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Weighted Temporal Utility

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Abstract

We propose a utility representation for preferences over risky timed outcomes, the weighted temporal utility model. It separates subjective evaluations of outcomes from attitudes towards psychological distance induced by risks and delays. Subjective evaluations of outcomes may depend on the time of receipt. A natural special case of our model arises when decision makers evaluate an outcome according to the extra utility it generates on top of expected baseline consumption, which can be interpreted as the status quo. Thus, deviations from stationarity can be driven by expected changes in baseline consumption, and need not be irrational. Moreover, a decision maker with a weighted temporal utility function can have time-consistent yet non-stationary preferences or stationary yet time-inconsistent preferences. We provide a characterization of our model and propose a non-parametric approach to elicit a weighted temporal utility function.

Keywords: Intertemporal choice, stationarity, time-(in)consistency
JEL-Classification: D91, D81

1 Introduction

Most decisions we take today involve an uncertain outcome at some point in the future. This is not only true for investments and savings, but also for daily decisions about, for instance, what to eat and whether or not to go to the gym. Empirical evidence shows that many decisions are time-inconsistent in the sense that the mere passage of time makes people change their plans (Frederick, Loewenstein, and O'Donoghue, 2002). Such time-inconsistencies can cause under-investment and unhealthy lifestyles, which impose large costs on society. A good understanding of the drivers of these time-inconsistencies can help to provide solutions to overcome them and to reduce the associated costs.

The literature on intertemporal choice has mostly abstracted from uncertainty in order to focus exclusively on pure time preference (for a survey see Frederick, Loewenstein, and O'Donoghue, 2002). However, as the future is inherently uncertain, it is also important to examine how pure time preference and risk attitudes interact. This paper proposes a model of intertemporal choice which contributes to the literature in two ways. First, unlike many existing models of intertemporal choice, our model allows for an interaction between pure time preference and risk attitude. Second, we separate two effects the delay of an outcome can have on its evaluation. On the one hand the delay of an outcome makes the decision maker project himself to the future and imagine how much he will appreciate the outcome once he receives it. On the other hand, the delay of the outcome makes its receipt psychologically more distant, and thereby less salient, than immediate outcomes.

This paper considers preferences over single outcomes to be received with a particular probability at a particular point in time. Our weighted temporal utility model evaluates such outcomes as follows. First the time-dependent utility of the outcome is determined. Then this utility is discounted by a weight, which depends on the psychological distance induced by the probability and the time at which the outcome is received. Keren and Roelofsma (1995), Abdellaoui, Diecidue, and Öncüler (2011), and Baucells and Heukamp

(2012) provide empirical evidence for the weight given to a probability to depend on the timing of the outcome. Therefore, our model of psychological distance does not require probability and time to be separable. Moreover, the magnitude effect, which shows that larger outcomes are discounted at a lower rate than smaller outcomes, suggests that an outcome and its timing are also not separable (Frederick, Loewenstein, and O'Donoghue, 2002). Thus, in line with the empirical evidence our model allows for interactions between probabilities and time on the one hand and between outcomes and time on the other hand while outcomes and probabilities are assumed to be separable.

The weighted temporal utility model is an alternative to the probability time-tradeoff model which was recently introduced by Baucells and Heukamp (2012). In their model the utility derived from an outcome is independent of time and probability, but the interaction between time and probability depends on outcomes through an outcome-dependent weighting function. They impose a restriction on the tradeoff between probability and time through their probability-time trade-off condition, which requires the tradeoff between time and probability to be constant for given outcomes. Our weighted temporal utility model is an alternative to the model of Baucells and Heukamp (2012). We impose weaker restrictions on some parts of the model and stronger restrictions on others. To be precise, we restrict the weighting function to be independent of outcomes and instead allow utility to depend on time. We do not impose any restrictions on the interaction between probability and time in the weighting function. The weighted temporal utility model can accommodate the empirical findings supporting the model of Baucells and Heukamp (2012). Moreover, for single outcomes our model accommodates rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting as special cases.

This paper contributes to the understanding of time-inconsistent behavior, by separating the two channels through which a delay can impact the evaluation of an outcome: (1) through its impact on the utility derived from the outcome, and (2) through its impact on the psychological distance to the outcome. The extent to which each of these two channels contribute to time-inconsistent behavior determines what type of policies can be effective to overcome such inconsistencies.

The literature on intertemporal choice has focussed almost exclusively on one potential driver of time-inconsistencies: non-stationarity. Stationarity holds if a preference between outcomes to be received at different points in time is unaffected by a common additional delay of all outcomes. Deviations from stationarity are often thought to be driven by pure time preference, being the way people weight future points in time, irrespective of the outcomes received at these points in time. Accordingly, hyperbolic discounting models were proposed to accommodate non-stationary behavior (Loewenstein and Prelec, 1992; Harvey, 1986, 1995; Mazur, 1987; and Phelps and Pollak, 1968). These models can be given a psychological foundation by construal-level theory (Trope and Liberman, 2010) and the non-linear manner in which humans perceive temporal distance (Zauberman et al., 2009). To the extent that these non-linear perceptions of time are irrational, we can view deviations from stationarity caused by pure time preference as irrational.

Deviations from stationarity are a potential cause of time-inconsistencies. Yet, Halevy (2014) provided empirical evidence that non-stationary behavior neither implies nor is implied by time-inconsistency. In his study only two-thirds of the subjects who exhibit time-consistency also exhibit stationarity and half of the subjects whose choices are time-inconsistent exhibit stationarity. These findings show that deviations from stationarity are not the sole drivers of time-inconsistent behavior and they cast doubt on the extent to which such deviations are irrational.

In our model deviations from stationarity are not only caused by pure time preference and non-linear perception of time, but also by the time-dependence of the utility of an outcome. Such time-dependence naturally arises whenever the decision maker evaluates an outcome according to the extra utility it generates on top of some baseline consumption (cf. Noor, 2009, and Gerber and Rohde, 2010). If the decision maker expects his baseline consumption to change over time, then the utility of an outcome depends on its timing irrespective of pure time preference. This dependency, which can induce non-stationarity, can be viewed as foresight of future utility and, thereby, is not irrational as long as it is perfect foresight.

As the timing of an outcome influences both its utility and the weight given to the probability that it will be received, measuring the weighting and utility functions of the weighted temporal utility model may seem difficult at first sight. We will show how this can be accomplished in a non-parametric way. This non-parametric approach does not require any assumption about the shape of the utility and weighting functions. In particular, it does not require an assumption of linear utility, which is often used in the literature.

