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Nontransferable Utility Bankruptcy Games

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Abstract:

In this paper, we analyze bankruptcy problems with nontransferable utility (NTU) from a game theoretical perspective by redefining corresponding NTU-bankruptcy games in a tailor-made way. It is shown that NTU-bankruptcy games are both coalitional merge convex and ordinal convex. Generalizing the notions of core cover and compromise stability for transferable utility (TU) games to NTU-games, we also show that each NTU-bankruptcy game is compromise stable. Thus, NTU-bankruptcy games are shown to retain the two characterizing properties of TU-bankruptcy games: convexity and compromise stability. As a first example of a game theoretical NTU-bankruptcy rule, we analyze the NTU-adjusted proportional rule and show that this rule corresponds to the compromise value of NTU-bankruptcy games.

Keywords: NTU-bankruptcy problem, NTU-bankruptcy game, Coalitional merge convexity, Ordinal convexity, Compromise stability, Core cover, Adjusted proportional rule.

JEL classification: C71

1 Introduction

In a (TU-)bankruptcy problem there is a group of agents with legal monetary claims over a estate, which is not large enough to satisfy the total claim. Bankruptcy problems were first analyzed in a game theoretical framework in O'Neill (1982). O'Neill (1982) defines associated bankruptcy games and shows that these are convex games; Aumann and Maschler (1985) propose the Talmud rule as a solution to bankruptcy problems and show that this rule corresponds to the nucleolus of the associated bankruptcy game; Curiel, Maschler

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and Tijs (1987) show that the (nonempty) core and the core cover of bankruptcy games coincide. In the (later) terminology of Quant, Borm, Reijnierse and van Velzen (2005), this means that bankruptcy games are compromise stable. Moreover, Curiel et al. (1987) show that the compromise value of bankruptcy games can be interpreted as an adjusted proportional rule for the underlying bankruptcy problem. Quant et al. (2005) show that the class of bankruptcy games is the only class of games that satisfies both convexity and compromise stability, up to S-equivalence. For a survey on TU-bankruptcy, we refer to Thomson (2003).

In this paper, we investigate bankruptcy problems with nontransferable utility, in short NTU-bankruptcy problems, which are a generalization of bankruptcy problems. An NTU-bankruptcy problem consists, first of all, of a set of agents N who each claim an individual level of utility over a estate. The corresponding vector of utility claims is summarized by $c \in \mathbb{R}^N_+$. Secondly, an NTU-bankruptcy problem specifies the estate set E of vectors of attainable utility with, typically, $c \notin E$. Although formally in a different setting, Chun and Thomson (1992) is one of the first papers that relates to NTU-bankruptcy problems. Orshan, Valenciano and Zarzuelo (2003) are the first to analyze NTU-bankruptcy problems in a game theoretical framework. They associate an NTU-game to an NTU-bankruptcy problem and show that the intersection of the core and the bilateral consistent prekernel of such a game is nonempty. However, they provide an example which illustrates that their game need not be ordinal convex. The reason for this is that the value of a coalition may contain elements that are not in the comprehensive hull of the set E. This, in our view, departs from the original idea in O'Neill (1982), where the value of a coalition is defined taking into account that the estate is going to be exactly distributed among the agents.

This paper redefines NTU-bankruptcy games, staying in line with the idea of O'Neill (1982). To this aim, we use a specific definition of an NTU-game which uses a slightly weaker notion of comprehensiveness than usual, while still in accordance with the general requirements imposed on an NTU-game by Osborne and Rubinstein (1994). It turns out that this type of NTU-bankruptcy games does satisfy ordinal convexity (cf. Vilkov, 1977) together with coalitional merge convexity (cf. Hendrickx, Borm and Timmer, 2002). Moreover, inspired by Tijs and Lipperts (1982) and Borm, Keiding, McLean, Oortwijn and Tijs (1992), we introduce the core cover for NTU-games. In line with Estévez-Fernández, Fiestras-Janeiro, Mosquera and Sánchez (2012), we show that the core cover of a compromise admissible NTU-game can be obtained as the translation of the core cover of an associated NTU-bankruptcy game. Following Quant et al. (2005), we define compromise stable NTU-games as those NTU-games with nonempty core for which the core and the core cover coincide. We show that NTU-bankruptcy games are compromise stable. Therefore, the characterizing properties of convexity and compromise stability for TU-bankruptcy games carry over to NTU-bankruptcy games. The game theoretical framework for NTU-bankruptcy problems also enables the analysis of game theoretical NTU-bankruptcy rules. As an example, we characterize the NTU-compromise value as defined in Borm et al. (1992) for bankruptcy games as an adjusted proportional rule.

The paper is structured as follows. Section 2 provides notions used throughout the paper. In Section 3 we introduce NTU-bankruptcy problems and NTU-games, discuss the requirements that we impose, and redefine NTU-bankruptcy games. Section 4 analyzes the notions of convexity and compromise stability for general NTU-games. Section 5 provides our main results: NTU-bankruptcy games are ordinal convex, coalitional merge convex, and compromise stable. Section 6 characterizes the compromise value of NTU-bankruptcy games as an adjusted proportional bankruptcy rule.

2 Preliminaries

Let N be a finite set, let $x, y \in \mathbb{R}^N$, $U \subset \mathbb{R}^N$ be closed and $S \in 2^{\mathrm{N}} \setminus \{\emptyset\}$. We denote $x_S = (x_i)_{i \in S}$, $x_S \geq y_S$ if $x_i \geq y_i$ for every $i \in S$, and $x_S > y_S$ if $x_i > y_i$ for every $i \in S$. We call U *S-comprehensive* if for all $a, b \in \mathbb{R}^N$ with $a \in U$, $b_S \leq a_S$, and $b_{N \setminus S} = a_{N \setminus S}$, it follows that $b \in U$. By $\mathrm{comp}^S(U)$ we denote the S-comprehensive hull of U and set $\mathrm{comp}(U) = \mathrm{comp}^N(U)$. The set of *$S$-weakly Pareto elements of U*, $\mathrm{WP}^S(U)$, is defined by

$$\mathrm{WP}^S(U) = \{x \in U \ : \ \text{there is no } y \in U \text{ such that } y_S > x_S\}$$

and set $\mathrm{WP}(U) = \mathrm{WP}^N(U)$. Related to the set of S-weakly Pareto elements, the set of *S-(strictly) dominated elements of U*, $\mathrm{Dom}^S(U)$, is defined by

$$\mathrm{Dom}^S(U) = \left\{x \in \mathbb{R}^N \ : \ \text{there is } y \in U \text{ such that } y_S > x_S\right\}$$

and we set $\mathrm{Dom}(U) = \mathrm{Dom}^N(U)$.

3 Modeling NTU-bankruptcy problems

In this section, we introduce NTU-bankruptcy problems, discuss the approach of Orshan et al. (2003) to model them as an NTU-game, and propose an alternative NTU-bankruptcy game.

In a TU-bankruptcy problem there is a group of agents that have claims over a estate which is not high enough to satisfy all claims. Formally, a *bankruptcy problem* is a tuple (N, e, c) where N is the set of agents, $e \in \mathbb{R}_+$ is the available estate, and $c \in \mathbb{R}^N_+$ is the vector of claims, with[1] $\sum_{i \in N} c_i \geq e$.

NTU-bankruptcy problems generalize bankruptcy problems. Our definition of NTU-bankruptcy problems is inspired by Chun and Thomson (1992). An *NTU-bankruptcy problem* is a tuple (N, E, c) where N is a finite set of agents while $E \subset \mathbb{R}^N_+$ and $c \in \mathbb{R}^N_+$ are such that the following five conditions are satisfied:

(E1) E is closed, convex, and $E \cap \mathbb{R}^N_{++} \neq \emptyset$,

[1] For technical reasons, it is imposed that $\sum_{i \in N} c_i \geq e$ instead of $\sum_{i \in N} c_i > e$.

(E2) there exist $p \in \mathbb{R}^N_{++}$ and $r \in \mathbb{R}$ such that for all $x \in E$, $\sum_{i \in N} p_i x_i \leq r$,

(E3) $E = \text{comp}(E) \cap \mathbb{R}^N_+$,

(E4) if $x, y \in \text{WP}(E) \cap \mathbb{R}^N_+$ and $x \geq y$, then, $x = y$,

(E5) $c \notin \text{int}(E)$.[2]

Condition (E2) implies that E is bounded and condition (E4) is a non-levelness property which is not required in Chun and Thomson (1992). It imposes that the weak Pareto boundary of E does not have segments parallel to a coordinate hyperplane. Note that $\text{WP}(E)$ is the closure of $\partial E \cap \mathbb{R}^N_{++}$.

As pointed out in Orshan et al. (2003), E can be interpreted to represent the set of utility vectors that the agents may achieve by means of efficient allocations of the available estate and c represents the utility levels claimed by the agents.

TU-bankruptcy problems have been studied in the literature along two different lines. One of the lines is the axiomatic study of bankruptcy rules, the other is the analysis of bankruptcy problems from a game theoretical perspective. This article will focus on the second line of research for NTU-bankruptcy problems.

We first recall the definitions of TU-bankruptcy games and of NTU-games. TU-bankruptcy games were first analyzed in O'Neill (1982). A *transferable utility game* (TU-game) is given by a pair (N, v) where N is the finite set of agents (or players) and $v : 2^N \to \mathbb{R}$ is a characteristic function satisfying $v(\emptyset) = 0$. Given a bankruptcy problem (N, e, c), the associated bankruptcy game, $(N, v_{e,c})$, is defined by $v_{e,c}(S) = \max\{0, e - \sum_{i \in N \setminus S} c_i\}$ for every $S \subset N$. The interpretation of the coalitional value $v_{e,c}(S)$ is as follows. Coalition S decides to leave the negotiations before the sharing of the estate e. This implies that if the total claim of the players in $N \setminus S$ exceeds e, then, S leaves with nothing and the agents in $N \setminus S$ continue the negotiations for e. Otherwise, each agent in $N \setminus S$ gets exactly his claim, c_i, and the agents in S start negotiations for the remaining of the estate, $e - \sum_{i \in N \setminus S} c_i$. The study of bankruptcy games and game theoretical bankruptcy rules has been further developed in Aumann and Maschler (1985) and Curiel et al. (1987).

