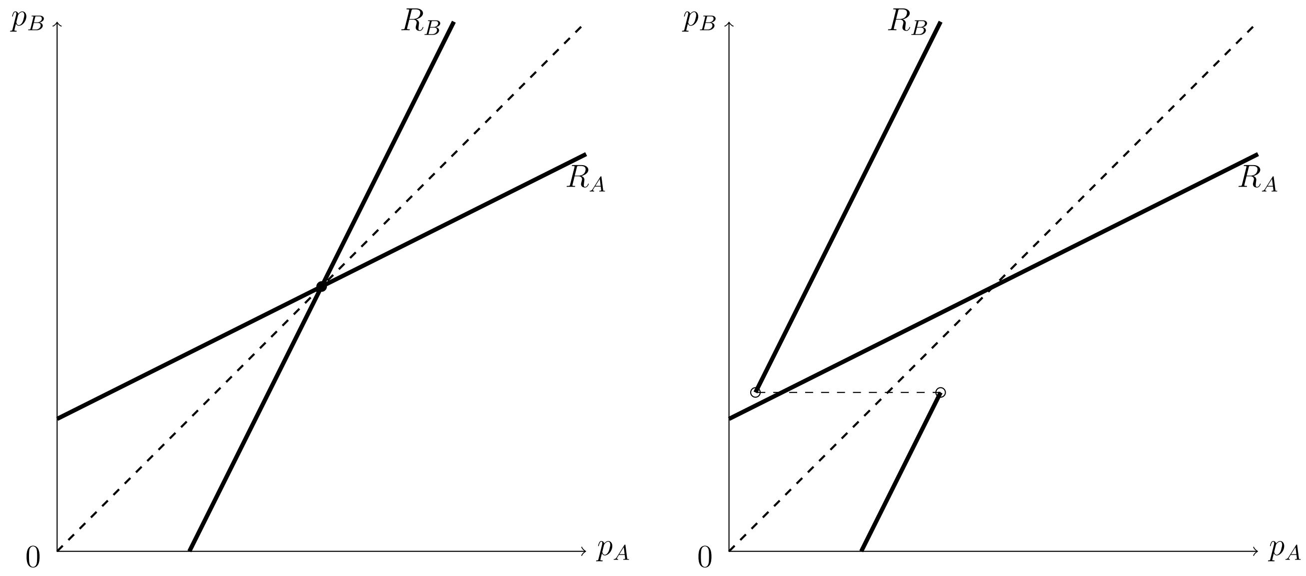
Using the implicit function theorem, we get

$$\frac{dp_A}{dp_B} = \frac{D''(p_B - p_A)p_A - D'(p_B - p_A)}{D''(p_B - p_A)p_A - 2D'(p_B - p_A)} = \frac{1}{2}$$

since demand is (piece-wise) linear.

Together with continuity of the reaction function this implies the existence of pure-strategy Nash-equilibria as Figure 5 illustrates. However, to obtain a continuous reaction function, we need to make strong assumptions on the demand function such as quasiconcavity or convexity of the inverse demand function (Caplin and Nalebuff, 1991). This is in general not met for the type of games discussed in this paper.

In Section 4.1, we show by example that even when the demand function is continuous, pure-strategy equilibria can fail to exist. For the case of discontinuous demand at $\mu = 0$, that is, when firms set equal prices, Section 4.2 will give sufficient conditions for the non-existence of pure-strategy equilibria. Section 4.3 will sketch the implications of these non-existence results for graphs: for which graphs and firm locations on these graphs can pure-strategy equilibria be ruled out? Finally, Section 4.4 will state, for the sake of completeness, the rather obvious result that any two-firm graph model will have a



(a) Pure-strategy equilibrium (b) No pure-strategy equilibrium

Figure 5: Downward jumps in the best-response.

equilibrium in mixed strategies.

4.1 Non-existence of pure-strategy Nash-equilibrium when demand is continuous

Figure 6 shows a graph that does not strictly adhere to our model, but illustrates how continuity of demand is not sufficient to guarantee the existence of pure-strategy equilibria. Note that if the indifference point is between A and C or between D and B, it is unique. If there is an indifference point between C and D instead, there necessarily are m of these points. Consequently, the slope of the demand curve is m times greater when the price differential μ is such that the indifferent consumers are located between C and D.

