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Reserve-Dependent Benefits and Costs in Life and Health Insurance Contracts

Marcus C. Christiansen[1]
Michel M. Denuit[2]
Jan Dhaene[3]

[1] *University of Ulm, Germany;*
[2] *Université Catholique de Louvain, Belgium;*
[3] *Katholieke Universiteit Leuven, Belgium, and University of the Free State, South Africa.*

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Duisenberg school of finance
Gustav Mahlerplein 117
1082 MS Amsterdam
The Netherlands
Tel.: +31(0)20 525 8579

Reserve-Dependent Benefits and Costs in Life and Health Insurance Contracts

MARCUS C. CHRISTIANSEN
Institute of Insurance Science
University of Ulm
Ulm, Germany

MICHEL M. DENUIT
Institut de Statistique, Biostatistique et Sciences Actuarielles
Université Catholique de Louvain
Louvain-la-Neuve, Belgium

JAN DHAENE
Actuarial Research Group, AFI
Faculty of Business and Economics
Katholieke Universiteit Leuven
Leuven, Belgium
Department of Mathematical Statistics and Actuarial Science
University of the Freestate
Bloemfontein, South Africa

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Abstract

Premiums and benefits associated with traditional life insurance contracts are usually specified as fixed amounts in policy conditions. However, reserve-dependent surrender values and reserve-dependent expenses are common in insurance practice. The famous Cantelli theorem in life insurance ensures that under appropriate assumptions surrendering can be ignored in reserve calculations provided the surrender payment equals the accumulated reserve. In this paper, more complex reserve-dependent payment patterns are considered, in line with insurance practice. Explicit formulas are derived for the corresponding reserve.

KEY WORDS: life insurance, multistate models, Markov process, surrender value, Cantelli theorem.

1 Introduction

Multistate models provide a convenient representation for generalized life insurance contracts, including life insurance policies, disability insurance policies and permanent health insurance policies, for instance. Each state represents a particular status for the policyholder. The benefits comprised in the contract are associated to sojourns in, or transitions between states. See, e.g., Chapter 8 in Dickson et al. (2009) for an introduction.

Under the Markovian assumption, Thiele's differential equation describes the dynamics of the accumulated reserve. As it can easily be solved numerically, using Euler's method for instance, it provides an efficient tool to perform actuarial calculations. The situation becomes nevertheless more difficult when benefits are expressed in terms of the reserves, as in the case of surrendering for instance. The famous Cantelli theorem ensures that under appropriate conditions surrendering can be ignored in the reserve calculations provided the surrender payment equals the reserve. This is true from a prospective perspective as well as from a retrospective perspective. However, the insurer generally applies a penalty when the policyholder cancels the contract so that this result is of little practical use.

In this paper, we consider reserve-dependent payment patterns and we derive explicit expressions for the reserve. Typical examples of reserve-dependent insurance benefits include:

- Surrender payments, with the surrender value equal to the accumulated reserve minus a cancellation fee.
- Capital management fees proportional to the reserves.
- Profit participation, with surplus dividends depending on the accumulated reserve.

We show that, under fairly general conditions, one can still apply Cantelli's theorem to derive an explicit expression for the reserves provided the structure of the benefits and premiums is appropriately modified. Several examples are discussed to illustrate the applicability of the approach proposed in the present paper.

The topic investigated here has already been examined in the literature. For instance, Norberg (1991) studies general multistate life insurance products and points out to the fact that Thiele's differential equation can also cope with payments depending on the reserves in a linear way. This author derives explicit expressions for the accumulated reserves in two particular cases:

(a) a widow's pension where the retrospective reserve is paid back to the husband in case the wife dies first, and

(b) a widow's pension with administration expenses expressed as a linear function of the reserve.

The present paper expands on the ideas of Norberg (1991) and presents explicit expressions for more general contracts. Milbrodt and Helbig (1999) also mention the key role played by

Thiele's equations if benefits are reserve-dependent. They discuss an annuity insurance with flexible time of retirement and death benefits and calculate accumulated reserves when surrender payments equal a fixed proportion of the accumulated reserve.

Notice that the problem considered here has also been discussed in several textbooks. For instance, in the multiple decrement model, Bowers et al. (1997, Section 11.4) show that as long as the withdrawal benefit in a double decrement model whole life insurance is equal to the reserve under the associated single decrement model, premiums and reserves coincide under the single and double decrement model. The present paper revisits this problem in a more general framework. It is shown that the conclusion drawn e.g. by Bowers et al. (1997) can be generalized, provided mild conditions are fullfilled.