A *non-transferable utility cooperative game* (NTU-game) is a pair (N, V) where N is a finite set of players and V is a set valued function that assigns to each $S \subset N$ a set $V(S) \subset \mathbb{R}^N$ of attainable payoff vectors satisfying

(1) $V(\emptyset) = \emptyset$;

(2) for every $S \in 2^{\text{N}} \setminus \{\emptyset\}$, $V(S)$ is nonempty, convex, closed and S-comprehensive;

(3) for every $S \in 2^{\text{N}} \setminus \{\emptyset\}$, the S-projection of the set $V(S) \cap (x + \mathbb{R}^N_+)$ is bounded for every $x \in \mathbb{R}^N$;

[2] For technical reasons, it is imposed that $c \notin \text{int}(E)$ instead of $c \notin E$.

(4) if $x, y \in \partial V(N)$ with $y_i \geq \max\{z_i : z \in V(\{i\})\}$ for every $i \in N$ and $x \geq y$, then, $x = y$.

Condition (4) is a weaker version of the non-levelness property in Aumann (1985) where the restriction $y_i \geq \max\{z_i : z \in V(\{i\})\}$ for every $i \in N$ is not required. From now on, we denote

$$v(i) = \max\{z_i : z \in V(\{i\})\}$$

for every $i \in N$. Note that $v(i)$ is well defined since $V(\{i\})$ is closed by condition (2) and the $\{i\}$-projection of $V(\{i\})$ is bounded by (3). Importantly, note that the concept of S-comprehensiveness in (2) is slightly weaker than the usual one. Although nonstandard, our definition of NTU-game is in line with the general definition provided in Osborne and Rubinstein (1994).

The following example illustrates a drawback of the game theoretical modeling of NTU-bankruptcy problems in Orshan et al. (2003).

Example 3.1. Consider the NTU-bankruptcy problem (N, E, c) with $N = \{1, 2\}$, $E = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$, and $c = (1, 2.2)$. The associated game, (N, V), in Orshan et al. (2003) is defined by $V(N) = \text{comp}(E)$ and $V(S) = \text{comp}(\{x \in \mathbb{R}^N_+ : (x_S, c_{N\setminus S}) \in E \text{ or } x_S = 0\})$ for all $S \subset N$, $S \neq N$. The coalitional values of this game for this example are given in Figure 1.

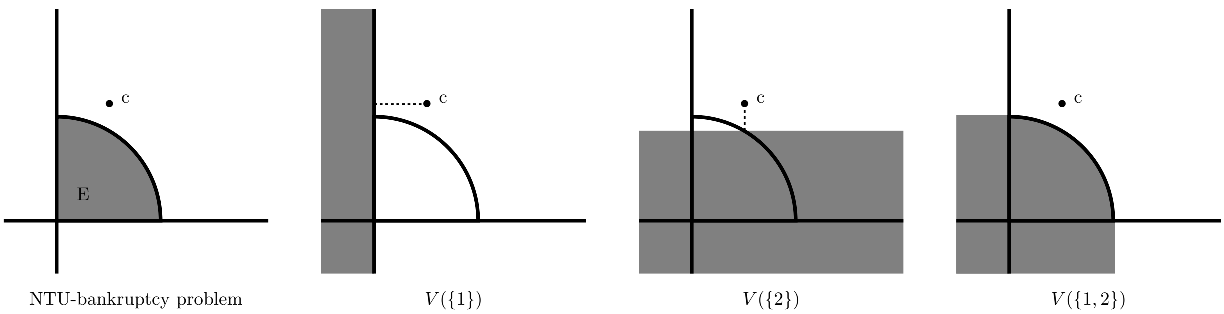

Figure 1: The NTU-game of Orshan et al. (2003) with respect to Example 3.1.

Note that points in $V(\{1\})$ and $V(\{2\})$ may fall outside of the comprehensive hull of E. For instance, since $(1, \sqrt{3}) \in E$, we have that $(100, \sqrt{3}) \in V(\{2\})$ although $(100, \sqrt{3}) \notin \text{comp}(E)$. In our opinion, this goes against the idea behind the definition of bankruptcy games in the transferable utility setting. If one considers the classical situation of (TU) bankruptcy games, coalition S has some idea of what the players in $N \setminus S$ will get: they either get c_i each, $i \in N \setminus S$, (and S shares $E - \sum_{i \in N\setminus S} c_i$) if $\sum_{i \in N \setminus S} c_i \leq E$, or they share exactly E among themselves (and S gets nothing) if $\sum_{i \in N\setminus S} c_i > E$. ◇

Next, we propose an alternative way to define NTU-bankruptcy games.

Definition 3.1. Let (N, E, c) be an NTU-bankruptcy problem. Then, the associated NTU-bankruptcy game, $(N, V_{E,c})$, is defined by $V_{E,c}(\emptyset) = \emptyset$, $V_{E,c}(N) = \text{comp}(E)$, and

$$V_{E,c}(S) = \begin{cases} \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S = 0,\, x_{N\setminus S} \le c_{N\setminus S}\right\}\right) & \text{if } (0_S, c_{N\setminus S}) \notin E, \\ \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S \le c_S,\, x_{N\setminus S} = c_{N\setminus S}\right\}\right) & \text{if } (0_S, c_{N\setminus S}) \in E. \end{cases}$$

for every $S \in 2^N \setminus \{\emptyset, N\}$.

Note that, for every $i \in N$,

$$v_{E,c}(i) = \begin{cases} 0 & \text{if } (0_{\{i\}}, c_{N\setminus\{i\}}) \notin E, \\ \max\left\{t \in \mathbb{R} \,:\, (t, c_{N\setminus\{i\}}) \in E\right\} & \text{if } (0_{\{i\}}, c_{N\setminus\{i\}}) \in E. \end{cases}$$

Note that $c_i \ge v_{E,c}(i) \ge 0$ for every $i \in N$. Moreover, note that NTU-bankruptcy games as defined above indeed satisfy the conditions of NTU-games. In fact, given $S \in 2^{\text{N}} \setminus \{\emptyset\}$, $V(S)$ is obviously non-empty, closed (because E is closed according to condition (E1) of NTU-bankruptcy problems), convex and S-comprehensive by definition. Moreover, the S-projection of the set $V_{E,c}(S) \cap (x + \mathbb{R}^N_+)$ is bounded for every $x \in \mathbb{R}^N$ by condition (E2) of NTU-bankruptcy problems. Further, (weak) non-levelness of $V_{E,c}(N) = \text{comp}(E)$ follows from condition (E4) of NTU-bankruptcy problems.

Example 3.2. Reconsider the NTU-bankruptcy problem (N, E, c) of Example 3.1. All coalitional values of the NTU-bankruptcy game, as provided by Definition 3.1, are given in Figure 2.

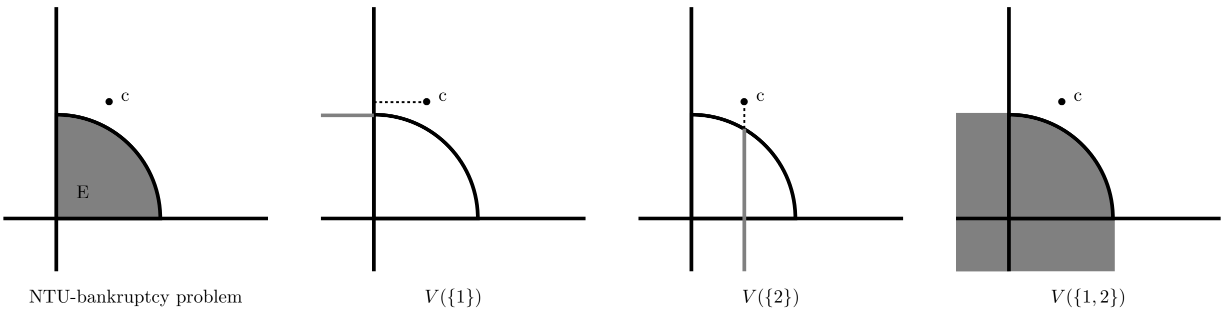

Figure 2: The NTU-bankruptcy game with respect to Example 3.1. ◇

4 Convexity and core cover for NTU-games

Convexity and compromise stability are two important concepts for TU-games. Convex games were first introduced in Shapley (1971) and correspond to those games for which the core and the convex hull of all marginal vectors of the game coincide (see Weber, 1988). Compromise stable games were first introduced in

Quant et al. (2005) as those with nonempty core for which the core and core cover (cf. Tijs and Lipperts, 1982) coincide. Quant et al. (2005) show that bankruptcy games are convex and compromise stable and that any TU-game that is convex and compromise stable is S-equivalent[3] to a bankruptcy game.

We will prove in Section 5 that NTU-bankruptcy games inherit analogues of the properties of convexity and compromise stability. To this end, this section focuses on the main notions of convexity for NTU-games as discussed in the literature and on introducing an analogue of compromise stability for NTU-games.

An NTU-game (N,V) is *superadditive* if for all $S,T \in 2^{\mathrm{N}} \setminus \{\emptyset\}$ with $S \cap T = \emptyset$,

$$V(S) \cap V(T) \subset V(S \cup T).$$

The set of *individually rational allocations* for S, $\mathrm{IR}(S)$, is defined by

$$\mathrm{IR}(S) = \{x \in V(S) \,:\, x_i \geq v(i) \text{ for every } i \in S\}\,.$$

There are several convexity notions for NTU-games in the literature. An NTU-game (N,V) is *ordinally convex* (cf. Vilkov, 1977) if for all $S,T \in 2^{\mathrm{N}} \setminus \{\emptyset\}$,

$$V(S) \cap V(T) \subset V(S \cup T) \cup V(S \cap T)$$

and is *coalitional merge convex* (cf. Hendrickx et al., 2002) if it is superadditive and for all $R \in 2^{\mathrm{N}} \setminus \{\emptyset\}$ and all $S,T \in 2^N$ such that $S \subset T \subset N \setminus R$, with $S \neq T$, the following statement is true:

For all $x \in \mathrm{WP}^S(V(S)) \cap \mathrm{IR}(S)$, all $y \in V(T)$, and all $z \in V(S \cup R)$ such that $z_S \geq x_S$, there exists an $a \in V(T \cup R)$ such that $a_T \geq y_T$ and $a_R \geq z_R$.