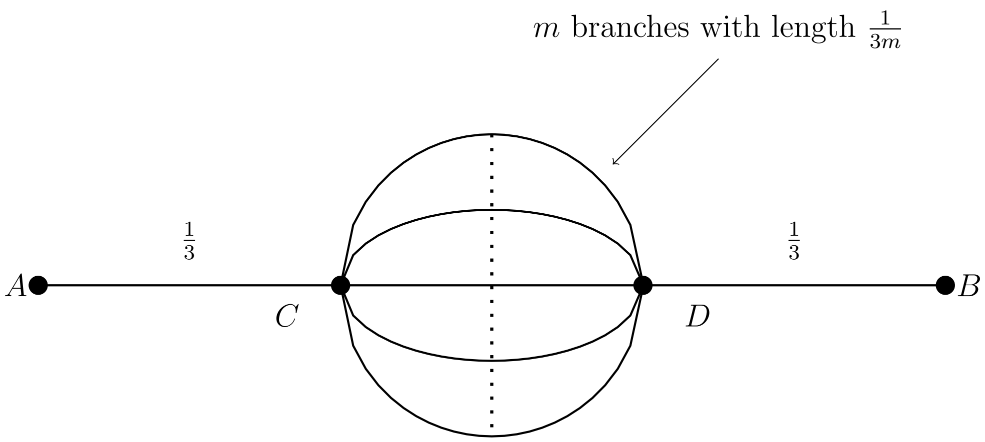


Figure 6: An example where demand is continuous

In order to analyze this game, we introduce a coordinate x, such that firm A is located at $x = 0$, node C at $x = \frac{1}{3}$, node D at $x = \frac{1}{3} + \frac{1}{3m}$ and firm B at $x = \frac{2}{3} + \frac{1}{3m}$. For $\frac{1}{3} \leq x \leq \frac{1}{3} + \frac{1}{3m}$, x represents a location on each of the m branches. It is straightforward but tedious to show that:

$$D(\mu) = \begin{cases} 0 & \text{if } \mu \leq -\left(\frac{2m+1}{3m}\right)^2 \\ \hat{x} & \text{if } -\left(\frac{2m+1}{3m}\right)^2 < \mu \leq -\frac{2m+1}{(3m)^2} \\ \frac{1}{3} + m\left(\hat{x} - \frac{1}{3}\right) & \text{if } -\frac{2m+1}{(3m)^2} < \mu \leq \frac{2m+1}{(3m)^2} \\ \frac{1}{3} + m\left(\hat{x} - \frac{1}{3} - \frac{1}{3m}\right) & \text{if } \frac{2m+1}{(3m)^2} < \mu \leq \left(\frac{2m+1}{3m}\right)^2 \\ 1 & \text{if } \mu > \left(\frac{2m+1}{3m}\right)^2 \end{cases} \quad (7)$$

where

$$\hat{x} = \frac{2m+1}{6m} + \frac{3m}{2(2m+1)}\mu.$$

Denote the equilibrium prices by p_A^* and p_B^* and the price difference $\mu^* = p_B^* - p_A^*$. If the equilibrium prices are at a point where the demand function is differentiable, then the first-order conditions for profit maximization implies

$$D'(\mu^*)p_A^* = D(\mu^*) \text{ and } D'(\mu^*)p_B^* = 1 - D(\mu^*).$$

By taking the difference of these first-order conditions, we see that:

$$D'(\mu^*)\mu^* = 1 - 2D(\mu^*). \tag{8}$$

Note that if $\mu^* < 0$, then $D(\mu^*) > \frac{1}{2}$ if (8) is supposed to hold. However, $\mu^* < 0$ implies that $p_A^* > p_B^*$ such that, due to the firms' position in the graph being symmetric, $D(\mu^*) < \frac{1}{2}$. Hence $\mu^* = 0$ is the only solution at a point where the demand function is differentiable. One can verify that

$$p_A^* = p_B^* = \frac{2m+1}{3m^2},$$

which tends to zero as $m \to \infty$. Since demand is equal to $\frac{1}{2}$, we see that profit tends to zero as well. Clearly if $m = 1$, this is a Nash-equilibrium (this is the Hotelling-line). However, if m is sufficiently large, then it will be profitable for firm A to deviate to $p_A = \frac{2}{9}$ (for instance). Note that as $m \to \infty$, $D\left(-\frac{2}{9}\right) = \hat{x} = \frac{1}{6}$. Hence the profit of deviating is $\frac{1}{6} \times \frac{2}{9} > 0$. So given sufficiently many branches, there is no Nash-equilibrium at a point where the demand function is differentiable.

The final possibility is that the Nash-equilibrium is at a point where the demand function is not differentiable. The only relevant non-differentiable point is at the point where demand for firm A is equal to $\frac{1}{3}$.[13] If demand for firm A is $\frac{1}{3}$, then $p_B - p_A = -\frac{2m+1}{(3m)^2}$. Note that profit for firm A in this case equals $\frac{1}{3}p_A$. If firm A deviates and sets the same

[13]There are four non-differentiable points: two of them correspond to points where demand is zero for one of the firms (and can be discarded on this ground). The main text discusses the case where demand for firm A is equal to $\frac{1}{3}$, the remaining point is the case where firm B's demand is $\frac{1}{3}$, which is not a Nash-equilibrium for similar reasons.

price as firm B, its profit would be

$$\frac{1}{2}\left(p_A - \frac{2m+1}{(3m)^2}\right),$$

which converges to $\frac{1}{2}p_A$ as $m \to \infty$. For m sufficiently, such a deviation will always be profitable.

This establishes that given sufficiently many branches, there is no pure-strategy Nash-equilibrium. The reason is that as m gets large, the demand function around $\mu = 0$ becomes very steep. This means that the equilibrium price must decrease (since demand is very elastic). But at the same time firms can keep the demand of a non-negligible part of the consumers by raising the price. This destabilizes the pure-strategy equilibria.