The remainder of this paper is organized as follows. Section 2 briefly recalls the multi-state Markovian setting for describing generalized life insurance contracts. In particular, definitions for the prospective and retrospective reserves are provided. Section 3 is devoted to Cantelli's fundamental theorem, which provides the technical argument used in Section 4 to derive the results in case of reserve-dependent insurance payments. Section 5 discusses several examples of practical relevance before the final Section 6 concludes the paper.

2 Multistate life insurance

Consider a k-state Markov transition model describing some insurance contract. The initial state is numbered 1. The policyholder is aged x at policy issue and time t measures the seniority of the contract, $t = 0$ corresponding to policy issue. The transition intensity functions are indexed by attained age and denoted as $\mu_{x+t}^{(ij)}$ for different states $i \neq j \in \{1, 2, \ldots, k\}$. All transition intensities have to be integrable functions. We define

$$\mu_{x+t}^{(ii)} = -\sum_{j:j\neq i} \mu_{x+t}^{(ij)}.$$

The corresponding transition probabilities between states i and j over the time interval (s, t) are denoted as ${}_{t-s}p_{x+s}^{(ij)}$. Clearly,

$${}_{t-s}p_{x+s}^{(ii)} = 1 - \sum_{j:j\neq i} {}_{t-s}p_{x+s}^{(ij)}.$$

These probabilities can be obtained as the unique solution of Kolmogorov's backward equations

$$\frac{\mathrm{d}}{\mathrm{d}s} {}_{t-s}p_{x+s}^{(ij)} = -\sum_{l\in\{1,2,\ldots,k\}} {}_{t-s}p_{x+s}^{(il)}\, \mu_{x+s}^{(lj)}$$

with initial condition ${}_0p_{x+s}^{(ij)} = 0$ for $i \neq j$ and ${}_0p_{x+s}^{(ii)} = 1$. Provided the transition intensity functions are piecewise constant, the Cox-Miller formula gives the explicit solution to this system. Likewise the transition probabilities also uniquely solve Kolmogorov's forward equations.

The interest earned on the savings account is modeled by the cumulative interest intensity function Δ_t. We assume that Δ_t has finite variation on compacts. Usually we have $\mathrm{d}\Delta_t = \delta_t \mathrm{d}t$ for some interest intensity function δ_t. The corresponding discount factor ${}_{t-s}v_s$ is the unique solution of

$$ {}_{t-s}v_s = 1 - \int_{(s,t]} {}_{t-u}v_u \, \mathrm{d}\Delta_u. $$

Let $c^{ij}(t)$ be the benefit paid by the insurer upon a transition from state i to state $j \neq i$ occurring at time $t \in (0, n]$. We assume that the functions $t \mapsto c^{ij}(t)$ are Borel-measurable and bounded. Let $\mathrm{d}B^i(t)$ be the sojourn benefit (net of premiums) in state i at time $t \in [0, n]$. Then $B_i(t)$ describes the accumulated sojourn payments (minus premiums paid) in state i in the time interval $[0, t]$. We assume that the functions $t \mapsto B_i(t)$ have finite variation.

The appropriate definition of the reserves has been investigated by Wolthuis and Hoem (1990). The prospective reserve at time t in state i is clearly given by

$$ V_+^i(t) = \sum_{j=1}^{k} \int_{(t,n]} {}_{s-t}v_t \, {}_{s-t}p_{x+t}^{(ij)} \mathrm{d}B^j(s) + \sum_{\substack{j,l=1 \\ j \neq l}}^{k} \int_t^n {}_{s-t}v_t \, {}_{s-t}p_{x+t}^{(ij)} \mu_{x+s}^{(jl)} c^{jl}(s) \mathrm{d}s. \tag{1} $$

The reserve (1) can also be obtained as the solution of Thiele's differential equation, which is given by

$$ \mathrm{d}V_+^i(t) = V_+^i(t-)\mathrm{d}\Delta_t - \mathrm{d}B^i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{k} \mu_{x+t}^{(ij)} \big(c^{ij}(t) + V_+^j(t) - V_+^i(t) \big) \mathrm{d}t \tag{2} $$

with terminal condition $V_+^i(n) = 0$, $i = 1, ..., k$. Milbrodt and Helbig (1999) show uniqueness of the solution of (2), but only for payment functions that do not depend on the reserve. In Section 4, we provide a uniqueness result also for reserve-dependent payments, given that the dependence is linear.