The outline of this paper is as follows. Section 2 introduces the weighted temporal utility (WTU) model and provides a characterization result. Section 3 discusses the specific version of WTU which accommodates for baseline consumption to which outcomes are added. Section 4 shows how WTU can accommodate all possible combinations of non-stationarity, time-inconsistency and time-invariance. Section 5 shows how the weighting and utility function of WTU can be measured. Finally, Section 6 concludes.

2 The Model

This paper considers preferences $\succcurlyeq$ over *risky timed outcomes* (x, p, t) which give *outcome* $x \in \mathbb{R}_+ = [0, \infty)$ with *probability* $p \in [0, 1]$ at *time* $t \in \mathbb{R}_+$. We assume that $\succcurlyeq$ is a continuous weak order. Strict preference $\succ$ and indifference $\sim$ are defined as usual. We further assume that $(x, p, t) \sim (y, q, s)$ whenever $px = qy = 0$.

Impatience holds if for all x, p, s, t with $s < t$ and $px > 0$ we have $(x, p, s) \succ (x, p, t)$. *Monotonicity in probabilities* holds if for every outcome $x > 0$, every time t, and all probabilities $p > q$ we have $(x, p, t) \succ (x, q, t)$. *Monotonicity in outcomes* holds if for every probability $p > 0$, time t, and outcomes $x > y$ we have $(x, p, t) \succ (y, p, t)$. *Monotonicity* holds if both monotonicity in probabilities and monotonicity in outcomes holds.

Weighted temporal utility (WTU) holds if $\succcurlyeq$ can be represented by

$$V(x, p, t) = w(p, t)v(x, t),$$

where w is a *weighting function* and v is a *utility function.* Under WTU a decision maker evaluates a risky timed outcome (x, p, t) by first determining the utility $v(x, t)$ that outcome x will yield at time t, irrespective of the probability that it will be received, and then discounting this temporal utility by a weight $w(p,t)$, which can be viewed as a time-dependent probability weighting function.

WTU captures the two ways in which the time at which a risky outcome is received, can influence its evaluation. First of all, the instantaneous utility derived from outcome x may depend on time t. This will be the case, if, for instance, a decision maker expects to be much wealthier in the future and therefore expects €100 to generate much less utility in the future than now. Second, as the instantaneous utility is generated in the future and only with a probability p, it can be viewed as a psychologically distant utility. The weighting function $w(p,t)$ transforms the two components, p and t, of this psychological distance into a discount which is applied to the instantaneous utility $v(x,t)$.

In psychology, construal level theory (Trope and Liberman, 2010) has been proposed as a theory which shows how psychological distance resulting from a.o. risk and time, influences decision making. Prelec and Loewenstein (1991) showed that there are many parallels between the impacts of risk and time on decision making, which supports the idea that risk and time can be summarized into one variable: psychological distance. Our model puts construal level theory into a (mathematical) weighting function. The weighting function w can be thought of as a function that first combines probability and delay into psychological distance, and then gives a weight to this distance. Keren and Roelofsma (1995), Abdellaoui, Diecidue, and Öncüler (2011), and Baucells and Heukamp (2012) provide empirical evidence for the non-separability of probability and time. Hence, we do not assume that $w(p,t)$ can be written as $w(p,t) = f(p)g(t)$ for some functions f and g. Yet, for single outcomes rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting are special cases of WTU.

WTU is an alternative to the probability and time tradeoff model of Baucells and Heukamp (2012), which is given by $V(x,p,t) = w(pe^{-r_x t})v(x)$. In their model the tradeoff between probability p and time t may depend on the outcome x. More precisely, they provide empirical evidence that the willingness to wait in exchange for a higher probability to receive a reward increases in the size of the reward. They use the term *subendurance* for this behavioral pattern and show that it can be rationalized by a weighting function w which also depends on the outcome x. In our model the dependence on outcomes of the tradeoff between probability and time is captured by the temporal utility function v. Hence, our model clearly separates attitudes towards psychological distance and attitudes towards outcomes, where the former are captured by the temporal weighting function w and the latter by the temporal utility function v. The following example shows that our model is compatible with all the empirical findings that Baucells and Heukamp (2012) use to justify their model.

Example 2.1 Let time be denoted in weeks. Consider a decision maker with WTU function $V(x,p,t) = w(p,t)v(x,t)$, where $w(p,t) = e^{-(-\ln(p)+0.023t)^{0.65}}$ for all p,t, $v(x,t) = \sqrt{100+\frac{5}{4}t+x} - \sqrt{100+\frac{5}{4}t}$ for all x and all $t < 4$, and $v(x,t) = \sqrt{105+x} - \sqrt{105}$ for all x and $t \geq 4$. The weighted temporal utilities of the prospects in Baucells and Heukamp (2012, Table 1) are summarized in Table 1. These utility levels yield modal choices as reported in Baucells and Heukamp (2012). Thus, WTU can account for their empirical findings.

In the remainder of this section we will provide a characterization of WTU. In addition to impatience and monotonicity two conditions are necessary and sufficient for WTU to hold.

The *hexagon condition at time* t holds if for all outcomes $x,y,z > 0$ and all probabilities $p,q,l > 0$ we have that

$$(y,p,t) \sim (x,q,t) \quad \text{and} \quad (z,p,t) \sim (y,q,t) \text{ and}$$
$$(y,q,t) \sim (x,l,t) \quad \text{imply}$$
$$(z,q,t) \sim (y,l,t).$$

The hexagon condition can be interpreted as follows. Assume that the tradeoff between p and q equals the tradeoff between q and l in the sense that they both offset the tradeoff

	Prospect A	Prospect B	V(A)	V(B)
1.	(€9, 100%, now)	(€12, 80%, now)	0.4403	0.3998
2.	(€9, 10%, now)	(€12, 8%, now)	0.0789	0.0939
3.	(€9, 100%, 3 months)	(€12, 80%, 3 months)	0.2789	0.3014
4.	(fl.100, 100%, now)	(fl.110, 100%, 4 weeks)	2.0603	1.7820
5.	(fl.100, 100%, 26 weeks)	(fl.110, 100%, 30 weeks)	0.9867	1.0041
6.	(fl.100, 50%, now)	(fl.110, 50%, 4 weeks)	0.9369	0.9373
7.	(€100, 100%, 1 month)	(€100, 90%, now)	3.2930	3.2858
8.	(€5, 100%, 1 month)	(€5, 90%, now)	0.1951	0.1959

Table 1: Prospects and utility values for Example 2.1. $V(A)$ and $V(B)$ denote the weighted temporal utility of prospect A and B, respectively. *Note:* Rows 4-6 have outcomes denoted in Dutch Guilders. We transformed them into Euro by using the conversion rate at the introduction of the Euro: fl.100 is approximately €45.45 and fl.110 is approximately €50. We set 1 month equal to 4 weeks and 3 months equal to 12 weeks.

between y and x at time t (the upper and lower left indifferences of the definition). If the tradeoff between p and q also offsets the tradeoff between z and y at time t, then the hexagon condition implies that the tradeoff between q and l offsets the tradeoff between z and y at time t as well (the upper and lower right indifferences of the definition). Thus, the hexagon condition allows us to claim that, at time t, the tradeoff between p and q equals the tradeoff between q and l, irrespective of the outcomes. Wakker (1989) showed that the hexagon condition is weaker than the often used Thomsen condition (Thomsen, 1927).