4.2 Non-existence of pure-strategy Nash-equilibrium when demand is discontinuous at $\mu = 0$

As in the previous subsection, we have to distinguish between two cases: the Nash-equilibrium is at a point where the demand function is differentiable or at a point where it is not. The first case is straightforward. From (8), we have a necessary condition for the price difference in a Nash-equilibrium. This implies that $\mu^* > 0$ if and only if $D(\mu^*) < \frac{1}{2}$ (and vice versa for $\mu^* < 0$). Hence there are no equilibria in the first case when $D(0^-) = \lim_{\mu \uparrow 0} D(\mu) < \frac{1}{2}$ and $D(0^+) = \lim_{\mu \downarrow 0} D(\mu) > \frac{1}{2}$.

The second case requires a bit more work.

First, suppose that μ^* and p_A^* are given and $\mu^* < 0$. By definition $p_B^* = p_A^* + \mu^*$. Profits in equilibrium for firm A and B are

$$D(\mu^*)p_A^* \text{ and } [1 - D(\mu^*)](p_A^* + \mu^*).$$

One possible deviation is to set the same price as the other firm. Then demand for firm A is $D(0)$ and demand for firm B is $1 - D(0)$. Profit levels after deviation are:

$$D(0)(p_A^* + \mu^*) \text{ and } [1 - D(0)]p_A^*.$$

By imposing that profit in equilibrium needs to be at least as high as profit after deviation, we get an interval of possible values of p_A^*:

$$\frac{-\mu^*(1-D(\mu^*))}{D(0)-D(\mu^*)} \leq p_A \leq \frac{-\mu^* D(0)}{D(0)-D(\mu^*)}.$$

Note that the lower boundary is only below the upper boundary if $D(0)+D(\mu^*) \geq 1$. Define $D(0^-) \equiv \sup\{D(\mu) \mid \mu < 0\}$. Hence if $D(0)+D(0^-) < 1$, then there cannot be a Nash-equilibrium for this value of μ^*.

Second, suppose that μ^* and p_A^* are given and $\mu^* > 0$. Following the same derivations as for the first case, we get an interval of possible values of p_A^*:

$$\frac{\mu^* D(0)}{D(0)-D(\mu^*)} \leq p_A \leq \frac{\mu^*(1-D(\mu^*))}{D(0)-D(\mu^*)}.$$

Note that the lower boundary is only below the upper boundary if $D(0)+D(\mu^*) \leq 1$. Define $D(0^+) \equiv \inf\{D(\mu) \mid \mu > 0\}$. Hence if $D(0)+D(0^+) > 1$, then there cannot be a Nash-equilibrium for this value of μ^*. Hence $D(0)+D(0^-) < 1$ and $D(0)+D(0^+) > 1$ is a sufficient condition for the non-existence of equilibria. Finally, we observe that $D(0) = \frac{1}{2}D(0^-) + \frac{1}{2}D(0^+)$. This establishes the following result.

Proposition 6. *Suppose firms are positioned such that the demand function $D(\cdot)$ has a discontinuity at $\mu = 0$.If $D(\cdot)$ is differentiable on $\{\mu \mid D(\mu) \in (0,1)\}\backslash\{0\}$, sufficient conditions for the non-existence of pure-strategy Nash-equilibria are that*

$$D(0^-) \equiv \lim_{\mu \uparrow 0} D(\mu) < \frac{1}{2} \text{ and}$$
$$D(0^+) \equiv \lim_{\mu \downarrow 0} D(\mu) > \frac{1}{2}.$$

Otherwise there exists no pure-strategy Nash-equilibrium if the following, somewhat more stringent conditions hold:

$$D(0^-) + 3D(0^+) > 2 \text{ and}$$
$$3D(0^-) + D(0^+) < 2.$$

Note that for graphs, where the demand function is differentiable at points where both firms have nonzero demand, the condition for non-existence is weak. Then the only requirement is that there is a discontinuity and, at the discontinuity, demand jumps from below one half to above one half. For graphs where the demand function is non-differentiable, the requirements are stronger. However, for graphs that are symmetric in the sense that $D(\mu) = 1 - D(-\mu)$, these conditions are always satisfied. In the next Section, we will use these criteria to find firm locations for which pure-strategy Nash-equilibria do not exist.

4.3 Pure-strategy price equilibria and types of graphs

In graph-theoretic terms, the Hotelling line is a complete bipartite $K_{1,2}$ graph, that is, a tree with one internal node and 2 leaves (edges).[14] The most straightforward extension of the Hotelling line is the $K_{1,3}$ graph of which Figure 1 depicts an example: a star graph that more informally can be described as a "Hotelling line with a junction". In this section, we consider sets of graphs that are identical except for the location of firm A and B, e.g. the graph depicted in Figure 7, where firm A and B are located a distance ε from the central node. We can show that there is a subset of graphs (with nonzero mass) for which pure-strategy Nash-equilibria do not exist.