Provided that the transition matrix $\boldsymbol{P}(s,t) = ({}_{t-s}p_{x+s}^{(ij)})_{i,j=1,\dots,k}$ is regular, the retrospective reserve is defined according to Wolthuis and Hoem (1990) as

$$ \begin{aligned} V_-^i(t) \quad = \quad & -\sum_{j=1}^{k} \int_{[0,t]} ({}_{t-s}v_s)^{-1} (\boldsymbol{P}(s,t))_{ij}^{-1} \mathrm{d}B^j(s) \\ & -\sum_{\substack{j,l=1 \\ j \neq l}}^{k} \int_0^t ({}_{t-s}v_s)^{-1} (\boldsymbol{P}(s,t))_{ij}^{-1} \mu_{x+s}^{(jl)} c^{jl}(s) \mathrm{d}s \end{aligned} \tag{3} $$

where $(\boldsymbol{P}(s,t))_{ij}^{-1}$ is element (i, j) of the inverse of the transition matrix $\boldsymbol{P}(s,t)$. It can also be obtained as the solution of Thiele's equation (2) recalled above but with initial condition $V_-^i(0-) = 0$ and V_+^i replaced by V_-^i.

Formulas (1) and (3) are true regardless of whether the payment functions are reserve-dependent or not. However, in case of reserve-dependent payments, (1) and (3) only implicitly describe the prospective and the retrospective reserves, and it is more convenient to work with Thiele's equation (2).

In order to calculate prospective and retrospective reserves, interest and transition intensity functions have to be chosen. Note that in the retrospective view the interest and transition intensity functions relate to the past and, thus, their realized values can be observed. This implies that the observed basis can be used as basis for the retrospective calculations. In the prospective view, however, interest and transition intensities relate to the future and therefore the prospective calculations are always performed with an assumed basis. Under this assumed basis, the values of retrospective and prospective reserves are equal provided premiums are determined by the equivalence principle (using the same basis). Let us mention that the International Association of Insurance Supervisors (IAIS) recommends that the liabilities of insurance companies should be evaluated on consistent bases, i.e. by means of an economic valuation that reflects the prospective future cash flows. In Europe, Solvency II determines that the best estimate of the provision for future commitments must be measured based on current information and realistic predictions.

3 Cantelli's Theorem

The results in this section are true for both retrospective and prospective reserves. In the remainder of this paper, we write $V^i(t)$ if a statement is true for both $V^i_-(t)$ and $V^i_+(t)$.

Consider two insurance contracts. The first one has benefits described by the functions $c^{ij}(t)$ and $B^i(t)$ and relies on actuarial assumptions $\mu^{(ij)}_{x+t}$ and Δ_t. The corresponding (prospective or retrospective) reserves are denoted as $V^i(t)$. The second insurance contract has payment functions $\overline{c}^{ij}(t)$ and $\overline{B}^i(t)$ and actuarial assumptions $\overline{\mu}^{(ij)}_{x+t}$ and $\overline{\Delta}_t$. Its reserves are denoted as $\overline{V}^i(t)$. The next result, which originally goes back to Cantelli (1914), states the conditions ensuring that the accumulated reserves of these two contracts coincide. To the authors' knowledge, the most general version of Cantelli's result can be found in Milbrodt and Stracke (1997). We further extend that result here by allowing for different interest rates.

Theorem 1 (Cantelli's Theorem) *We have $V^i(t) = \overline{V}^i(t)$ for all $i = 1, ..., k$ and all $t \in [0, n]$ if, and only if,*

$$\begin{aligned} &V^i(t-)\,d\Delta_t - dB^i(t) - \sum_{\substack{j=1\\ j\neq i}}^{k} \mu^{(ij)}_{x+t}(c^{ij}(t) + V^j(t) - V^i(t))\,dt \\ &= V^i(t-)\,d\overline{\Delta}_t - d\overline{B}^i(t) + \sum_{\substack{j=1\\ j\neq i}}^{k} \overline{\mu}^{(ij)}_{x+t}(\overline{c}^{ij}(t) + V^j(t) - V^i(t))\,dt \end{aligned} \tag{4}$$

for all $i = 1, ..., k$ and all $t \in [0, n]$.

Proof. If $V^i(t) = \overline{V}^i(t)$ for all states i and all times t then the Thiele equations for $V^i(t)$ and $\overline{V}^i(t)$ are equivalent and (4) is valid. On the other hand, if (4) holds, then the reserves $V^i(t)$ and $\overline{V}^i(t)$ have equivalent Thiele equations and, thus, the solutions $V^i(t)$ and $\overline{V}^i(t)$ are equal. Note that $V^i(t)$ and $\overline{V}^i(t)$ have the same initial/terminal condition. In case of the retrospective reserves we have $V^i_-(0-) = 0 = \overline{V}^i_-(0-)$ and in case of the prospective reserves we have $V^i_+(n) = 0 = \overline{V}^i_+(n)$, regardless of the choice of the payment functions and intensity functions. ■

Note that for $t = 0$ equation (4) is equivalent to $B^i(0) = \overline{B}^i(0)$. Some authors prefer to write the case $t = 0$ as an initial condition and assume (4) only for $t > 0$. Analogously, the case $t = n$ can be interpreted as a terminal condition.