Probability-independent time-outcome tradeoff holds if for all outcomes $x, y, x_0, y_0 > 0$, all probabilities p, p_0, and every time t we have that

$$(x,1,t) \sim (x_0,1,0) \quad \text{and} \quad (x,p,t) \sim (x_0,p_0,0) \text{ and}$$
$$(y,1,t) \sim (y_0,1,0) \quad \text{imply}$$
$$(y,p,t) \sim (y_0,p_0,0).$$

Probability-independent time-outcome tradeoff can be interpreted as follows. Assume that the tradeoff between x for sure and x_0 for sure equals the tradeoff between y for sure and y_0 for sure in the sense that they both offset the tradeoff between time t and time 0 (the upper and lower left indifferences of the definition). Assume that the tradeoff between x with probability p and x_0 with probability p_0 also offsets the tradeoff between time t and time 0. Then probability-independent time-outcome tradeoff implies that the tradeoff between y with probability p and y_0 with probability p_0 offsets the tradeoff between time t and time 0 as well (the upper and lower right indifferences of the definition).

The hexagon condition at time 0 and probability-independent time-outcome tradeoff imply WTU, as is shown in the following theorem. The proof is in the Appendix.

Theorem 2.2 *Under impatience and monotonicity the following statements are equivalent:*

(i) *Probability-independent time-outcome tradeoff and the hexagon condition at time* 0 *hold.*

(ii) *Preferences* $\succcurlyeq$ *can be represented by*

$$V(x, p, t) = w(p, t)v(x, t)$$

with $w(0, t) = v(0, t) = 0$ *for all* t*. Moreover, for all* x, p, t *we have* $w(p, t) \geq 0$ *and* $v(x, t) \geq 0$ *with* w *increasing in* p *and* v *increasing in* x*.*

Baucells and Heukamp (2012) found evidence for subendurance, which they define as follows. *Subendurance* holds if for all (x, p, t) with $px > 0$, all $\theta \in (0, 1)$, all $\Delta \in (0, \infty)$, and all $y \in (0, x)$ we have that $(x, p, t + \Delta) \sim (x, p\theta, t)$ implies $(y, p, t + \Delta) \preccurlyeq (y, p\theta, t)$. *Isoendurance* holds if the implied weak preference $\preccurlyeq$ is always an indifference $\sim$. The following proposition follows immediately.

Proposition 2.3 *Under weighted temporal utility subendurance (isoendurence) is equivalent to*

$$v(x,t)/v(x,t+\Delta)$$

being weakly decreasing (constant) in x *for all* $x > 0$, *all* t, *and all* $\Delta \in (0, \infty)$.

Under weighted temporal utility, isoendurance implies that the utility function is independent of time, as shown in the following proposition.

Proposition 2.4 *Under impatience and monotonicity the following statements are equivalent:*

(i) *Probability-independent time-outcome tradeoff, the hexagon condition at time* 0, *and isoendurance hold.*

(ii) *Preferences* $\succcurlyeq$ *can be represented by*

$$V(x,p,t) = w(p,t)v(x)$$

with $w(0,t) = v(0) = 0$ *for all* t. *Moreover, for all* x, p, t *we have* $w(p,t) \geq 0$ *and* $v(x) \geq 0$ *with* w *increasing in* p *and* v *increasing in* x.

3 Baseline Consumption

A special case of WTU naturally arises if the decision maker has a discounted expected utility function and adds any outcome to his baseline consumption at the time when the outcome is received. If b_t is baseline consumption at time t and u is the decision maker's utility function, then the discounted expected utility of a risky timed outcome (x, p, t) is

$$V(x,p,t) = p\delta(t)\left(u(b_t + x) - u(b_t)\right), \tag{1}$$

where δ is the time discount function. The utility generated by receiving outcome x at time t, thereby, is the extra utility outcome x generates on top of the utility derived from baseline consumption at time t. This model is consistent with the one proposed by

Noor (2009). Gerber and Rohde (2014) derive testable hypotheses on the discount function when baseline consumption is unobserved. Baseline consumption can be interpreted as any status quo to which additional outcomes are added. Gerber and Rohde (2010) considered a more general case where baseline consumption can be stochastic rather than deterministic. Given the empirical evidence against the separability of probability and time (Keren and Roelofsma, 1995; Abdellaoui, Diecidue, and Öncüler, 2011; Baucells and Heukamp, 2012), we will focus on the following more general version of (1):

$$V(x,p,t) = w(p,t)\left(u(b_t + x) - u(b_t)\right), \tag{2}$$

where w is the weighting function that we already know from the WTU model. The question is, under which conditions the temporal utility function $v(x,t)$ can be written as a utility difference $u(b_t + x) - u(b_t)$ for some b_t, i.e. under which conditions the decision maker behaves as if he has a baseline consumption and evaluates an outcome by the extra utility it generates on top of this baseline consumption. It turns out that the following condition is vital.

Baseline consumption $b_t \geq 0$ *exists at time* $t > 0$ if preferences at time 0 can be represented by $V(x,p,0) = w(p)u(x)$ with w and u nonnegative, and if for all $x, y, x_0, y_0 > 0$ and $p > 0$ we have that

$$\begin{aligned} (x,p,t) &\sim (x_0,1,0) \text{ and} \\ (y,p,t) &\sim (y_0,1,0) \\ &\text{imply} \\ \frac{u(x_0)}{u(y_0)} &= \frac{u(b_t + x) - u(b_t)}{u(b_t + y) - u(b_t)} \end{aligned}$$

The following theorem provides a characterization of (2). The proof is in the Appendix.

Theorem 3.1 *Under impatience and monotonicity the following statements are equivalent:*

(i) *Baseline consumption* $b_t \geq 0$ *exists for every* $t > 0$.

(ii) *Preferences* $\succcurlyeq$ *can be represented by*

$$V(x, p, t) = w(p, t)\,(u(b_t + x) - u(b_t))$$

with $b_0 = 0$ *and* $w(0, t) = u(0) = 0$ *for all* t. *Furthermore,* $w(p, t) \geq 0$, $u(x) \geq 0$, *and* $b_t \geq 0$ *for all* x, p, t *and* $u(b_t + x) - u(b_t) > 0$ *for all* $x > 0$.