Theorem 7. *For every $K_{1,3}$ graph, there exists a continuum of firm locations for which the price competition game does not possess a pure-strategy Nash equilibrium.*

Proof. Consider the graph in Figure 7. Without loss of generality, $\delta \geq \max\{\xi + \varepsilon, \varphi + \varepsilon\}$. Observe that demand is differentiable on $\{\mu \mid D(\mu) \in (0,1)\}\backslash\{0\}$. Note that we have $D(0^-) = \varphi + \varepsilon$ and $D(0^+) = \varphi + \delta + \varepsilon$. Since $\delta \geq \max\{\xi + \varepsilon, \varphi + \varepsilon\}$, we see that $D(0^-) < \frac{1}{2}$ and $D(0^+) > \frac{1}{2}$. From Proposition 6, we see that a pure-strategy Nash equilibrium does not exist. ∎

Thus, the most minor graph-theoretic extension of the D'Aspremont *et al.* (1979) two-firm line model with quadratic transportation costs is sufficient to lead to a non-existence result. The proof shows that no pure-strategy equilibrium exists if both firms

[14] A graph $\mathcal{G}$ is bipartite if its vertices can be divided into two classes N_1 and N_2 such that $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$ and every edge joins a node of N_1 to a node of N_2. A bipartite graph $\mathcal{G}$ is called a complete bipartite graph is the graph contains all possible edges joining edges in the two distinct classes. Star graphs with $k+1$ nodes are complete bipartite graphs with k leaves and are also denoted as S_k. The star S_3 (or $K_{1,3}$) with three edges is also called a claw. Note that the Hotelling line may as well be described as a $K_{1,1}$ graph: two nodes joined by one edge. See e.g. Bollobás (1998) for a formal treatment of graph theory.

are at equidistance from the junction and the longest edge is not inhabited by any of the firms.

A brief comparison of this result with Varian's non-existence result (1980, Proposition 2) is appropriate. In Varian, the absence of symmetric price equilibria is due to the assumption of declining average cost curves and the fact that a slight price cut by one of the stores leads this store to capture all informed consumers. The behavior of these informed customers is akin to the flocking of consumers along the junction of the $K_{1,3}$-graph to the firm charging the lowest price including transportation cost. Other than in Varian's model, a noncooperative equilibrium in pure strategies may therefore exist unless the fraction of informed customers – i.e. the length of the junction – is sufficiently large for firms to start a price war. The proof essentially shows that in every $K_{1,3}$ graph one can position the firms such that this condition holds.[15]

4.3.1 Example: Hinterlands

In order to provide further intuition on how the non-existence result of Proposition 6 relates to actual graphs, we give the next example that shows the importance of *hinterlands*. Figure 8(a) shows a graph where firm A has a *hinterland*, i.e. the consumers on the most left edge can only reach firm B by traveling via firm A. The parameter δ is a measure of the size of the hinterland. Firm A has an incentive to exploit these customers: especially if they are located deep in the hinterland, firm A can charge a substantial premium before these consumers switch to firm B. One can check that for $\delta < \frac{1}{2}$ all conditions in Proposition 6 are met, and therefore there will not be a pure strategy equilibrium.[16] In Figure 8(b), we see the reaction functions when $\delta = 8$. For firm A, the best-response is to set a higher price than firm B if firm B's price is below a certain threshold. However firm A will just undercut firm B when p_B is high. Firm B's reaction function is similar, but in addition it will set a higher price than firm A if firm A sets a really low price. The

[15]In Economides (1986b), consumers are evenly distributed on a surface. He shows that demand and profit functions are continuous for fairly general distance functions including the Euclidean metric but not for the block metric. In our application, the distance between two points x and y is determined by the length of the shortest path between these two points. That is, we cannot use a different distance function to remove the "thickness" of consumers at the boundary to restore existence.

[16]For $\delta < \frac{1}{2}$, $D(0^-) + 3D(0^+) = \frac{7+4\delta}{3+\delta} > 2$ and $3D(0^-) + D(0^+) = \frac{5+4\delta}{3+\delta} < 2$.

equilibrium prices are $p_A^* = 26\frac{2}{3}$ and $p_B^* = 17\frac{1}{3}$.[17] Note that the location of the jump in firm A's reaction function is crucial in determining whether a pure-strategy equilibrium exists or not. If the jump had occurred for a lower value of p_B, then it would have jumped over the reaction function of firm B.

4.3.2 Beyond $K_{1,3}$-graphs

We conjecture, but do not prove, that the non-existence result holds for all graph models of price competition involving two or more firms:

Conjecture 8. *For every graph $\mathcal{G}$ with a least one node having degree 3 or higher, there exists firm locations for which the price competition game does not possess a pure-strategy Nash equilibrium.*

The condition that at least of the graph's has to be of degree 3 or higher rules out the structures for which we know that they do have a pure-strategy equilibrium, such as the line and circle.