The general result stated under Theorem 1 allows us to precisely state when surrendering can be ignored in the actuarial computations, as explained in the next example.

Example 1 *Suppose that state k corresponds to the cancellation of the policy so that $V^k(t) = 0$. The surrender value is assumed to be equal to $c^{ik}(t) = V^i(t)$ for $i = 1, ..., k-1$ and $t \in (0, n]$. If we set $\overline{\mu}^{(ik)}_{x+t} = 0$ (i.e. surrender never occurs) and let all other parameters of the alternative model be equal to the original model, then the alternative model has the same reserves $V^i(t)$ for all $t \in [0, n]$.*

4 Reserve-dependent benefits

If the insurer's payments are reserve-dependent, formulas (1) and (3) do not explicitly define the reserves anymore, since their right-hand sides depend on the reserves entering $c^{ij}(t)$ and/or $B^i(t)$. In this section we show how to obtain explicit formulas for the reserves if the payment functions are linear functions of the reserves.

Suppose that the insurer's payments in case of transition at time t can be decomposed into a time-dependent lump sum $c_0^{ij}(t)$ plus a time-dependent share $c_1^{ij}(t)$ of the change $V^i(t) - V^j(t)$ in the reserve due to the transition from state i to state j, i.e.

$$c^{ij}(t) = c_0^{ij}(t) + c_1^{ij}(t)\big(V^i(t) - V^j(t)\big) \tag{5}$$

with $0 \le c_1^{ij}(t) \le 1$. Here, the c_0^{ij} are reserve-independent, that is, the functions $c_0^{ij}(t)$ are completely specified at policy issue without reference to the contract reserves.

Notice that $V^i(t) - V^j(t)$ in (5) may be negative for some transitions. Payments of the form (5) are particularly useful to represent benefits paid in case of policyholder's death or in case of policy cancellation. If the attained state j corresponds to death or policy cancellation then $V^j(t) = 0$ and the benefit paid in case such a transition occurs at time t is equal to the lump sum $c_0^{ij}(t)$ plus a share $c_1^{ij}(t)$ of the reserve $V^i(t)$. Specification (5) is in line with insurance practice where the penalty in case the contract is surrendered is often expressed as a decreasing percentage of the reserve: if the policy is cancelled soon after it has been issued then the policyholder gets a smaller share c_1^{ij} of the reserve compared to the case where the contract is surrendered near to maturity. The time-varying percentage $c_1^{ij}(t)$ involved in (5) appropriately deals with this kind of rule.

Similarly, assume that the sojourn benefits can be represented as

$$\mathrm{d}B^i(t) = \mathrm{d}B_0^i(t) + V^i(t-)\mathrm{d}B_1(t), \tag{6}$$

where $B_0^i(t)$ and $B_1(t)$ are finite variation functions. Also, B_0^i is the accumulated value of reserve-independent benefits net of premiums paid, so that premiums are only taken into account in B_0^i and not in the reserve-dependent part of B^i. Notice that the share $\mathrm{d}B_1(t)$ of the reserve in (6) does not depend on the state occupied by the policyholder. Extensions to state-dependent shares are discussed in the final Section 6.

Proposition 1 *Given that the transition and sojourn benefits are of the form* (5)-(6), *Thiele's equations can be rewritten as*

$$dV^i(t) = V^i(t-)d\overline{\Delta}_t - d\overline{B}^i(t) - \sum_{\substack{j=1 \\ j\neq i}}^{k} \overline{\mu}_{x+t}^{(ij)}\big(\overline{c}^{ij}(t) + V^j(t) - V^i(t)\big)\,dt \tag{7}$$

where

$$\begin{aligned} d\overline{\Delta}_t &= d\Delta_t - dB_1(t) \\ \overline{c}^{ij}(t) &= \frac{c_0^{ij}(t)}{1 - c_1^{ij}(t)} \\ d\overline{B}^i(t) &= dB_0^i(t) \\ \overline{\mu}_{x+t}^{(ij)} &= \mu_{x+t}^{(ij)}(1 - c_1^{ij}(t)). \end{aligned}$$