Moreover, the $\{b_t\}_{t>0}$ *in* (i) *are the same as in* (ii).

The existence of baseline consumption is defined in terms of the utility function at time 0, and not expressed entirely in terms of preference conditions. The reason for this is that for the representation to hold, one needs a preference condition that imposes structure on utility differences. For the risky timed outcomes we consider, a condition on utility differences cannot immediately be translated into a preference condition, as the utility function in WTU is not unique up to both level and unit. In a richer domain where one could receive more than one outcome with a positive probability, utility differences would have immediate empirical meaning and a condition for baseline consumption could be expressed entirely in terms of preference conditions. Yet, for this paper we chose the domain of risky timed outcomes, in line with Baucells and Heukamp (2012).

The weighted temporal utility function with baseline consumption in Eq.(2) satisfies several properties that are worth mentioning. First of all, the utility derived from a strictly positive outcome is always positive, irrespective of its timing. Second, for a concave utility function u, increasing baseline consumption over time means that the utility $u(b_t+x)-u(b_t)$ derived from receiving outcome x is decreasing in time. Thus, for a concave utility function u the utility $u(b_t + x) - u(b_t)$ derived from receiving outcome x at time t can never be larger than the utility derived from outcome x at time $t = 0$.

4 Stationarity, Time Invariance and Time Consistency

In the last few decades a substantial part of the literature on intertemporal choice has analyzed deviations from stationarity as a potential driver of time-inconsistent behavior (Frederick, Loewenstein, and O'Donoghue, 2002). Stationarity requires preferences between timed outcomes to remain unchanged when all outcomes are additionally delayed by an equal amount of time. Together with time-invariance, a property that we will explain later, deviations from stationarity imply time-inconsistent behavior in the sense that the mere passage of time can make people change their plans. Such inconsistencies are undesirable if they result in losses of welfare associated with, for instance, too little savings and unhealthy lifestyles.

The literature on intertemporal choice has focussed almost exclusively on deviations from stationarity, and not, or much less, on other potential causes of time-inconsistencies. Yet Halevy (2014) points out that, unlike the literature may seem to suggest, deviations from stationarity need not imply time-inconsistencies, and stationarity need not imply time-consistency. The WTU model indeed suggests that deviations from stationarity need not be irrational.

The WTU model separates two channels through which deviations from stationarity may arise. The first channel is the way in which time and probability are combined into psychological distance and the corresponding weighting function. In a setting without uncertainty and with time-independent utilities this channel is the usual discount function. This is the setting most intertemporal studies considered so far (Frederick, Loewenstein, and O'Donoghue, 2002). Deviations from stationarity in this setting are related to non-linear perception of temporal distance (Takahashi et al. 2008). One may argue that such non-linear perception is irrational, which gives the resulting non-stationarity a flavor of irrationality as well. Thus, non-stationarities arising through the weighting function in WTU may well be viewed as irrational.

The second channel through which non-stationarities can be generated in the WTU model, is the time-dependence of the utility derived from an outcome. Such non-stationarities, for instance, arise in the model with baseline consumption if baseline consumption is expected to change over time (Gerber and Rohde, 2010, 2014, and Noor, 2009). If these expectations about baseline consumptions are perfect foresight, the resulting deviations from stationarity are perfectly rational. Thus, non-stationarities arising through the time-dependence of utility in WTU need not be irrational.

In this section we will elaborate further on the relation between stationarity and time-inconsistency in the WTU model. Two points in time are crucial when choosing between risky timed outcomes: *consumption time* – the time at which the outcome is received - and *decision time* – the time at which the decision is made. So far we have only varied consumption time while the decision time was fixed at $t = 0$. In order to shed light on time-inconsistencies this section will also consider changes in the decision time. Varying the decision time and the consumption time gives rise to three notions of consistent behavior depending on whether only one or both of them are varied. We will discuss the three notions of consistent behavior.

This section assumes that for every decision time τ the decision maker has a preference relation $\succcurlyeq^\tau$ over risky timed outcomes to be received from time τ onwards. By $\{\succcurlyeq^\tau\}_\tau$ we denote the set of preferences for all decision times τ. Strict preference $\succ^\tau$ and indifference $\sim^\tau$ are defined as usual. In line with Halevy (2014) we define stationarity, time invariance and time-consistency as follows.

Definition 4.1 *Preferences* $\{\succcurlyeq^\tau\}_\tau$ *are* ***stationary*** *if for every* x, y, p, q, τ, s, t, *with* $0 \leq \tau \leq s, t$, *and for every* $\Delta \geq 0$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t + \Delta) \sim^\tau (y, q, s + \Delta)$$

Stationarity means that preferences remain unchanged if the decision time remains unchanged and all consumption times are delayed by a common time interval.

Definition 4.2 *Preferences* $\{\succcurlyeq^\tau\}_\tau$ *are* ***time invariant*** *if for every* x, y, p, q, τ, s, t *with* $0 \leq \tau \leq s, t$, *and for every* $\Delta \geq 0$,

$$(x, p, t) \sim^\tau (y, q, s) \iff (x, p, t + \Delta) \sim^{\tau+\Delta} (y, q, s + \Delta)$$

Time invariance means that preferences remain unchanged if the distance from consumption time to decision time remains unchanged.

Definition 4.3 *Preferences* $\{\succsim^{\tau}\}_{\tau}$ *are* ***time-consistent*** *if for every* $x, y, p, q, \tau, \tau', s, t$, *with* $0 \leq \tau, \tau' \leq s, t$,

$$(x, p, t) \sim^{\tau} (y, q, s) \iff (x, p, t) \sim^{\tau'} (y, q, s)$$

Time-consistency means that the preference over a pair of risky timed outcomes is independent of the decision time.

It is straightforward to show that any two of the three properties (stationarity, time invariance, and time-consistency) imply the third and hence either none or at least two of the properties must be violated. Thus, there are five possible preference types as summarized in Table 2.

Type	Stationary	Time Invariant	Time Consistent
I	Yes	Yes	Yes
II	Yes	No	No
III	No	Yes	No
IV	No	No	Yes
V	No	No	No

Table 2: The five possible preference types. “Yes” (“No”) means that preferences have (do not have) the corresponding property.

Halevy (2014) provides experimental evidence for all five preference types in Table 2. In particular, in his experiment only two-thirds of the subjects who exhibit time-consistency also exhibit stationarity and half of the subjects whose choices are time-inconsistent exhibit stationarity. This shows that non-stationary behavior, e.g. due to decreasing impatience, is not equivalent to time-inconsistency. We will now demonstrate that an extension of our preference model to arbitrary decision times $\tau \geq 0$ can account for all five preference types in Table 2. We summarize these findings in the following theorem which is proved in the Appendix.