The sufficient condition in Proposition 6 allow us to find locations for which the price competition game does not possess a pure-strategy Nash equilibrium. Figure 9 shows a more complicated graph, where there are several nodes with degree 3. Our aim is to find locations for firm A and B such that the conditions of Proposition 6 hold. The arrows in Figure 9 give the direction of the shortest path to node i. Let $m_{i,\alpha}$ denote the fraction of consumers who are located on the edge connecting i with α plus the consumers whose shortest path to i goes via α. Note that the last category are the consumers located on the edge connecting α and β, but closer to α than to β. Observe that $m_{i,\alpha} = 15.6/94.54 \approx 0.165$. Let $m_{i,\beta}$ and $m_{i,\gamma}$ be similarly defined. Calculations show that $m_{i,\beta} \approx 0.257$ and $m_{i,\gamma} \approx 0.578$. Assume that Firm A is located on the edge $i\alpha$ and firm B on the edge $i\beta$. Let $d_{Ai} = d_{Bi}$. Moreover assume that A and B are sufficiently close to node i such that if the shortest path to i goes via α, β or γ (for consumers not located on edges $i\alpha$, $i\beta$ or $i\gamma$), then the shortest path to firm A or B takes the same route. Then all consumers whose shortest path to node i goes via γ are indifferent between firm A and firm B when they set the same price. The fraction of consumers with a strict preference

[17] The reaction functions are computed numerically: the Matlab-files are available on request.

for firm A's product is $m_{i,\alpha}$ and a fraction $m_{i,\beta}$ of consumers strictly prefer B. Therefore $D(0^-) = m_{i,\alpha} \approx 0.165$ and $D(0^+) = m_{i,\alpha} + m_{i,\gamma} \approx 0.743$, which satisfy the conditions in Proposition 6. For this graph, we obtain the same result as for the $K_{1,3}$-graph. Note that the locations of the firms are relatively close and the mass of consumers that chooses the firm that offers the lowest price ($m_{i,\gamma}$) is large. Moreover, the firms have no hinterland (i.e. a base of "captured customers"). The (mixed strategy) Nash-equilibrium is probably characterized by intense price competition. If firms could endogenously choose their location, this seems like an unlikely candidate for an equilibrium. Yet, in order to find an equilibrium of the location game, we need to know the (expected) profit for this location pair.

4.4 Mixed-strategy price equilibria

Demand discontinuities lead to the non-existence of equilibria in pure strategies. However, for the model with two firms, mixed-strategy price equilibria do exist because the profit functions $\pi_i(p_A, p_B)$ $(i = A, B)$ are bounded and weakly lower semi-continuous in p_i and $\sum_{i=1}^{2} \pi_i(\mathbf{p})$ is upper-semicontinuous (Dasgupta and Maskin, 1986a, Theorem 5). For the model with two firms, we therefore have the following positive result, which essentially extends Theorem 3 in Dasgupta and Maskin (1986b).

Theorem 9. *The two-firm graph model of price competition has a mixed-strategy equilibrium.*

This theorem does not use the assumption of quadratic transportation cost. This implies that the result extends to graph models where consumers face other forms of nonlinear or linear transportation cost.

5 Summary and discussion

This paper is a first contribution to the analysis of graph models of price competition. The algorithm introduced allows one to numerically evaluate firm-level demand and profits for all graphs where consumers are uniformly distributed along the edges and face quadratic transportation cost and where two firms compete in prices conditional on their location.

One important phenomenon for this type of models is that spatial discontinuities in demand may occur. The most important result is that the existence result by D'Aspremont *et al.* (1979) for the $K_{1,2}$ graph does not extend to the $K_{1,3}$ graph, arguably the most straightforward extension of the original model.

We believe that the framework presented in this paper offers ample scope for future research. Besides proving or falsifying the conjecture on the non-existence of pure-strategy price equilibria for graphs, natural directions for further investigation include the analysis of markets with three or more firms, issues related to endogenous entry and markets where consumers face non-linear, but not necessarily quadratic transportation costs. Furthermore, whereas the present paper presents numerical evaluations for a number of specific graphs, it is worthwhile to investigate more systematically the relationship between graph characteristics, firm locations within the graph and pricing equilibria. One of the results in D'Aspremont *et al.* (1979) is that for the line model with linear transportation cost, pure-strategy equilibria exist if firms are far enough apart. Are there classes of graphs for which a similar result can be obtained?

Another avenue for research is the study of the relationship between graph characteristics, firm location and the occurrence and characteristics (length, amplitude, symmetry) of price cycles. Theoretical Edgeworth cycles, first described by Edgeworth (1925) and given a solid game-theoretic foundations by Maskin and Tirole (1988), are characterized by strongly asymmetric periods of price cuts followed by a rapid price increase. Theoretically, Edgeworth price cycles are most likely to occur in markets characterized by homogenous goods and extremely price-sensitive consumers. Consistent with this, one particular market in which asymmetric price cycles have been consistently found is the market for retail gasoline. Typically, these studies start with the observation of price cycles in a certain market, verify whether or not the cycles are asymmetric, and, conditional on finding asymmetries, look for the possible causes.[18] Noel (2009) for example decomposes asymmetric price cycles into a part that can be explained by Edgeworth cycles and a part driven by other unknown sources. Less attention has been paid to why

[18]These empirical studies give evidence for price cycles in the US (Castanias and Johnson, 1993; Lewis (2011); Lewis and Noel (2011)), Canada (Noel, 2007a, 2007b; Eckert, 2003), Australia (Wang, 2009; De Roos and Katayama, 2013). Bachmeier and Griffin, 2003 do not uncover asymmetric cycles.

some firms are cycling and other are not. Exceptions are Noel (2007a) and De Roos and Katayama (2013) who use a Markov switching-regression model and allow for differences in the price cycles of major firms and independents. The location of a firm on a given road network relative the location of its competitors might be an important additional variable explaining the occurrence and shape of these price cycles.