Proof. Plugging (5) and (6) into Thiele's equation (2) yields

$$\begin{aligned} \mathrm{d}V^i(t) = & V^i(t-)\mathrm{d}\Delta_t - \mathrm{d}B_0^i(t) - V^i(t-)\mathrm{d}B_1(t) \\ & - \sum_{\substack{j=1 \\ j\neq i}}^{k} \mu_{x+t}^{(ij)}\Big(c_0^{ij}(t) + c_1^{ij}(t)\big(V^i(t) - V^j(t)\big) + V^j(t) - V^i(t)\Big)\mathrm{d}t. \end{aligned}$$

Rearranging terms and using the definitions of $\overline{\Delta}_t$, $\overline{c}^{ij}(t)$, $\overline{B}^i(t)$ and $\overline{\mu}_{x+t}^{(ij)}$, we obtain equation (7). ■

Equation (7) is in fact the equivalence condition (4) in Cantelli's Theorem. So Proposition 1 presents us a specific alternative model that has equivalent reserves.

If the insurer's payments are reserve-dependent, it is not obvious whether Thiele's equation really has a solution. The following corollary gives an answer.

Corollary 1 *Given that the transition and sojourn benefits are of the form* (5)-(6), *Thiele's equation for the prospective reserve and Thiele's equation for the retrospective reserve have unique solutions.*

Proof. Because of Proposition 1, Thiele's equation is equivalent to a differential equation of Thiele-type where the payment functions do not depend on the reserve. Using the uniqueness result of Milbrodt and Helbig (1999) for the latter equation, we get also uniqueness for the former equation. ∎

The uniqueness property according to Corollary 1 is particularly important whenever we want to calculate the reserves by applying numerical methods for solving Thiele's differential equations. However, we will show that such numerical procedures are not needed here as we can calculate the reserves from explicit formulas.

Taking into account Theorem 1, we learn from Proposition 1 that substituting Δ_t, $c^{ij}(t)$, $B^i(t)$, $\mu_{x+t}^{(ij)}$ with $\overline{\Delta}_t$, $\overline{c}^{ij}(t)$, $\overline{B}^i(t)$, $\overline{\mu}_{x+t}^{(ij)}$ as defined above leaves the reserves unchanged. In particular:

- A reserve-dependent sojourn payment of $V^i(t-)\mathrm{d}B_1(t)$ can be compensated by reducing the interest intensity $\mathrm{d}\Delta_t$ by $\mathrm{d}B_1(t)$.
- A reserve-dependent transition payment of $c_1^{ij}(t)(V^i(t)-V^j(t))$ can be compensated by reducing the transition intensity $\mu_{x+t}^{(ij)}$ by the factor $(1-c_1^{ij}(t))$ and at the same time multiplying the constant transition payment $c_0^{ij}(t)$ by the factor $(1-c_1^{ij}(t))^{-1}$. Note that the factor $(1-c_1^{ij}(t))$ represents the percentage of the reserves that are inherited by the insurance portfolio when the policyholder moves from state i to state j.

Combining Cantelli's Theorem and Proposition 1, we can derive explicit representation formulas for the reserves.

Proposition 2 *Given that the transition and sojourn payments are of the form* (5) *and* (6)*, the prospective reserve has the explicit representation*

$$V_+^i(t) = \sum_{j=1}^{k} \int_{(t,n]} {}_{s-t}\overline{v}_t \, {}_{s-t}\overline{p}_{x+t}^{(ij)} dB_0^j(s) + \sum_{\substack{j,l=1 \\ j\neq l}}^{k} \int_t^n {}_{s-t}\overline{v}_t \, {}_{s-t}\overline{p}_{x+t}^{(ij)} \mu_{x+s}^{(jl)} c_0^{jl}(s)\, ds$$

and the retrospective reserve has the explicit representation

$$V_-^i(t) = -\sum_{j=1}^{k} \int_{[0,t]} ({}_{t-s}\overline{v}_s)^{-1} (\overline{\boldsymbol{P}}(s,t))_{ij}^{-1} dB_0^j(s) - \sum_{\substack{j,l=1 \\ j\neq l}}^{k} \int_0^t ({}_{t-s}\overline{v}_s)^{-1} (\overline{\boldsymbol{P}}(s,t))_{ij}^{-1} \mu_{x+s}^{(jl)} c_0^{jl}(s)\, ds,$$

where the ${}_{s-t}\overline{p}_{x+t}^{(ij)}$ *are the transition probabilities corresponding to the transition intensities* $\overline{\mu}_{x+s}^{(ij)} = \mu_{x+t}^{(ij)}(1-c_1^{ij}(t))$, $\overline{\boldsymbol{P}}(s,t)$ *is the associated transition matrix, and* ${}_{s-t}\overline{v}_t$ *is the discounting factor corresponding to the cumulative interest intensity* $\overline{\Delta}_t = \Delta_t - B_1(t)$.