Theorem 4.4 *Assume that time-τ preferences $\succsim^\tau$ are represented by the utility function*

$$V_\tau(x, p, t) = w_\tau(p, t) v_\tau(x, t)$$

for all x, p, t, τ, for some functions w_τ and v_τ. Then the following specifications of the functions w_τ and v_τ yield the preference types in Table 2.

1. ***Type I*** *If $v_\tau(x, t) = \bar{v}(x)$ for some function $\bar{v}$ and if $w_\tau(p, t) = \delta^{t-\tau} w(p)$ for some function w and some $\delta > 0$, then preferences are stationary, time invariant and time-consistent.*

2. ***Type II*** *If $v_\tau(x, t) = \bar{v}_\tau(x)$ for some function $\bar{v}_\tau$ and if $w_\tau(p, t) = \delta^{t-\tau} w(p)$ for some function w and some $\delta > 0$, then preferences are stationary, but neither time invariant nor time-consistent unless $\bar{v}_\tau$ satisfies*
$$\frac{\bar{v}_\tau(x)}{\bar{v}_\tau(y)} = \frac{\bar{v}_{\tau+\Delta}(x)}{\bar{v}_{\tau+\Delta}(y)} \tag{3}$$
for all $x, y, \tau, \Delta \geq 0$.

3. ***Type III*** *If $v_\tau(x, t) = \bar{v}(x)$ for some function $\bar{v}$ and if $w_\tau(p, t) = w(p)[1 + \alpha(t - \tau)]^{-1}$ for some function w and some $\alpha > 0$, then preferences are time invariant, but neither stationary nor time-consistent.*

4. ***Type IV*** *If $v_\tau(x, t) = \bar{v}(x, t)$ for some function $\bar{v}$ and if $w_\tau(p, t) = w(p)\delta^{t-\tau}$ for some function w and some $\delta > 0$, then preferences are time-consistent, but neither stationary nor time invariant unless $\bar{v}$ satisfies*
$$\frac{\bar{v}(x, t)}{\bar{v}(y, s)} = \frac{\bar{v}(x, t + \Delta)}{\bar{v}(y, s + \Delta)} \tag{4}$$
for all $x, y, t, s, \Delta \geq 0$.

5. ***Type V*** *If* $v_\tau(x,t) = \bar{v}_\tau(x)$ *for some function* $\bar{v}_\tau$ *and if* $w_\tau(p,t) = w(p)[1+\alpha(t-\tau)]^{-1}$ *for some function* w *and some* $\alpha > 0$*, then preferences violate stationarity. Moreover, preferences are not time invariant unless* $\bar{v}_\tau$ *satisfies*

$$\frac{\bar{v}_\tau(x)}{\bar{v}_\tau(y)} = \frac{\bar{v}_{\tau+\Delta}(x)}{\bar{v}_{\tau+\Delta}(y)} \tag{5}$$

for all $x, y, \tau, \Delta \geq 0$*. Finally, preferences are not time-consistent unless* $\bar{v}$ *satisfies*

$$\frac{\bar{v}_\tau(x)(1+\alpha(s-\tau))}{\bar{v}_\tau(y)(1+\alpha(t-\tau))} = \frac{\bar{v}_{\tau'}(x)(1+\alpha(s-\tau'))}{\bar{v}_{\tau'}(y)(1+\alpha(t-\tau'))} \tag{6}$$

for all x, y, τ, τ', s, t*, with* $0 \leq \tau, \tau' \leq s, t$*.*

In the example of Type III of Theorem 4.4 non-stationarities are driven purely by the weighting function, which depends only on the (psychological) distance between decision and consumption time. Here time invariance holds because preferences only depend on time through this distance. This case is given most attention in the literature.

In the example of Type IV in Theorem 4.4 non-stationarities are driven purely by the time-dependence of the utility function. This case would, for instance, arise if utilities depend on baseline consumption that is perfectly foreseen to change over time. The resulting non-stationarity would be the result of the decision maker projecting himself to the future and imagining how much he will enjoy an outcome in the future.

Theorem 4.4 shows that WTU is sufficiently rich to cover all possible preference types concerning stationarity, time invariance and time-consistency. Moreover, it clearly demonstrates that violations of stationarity are by no means the only source of time-inconsistent behavior. Hence, we have to measure both w and v in order to get a complete picture of how a decision maker's preference responds to changes in decision or consumption time. This is the topic of the next section.

5 Parameter-Free Elicitation of $V(x,p,t)$

This section presents a parameter-free method for eliciting the weighting function $w(p,t)$ and the utility function $v(x,t)$ of WTU for a given continuous preference relation $\succcurlyeq$ over risky timed outcomes which satisfies impatience and monotonicity. We start with an elicitation of $w(p, 0)$.

Elicitation of $w(p, 0)$

Fix an arbitrary outcome $x > 0$, an arbitrary probability p_0 with $0 < p_0 < 1$, and a parameter κ with $0 < \kappa < 1$. Without loss of generality we can normalize w so that

$$w(p_0, 0) = \kappa \text{ and } w(1, 0) = 1.^{1}$$

Elicit y_1 such that

$$(x, p_0, 0) \sim (y_1, 1, 0) \tag{7}$$

and p_1 such that

$$(x, p_1, 0) \sim (y_1, p_0, 0). \tag{8}$$

By monotonicity y_1 and p_1 are unique and satisfy $y_1 < x$ and $p_1 < p_0$. Indifference (7) is equivalent to

$$\kappa v(x, 0) = v(y_1, 0) \tag{9}$$

and (8) is equivalent to

$$w(p_1, 0)v(x, 0) = \kappa v(y_1, 0). \tag{10}$$

From (9) and (10) it follows that

$$w(p_1, 0) = \kappa^2.$$

We can continue like this and elicit y_i and p_i for $i = 2, 3, \ldots,$ such that

$$(x, p_{i-1}, 0) \sim (y_i, 1, 0) \tag{11}$$

and

$$(x, p_i, 0) \sim (y_i, p_{i-1}, 0). \tag{12}$$

---

[1] Observe that $V(x, p, t) = w(p, t)v(x, t)$ and $V'(x, p, t) = w'(p, t)v'(x, t)$ both represent $\succcurlyeq$ if and only if there exist $\alpha_1, \alpha_2, \beta > 0$ such that $w'(p, t) = (\alpha_1 w(p, t))^{\beta}$ and $v'(x, t) = (\alpha_2 v(x, t))^{\beta}$.