References

Anderson, Simon P., Jacob K. Goeree, and Roald Ramer, "Location, Location, Location," *Journal of Economic Theory*, 1997, *77*, 102–127.

Bachmeier, Lance J. and James M Griffin, "New Evidence on Asymmetric Gasoline Price Responses," *Review of Economics and Statistics*, August 2003, *85* (3), 772–776.

Bertsekas, Dimitri P., *Linear Network Optimization: Algorithms and Codes*, Cambridge, Massachusetts: MIT Press, 1991.

Bloch, Francis and Nicolas Quérou, "Pricing in Social Networks," *Games and Economic Behavior*, July 2013, *80*, 243–261.

Bollobás, Béla, *Modern Graph Theory* Graduate Texts in Mathematics, New York: Springer, 1998.

Bollobás, Belá and Nicolas Stern, "The Optimal Structure of Market Areas," *Journal of Economic Theory*, 1972, *4*, 174–179.

Buechel, Berno and Nils Röhl, "Robust Equilibria in Location Games," *European Journal of Operations Research*, January 2015, *240* (2-3), 505–517.

Caplin, Andrew and Barry Nalebuff, "Aggregation and Imperfect Competition: On the Existence of Equilibrium," *Econometrica*, January 1991, *59* (1), 25–59.

Castanias, Rick and Herb Johnson, "Gas Wars: Retail Gasoline Price Fluctuations," *Review of Economics and Statistics*, February 1993, *75* (1), 171–174.

Claycombe, Richard J. and Tamara E. Mahan, "Spatial Aspects of Retail Market Structure: Beef Pricing Revisited," *International Journal of Industrial Organization*, June 1991, *11*, 283–291.

Dasgupta, Partha and Eric Maskin, "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *The Review of Economic Studies*, January 1986a, *53* (1), 1–26.

___ **and** ___ , "The Existence of Equilibbrium in Discontinous Economic Games, II: Applications," *Review of Economic Studies*, January 1986b, *53* (1), 27–41.

D'Aspremont, C., J. Jaskold Gabszewicz, and J.-F. Thisse, "On Hotelling's "Stability in Competition"," *Econometrica*, September 1979, *47* (5), 1145–1150.

de Roos, Nicolas and Hajime Katayama, "Gasoline Price Cycles Under Discrete Time Pricing," *Economic Record*, June 2013, *89* (285), 175–193.

Eaton, B. Curtis and Richard G. Lipsey, "The Non-Uniqueness of Equilibrium in the Loschian Location Model," *American Economic Review*, March 1976, *66* (1), 77–93.

Eckert, Andrew, "Retail Price Cycles and the Presence of Small Firms," *International Journal of Industrial Organization*, February 2003, *21* (2), 151–170.

Economides, Nicolas, "Nash equilibrium in duopoly with products defined by two characteristics," *RAND Journal of Economics*, Autumn 1986b, *17* (3), 431–439.

Edgeworth, Francis, "The Pure Theory of Monopoly," in "Papers Relating to Political Economy," Vol. 1, London: MacMillan, 1925, pp. 111–142.

Fournier, Gaëton and Marco Scarsini, "Hotelling Games on Networks: Efficiency on Equilibria," CES Working Papers no. 2014-33 2014.

Greenhut, M. L., M. J. Hwang, and H. Ohta, "An Empirical Evaluation of the Equilibrium Size and Shape of Market Areas," *International Economic Review*, February 1976, *17* (1), 172–190.

Holahan, William L. and Richard E. Schuler, "The Welfare Effects of Market Shapes in the Loschian Location Model: Squares Vs. Hexagons," *American Economic Review*, September 1981, *71* (4), 738–746.

Hotelling, Harold, "Stability in Competition," *Economic Journal*, March 1929, *39* (153), 41–57.

Houde, Jean-François, "Spatial Differentiation and Vertical Mergers in Retail Markets for Gasoline," *American Economic Review*, August 2012, *102* (5), 2147–2182.

Irmen, Andreas and Jacques-François Thisse, "Competition in Multi-Characteristics Spaces: Hotelling Was Almost Right," *Journal of Economic Theory*, 1998, *78*, 76–102.

Lewis, Matthew S., "Asymmetric Price Adjustment and Consumer Search: An Examination of the Retail Gasoline Market," *Journal of Economics and Management Strategy*, Summer 2011, *20* (2), 409–449.

___ **and Michael D. Noel**, "The Speed of Gasoline Price Response in Markets with and without Edgeworth Cycles," *Review of Economics and Statistics*, May 2011, *93* (2), 672–682.