It turns out that ordinally convexity and coalitional merge convexity are strong, but independent, properties. For this and for a summary of convexity notions in the literature and their relations, we refer to Hendrickx et al. (2002).

Borm et al. (1992) generalize the concepts of utopia payoff and minimal right of a player for TU-games (see Tijs, 1981, Tijs and Lipperts, 1982) to NTU-games. Let (N,V) be an NTU-game and let $i \in N$. The *utopia payoff* to player i, $K_i(V)$, is defined by[4]

$$K_i(V) = \sup \left\{ t \in \mathbb{R} \,:\, \begin{array}{l} \text{there exists } a \in \mathbb{R}^{N\setminus\{i\}} \text{ with } (a,t) \in V(N), \\ (a,t) \notin \mathrm{Dom}^{N\setminus\{i\}}(V(N \setminus \{i\})), \text{ and} \\ a \geq (v(j))_{j \in N\setminus\{i\}} \end{array} \right\}.$$

[3]Two TU-games (N,v) and (N,w) are S-equivalent if there exist $a \in \mathbb{R}^N$ and $k > 0$ such that $w(S) = kv(S) + \sum_{i\in S} a_i$ for every $S \in 2^{\mathrm{N}} \setminus \{\emptyset\}$.

[4]Note that Borm et al. (1992) use an alternative definition of NTU-games with $V(S) \subset \mathbb{R}^S$. We have adapted their definitions to our setting.

Note that $K_i(V) < \infty$ for every $i \in N$ by condition (3) of an NTU-game.

In order to define the minimal right of player i, we first introduce some extra notation. We denote $\rho_i^{\{i\}}(V) = v(i)$ and for every $S \in 2^N \setminus \{\{i\}\}$ with $S \ni i$, $\rho_i^S(V)$ is the highest amount that player i can obtain if coalition S forms by giving the players in $S \setminus \{i\}$ (slightly) more than their utopia payoffs. Formally,

$$\rho_i^S(V) = \sup\left\{t \in \mathbb{R} \,:\, \text{there exists } a \in \mathbb{R}^{N\setminus\{i\}} \text{ with } (a,t) \in V(S) \text{ and } a_{S\setminus\{i\}} > K_{S\setminus\{i\}}(V)\right\}.$$

Note that $\rho_i^S(V)$ might equal $-\infty$. The *minimal right* of player $i \in N$, $k_i(V)$, is defined by

$$k_i(V) = \max_{S\in 2^N : S\ni i} \left\{\rho_i^S(V)\right\}.$$

The *core* of an NTU-game (N,V), $\text{Core}(V)$, is defined by

$$\text{Core}(V) = \left\{x \in V(N) \,:\, \text{there is no } S \in 2^{\text{N}} \setminus \{\emptyset\} \text{ with } x \in \text{Dom}^S(V(S))\right\}.$$

Note that $\text{Core}(V) \subset \text{WP}(V(N))$.

Theorem 4.1 (cf. Borm et al. (1992))**.** Let (N,V) be an NTU-game with $x \in \text{Core}(V)$. Then,

$$k(V) \leq x \leq K(V).$$

The core cover for TU-games is introduced in Tijs and Lipperts (1982) as the set of allocations that are efficient and bounded by the vector of minimal rights from below and by the vector of utopia payoffs from above. Here, we generalize this concept to NTU-games. The *core cover* of an NTU-game (N,V), $\text{CC}(V)$, is defined by

$$\text{CC}(V) = \{x \in \text{WP}(V(N)) \,:\, k(V) \leq x \leq K(V)\}.$$

By Theorem 4.1, it follows that the core of an NTU-game is contained in its core cover.

We say that an NTU-game (N,V) is *compromise admissible* if it has a nonempty core cover. Clearly, if the core of the game is nonempty, then, the game is NTU-compromise admissible.

Following the concept of compromise stable TU-game (see Quant et al., 2005), we say that an NTU-game (N,V) is *compromise stable* if it is compromise admissible and $\text{CC}(V) = \text{Core}(V)$.

Let (N,V) be an NTU-compromise admissible game. Following Borm et al. (1992), the *compromise value*, $T(V)$, is defined as the unique vector on the line segment between $k(V)$ and $K(V)$ which lies in $V(N)$ and is closest to the utopia vector $K(V)$. Formally,

$$T(V) = \lambda K(V) + (1-\lambda)k(V)$$

where $\lambda = \max\left\{\tilde{\lambda} \in [0,1] \,:\, \tilde{\lambda}K(V) + (1-\tilde{\lambda})k(V) \in V(N)\right\}$. Note that λ is well-defined because $k(V) \in V(N)$[5] and $V(N)$ is closed and comprehensive.

As an illustration of a compromise stable game, consider the following example.

[5]Note that for a compromise admissible game, compromise admissibility implies $k(V) \in V(N)$.

Example 4.1. Let $N = \{1, 2, 3\}$. The NTU-game (N, V) is defined by

$$\begin{aligned} V(\{i\}) &= \left\{x \in \mathbb{R}^3 \,:\, x_i \leq 0\right\}, \\ V(\{1,2\}) &= \text{comp}^{\{1,2\}}\left(\left\{x \in \mathbb{R}^3 \,:\, 4x_1^2 + 4x_2^2 = 1\right\}\right), \\ V(\{1,3\}) &= \text{comp}^{\{1,3\}}\left(\left\{x \in \mathbb{R}^3 \,:\, x_1 = \frac{1}{4},\, x_3 = \frac{1}{2}\right\}\right), \\ V(\{2,3\}) &= \text{comp}^{\{2,3\}}\left(\left\{x \in \mathbb{R}^3 \,:\, x_2 = \frac{1}{4},\, x_3 = \frac{1}{2}\right\}\right), \\ V(N) &= \text{comp}\left(\left\{x \in \mathbb{R}^3_+ \,:\, 4x_1^2 + 4x_2^2 + x_3^2 = 1\right\}\right). \end{aligned}$$

Then,

$$\text{Core}(V) = \left\{x \in \mathbb{R}^3_+ \,:\, x_1 \geq \frac{1}{4}, x_2 \geq \frac{1}{4}, x_3 = 0, 4x_1^2 + 4x_2^2 = 1\right\},$$

$$K(V) = \left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, 0\right),\ k(V) = \left(\frac{1}{4}, \frac{1}{4}, 0\right),\ T(V) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0\right)$$

and

$$\text{CC}(V) = \left\{x \in \text{WP}(V(N)) \,:\, \left(\frac{1}{4}, \frac{1}{4}, 0\right) \leq x \leq \left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, 0\right) \text{ and } 4x_1^2 + 4x_2^2 + x_3^2 = 1\right\} = \text{Core}(V).$$

Therefore, (N, V) is compromise stable. ◇

Quant et al. (2005) characterize a compromise stable TU-game as a compromise admissible game for which the coalitional values never exceed the maximum between the sum of the minimal rights of the members of the coalition and the difference between the value of the grand coalition and the sum of the utopia value of the players outside the coalition. We partially generalize this result to NTU-games in Theorem 4.2.

Theorem 4.2. Let (N, V) be a compromise admissible NTU-game. If for every $S \in 2^N \setminus \{\emptyset\}$, at least one of the following two conditions is satisfied,

(i) $V(S) \subset \left\{x \in \mathbb{R}^N \,:\, x_S \leq k_S(V)\right\}$,

or

(ii) $V(S) \subset \text{comp}(\left\{x \in \text{WP}(V(N)) \,:\, x_{N\setminus S} \geq K_{N\setminus S}(V)\right\})$,

then, (N, V) is compromise stable.

Proof: We have to show that (N, V) is compromise stable, that is, $\text{Core}(V) = \text{CC}(V)$. By Theorem 4.1, we know that $\text{Core}(V) \subset \text{CC}(V)$; therefore, we only have to show that $\text{Core}(V) \supset \text{CC}(V)$.

Let $z \in \text{CC}(V)$. Then,

$$z \in \text{WP}(V(N)) \text{ and } k(V) \leq z \leq K(V).$$

Moreover, since $k_i(V) \geq v(i)$ for all $i \in N$, we have that

$$z \in \mathrm{IR}(S) \text{ for every } S \in 2^N \setminus \{\emptyset\}.$$

We prove that $z \in \mathrm{Core}(V)$, that is, for every $S \in 2^N \setminus \{\emptyset\}$, $z \notin \mathrm{Dom}^S(V(S))$. Let $S \in 2^{\mathrm{N}} \setminus \{\emptyset\}$.

First, assume $V(S) \subset \{x \in \mathbb{R}^N : x_S \leq k_S(V)\}$. Then, $z_S \geq k_S(V) \geq y_S$ for every $y \in V(S)$. Therefore, $z \notin \mathrm{Dom}^S(V(S))$.

Second, assume $V(S) \subset \mathrm{comp}\left(\{x \in \mathrm{WP}(V(N)) : x_{N\setminus S} \geq K_{N\setminus S}(V)\}\right)$. We proceed by contradiction. Suppose that $z \in \mathrm{Dom}^S(V(S))$. Then, there exists $y \in \mathrm{WP}(V(S))$ with $y_S > z_S$ and, by assumption, there exists $\tilde{y} \in \mathrm{WP}(V(N))$ such that $\tilde{y}_S \geq y_S > z_S$ and $\tilde{y}_{N\setminus S} \geq K_{N\setminus S}(V) \geq z_{N\setminus S}$. Then, $z \in \mathrm{IR}(N) \cap \mathrm{WP}(V(N))$ and condition (4) of an NTU-game imply $\tilde{y} = z$. This establishes a contradiction to our premise that $\tilde{y}_S > z_S$.