Proof. In case of payment functions that are not reserve-dependent, the solutions (1) and (3) of Thiele's equations are explicit formulas. In the setting of Proposition 1, the payment functions $B^i(t)$ and $c^{ij}(t)$ are reserve-dependent while the alternative payment functions $\overline{B}^i(t)$ and $\overline{c}^{ij}(t)$ are not. Thus, using that $V^i(t) = \overline{V}^i(t)$ because of Cantelli's Theorem and Proposition 1, we obtain explicit formulas for $V^i(t)$ by using the solutions (1) and (3) for Thiele's equations for $\overline{V}^i(t)$. ■

Corollary 2 *Consider a multiple decrement model, which means that all transition intensities other than $\mu_{x+s}^{(1j)}$, $j = 2, ..., k$ are zero and sojourn benefits as well as transition payments are only made in and from state* 1*. Further, let*

$$d(\Delta_t - B_1(t)) = (\delta_t - b_1(t))\,dt.$$

Then under assumptions (5) *and* (6) *the prospective reserve in state* 1 *has the representation*

$$\begin{aligned} V_+^1(t) = & \int_{(t,n]} \exp\left(-\int_t^s (\delta_u - b_1(u))\,du - \sum_{l=2}^k \int_t^s \mu_{x+u}^{(1l)} (1 - c_1^{1l}(u))\,du \right) dB_0^1(s) \\ & + \sum_{j=2}^k \int_t^n \exp\left(-\int_t^s (\delta_u - b_1(u))\,du - \sum_{l=2}^k \int_t^s \mu_{x+u}^{(1l)} (1 - c_1^{1l}(u))\,du \right) \mu_{x+s}^{(1j)}\, c_0^{1j}(s)\,ds \end{aligned}$$

and the retrospective reserve in state 1 *has the representation*

$$\begin{aligned} V_-^1(t) = & -\int_{[0,t]} \exp\left(\int_s^t (\delta_u - b_1(u))\,du + \sum_{l=2}^k \int_s^t \mu_{x+u}^{(1l)} (1 - c_1^{1l}(u))\,du \right) dB_0^1(s) \\ & - \sum_{j=2}^k \int_0^t \exp\left(\int_s^t (\delta_u - b_1(u))\,du + \sum_{l=2}^k \int_s^t \mu_{x+u}^{(1l)} (1 - c_1^{1l}(u))\,du \right) \mu_{x+s}^{(1j)}\, c_0^{1j}(s)\,ds. \end{aligned}$$

Proof. The sojourn probability in state 1 has the representation

$${}_{t-s}\overline{p}_{x+s}^{(11)} = \exp\left(-\int_s^t \sum_{l=2}^k \overline{\mu}_{x+u}^{(1l)} \mathrm{d}u \right),$$

and for an interest intensity of $(\delta_t - b_1(t))$ the discounting factor has the form

$${}_{t-s}\overline{v}_s = \exp\left(-\int_s^t \Big(\delta_u - b_1(u)\Big) \mathrm{d}u \right).$$

The announced result then follows from Proposition 2. ■

5 Examples

In this section, we illustrate the use of Propositions 1 and 2 for some practical situations. We first consider reserve-dependent capital management fees and surplus dividends. The case of reserve-dependent surrender values is then investigated.

5.1 Capital management charges and surplus dividends

Suppose that a proportional capital management fee of β and a constant capital management fee of α are charged for the investment of the reserve, i.e. the insurer charges additional costs of $\beta V_-^i(t-)+\alpha$ in state i at time t. Then, we have sojourn benefits of the form (6) with $\mathrm{d}B_0^i(t)=\mathrm{d}B^i(t)+\alpha\mathrm{d}t$ and $\mathrm{d}B_1(t)=\beta\mathrm{d}t$. Here, these fees are considered as "benefits", not paid to the beneficiary but to the insurer's cost department. According to Proposition 2 the retrospective reserve has the representation

$$\begin{aligned}V_-^i(t)=&-\sum_{j=1}^{k}\int_{[0,t]}\exp\left(\int_s^t(\delta_u-\beta)\mathrm{d}u\right)(\boldsymbol{P}(s,t))_{ij}^{-1}\big(\mathrm{d}B^j(s)+\alpha\mathrm{d}s\big)\\&-\sum_{\substack{j,l=1\\j\neq l}}^{k}\int_0^t\exp\left(\int_s^t(\delta_u-\beta)\mathrm{d}u\right)(\boldsymbol{P}(s,t))_{ij}^{-1}\mu_{x+s}^{(jl)}c^{jl}(s)\mathrm{d}s\end{aligned}$$

and the prospective reserve has the representation

$$\begin{aligned}V_+^i(t)=&\sum_{j=1}^{k}\int_{(t,n]}\exp\left(-\int_t^s(\delta_u-\beta)\mathrm{d}u\right)\,{}_{s-t}p_{x+t}^{(ij)}\big(\mathrm{d}B^j(s)+\alpha\mathrm{d}s\big)\\&+\sum_{\substack{j,l=1\\j\neq l}}^{k}\int_t^n\exp\left(-\int_t^s(\delta_u-\beta)\mathrm{d}u\right)\,{}_{s-t}p_{x+t}^{(ij)}\mu_{x+s}^{(jl)}c^{jl}(s)\mathrm{d}s.\end{aligned}$$