Lösch, Auguste, *The Economics of Location*, New Haven, Connecticut: Yale University Press, 1954.

Maskin, Eric and Jean Tirole, "A Theory of Dynamic Oligopoly, II: Price Competition, Kinked Demand Curves, and Edgeworth Cycles," *Econometrica*, May 1988, *56* (3), 571–599.

Mills, Edwin S. and Michael R. Lav, "A Model of Market Areas with Free Entry," *The Journal of Political Economy*, June 1964, *72* (3), 278–288.

Noel, Michael D., "Edgeworth Price Cycles: Evidence from the Toronto Retail Gasoline Market," *Journal of Industrial Economics*, March 2007a, *LV* (1), 69–92.

___ , "Edgeworth Price Cycles, Cost-Based Pricing and Sticky Pricing in Retail Gasoline Markets," *Review of Economics and Statistics*, May 2007b, *89* (2), 224–234.

___ , "Do Retail Gasoline Prices Respond Asymmetrically to Cost Shocks? The Influence of Edgeworth Cycles," *RAND Journal of Economics*, Autumn 2009, *40* (3), 582–595.

Osborne, Martin J. and Carolyn Pitchik, "Equilibrium in Hotelling's Model of Spatial Competition," *Econometrica*, July 1987, *55* (4), 911–922.

Picone, Gabriel A., David B. Ridley, and Paul A. Zandbergen, "Distance Decreases with Differentiation: Strategic Agglomeration by Retailers," *International Journal of Industrial Organization*, 2009, *27*, 463–473.

Raith, Michael, "Spatial Retail Markets with Commuting Consumers," *International Journal of Industrial Organization*, June 1996, *14* (4), 447–463.

Salop, Steven C., "Monopolistic Competition with Outside Goods," *Bell Journal of Economics*, Spring 1979, *10* (1), 141–156.

Varian, Hal R., "A Model of Sales," *American Economic Review*, September 1980, *70* (4), 651–659.

Von Ungern-Sternberg, Thomas, "Monopolistic Competition on the Pyramid," *Journal of Industrial Economics*, June 1991, *39* (4), 355–368.

Wang, Zhongmin, "(Mixed) Strategy in Oligopoly Pricing: Evidence from Gasoline Price Cycles Before and under a Timing Regulation," *Journal of Political Economy*, December 2009, *117* (6), 987–1030.

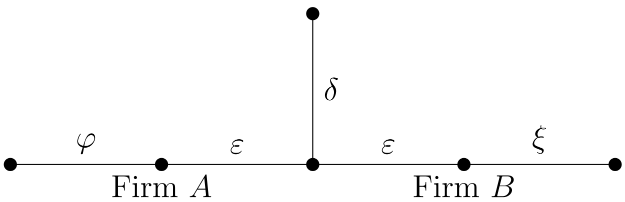
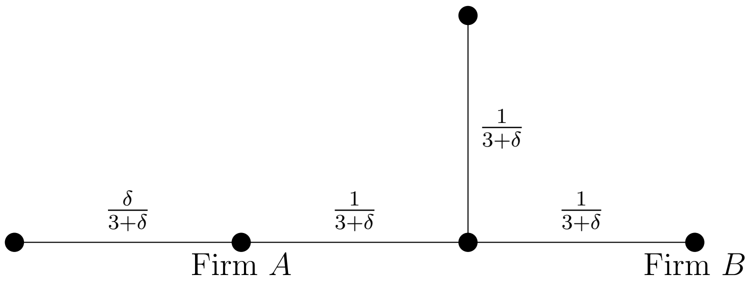
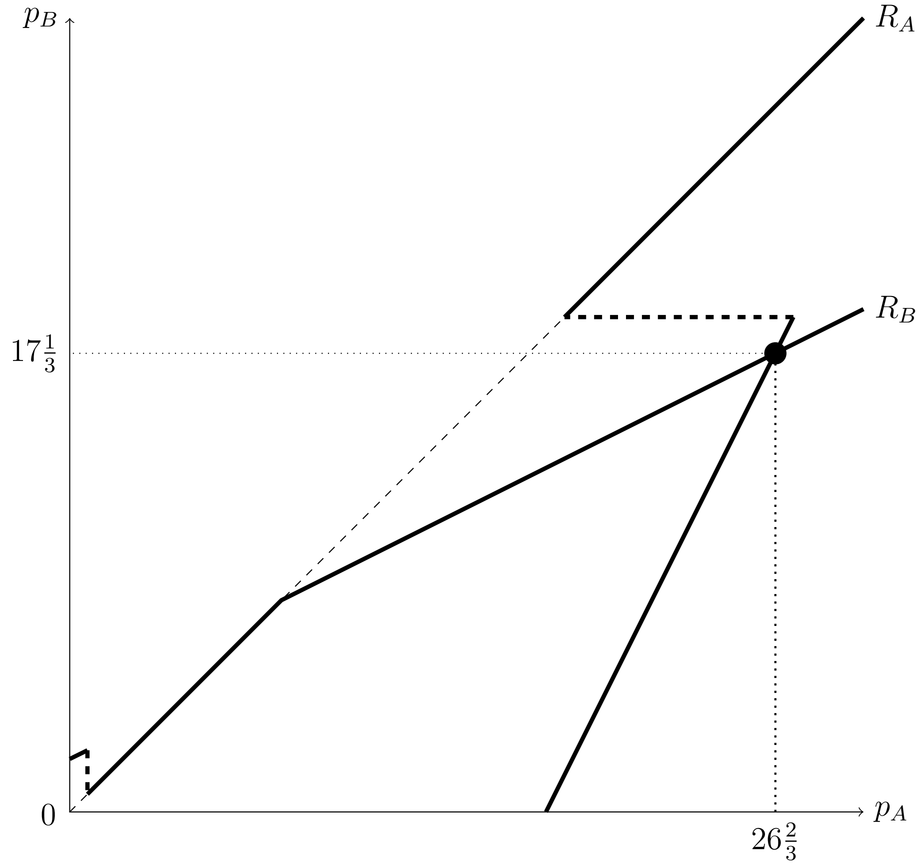