It follows that

$$w(p_i, 0) = \kappa^{2^i} \text{ for all } i, \tag{13}$$

which can be shown as follows. For $i = 1$ we already verified that (13) holds. Now suppose that $w(p_{i-1}, 0) = \kappa^{2^{i-1}}$. From indifference (11) we have

$$w(p_{i-1}, 0)v(x, 0) = v(y_i, 0)$$

From indifference (12) we have

$$w(p_i, 0)v(x, 0) = w(p_{i-1}, 0)v(y_i, 0).$$

It follows that

$$w(p_i, 0) = (\kappa^{2^{i-1}})^2 = \kappa^{2^i}.$$

By choosing the starting point p_0 arbitrarily close to 1 we can make the grid on which we determine the weighting function $w(p, 0)$ arbitrarily fine.

Elicitation of $v(x, 0)$

Given $w(p, 0)$ with $w(1, 0) = 1$ it is straightforward to elicit $v(x, 0)$. Fix an arbitrary outcome $x > 0$. Without loss of generality we can normalize v so that

$$v(x, 0) = 1.$$ [2]

Then, for any outcome y with $y < x$ elicit p such that

$$(y, 1, 0) \sim (x, p, 0).$$

Then we have $v(y, 0) = w(p, 0)$. Similarly, for any outcome y with $y > x$ elicit q such that

$$(x, 1, 0) \sim (y, q, 0).$$

It follows that $v(y, 0) = \frac{1}{w(q,0)}$.

[2] See Footnote 1.

Elicitation of $w(p,t)$ and $v(x,t)$ for $t > 0$

In order to elicit $w(p,t)$ and $v(x,t)$ for $t > 0$ we use the method in the proof of Theorem 2.2. For every $x > 0$ elicit $x_0(x,t)$ such that

$$(x, 1, t) \sim (x_0(x,t), 1, 0)$$

and define

$$v(x,t) = v(x_0(x,t), 0).$$

Fix $x > 0$. For every $p > 0$ elicit $p_0(p,t)$ such that

$$(x, p, t) \sim (x_0(x,t), p_0(p,t), 0)$$

and define

$$w(p,t) = w(p_0(p,t), 0).$$

6 Conclusion

We introduced the weighted temporal utility model, which evaluates risky timed outcomes by the product of time-dependent utility generated by this outcome and a time-dependent probability weight. The model is consistent with empirical findings suggesting that probability and time as well as outcome and time are not separable. For single outcomes to be received with a specific probability at a single point in time, weighted temporal utility covers rank-dependent utility, prospect theory, exponential discounting, and hyperbolic discounting as special cases.

Another special case of weighted temporal utility arises when the decision maker evaluates an outcome at a specific point in time by the extra utility it generates on top of the utility derived from baseline consumption. If baseline consumption is expected to change over time, then the utility generated by an outcome is indeed time-dependent.

We showed that the time-dependency of the utility generated by an outcome can give rise to non-stationarities even if probabilities are weighted linearly and time is discounted exponentially. We also showed that this type of non-stationarity does not necessarily induce time-inconsistent behavior. Our model can therefore explain the findings of Halevy (2014).

It is important to note that the deviations from stationarity which are induced by the time-dependence of utilities, are not necessarily irrational. If one, for instance, considers the special case with baseline consumption, then a perfect foresight of changes in baseline consumption induces deviations from stationarity. Yet, these deviations are driven by perfect foresight, and, thereby, perfectly rational.

7 References

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Appendix

Lemma 1 *Under monotonicity the following statements are equivalent:*

(i) *The hexagon condition at time* 0 *holds.*

(ii) *Preferences at time* 0 *can be represented by*

$$V(x, p, 0) = w_0(p)u_0(x).$$

with $w_0(p) \geq 0$ *and* $u_0(x) \geq 0$ *for all* x, p, *and* $w_0(0) = u_0(0) = 0$. *Moreover,* w_0 *and* u_0 *are increasing.*

Proof of Lemma 1: The fact that *(ii)* implies *(i)* can easily be shown and also follows directly from Theorem III.4.1 in Wakker (1989).

Now assume that *(i)* holds. Then Theorem III.4.1 in Wakker (1989) shows that we have a representation

$$V(x, p, 0) = w_0(p)u_0(x)$$

for all positive outcomes and positive probabilities. Moreover, $w_0(p) > 0$ for all $p > 0$ and $u_0(x) > 0$ for all $x > 0$. We will first show that $w_0(p)$ goes to zero as p goes to zero. Suppose that this were not the case and that $w_0(p)$ would go to a positive number W as p goes to zero. Now consider any two outcomes $y > x > 0$ and a very small probability $\varepsilon > 0$. Then

$$(y, \varepsilon, 0) \succ (x, \varepsilon, 0) \succ (0, \varepsilon, 0) \sim (y, 0, 0)$$

By continuity there must be a probability $\kappa > 0$ with

$$(x, \varepsilon, 0) \sim (y, \kappa, 0),$$

which implies

$$w_0(\varepsilon)u_0(x) = w_0(\kappa)u_0(y).$$

Note that $\kappa < \varepsilon$. Yet, when ε is small enough, we have that $\frac{w_0(\varepsilon)}{w_0(\kappa)} \approx \frac{W}{W} = 1$, which contradicts the fact that we can find such a κ with $w_0(\varepsilon)u_0(x) = w_0(\kappa)u_0(y)$ for all outcomes $y > x$. Thus, it must be the case that $w_0(p)$ goes to zero as p goes to zero. A similar argument shows that $u_0(x)$ goes to zero as x goes to zero.

Define $u_0(0) = 0$ and $w_0(0) = 0$. Consider $(x, p, 0) \succcurlyeq (y, q, 0)$. If x, y, p, and q are all positive then we have that

$$w_0(p)u_0(x) \geq w_0(q)u_0(y).$$

If $x = 0$ or $p = 0$ then we must have $y = 0$ or $q = 0$, which implies that $w_0(p)u_0(x) \geq w_0(q)u_0(y)$. If $y = 0$ or $q = 0$ and $x > 0$ and $p > 0$, then $w_0(p)u_0(x) \geq w_0(q)u_0(y)$ follows as well. This shows that preferences at time 0 can be represented by $V(x, p, 0) = w_0(p)u_0(x)$. Since $\succcurlyeq$ satisfies monotonicity and $w_0(p) > 0$ for all $p > 0$ and $u_0(x) > 0$ for all $x > 0$ it is straightforward to show that w_0 and u_0 are increasing. This proves the Lemma. □

Proof of Theorem 2.2

We first prove that *(i)* implies *(ii)*. Assume that probability-independent time-outcome tradeoff and the hexagon condition at time 0 hold. By Lemma 1 preferences at time 0 can be represented by $V(x, p, 0) = w_0(p)u_0(x)$ with w_0 and u_0 nonnegative and increasing, and $w_0(0) = u_0(0) = 0$. For every outcome $x \geq 0$ and time t define the outcome $x_0(x, t)$ by $(x, 1, t) \sim (x_0(x, t), 1, 0)$. By impatience, monotonicity and continuity $x_0(x, t)$ is always well-defined. For every $x > 0$ and every p, t define the probability $p_0(x, p, t)$ by $(x, p, t) \sim (x_0(x, t), p_0(x, p, t), 0)$. By impatience, monotonicity, and continuity $p_0(x, p, t)$ is always defined for $x > 0$. Probability-independent time-outcome tradeoff implies that $p_0(x, p, t) = p_0(y, p, t)$ for all $x, y > 0$. Thus, we define $p_0(p, t) = p_0(x, p, t)$.