□

The following example however illustrates that the sufficient conditions in Theorem 4.2 are not necessary ones to achieve compromise stability.

Example 4.2. Reconsider the compromise stable NTU-game of Example 4.1. For $S = \{1, 3\}$, however, conditions (i) and (ii) in Theorem 4.2 are not satisfied: $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right) \in V(\{1, 3\})$ and

$$\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right) \notin \{x \in \mathbb{R}^3 : x_{\{1,3\}} \leq k_{\{1,3\}}(V)\} = \left\{x \in \mathbb{R}^3 : x_1 \leq \frac{1}{4},\ x_3 \leq 0\right\}$$

and

$$\begin{aligned} \left(\tfrac{1}{4}, \tfrac{\sqrt{3}}{4}, \tfrac{1}{2}\right) &\notin \mathrm{comp}\left(\left\{x \in \mathrm{WP}(V(N)) : x_2 \geq K_2(V) = \tfrac{\sqrt{3}}{4}\right\}\right) \\ &= \mathrm{comp}\left(\left\{x \in \mathbb{R}^3_+ : 4x_1^2 + 4x_2^2 + x_3^2 = 1 \text{ and } x_2 \geq \tfrac{\sqrt{3}}{4}\right\}\right). \end{aligned}$$

◇

Note that the non-levelness requirement is key in the proof of Theorem 4.2. In fact, Theorem 4.2 need not hold in games not satisfying the non-levelness condition, as the following example illustrates.

Example 4.3. Consider the 4-players NTU-game (N, V) defined by

$$\begin{aligned} V(\{i\}) &= \{x \in \mathbb{R}^N : x_i \leq 0\} && \text{for every } i \in N, \\ V(S) &= \{x \in \mathbb{R}^N : x_i \leq 2 \text{ for every } i \in S\} && \text{for every } S \in 2^N,\ |S| = 2, \\ V(S) &= \{x \in \mathbb{R}^N : x_i \leq 4 \text{ for every } i \in S\} && \text{for every } S \in 2^N,\ |S| = 3, \\ V(N) &= \{x \in \mathbb{R}^N : x_i \leq 7 \text{ for every } i \in N\}. \end{aligned}$$

This game satisfies conditions (1), (2), and (3) of an NTU-game, but not (4). Here, $K(V) = (7, 7, 7, 7)$ and $k(V) = (0, 0, 0, 0)$. Therefore, the game satisfies the conditions in Theorem 4.2 since

$$V(S) \subset \mathrm{comp}^N\left(\{x \in \mathrm{WP}(V(N)) : x_{N\setminus S} \geq K_{N\setminus S}(V)\}\right)$$

for every $S \in 2^N \setminus \{\emptyset\}$. However, $\text{Core}(V) \neq \text{CC}(V)$. To see this, note that

$$\text{CC}(V) = \{x \in \mathbb{R}^N \,:\, 0 \leq x_i \leq 7 \text{ for every } i \in N \text{ and } x_j = 7 \text{ for some } j \in N\},$$

$(1,1,7,7) \in \text{CC}(V)$, and $(1,1,7,7) \in \text{Dom}^{\{1,2\}}(V(\{1,2\}))$, which implies $(1,1,7,7) \notin \text{Core}(V)$. $\diamond$

Next, we generalize some results of Curiel et al. (1987) on truncated claims, utopia vectors, and minimal rights to the NTU-setting.

Let (N,E,c) be an NTU-bankruptcy problem. We denote by $c^t \in \mathbb{R}^N$ the *truncated vector of claims* defined, for every $i \in N$, by

$$c_i^t = \begin{cases} c_i & \text{if } (c_i, 0_{N\setminus\{i\}}) \in E, \\ \max\{\tilde{t} \in \mathbb{R} \,:\, (\tilde{t}, 0_{N\setminus\{i\}}) \in E\} & \text{otherwise.} \end{cases}$$

Note that $c^t \in \mathbb{R}^N_+$ and $(c_i^t, 0_{N\setminus\{i\}}) \in \text{WP}(E)$ for every $i \in N$ with $c_i^t < c_i$. Next, we show that the properties related to claims truncation shown in Curiel et al. (1987) for TU-bankruptcy games are also satisfied for NTU-bankruptcy games. The proof of this proposition uses the following lemma.

Lemma 4.3. Let (N,E,c) be an NTU-bankruptcy problem, let $i \in N$, and let $x, y \in \mathbb{R}^{N\setminus\{i\}}$ with $x \leq y$, $x \neq y$, and $(x,0),(y,0) \in E$. Then, with $t_x = \max\{t \in \mathbb{R} \,|\, (x,t) \in E\}$ and $t_y = \max\{t \in \mathbb{R} \,|\, (y,t) \in E\}$,

$$t_x > t_y.$$

Proof: On the contrary, suppose that $t_x \leq t_y$. Note that $x, y \in \mathbb{R}^{N\setminus\{i\}}_+$ since $(x,0),(y,0) \in E$ and $E \subset \mathbb{R}^N_+$. Since E is closed and bounded, we have that E is compact, tx, $ty \in \mathbb{R}_+$ and

$$(x,t_x),(y,t_y) \in \text{WP}(E).$$

Since $x \leq y$, $t_x \leq t_y$, and $(x,t_x),(y,t_y) \in \text{WP}(E)$, we have $(x,t_x) = (y,t_y)$ by condition (E4). This establishes a contradiction to our premise that $x \neq y$. $\square$

Proposition 4.4. Let (N,E,c) be an NTU-bankruptcy problem and let $(N,V_{E,c})$ be the associated NTU-bankruptcy game. Then, (i) $V_{E,c} = V_{E,c^t}$, (ii) $K(V_{E,c}) = c^t$, and (iii) $k(V_{E,c}) = (v_{E,c}(i))_{i\in N}$.

Proof:

(i) $V_{E,c} = V_{E,c^t}$.

First, note that $V_{E,c}(\emptyset) = \emptyset = V_{E,c^t}(\emptyset)$ and $V_{E,c}(N) = \text{comp}(E) = V_{E,c^t}(N)$ by definition of NTU-bankruptcy games. Second, let $S \in 2^N \setminus \{\emptyset, N\}$. We distinguish between two cases: $(0_S, c_{N\setminus S}) \in E$ and $(0_S, c_{N\setminus S}) \notin E$.

Case 1: $(0_S, c_{N\setminus S}) \in E$.

Note that by condition (E3) of NTU-bankruptcy problems and by definition of c^t, $c^t_{N\setminus S} = c_{N\setminus S}$ and $(0_S, c^t_{N\setminus S}) \in E$. If $c^t_S = c_S$, then,

$$\begin{aligned} V_{E,c^t}(S) &= \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S \le c^t_S,\, x_{N\setminus S} = c^t_{N\setminus S}\right\}\right) \\ &= \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S \le c_S,\, x_{N\setminus S} = c_{N\setminus S}\right\}\right) = V_{E,c}(S). \end{aligned}$$

If $c^t_S \ne c_S$, since $c^t_{N\setminus S} = c_{N\setminus S}$ and $c^t_S \le c_S$, we have

$$\begin{aligned} V_{E,c^t}(S) &= \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S \le c^t_S,\, x_{N\setminus S} = c^t_{N\setminus S}\right\}\right) \\ &\subset \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S \le c_S,\, x_{N\setminus S} = c_{N\setminus S}\right\}\right) = V_{E,c}(S). \end{aligned}$$

To show the other inclusion, we proceed by contradiction. Suppose that $x \in V_{E,c}(S)$ and $x \notin V_{E,c^t}(S)$. Then, there exists $i \in S$ such that $x_i > c^t_i$. Since $x \in V_{E,c}(S)$, we have that $c^t_i < x_i \le c_i$. Note that $x_{N\setminus S} = c_{N\setminus S} = c^t_{N\setminus S}$ by definition of $V_{E,c}(S)$ and our assumption. Moreover, if $x_j < 0$ for some $j \in S$, we have that $x^+ \in \mathbb{R}^N_+$ defined by $x^+_j = \max\{x_j, 0\}$ for every $j \in N$ belongs to E by definition of $V_{E,c}(S)$. By condition (E3) of NTU-bankruptcy problems, we have that $(x_i, 0_{N\setminus\{i\}}) \in E$ with $x_i > c^t_i$. This establishes a contradiction to the definition of c^t_i.

Case 2: $(0_S, c_{N\setminus S}) \notin E$.

In this case, we need to distinguish between two new situations: $(0_S, c^t_{N\setminus S}) \notin E$ and $(0_S, c^t_{N\setminus S}) \in E$.

Case 2.a: $(0_S, c^t_{N\setminus S}) \notin E$.

Then,

$$\begin{aligned} V_{E,c^t}(S) &= \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S = 0,\, x_{N\setminus S} \le c^t_{N\setminus S}\right\}\right) \\ &= \text{comp}^S\left(\left\{x \in \text{WP}(E) \,:\, x_S = 0,\, x_{N\setminus S} \le c_{N\setminus S}\right\}\right) = V_{E,c}(S) \end{aligned}$$

where the second equality follows because $(c^t_i, 0_{N\setminus\{i\}}) \in \partial E$ for every $i \in N \setminus S$ with $c^t_i < c_i$.

Case 2.b: $(0_S, c^t_{N\setminus S}) \in E$.

Note that, in this case, $c^t_{N\setminus S} \ne c_{N\setminus S}$. Moreover, there is $i \in N \setminus S$ such that either $N \setminus S = \{i\}$, or $c^t_j = 0$ for every $j \in N \setminus S$, $j \ne i$.