Figure 7: Location of firms; numbers indicate the length of the edges, the total mass of consumers is $2\varepsilon + \varphi + \delta + \xi = 1$.



(a) Graph with a hinterland for firm A



(b) Reaction curves for firm A and B ($\delta = 8$)

Figure 8: Example with hinterland

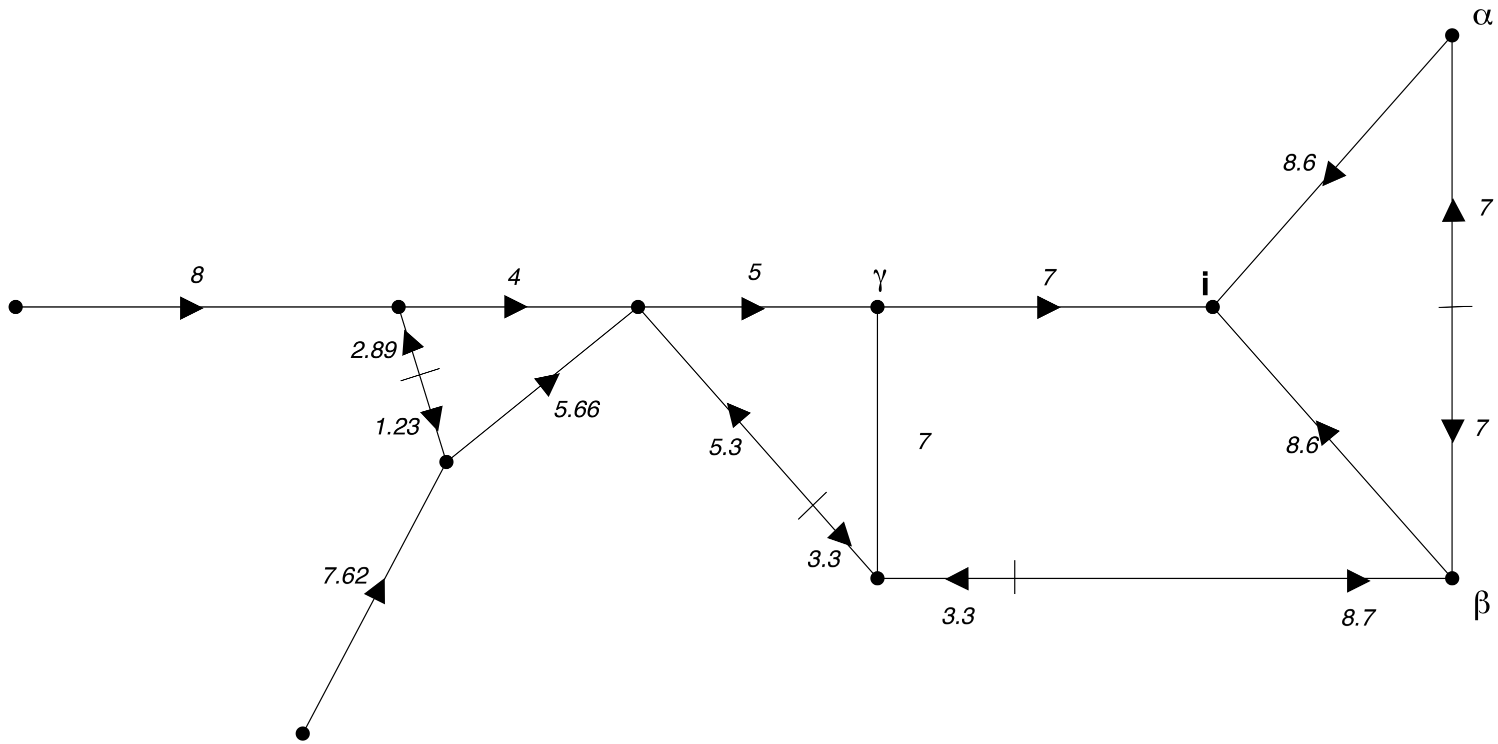


Figure 9: Arrows denote direction of shortest path to node i, numbers denote the length of edges and bars on edges denote a change points.