Thus, we can disregard the capital management charges β and α if we

- decrease the interest intensities by the proportional fee β,
- charge additional premiums of α.

Hence, for calculating the reserve explicitly, we have to replace the actual model, specified by $\mathrm{d}\Delta_t$, $c^{ij}(t)$, $\mu_{x+t}^{(ij)}$ and $\mathrm{d}B^i(t)$, by the model

$$\begin{aligned}\overline{c}^{ij}(t)&=c^{ij}(t)\\\mathrm{d}\overline{B}^i(t)&=\mathrm{d}B^i(t)+\alpha\mathrm{d}t\\\overline{\mu}_{x+t}^{(ij)}&=\mu_{x+t}^{(ij)}\\\mathrm{d}\overline{\Delta}_t&=\mathrm{d}\Delta_t-\beta\mathrm{d}t.\end{aligned}$$

The adapted model leads to explicit expressions for the reserves, while the original model only leads to implicit expressions.

The case of surplus dividends depending on the accumulated reserve can be treated in a similar way. Assume that technical profits expected to be distributed to the policyholders amount to a time-varying percentage $\beta(t)$ of the retrospective reserve $V_-^i(t-)$ so that we have sojourn benefits of the form (6) with $\mathrm{d}B_0^i(t)=\mathrm{d}B^i(t)$ and $\mathrm{d}B_1(t)=\beta(t)\mathrm{d}t$. Proceeding as before for the capital management fees, we can derive the resulting expression for the reserves.

5.2 Surrender payments

Let state k denote surrender. Suppose that the surrender payments are equal to the prospective reserve minus a proportional and a constant cancellation fee of β and α, respectively, i.e. the transition benefits are as stated under (5) with

$$c^{ik}(t) = c_0^{ik}(t) + c_1^{ik}(t)V^i(t) = -\alpha + (1-\beta)V^i(t), \quad i = 1, \dots, k-1. \tag{8}$$

Depending on whether surrender payments shall relate to the assets or the liabilities assigned to the insurance contract, the reserve in equation (8) is either the retrospective or the prospective reserve. In the prospective case, the surrender value corresponds to the liabilities that are freed by terminating the contract. In the retrospective case, the surrender value corresponds to the assets accrued by the contract.

According to Proposition 2, in the retrospective case the retrospective reserve has the representation

$$\begin{aligned} V_+^i(t) = & \sum_{j=1}^{k-1} \int_{(t,n]} {}_{s-t}v_t \, {}_{s-t}\overline{p}_{x+t}^{(ij)} \big(\mathrm{d}B_0^j(s) - \alpha\mu_{x+s}^{(jk)}\mathrm{d}s\big) \\ & + \sum_{\substack{j,l=1 \\ j\neq l}}^{k-1} \int_t^n {}_{s-t}v_t \, {}_{s-t}\overline{p}_{x+t}^{(ij)} \mu_{x+s}^{(jl)} c_0^{jl}(s)\mathrm{d}s, \end{aligned}$$

and in the prospective case the prospective reserve satisfies the equation

$$\begin{aligned} V_-^i(t) = & -\sum_{j=1}^{k-1} \int_{[0,t]} ({}_{t-s}v_s)^{-1} (\overline{\boldsymbol{P}}(s,t))_{ij}^{-1} \big(\mathrm{d}B_0^j(s) - \alpha\mu_{x+s}^{(jk)}\mathrm{d}s\big) \\ & - \sum_{\substack{j,l=1 \\ j\neq l}}^{k-1} \int_0^t ({}_{t-s}v_s)^{-1} (\overline{\boldsymbol{P}}(s,t))_{ij}^{-1} \mu_{x+s}^{(jl)} c_0^{jl}(s)\mathrm{d}s, \end{aligned}$$