> We show this claim by contradiction. Assume that $|N \setminus S| > 1$ and that there are $i, j \in N \setminus S$ with $i \ne j$ and $c^t_i, c^t_j > 0$. By our assumption and condition (E3), we have that $(0_{N\setminus\{i,j\}}, c^t_j, c^t_i) \in E$. If $(0_{N\setminus\{i,j\}}, c^t_j, c^t_i) \in \text{WP}(E)$, then, $(0_{N\setminus\{i\}}, c^t_i) \in \text{WP}(E)$ and (E4) imply $(0_{N\setminus\{i,j\}}, c^t_j, c^t_i) = (0_{N\setminus\{j\}}, c^t_i)$. This establishes a contradiction to our premise that $c^t_j > 0$.
>
> If $(0_{N\setminus\{i,j\}}, c^t_j, c^t_i) \in E \setminus \text{WP}(E)$, then, by Lemma 4.3, $\max\left\{t \in \mathbb{R} \,|\, (0_{N\setminus\{i,j\}}, c^t_j, t) \in E\right\} <$

$\max\left\{t \in \mathbb{R} \mid (0_{N\setminus\{i\}}, t) \in E\right\} = c_i^t$. This establishes a contradiction to our premise that $(0_{N\setminus\{i,j\}}, c_j^t, c_i^t) \in E$.

Then,

$$\begin{aligned} V_{E,c^t}(S) &= \operatorname{comp}^S\left(\left\{x \in \mathrm{WP}(E) \,:\, x_S = 0,\, x_{N\setminus S} \le c^t_{N\setminus S}\right\}\right) \\ &= \operatorname{comp}^S\left(\left\{x \in \mathrm{WP}(E) \,:\, x_{N\setminus\{i\}} = 0,\, x_i \le c^t_i\right\}\right) \\ &= \operatorname{comp}^S\left(\left\{x \in \mathrm{WP}(E) \,:\, x_{N\setminus\{i\}} = 0,\, x_i \le c_i\right\}\right) \\ &= \operatorname{comp}^S\left(\left\{x \in \mathrm{WP}(E) \,:\, x_S = 0,\, x_{N\setminus S} \le c_{N\setminus S}\right\}\right) = V_{E,c}(S) \end{aligned}$$

where the third equality follows because $(c_i^t, 0_{N\setminus\{i\}}) \in \mathrm{WP}(E)$.

(ii) $\boldsymbol{K(V_{E,c}) = c^t}$.

Since $V_{E,c} = V_{E,c^t}$, it suffices to show that $K(V_{E,c^t}) = c^t$. Let $i \in N$. Note that if $(0_{N\setminus\{i\}}, c_i) \in E$, then, $c_i^t = c_i$. If $(0_{N\setminus\{i\}}, c_i) \notin E$, then, $c_i^t < c_i$ and $(0_{N\setminus\{i\}}, c_i^t) \in \mathrm{WP}(E)$. Then,

$$V_{E,c^t}(N \setminus \{i\}) = \operatorname{comp}^{N\setminus\{i\}}(\{x \in \mathrm{WP}(E) \,:\, x_{N\setminus\{i\}} \le c^t_{N\setminus\{i\}}, x_i = c_i^t\}).$$

First, we show that $K_i(V_{E,c^t}) \le c_i^t$ by contradiction. Suppose that $K_i(V_{E,c^t}) > c_i^t$. From the definition of $K_i(V_{E,c^t})$, we can choose $b \in \mathbb{R}^{N\setminus\{i\}}$ such that

(a) $(b, K_i(V_{E,c^t})) \in \operatorname{comp}(E)$,

(b) $(b, K_i(V_{E,c^t})) \notin \operatorname{Dom}^{N\setminus\{i\}}(V_{E,c^t}(N \setminus \{i\}))$, and

(c) $b_j \ge v_{E,c^t}(j)$ for all $j \in N \setminus \{i\}$.

Clearly, $(b, K_i(V_{E,c^t})) \in \mathrm{WP}(E) \subset \partial E$. Moreover, $K_i(V_{E,c^t}) > c_i^t$ and $(b, K_i(V_{E,c^t})) \in \partial E$ imply $(b, c_i^t) \in \operatorname{comp}(E)$. Since $c_i^t \ge v_{E,c^t}(i)$, it follows that $(b, c_i^t) \in \mathrm{IR}(N)$. Then, condition (4) of the NTU-game V_{E,c^t} implies $(b, c_i^t) \notin \partial E$. Therefore, there is a $y \in \partial E$ such that $y > (b, c_i^t)$. Hence, $(y_{N\setminus\{i\}}, c_i^t) \in V_{E,c^t}(N \setminus \{i\})$ with $y_{N\setminus\{i\}} > b$. This establishes a contradiction to our premise that $(b, K_i(V_{E,c^t})) \notin \operatorname{Dom}^{N\setminus\{i\}}(V_{E,c^t}(N \setminus \{i\}))$.

Second, we show that $K_i(V_{E,c^t}) \ge c_i^t$. Note that

$$(0_{N\setminus\{i\}}, c_i^t) \in E \text{ implies that } ((v_{E,c^t}(j))_{j\in N\setminus\{i\}}, c_i^t) \in E.$$

To see this, observe that if $(0_j, c^t_{N\setminus\{j\}}) \in E$ for some $j \in N \setminus \{i\}$, then, $(v_{E,c^t}(j), c^t_{N\setminus\{j\}}) \in E$ by definition of $v_{E,c^t}(j)$ which, together with $v_{E,c^t}(k) \le c_k^t$ for all $k \in N \setminus \{i, j\}$, implies $((v_{E,c^t}(j))_{j\in N\setminus\{i\}}, c_i^t) \in E$ by condition (E3) of NTU-bankruptcy problems. If $(0_j, c^t_{N\setminus\{j\}}) \notin E$ for every $j \in N \setminus \{i\}$, then, $v_{E,c^t}(j) = 0$ for every $j \in N \setminus \{i\}$ and $((v_{E,c^t}(j))_{j\in N\setminus\{i\}}, c_i^t) = (0_{N\setminus\{i\}}, c_i^t) \in E$ by assumption.

Take $\tilde{t} = \max\{t \ge 0 \,:\, ((v_{E,c^t}(j) + t)_{j\in N\setminus\{i\}}, c_i^t) \in E\}$. It follows that

(a) $((v_{E,c^t}(j)+\tilde{t})_{j\in N\setminus\{i\}}, c_i^t) \in \text{comp}(E)$,

(b) $((v_{E,c^t}(j)+\tilde{t})_{j\in N\setminus\{i\}}, c_i^t) \notin \text{Dom}^{N\setminus\{i\}}(V_{E,c^t}(N\setminus\{i\}))$, and

(c) $v_{E,c^t}(j)+\tilde{t} \geq v_{E,c^t}(j)$ for all $j \in N\setminus\{i\}$.

Therefore, $K_i(V_{E,c^t}) \geq c_i^t$.

(iii) $k(V_{E,c}) = (v_{E,c}(i))_{i\in N}$.

Since $V_{E,c} = V_{E,c^t}$, it suffices to show that $k(V_{E,c^t}) = v_{E,c^t}(i)$. Let $i \in N$. By definition, $\rho_i^{\{i\}}(V_{E,c^t}) = v_{E,c^t}(i)$. Let $S \in 2^N \setminus \{\{i\}, N\}$ with $S \ni i$. Note that $\rho_i^S(V_{E,c^t})$ is given by

$$\rho_i^S(V_{E,c^t}) = \sup\left\{t \in \mathbb{R} : \text{ there exists } a \in \mathbb{R}^{N\setminus\{i\}} \text{ with } (a,t) \in V_{E,c^t}(S) \text{ and } a_{S\setminus\{i\}} > c^t_{S\setminus\{i\}}\right\}.$$

If $(0_S, c^t_{N\setminus S}) \notin E$, then, $\rho_i^S(V_{E,c^t}) = -\infty$ because for any $t \in \mathbb{R}$, we cannot find $a \in \mathbb{R}^{N\setminus\{i\}}$ with $(a,t) \in V_{E,c^t}(S)$ and $a_{S\setminus\{i\}} > c^t_{S\setminus\{i\}}$ since $(a,t) \in V_{E,c^t}(S)$ implies $a_{S\setminus\{i\}} \leq 0_{S\setminus\{i\}} \leq c^t_{S\setminus\{i\}}$.

If $(0_S, c^t_{N\setminus S}) \in E$, then, $\rho_i^S(V_{E,c^t}) = -\infty$ because for any $t \in \mathbb{R}$, we cannot find $a \in \mathbb{R}^{N\setminus\{i\}}$ with $(a,t) \in V_{E,c^t}(S)$ and $a_{S\setminus\{i\}} > c^t_{S\setminus\{i\}}$ since $(a,t) \in V_{E,c^t}(S)$ implies $a_{S\setminus\{i\}} \leq c^t_{S\setminus\{i\}}$.

Last, we analyze $\rho_i^N(V_{E,c^t})$. Note that $\rho_i^N(V_{E,c^t})$ is given by

$$\rho_i^N(V_{E,c^t}) = \sup\left\{t \in \mathbb{R} : \text{ there exists } a \in \mathbb{R}^{N\setminus\{i\}} \text{ with } (a,t) \in \text{comp}(E) \text{ and } a_{N\setminus\{i\}} > c^t_{N\setminus\{i\}}\right\}.$$

If $(0_i, c^t_{N\setminus\{i\}}) \notin E$, then, $\rho_i^N(V_{E,c^t}) = -\infty$ because $(0_i, c^t_{N\setminus\{i\}}) \notin E$ and, therefore, for any $t \in \mathbb{R}$, we cannot find $a \in \mathbb{R}^{N\setminus\{i\}}$ with $(a,t) \in \text{comp}(E)$ and $a_{N\setminus\{i\}} > c^t_{N\setminus\{i\}}$.