Now we define

$$v(x, t) = u_0(x_0(x, t))$$

for all x, t with $x > 0$ and set $v(0, t) = 0$. Further we define

$$w(p, t) = w_0(p_0(p, t))$$

for all p, t. Then we have for $x, y > 0$

$$\begin{aligned}
(x,p,t) &\succcurlyeq (y,q,s) \\
\iff (x_0(x,t), p_0(p,t), 0) &\succcurlyeq (x_0(y,s), p_0(q,s), 0) \\
\iff w_0(p_0(p,t))u_0(x_0(x,t)) &\geq w_0(p_0(q,s))u_0(x_0(y,s)) \\
\iff w(p,t)v(x,t) &\geq w(q,s)v(y,s).
\end{aligned}$$

If $x = 0$ or $y = 0$ then it is straightforward to verify that $(x, p, t) \succcurlyeq (y, q, s)$ if and only if $w(p,t)v(x,t) \geq w(q,s)v(y,s)$. Thus, $V(x,p,t) = w(p,t)v(x,t)$ represents $\succcurlyeq$. Moreover, by definition, $w(p,t)$ and $v(x,t)$ are nonnegative and increasing in p and x, respectively, and $w(0,t) = v(0,t) = 0$ for all t.

Now we need to prove that *(ii)* implies *(i)*. Assume that preferences $\succcurlyeq$ can be represented by

$$V(x,p,t) = w(p,t)v(x,t),$$

where $w(p,t)$ and $v(x,t)$ are nonnegative and increasing in p and x, respectively, and $w(0,t) = v(0,t) = 0$ for all t. The hexagon condition at time 0 follows from Lemma 1. Assume that $x, y > 0$ and $(x,1,t) \sim (x_0,1,0)$, $(x,p,t) \sim (x_0,p_0,0)$, and $(y,1,t) \sim (y_0,1,0)$. Then

$$\frac{v(x,t)}{v(x_0,0)} = \frac{v(y,t)}{v(y_0,0)}.$$

If $p > 0$, then

$$\frac{v(x,t)}{v(x_0,0)} = \frac{w(p_0,0)}{w(p,t)}$$

and it follows that

$$\frac{v(y,t)}{v(y_0,0)} = \frac{w(p_0,0)}{w(p,t)}$$

Thus, $(y,p,t) \sim (y_0,p_0,0)$ if $p > 0$. If $p = 0$, then $(x,p,t) \sim (x_0,p_0,0)$, implies that $p_0 = 0$ and hence $(y,p,t) \sim (y_0,p_0,0)$ holds in this case as well. □

Proof of Proposition 2.4

From Theorem 2.2 we know that *(i)* implies that preferences can be represented by $V(x, p, t) = w(p, t)v(x, t)$ with $w(p, t) \geq 0$ and $v(x, t) \geq 0$ for all x, p, t and with w increasing in p and v increasing in x. The next step is to show that isoendurance implies that we can find functions $\delta(t)$ and $v(x)$ such that $v(x, t) = \delta(t)v(x)$. Define $v(x) = v(x, 0)$ for all x and

$$\delta(x, t) = \frac{v(x, t)}{v(x, 0)}$$

for all $x > 0$ and all t. Isoendurance implies that the latter fraction is independent of x. Thus, $\delta(x, t)$ depends only on t, and can thereby also be written as $\delta(t)$. It follows that $V(x, p, t) = w(p, t)\delta(t)v(x)$. We can now define $\tilde{w}(p, t) = w(p, t)\delta(t)$, such that *(ii)* follows.

Now assume that *(ii)* holds so that preference can be represented by $V(x, p, t) = w(p, t)v(x)$. Given Theorem 2.2 it remains to show that isoendurence holds. Assume that $(x, p, t+\Delta) \sim (x, p\theta, t)$ for $\theta \in (0, 1)$, $\Delta \in (0, \infty)$, $p > 0$, and $x > 0$. Now consider outcome y such that $0 < y < x$. Indifference $(x, p, t + \Delta) \sim (x, p\theta, t)$ implies $w(p, t + \Delta)v(x) = w(p\theta, t)v(x)$. It follows that $w(p, t + \Delta) = w(p\theta, t)$. Thus, $w(p, t + \Delta)v(y) = w(p\theta, t)v(y)$, i.e., isoendurance holds. □

Proof of Theorem 3.1

We first prove that *(i)* implies *(ii)*. For every x, p, t with $x > 0$ define $e(x, p, t)$ by $(x, p, t) \sim (e(x, p, t), 1, 0)$. By impatience, monotonicity, and continuity $e(x, p, t)$ is well-defined and unique. Moreover, monotonicity implies that $u(x)$ is increasing in x.

Set $b_0 = 0$. Now define

$$w(x, p, t) = \frac{u(e(x, p, t))}{u(b_t + x) - u(b_t)}$$

for every $x > 0$ and every p, t. By the definition of baseline consumption we have that

$$\frac{u(e(x, p, t))}{u(b_t + x) - u(b_t)} = \frac{u(e(y, p, t))}{u(b_t + y) - u(b_t)}$$

for every $x, y > 0$. It follows that $w(x, p, t)$ is independent of x. Thus, we define $w(p, t) = w(x, p, t)$.

It remains to be shown that $w(p,t)\left(u(b_t+x)-u(b_t)\right)$ represents $\succcurlyeq$. For $x,y>0$, and $p,q,s,t\geq 0$, we have that

$$\begin{aligned} (x,p,t) &\succcurlyeq (y,q,s) \\ \iff (e(x,p,t),1,0) &\succcurlyeq (e(y,q,s),1,0) \\ \iff u(e(x,p,t)) &\geq u(e(y,q,s)) \\ \iff w(p,t)\left(u(b_t+x)-u(b_t)\right) &\geq w(q,s)\left(u(b_s+y)-u(b_s)\right). \end{aligned}$$

For $x=0$ we have $(x,p,t)\succcurlyeq(y,q,s)$ if and only if $y=0$ or $q=0$ if and only if $w(p,t)\left(u(b_t+x)-u(b_t)\right)\geq w(q,s)\left(u(b_s+y)-u(b_s)\right)$, as $w(0,s)=0$. For $y=0$ we have $w(p,t)\left(u(b_t+x)-u(b_t)\right)\geq 0=w(q,s)\left(u(b_s+y)-u(b_s)\right)$. This proves our result.