where ${}_{s-t}\overline{p}_{x+t}^{(ij)}$ and $(\overline{\boldsymbol{P}}(s,t))_{ij}$ are the transition probabilities of the multistate model where the surrender intensities are reduced by the factor β. That means that we can replace the reserve-dependent surrender payment with a constant penalty $\frac{\alpha}{\beta}$ in case of policy cancellation provided we multiply the surrender intensities by the factor β. Alternatively, this constant penalty can be viewed as additional premiums charged at a rate equal to α times the surrender intensities $\mu_{x+s}^{(jk)}$. Hence,

$$\begin{aligned} \overline{c}^{ij}(t) &= c^{ij}(t), \quad j \neq k \\ \overline{c}^{ik}(t) &= -\frac{\alpha}{\beta} \\ \mathrm{d}\overline{B}^i(t) &= \mathrm{d}B^i(t) \\ \overline{\mu}_{x+t}^{(ij)} &= \mu_{x+t}^{(ij)}, \quad j \neq k \\ \overline{\mu}_{x+t}^{(ik)} &= \beta\mu_{x+t}^{(ik)} \\ \mathrm{d}\overline{\Delta}_t &= \mathrm{d}\Delta_t. \end{aligned}$$

Using the adapted model leads to explicit expressions for the reserves $\overline{V}_+^i(t)$ and $\overline{V}_-^i(t)$, which are equal to $V_+^i(t)$ and $V_-^i(t)$, respectively.

6 Conclusion

In this paper, several reserve-dependent payment patterns have been considered and explicit expressions have been derived for the corresponding reserves. The key theoretical argument is that Cantelli's theorem still applies provided the benefit payments and the technical basis are modified appropriately. Several examples have been discussed to illustrate the wide applicability of the approach proposed in the present paper.

Some extensions are possible by letting the interest rate credited to the reserve vary according to the state occupied by the policyholder. For instance, if instead of applying the same coefficient $c_1^{ij}(t)$ to both $V^i(t)$ and $V^j(t)$ in (5), transition payments of the form

$$c^{ij}(t) = c_0^{ij}(t) - c_1^{ij}(t)\, V^j(t) + c_2^{ij}(t)\, V^i(t) \tag{9}$$

are considered then it is still possible to identify the corresponding changes in the technical basis to get rid of reserve-dependent benefits. This specification can be explained as follows. In case of a transition from i to j at time t, the capital $V^i(t)$ is released and a subsequent reserving of $V^j(t)$ is needed. The factor $(1 - c_2^{ij})$ describes the percentage of $V^i(t)$ that is inherited to the insurance portfolio upon leaving the state i. The factor $(1-c_1^{ij})$ describes the percentage of the capital $V^j(t)$ that has to be raised by the insurance portfolio. Stated differently, $c_2^{ij}(t)$ is the portion that the policyholder benefits from the released capital $V^i(t)$, and $c_1^{ij}(t)$ is the portion that the policyholder has to contribute to the subsequent reserving of $V^j(t)$.

Reserve-dependent sojourn payments represented as

$$\mathrm{d}B^i(t) = \mathrm{d}B_0^i(t) + b_1^i(t)\, V^i(t-)\, \mathrm{d}t, \tag{10}$$

may also deserve consideration. Clearly, the specifications (9)-(10) can be handled by letting the interest rate $\overline{\delta}_t$ depend on the state occupied by the policyholder.

Notice that nonlinear expressions can also be of interest instead of (5). For instance, the policy conditions may specify that the accumulated reserve is paid in case of death, with a guaranteed minimum. This means that transition benefits of the form

$$c^{ij}(t) = \max\left\{c_0^{ij}(t), c_1^{ij}(t)(V^i(t) - V^j(t))\right\} \tag{11}$$

may also deserve interest. However, the specification (11) requires numerical procedures to reach a solution whereas the present paper rather focusses on explicit expressions of the solutions.

Despite their theoretical interest, we must acknowledge that the formulas involving lapse rates must be considered with great care as it is extremely difficult to estimate or forecast such rates: cancelling the contract is at the discretion of the policyholder and this decision may depend on individual factors as well as macroeconomic conditions (including current market interest rates compared to the technical guaranteed one). See, e.g., Eling and Kiesenbauer (2013) or Fier and Liebenberg (2013) as well as the references therein. This is why many insurers do not allow for lapses in premium calculation. When lapses are used to reduce the premium, the business is called lapse-supported and this may lead to severe adverse financial consequences due to the systematic risk involved.

Instead of reserve-dependent benefits, another popular policy condition is premium refund. In this case, premiums paid until death are refunded without interest or compounded at the technical interest rate if death occurs before maturity. In case of policy cancellation, the surrender value may be equal to a time-varying percentage of the total premiums paid so far (this time-varying percentage mimicking the evolution of the reserve). This approach avoids much of the technicalities developed in the present paper, while being attractive and transparent for the policyholders.

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