If $(0_i, c^t_{N\setminus\{i\}}) \in E$, then, $\rho_i^N(V_{E,c^t}) = -\infty$ if $(0_i, c^t_{N\setminus\{i\}}) \in \partial E$ and $\rho_i^N(V_{E,c^t}) = v_{E,c^t}(i)$ otherwise.

Therefore, it follows that $k_i(V_{E,c^t}) = v_{E,c^t}(i)$. □

Next, we generalize a result in Estévez-Fernández et al. (2012) that states that the core cover of a compromise admissible TU-game can be obtained as a translation of the core cover of a TU-bankruptcy game to the NTU-setting.

Theorem 4.5. *Let (N,V) be a compromise admissible NTU-game with $V(N)\cap\mathbb{R}^N_{++} \neq \emptyset$. Then,* $\text{CC}(V) = k(V) + \text{CC}(V_{E,c})$ *where $E = (V(N)-k(V))\cap\mathbb{R}^N_{++}$ and $c = K(V)-k(V)$.*

Proof: First, note that (N,E,c) with $E = (V(N)-k(V))\cap\mathbb{R}^N_{++}$ and $c = K(V)-k(V)$ is an NTU-bankruptcy problem. To see this, note that $E = V(N)-k(V)$ can be interpreted as a translation of the center of coordinates to $k(V)$. Then, (E1) follows by condition (2) of an NTU-game, (E2) is a direct consequence of condition (3) of an NTU-game, (E3) follows by condition (2) of an NTU-game, (E4) is a direct consequence of condition (4) of an NTU-game, (E5) follows because $\mathrm{CC}(V) \neq \emptyset$ and $K(V) \notin \mathrm{int}(V(N))$. Moreover, note that $c = K(V) - k(V) = c^t$ using the definition of $K(V)$.

Let $(N, V_{E,c})$ be its corresponding NTU-bankruptcy game. By Proposition 4.4, we know that

$$K(V_{E,c}) = K(V) - k(V) \quad \text{and} \quad k(V_{E,c}) = (v_{E,c}(i))_{i \in N}.$$

Note that

$$v_{E,c}(i) = \begin{cases} 0 & \text{if } (k_i(V), K_{N\setminus\{i\}}(V)) \notin V(N), \\ \sup\left\{t \in \mathbb{R} \,:\, \left(t + k_i(V), K_{N\setminus\{i\}}(V)\right) \in V(N)\right\} & \text{if } (k_i(V), K_{N\setminus\{i\}}(V)) \in V(N). \end{cases}$$

This implies that $v_{E,c}(i) = 0$ for every $i \in N$. To see this, let $(k_i(V), K_{N\setminus\{i\}}(V)) \in V(N)$. Clearly, since

$$\begin{aligned} k_i(V) &= \max_{S \in 2^N : S \ni i} \left\{\rho_i^S(V)\right\} \\ &\geq \rho_i^N(V) = \sup\left\{t \in \mathbb{R} \,:\, \text{there exists } a \in \mathbb{R}^{N\setminus\{i\}} \text{ with } (a,t) \in V(N) \text{ and } a > K_{N\setminus\{i\}}(V)\right\}, \end{aligned}$$

it must be the case that $(k_i(V), K_{N\setminus\{i\}}(V)) \in \partial V(N) \cap \mathrm{IR}(N)$ which implies that $(k_i(V), K_{N\setminus\{i\}}(V)) \in \mathrm{WP}(V(N))$ and $v_{E,c}(i) = 0$ for every $i \in N$.

Next, we show that $\mathrm{CC}(V) = k(V) + \mathrm{CC}(V_{E,c})$. First, we prove that $\mathrm{CC}(V) \subset k(V) + \mathrm{CC}(V_{E,c})$. Let $x \in \mathrm{CC}(V)$, then, $k(V) \leq x \leq K(V)$ and $x \in \mathrm{WP}(V(N))$. Thus, $0_N = (v_{E,c}(i))_{i \in N} \leq x - k(V) \leq K(V) - k(V)$ and $x - k(V) \in \mathrm{WP}(V_{E,c}(N))$. Hence, $x - k(V) \in \mathrm{CC}(V_{E,c})$. Last, we prove that $\mathrm{CC}(V) \supset k(V) + \mathrm{CC}(V_{E,c})$. Let $x \in \mathrm{CC}(V_{E,c})$. Then, $k(V_{E,c}) \leq x \leq K(V_{E,c})$ and $x \in \mathrm{WP}(V_{E,c}(N))$. By Proposition 4.4, we know that $k_i(V_{E,c}) = v_{E,c}(i)$ and $K_i(V_{E,c}) = c_i^t$ for every $i \in N$. Then, $k_i(V_{E,c}) = 0$ and $K_i(V_{E,c}) = K_i(V) - k_i(V)$ for every $i \in N$. Therefore, $k(V) \leq x + k(V) \leq K(V)$ and $x + k(V) \in \mathrm{WP}(V(N))$. As a result, $x + k(V) \in \mathrm{CC}(V)$. □

5 Properties of NTU-bankruptcy games

In this section, we show that NTU-bankruptcy games are coalitional merge convex, ordinal convex, and compromise stable. First, we prove that all NTU-bankruptcy games are superadditive.

Lemma 5.1. Every NTU-bankruptcy game is superadditive.

Proof: Let (N, E, c) be an NTU-bankruptcy problem and let $(N, V_{E,c})$ be the associated NTU-bankruptcy game. Let $S, T \in 2^{\mathrm{N}} \setminus \{\emptyset\}$ with $S \cap T = \emptyset$. We show that

$$V_{E,c}(S) \cap V_{E,c}(T) \subset V_{E,c}(S \cup T).$$

Let $z \in V_{E,c}(S) \cap V_{E,c}(T)$. Without loss of generality, it suffices to distinguish between two cases: (i) $(0_S, c_{N\setminus S}) \in E$ and (ii) $(0_S, c_{N\setminus S}) \notin E$, $(0_T, c_{N\setminus T}) \notin E$.

Case (i): $(0_S, c_{N\setminus S}) \in E$.

In this case,

$$V_{E,c}(S) = \mathrm{comp}^S \left(\left\{x \in \mathrm{WP}(E) \,:\, x_S \leq c_S,\, x_{N\setminus S} = c_{N\setminus S}\right\}\right).$$

Moreover, $(0_{S\cup T}, c_{N\setminus(S\cup T)}) \in E$ by condition (E3) of NTU-bankruptcy problems and, consequently,

$$V_{E,c}(S \cup T) = \mathrm{comp}^S \left(\left\{x \in \mathrm{WP}(E) \,:\, x_{S\cup T} \leq c_{S\cup T},\, x_{N\setminus(S\cup T)} = c_{N\setminus(S\cup T)}\right\}\right).$$

Therefore, we have $z_S \leq c_S$ and $z_{N\setminus S} = c_{N\setminus S}$. Consequently, $z_T = c_T$ and $z_{N\setminus(S\cup T)} = c_{N\setminus(S\cup T)}$ since $T \subset N \setminus S$. As a result, $z \in V_{E,c}(S \cup T)$.

Case (ii): $(0_S, c_{N\setminus S}) \notin E$, $(0_T, c_{N\setminus T}) \notin E$.

Recall that $E \subset \mathbb{R}^N_+$. By definition of $(N, V_{E,c})$, we have

$$V_{E,c}(R) = \mathrm{comp}^R \left(\left\{x \in \mathrm{WP}(E) \,:\, x_R = 0,\, x_{N\setminus R} \leq c_{N\setminus R}\right\}\right)$$

for $R \in \{S, T\}$. Since $z \in V_{E,c}(S)$, we have $z_S \leq 0$, $0 \leq z_{N\setminus S} \leq c_{N\setminus S}$, and since $z \in V_{E,c}(T)$, we have $z_T \leq 0$, $0 \leq z_{N\setminus T} \leq c_{N\setminus T}$. Consequently,

$$z = (0_{S\cup T}, z_{N\setminus(S\cup T)}) \text{ with } 0 \leq z_{N\setminus(S\cup T)} \leq c_{N\setminus(S\cup T)}$$

since $S \subset N \setminus T$ and $T \subset N \setminus S$. Moreover, since $z \in V_{E,c}(S)$ with $z_s = 0$ and $0 \leq z_{N\setminus S} \leq c_{N\setminus S}$, by definition of comp^S, we have that

$$z = (0_{S\cup T}, z_{N\setminus(S\cup T)}) \in \mathrm{WP}(E).$$

If $(0_{S\cup T}, c_{N\setminus(S\cup T)}) \notin E$, then,

$$z \in \mathrm{comp}^{S\cup T} \left(\left\{x \in \mathrm{WP}(E) : x_{S\cup T} = 0,\, x_{N\setminus(S\cup T)} \leq c_{N\setminus(S\cup T)}\right\}\right) = V_{E,c}(S \cup T).$$

If $(0_{S\cup T}, c_{N\setminus(S\cup T)}) \in E$, then,

$$z = (0_{S\cup T}, c_{N\setminus(S\cup T)}) \in \mathrm{comp}^{S\cup T} \left(\left\{x \in \mathrm{WP}(E) : x_{S\cup T} \leq c_{S\cup T},\, x_{N\setminus(S\cup T)} = c_{N\setminus(S\cup T)}\right\}\right) = V_{E,c}(S \cup T)$$

where the first equality follows by condition (E4) of NTU-bankruptcy problems since $z = (0_{S\cup T}, z_{N\setminus(S\cup T)}) \in \mathrm{WP}(E)$ and $z \leq (0_{S\cup T}, c_{N\setminus(S\cup T)}) \in E$. □

Theorem 5.2. Every NTU-bankruptcy game is coalitional merge convex.