The fact that *(ii)* implies *(i)* follows easily by substitution. □

Proof of Theorem 4.4

1. Let $V_\tau(x,p,t)=\delta^{t-\tau}w(p)\bar{v}(x)$. Then for every x,y,p,q,τ,s,t, with $0\leq\tau\leq s,t$, and for every $\Delta\geq 0$,

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}(y)}{\bar{v}(x)} \\ \iff (x,p,t+\Delta) &\sim^\tau (y,q,s+\Delta) \end{aligned}$$

Hence, preferences are stationary. Moreover, for every x,y,p,q,τ,s,t with $0\leq\tau\leq s,t$, and for every $\Delta\geq 0$,

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}(y)}{\bar{v}(x)} \\ \iff (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \end{aligned}$$

Thus, preferences are time invariant. From stationarity and time invariance it then follows that preferences are also time consistent.

2. Let $V_\tau(x,p,t) = \delta^{t-\tau} w(p)\bar{v}_\tau(x)$. Then for every x, y, p, q, τ, s, t, with $0 \leq \tau \leq s, t$, and for every $\Delta \geq 0$,

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}_\tau(y)}{\bar{v}_\tau(x)} \\ \iff (x,p,t+\Delta) &\sim^\tau (y,q,s+\Delta) \end{aligned}$$

Hence, preferences are stationary. To see that preferences are not time-invariant in general, observe that

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}_\tau(y)}{\bar{v}_\tau(x)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}_{\tau+\Delta}(y)}{\bar{v}_{\tau+\Delta}(x)} \end{aligned}$$

Therefore, preferences are time invariant if and only if for every $x, y, \tau, \Delta \geq 0$, (3) holds. If (3) is violated, then preferences are also not time consistent because otherwise, stationarity and time consistency would imply time invariance.

3. Let $V_\tau(x,p,t) = w(p)[1 + \alpha(t-\tau)]^{-1}\bar{v}(x)$. Then for every x, y, p, q, τ, s, t with $0 \leq \tau \leq s, t$, and for every $\Delta \geq 0$,

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \frac{w(p)\bar{v}(x)}{w(q)\bar{v}(y)} &= \frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \\ \iff (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \end{aligned}$$

Hence, preferences are time invariant. To see that preferences are not stationary, observe that

$$\begin{aligned}(x,p,t) &\sim^{\tau} (y,q,s)\\ \iff \frac{w(p)\bar{v}(x)}{w(q)\bar{v}(y)} &= \frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)}\end{aligned}$$

and

$$\begin{aligned}(x,p,t+\Delta) &\sim^{\tau} (y,q,s+\Delta)\\ \iff \frac{w(p)\bar{v}(x)}{w(q)\bar{v}(y)} &= \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)}\end{aligned}$$

Since $\frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \neq \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)}$ for $\Delta > 0$, preferences are not stationary. Hence, preferences are also not time consistent, because time invariance and time consistency implies stationarity.

4. Let $V_\tau(x,p,t) = \delta^{t-\tau} w(p)\bar{v}(x,t)$. Then for every $x, y, p, q, \tau, \tau', s, t$, with $0 \le \tau, \tau' \le s, t$,

$$\begin{aligned}(x,p,t) &\sim^{\tau} (y,q,s)\\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}(y,s)}{\bar{v}(x,t)}\\ \iff (x,p,t) &\sim^{\tau'} (y,q,s)\end{aligned}$$

Hence, preferences are time consistent. To see that preferences are not stationary in general, observe that

$$\begin{aligned}(x,p,t) &\sim^{\tau} (y,q,s)\\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}(y,s)}{\bar{v}(x,t)}\end{aligned}$$

and

$$\begin{aligned}(x,p,t+\Delta) &\sim^{\tau} (y,q,s+\Delta)\\ \iff \delta^{t-s}\frac{w(p)}{w(q)} &= \frac{\bar{v}(y,s+\Delta)}{\bar{v}(x,t+\Delta)}\end{aligned}$$

Therefore, preferences are stationary if and only if for every $x, y, s, t, \Delta \geq 0$, (4) holds. If (4) is violated, then preferences are also not time invariant because otherwise, time consistency and time invariance would imply stationarity.

5. Let $V_\tau(x,p,t) = w(p)[1+\alpha(t-\tau)]^{-1}\bar{v}_\tau(x)$. To see that preferences are not stationary, observe that

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \frac{w(p)\bar{v}_\tau(x)}{w(q)\bar{v}_\tau(y)} &= \frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^\tau (y,q,s+\Delta) \\ \iff \frac{w(p)\bar{v}_\tau(x)}{w(q)\bar{v}_\tau(y)} &= \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)} \end{aligned}$$

Since $\frac{1+\alpha(t-\tau)}{1+\alpha(s-\tau)} \neq \frac{1+\alpha(t+\Delta-\tau)}{1+\alpha(s+\Delta-\tau)}$ for $\Delta > 0$, preferences are not stationary. To see that preferences are not necessarily time invariant, observe that

$$\begin{aligned} (x,p,t) &\sim^\tau (y,q,s) \\ \iff \frac{w(p)(1+\alpha(s-\tau))}{w(q)(1+\alpha(t-\tau))} &= \frac{\bar{v}_\tau(y)}{\bar{v}_\tau(x)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t+\Delta) &\sim^{\tau+\Delta} (y,q,s+\Delta) \\ \iff \frac{w(p)(1+\alpha(s-\tau))}{w(q)(1+\alpha(t-\tau))} &= \frac{\bar{v}_{\tau+\Delta}(y)}{\bar{v}_{\tau+\Delta}(x)} \end{aligned}$$

Therefore, preferences are time invariant if and only if for every $x, y, s, t, \Delta \geq 0$, (5) holds. To see that preferences are not necessarily time consistent, observe that

$$\begin{aligned} (x,p,t) &\sim^{\tau} (y,q,s) \\ \iff \frac{\bar{v}_{\tau}(x)(1+\alpha(s-\tau))}{\bar{v}_{\tau}(y)(1+\alpha(t-\tau))} &= \frac{w(q)}{w(p)} \end{aligned}$$

and

$$\begin{aligned} (x,p,t) &\sim^{\tau'} (y,q,s) \\ \iff \frac{\bar{v}_{\tau'}(x)(1+\alpha(s-\tau'))}{\bar{v}_{\tau'}(y)(1+\alpha(t-\tau'))} &= \frac{w(q)}{w(p)} \end{aligned}$$

Therefore, preferences are time consistent if and only if for every x, y, s, t, τ, τ', with $0 \leq \tau, \tau' \leq s, t$, (6) holds.

□