Proof: Let (N, E, c) be an NTU-bankruptcy problem and let $(N, V_{E,c})$ be the associated NTU-bankruptcy game. By Lemma 5.1, we know that $(N, V_{E,c})$ is superadditive. Let $U \subset 2^{\mathrm{N}} \setminus \{\emptyset\}$ and $S \subset T \subset N \setminus U$ with $S \neq T$. Let $x \in \mathrm{WP}^S(V_{E,c}(S)) \cap \mathrm{IR}(S)$, $y \in V_{E,c}(T)$, and $z \in V_{E,c}(S \cup U)$ with $z_S \geq x_S$. We show that there exists an $a \in V_{E,c}(T \cup U)$ such that $a_T \geq y_T$ and $a_U \geq z_U$. We distinguish between two cases: (i) $(0_T, c_{N\setminus T}) \notin E$ and (ii) $(0_T, c_{N\setminus T}) \in E$.

Case (i): $(0_T, c_{N\setminus T}) \notin E$.

First, note that $V_{E,c}(T) = \mathrm{comp}^T\left(\left\{\tilde{x} \in \mathrm{WP}(E) \,:\, \tilde{x}_T = 0,\, \tilde{x}_{N\setminus T} \leq c_{N\setminus T}\right\}\right)$. Therefore,

$$y_T \leq 0 \text{ and, in particular, } y_S \leq 0.$$

Moreover, by condition (E3) of NTU-bankruptcy problems, we have that $(0_{\{i\}}, c_{N\setminus\{i\}}) \notin E$ for every $i \in T$ and, consequently, $v_{E,c}(i) = 0$ for every $i \in T$. Further, condition (E3) of NTU-bankruptcy problems and $S \subset T$ also imply $(0_S, c_{N\setminus S}) \notin E$, $V_{E,c}(S) = \mathrm{comp}^S\left(\left\{\tilde{x} \in \mathrm{WP}(E) \,:\, \tilde{x}_S = 0,\, \tilde{x}_{N\setminus S} \leq c_{N\setminus S}\right\}\right)$, and $v_{E,c}(i) = 0$ for every $i \in S$. Since $x \in \mathrm{WP}^S(V_{E,c}(S)) \cap \mathrm{IR}(S)$, we have $x_S = 0$. Then,

$$z_S \geq x_S = 0.$$

Consider three cases: (i.a) $(0_{S\cup U}, c_{N\setminus(S\cup U)}) \in E$, (i.b) $(0_{S\cup U}, c_{N\setminus(S\cup U)}) \notin E$, $(0_{T\cup U}, c_{N\setminus(T\cup U)}) \in E$, and (i.c) $(0_{S\cup U}, c_{N\setminus(S\cup U)}) \notin E$, $(0_{T\cup U}, c_{N\setminus(T\cup U)}) \notin E$.

(i.a) $(0_{S\cup U}, c_{N\setminus(S\cup U)}) \in E$.

Then, by condition (E3) of NTU-bankruptcy problems, $(0_{T\cup U}, c_{N\setminus(T\cup U)}) \in E$ and, consequently,

$$V_{E,c}(R \cup U) = \mathrm{comp}^{R\cup U}\left(\left\{\tilde{x} \in \mathrm{WP}(E) \,:\, \tilde{x}_{R\cup U} \leq c_{R\cup U},\, \tilde{x}_{N\setminus(R\cup U)} = c_{N\setminus(R\cup U)}\right\}\right)$$

for $R \in \{S, T\}$. Then, $z \in V_{E,c}(S \cup U)$ implies $z \in V_{E,c}(T \cup U)$. Choosing $a = z$, we have $a_U = z_U$ and $a_S = z_S \geq x_S = 0 \geq y_S$. We still need to show that $a_{T\setminus S} \geq y_{T\setminus S}$. Note that $(T \setminus S) \subset (N \setminus (S \cup U))$ and $a_{T\setminus S} = z_{T\setminus S} = c_{T\setminus S} \geq 0 \geq y_{T\setminus S}$.

(i.b) $(0_{S\cup U}, c_{N\setminus(S\cup U)}) \notin E$, $(0_{T\cup U}, c_{N\setminus(T\cup U)}) \in E$.

Then,

$$V_{E,c}(S \cup U) = \mathrm{comp}^{S\cup U}\left(\left\{\tilde{x} \in \mathrm{WP}(E) \,:\, \tilde{x}_{S\cup U} = 0,\, \tilde{x}_{N\setminus(S\cup U)} \leq c_{N\setminus(S\cup U)}\right\}\right) \text{ and}$$
$$V_{E,c}(T \cup U) = \mathrm{comp}^{T\cup U}\left(\left\{\tilde{x} \in \mathrm{WP}(E) \,:\, \tilde{x}_{T\cup U} \leq c_{T\cup U},\, \tilde{x}_{N\setminus(T\cup U)} = c_{N\setminus(T\cup U)}\right\}\right).$$

Choose $a = (0_{T\cup U}, c_{N\setminus(T\cup U)}) \in V_{E,c}(T\cup U)$. Note that $z_U \leq 0$ since $z \in V_{E,c}(S\cup U)$. Then, $a_U = 0 \geq z_U$ and $a_T = 0 \geq y_T$.

(i.c) $(0_{S\cup U}, c_{N\setminus(S\cup U)}) \notin E$, $(0_{T\cup U}, c_{N\setminus(T\cup U)}) \notin E$.

Then,

$$V_{E,c}(R\cup U) = \text{comp}^{R\cup U}\left(\left\{\tilde{x} \in \text{WP}(E) \,:\, \tilde{x}_{R\cup U} = 0,\, \tilde{x}_{N\setminus(R\cup U)} \leq c_{N\setminus(R\cup U)}\right\}\right)$$

for $R \in \{S, T\}$. Choose $a \in \text{WP}(E)$ with $a_{T\cup U} = 0$ and $0 \leq a_{N\setminus(T\cup U)} \leq c_{N\setminus(T\cup U)}$. Then, $a \in V_{E,c}(T\cup U)$. Note that $z_U \leq 0$ since $z \in V_{E,c}(S\cup U)$. Then, $a_U = 0 \geq z_U$ and $a_T = 0 \geq y_T$.

Case (ii): $(0_T, c_{N\setminus T}) \in E$.

By condition (E3) of NTU-bankruptcy problems, $(0_{T\cup U}, c_{N\setminus(T\cup U)}) \in E$ and, then,

$$V_{E,c}(R) = \text{comp}^{R}\left(\left\{\tilde{x} \in \text{WP}(E) \,:\, \tilde{x}_R \leq c_R,\, \tilde{x}_{N\setminus R} = c_{N\setminus R}\right\}\right)$$

for $R \in \{T, T\cup U\}$. Since $y \in V_{E,c}(T)$, we have $y_{N\setminus T} = c_{N\setminus T}$ and, in particular, $y_{N\setminus(T\cup U)} = c_{N\setminus(T\cup U)}$. Therefore, it follows that $y \in V_{E,c}(T\cup U)$ and $y_U = c_U$. Choosing $a = y$, we have $a_T = y_T$ and $a_U = y_U = c_U \geq z_U$ where the last inequality follows from the fact that $z \in V_{E,c}(S\cup U)$. □

Theorem 5.3. Every NTU-bankruptcy game is ordinal convex.

Proof: Let (N, E, c) be an NTU-bankruptcy problem and let $(N, V_{E,c})$ be the associated NTU-bankruptcy game. Let $S, T \in 2^{\mathrm{N}} \setminus \{\emptyset\}$. We show that

$$V_{E,c}(S) \cap V_{E,c}(T) \subset V_{E,c}(S\cup T) \cup V_{E,c}(S\cap T).$$

If $S\cap T = \emptyset$, the result follows by Lemma 5.1. Let $S\cap T \neq \emptyset$ and let $y \in V_{E,c}(S) \cap V_{E,c}(T)$. Without loss of generality, it suffices to distinguish between two cases: (i) $(0_S, c_{N\setminus S}) \in E$ and (ii) $(0_S, c_{N\setminus S}) \notin E$ and $(0_T, c_{N\setminus T}) \notin E$.

Case (i): $(0_S, c_{N\setminus S}) \in E$.

In this case, $(0_{S\cup T}, c_{N\setminus(S\cup T)}) \in E$ by condition (E3) of NTU-bankruptcy problems and, consequently,

$$V_{E,c}(R) = \text{comp}^{R}\left(\left\{\tilde{x} \in \text{WP}(E) \,:\, \tilde{x}_R \leq c_R,\, \tilde{x}_{N\setminus R} = c_{N\setminus R}\right\}\right)$$

for $R \in \{S, S\cup T\}$. Since $y \in V_{E,c}(S)$, $y_{N\setminus S} = c_{N\setminus S}$. Therefore, $y_{N\setminus(S\cup T)} = c_{N\setminus(S\cup T)}$ and $y \in V_{E,c}(S\cup T)$.

Case (ii): $(0_S, c_{N\setminus S}) \notin E$ and $(0_T, c_{N\setminus T}) \notin E$.

In this case, $(0_{S\cap T}, c_{N\setminus(S\cap T)}) \notin E$ by condition (E3) of NTU-bankruptcy problems and, consequently,

$$V_{E,c}(R) = \text{comp}^{R}\left(\left\{x \in \text{WP}(E) \,:\, x_R = 0,\, x_{N\setminus R} \leq c_{N\setminus R}\right\}\right)$$

for $R \in \{S, T, S\cap T\}$. Since $y \in V_{E,c}(S)$, we have that

$$y_S \leq 0 \text{ and } 0 \leq y_{N\setminus S} \leq c_{N\setminus S} \text{ with } (0_S, y_{N\setminus S}) \in \text{WP}(E),$$

and since $y \in V_{E,c}(T)$, we have that

$$y_T \leq 0 \text{ and } 0 \leq y_{N\setminus T} \leq c_{N\setminus T} \text{ with } (0_T, y_{N\setminus T}) \in \mathrm{WP}(E).$$

Therefore,

$$y_{S\cap T} \leq 0 \text{ and } 0 \leq y_{N\setminus (S\cap T)} \leq c_{N\setminus (S\cap T)} \text{ with } (0_{S\cap T}, y_{N\setminus (S\cap T)}) \in \mathrm{WP}(E).$$

Then, $y \in V_{E,c}(S \cap T)$. □

Next, we show that NTU-bankruptcy games are compromise stable.

Theorem 5.4. Every NTU-bankruptcy game is compromise stable.

Proof: Let (N, E, c) be an NTU-bankruptcy problem and let $(N, V_{E,c})$ be the associated NTU-bankruptcy game. By Theorem 4.2, it suffices to show that either $V_{E,c}(S) \subset \{x \in \mathbb{R}^N : x_S \leq k_S(V_{E,c})\}$, or $V_{E,c}(S) \subset \mathrm{comp}\left(\{x \in \mathrm{WP}(V_{E,c}(N)) : x_{N\setminus S} \geq K_{N\setminus S}(V_{E,c})\}\right)$ for every $S \in 2^N \setminus \{\emptyset\}$. Fix $S \in 2^N \setminus \{\emptyset\}$, we distinguish between two cases: (i) $(0_S, c_{N\setminus S}) \in E$ and (ii) $(0_S, c_{N\setminus S}) \notin E$.

Case (i): $(0_S, c_{N\setminus S}) \in E$.

Then, $(0_{N\setminus\{i\}}, c_i) \in E$ for every $i \in N \setminus S$ by condition (E3) of NTU-bankruptcy problems and, consequently, $c^t_{N\setminus S} = c_{N\setminus S}$. Then,

$$\begin{aligned} V_{E,c}(S) &= \mathrm{comp}^S\left(\{x \in \mathrm{WP}(E) : x_S \leq c_S,\, x_{N\setminus S} = c_{N\setminus S}\}\right) \\ &= \mathrm{comp}^S\left(\{x \in \mathrm{WP}(E) : x_S \leq c_S,\, x_{N\setminus S} = K_{N\setminus S}(V_{E,c})\}\right) \\ &\subset \mathrm{comp}^N\left(\{x \in \mathrm{WP}(V_{E,c}(N)) : x_{N\setminus S} \geq K_{N\setminus S}(V_{E,c})\}\right) \end{aligned}$$

where the second equality follows by Proposition 4.4 and $c^t_{N\setminus S} = c_{N\setminus S}$.

Case (ii): $(0_S, c_{N\setminus S}) \notin E$.

In this case, $(0_i, c_{N\setminus\{i\}}) \notin E$ for every $i \in S$ by condition (E3) of NTU-bankruptcy problems and $k_i(V_{E,c}) = v_{E,c}(i) = 0$ for every $i \in S$, where the first equality follows by Proposition 4.4. Moreover, $x \in V_{E,c}(S)$ implies $x_S \leq 0$. Therefore, $V_{E,c}(S) \subset \{x \in \mathbb{R}^N : x_S \leq k_S(V_{E,c})\}$. □

6 A game theoretical NTU-bankruptcy rule

The adjusted proportional rule was introduced for TU-bankruptcy problems in Curiel et al. (1987). The adjusted proportional rule assigns to each agent his minimal right first, and the remaining estate is proportionally shared with respect to the vector of updated claims, where the new claim takes into account that every agent has already obtained his minimal right and that nobody should get more than the remaining estate. Formally, given a TU-bankruptcy problem (N, e, c), the *adjusted proportional rule* assigns $\text{AProp}(N, e, c) = m(N, e, c) + \text{Prop}(N, \tilde{e}, \tilde{c})$ where $m_i(N, e, c) = \max\{0, e - \sum_{j \in N \setminus \{i\}} c_j\}$, $\tilde{e} = e - \sum_{i \in N} m_i(N, e, c)$, $\tilde{c}_i = \min\{\tilde{e}, c_i - m_i(N, e, c)\}$, and $\text{Prop}(N, \tilde{e}, \tilde{c}) = \frac{\tilde{e}}{\sum_{i \in N} \tilde{c}_i} \tilde{c}$. Note that $m_i(N, e, c) = v_{E,c}(\{i\})$. Curiel et al. (1987) show that the adjusted proportional rule and the compromise value of the associated TU-bankruptcy game lead to the same allocation for any TU-bankruptcy problem. Next, we show that this result can be generalized to the NTU-setting.

In order to generalize the adjusted proportional rule to NTU-bankruptcy problems, we first need to define the minimal right of an agent in an NTU-bankruptcy problem. Let (N, E, c) be an NTU-bankruptcy problem and let $i \in N$. The *minimal right* of agent i, $m_i(N, E, c)$, is defined by $m_i(N, E, c) = v_{E,c}(i)$. Then, for an NTU-bankruptcy problem (N, E, c), the *adjusted proportional rule* assigns

$$\text{AProp}(N, E, c) = m(N, E, c) + \text{Prop}(N, \tilde{E}, \tilde{c})$$

where $\text{Prop}(N, \tilde{E}, \tilde{c}) = t\tilde{c}$ with $t = \sup\left\{\tilde{t} \in \mathbb{R}_+ \,:\, \tilde{t}\tilde{c} \in \tilde{E}\right\}$, $\tilde{E} = E - \{m(N, E, c)\}$, and $\tilde{c} \in \mathbb{R}^N$ is defined by $\tilde{c} = c^t - m(N, E, c)$. It readily follows that $0 \leq \text{AProp}(N, E, c) \leq c$.

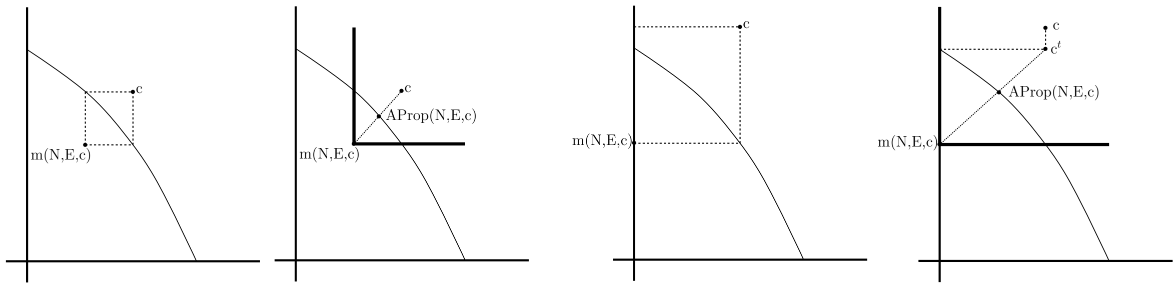

Figure 3: Graphical illustrations of the adjusted proportional rule.

Theorem 6.1. Let (N, E, c) be an NTU-bankruptcy problem. Then, $\text{AProp}(N, E, c) = T(V_{E,c})$.

Proof: We have to show that

$$\text{AProp}(N, E, c) = m(N, E, c) + \text{Prop}(N, \tilde{E}, \tilde{c}) = \lambda K(V_{E,c}) + (1 - \lambda)k(V_{E,c}) = T(V_{E,c})$$

with $\tilde{E} = E - m(N, E, c)$, $\tilde{c} = c^t - m(N, E, c)$, and $\lambda = \sup\left\{\tilde{\lambda} \in [0, 1] : \tilde{\lambda} K(V_{E,c}) + (1 - \tilde{\lambda})k(V_{E,c}) \in V_{E,c}(N)\right\}$. Note that $m(N, E, c) = (v_{E,c}(i))_{i \in N} = k(V_{E,c})$ by Proposition 4.4 and $T(V_{E,c}) = k(V_{E,c}) + \lambda(K(V_{E,c}) - k(V_{E,c}))$. Therefore, it suffices to show that

$$\text{Prop}(N, \tilde{E}, \tilde{c}) = \lambda(K(V_{E,c}) - k(V_{E,c})).$$

By Proposition 4.4, we have that $K(V_{E,c}) = c^t$. Then,

$$\text{Prop}(N, \tilde{E}, \tilde{c}) = t\tilde{c} = t(c^t - m(N, E, c)) = t(K(V_{E,c}) - k(V_{E,c})),$$

where $t = \sup\left\{\tilde{t} \in \mathbb{R}_+ \,:\, \tilde{t}\tilde{c} \in \tilde{E}\right\}$. Therefore, we only need to show that $t = \lambda$. It follows that

$$\begin{aligned}
t &= \sup\left\{\tilde{t} \in \mathbb{R}_+ \,:\, \tilde{t}\tilde{c} \in \tilde{E}\right\} \\
&= \sup\left\{\tilde{t} \in \mathbb{R}_+ \,:\, \tilde{t}(c^t - m(N, E, c)) \in (E - m(N, E, c))\right\} \\
&= \sup\left\{\tilde{t} \in \mathbb{R}_+ \,:\, \tilde{t}(K(V_{E,c}) - k(V_{E,c})) \in (V_{E,c}(N) - k(V_{E,c}))\right\} \\
&= \sup\left\{\tilde{t} \in \mathbb{R}_+ \,:\, k(V_{E,c}) + \tilde{t}(K(V_{E,c}) - k(V_{E,c})) \in V_{E,c}(N)\right\} \\
&= \sup\left\{\tilde{t} \in [0,1] \,:\, k(V_{E,c}) + \tilde{t}(K(V_{E,c}) - k(V_{E,c})) \in V_{E,c}(N)\right\} \\
&= \sup\left\{\tilde{t} \in [0,1] \,:\, \tilde{t}K(V_{E,c}) + (1 - \tilde{t})k(V_{E,c}) \in V_{E,c}(N)\right\} = \lambda
\end{aligned}$$

where the second equality follows by definition of $\tilde{c}$ and $\tilde{E}$; the third equality is a direct consequence of $K(V_{E,c}) = c^t$, $k(V_{E,c}) = m(N, E, c)$, and $V_{E,c}(E) = \text{comp}^N(E)$; the fifth equality follows because $k(V_{E,c}) + t(K(V_{E,c}) - k(V_{E,c}))$ is the unique vector on the line segment between $k(V_{E,c})$ and $K(V_{E,c})$ which lies in $V_{E,c}(N)$ and is closest to $K(V_{E,c})$. □

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