Middlemen: A Directed Search Equilibrium Approach

Makoto Watanabe

Faculty of Economics and Business Administration, VU University Amsterdam, and Tinbergen Institute.

Middlemen: A Directed Search Equilibrium Approach

Makoto Watanabe*

VU University Amsterdam, Tinbergen Institute†

December 9, 2012

Abstract

This paper studies an intermediated market operated by middlemen with high inventory holdings. I present a directed search model in which middlemen are less likely to experience a stockout because they have the advantage of inventory capacity, relative to other sellers. The model explains why popular items are sold at a larger premium, and everyday items at a larger discount, by large-scaled intermediaries. The concentration of middlemen's market, i.e., few middlemen, each with large capacity, can lead to a higher matching efficiency, but with a lower total welfare, compared to having many middlemen, each with small capacity.

Keywords: Directed Search, Intermediation, Inventory holdings

JEL Classification Number: D4, F1, G2, L1, L8, R1

1 Introduction

According to the "Used Car Market Report in 2007," produced by Manheim Consulting, about 65.7% of the 42.6 million U.S. used car sales in 2006 were from dealers. The average transacted price of a used vehicle was \$10,875 for franchised dealers, \$8,675 for independent dealers, and \$4,450 for private sellers. They note that the main reason for these differences was that large-scale dealers have greater control over their used vehicle inventory, as a result of their active stocking of popular brands.

This paper studies the functioning of markets with middlemen (e.g. retailers, wholesalers, trading entrepreneurs, dealers or brokers of services and durable goods and assets). One important feature of such a market is that a middleman's high inventory capacity can influence buyers' search decision: the stocks maintained by middlemen are valuable to those who wish to save time and effort that would have to be spent on searching on an individual basis.[1] The above evidence further suggests that the middlemen's capacity can not only attract many buyers, but also allow them to charge a premium for the immediacy service it provides. The immediacy premium appears more clearly for popular merchandise as shown by Dana and Spier (2001): Conducting a survey about four new releases at 20 rental video outlets within a local market, they find that Blockbuster Video (the leading intermediary in the rental video industry, who had adopted the policy to increase its stocks of new-releases) charged \$3.81 and had 86% availability on average, whereas the corresponding numbers were \$3.32 and 60% for other national chains, and \$2.62 and 48% for the independent stores.

In the existing literature, Rubinstein and Wolinsky (1987) is the first to show that an intermediated market can be active under frictions, when it is operated by middlemen who have an advantage in the meeting rate over the original suppliers. Given some exogenous meeting process, two main reasons have been considered for the middlemen's advantage in the rate of successful trades: a middleman may be able to guarantee the quality of goods (Biglaiser, 1993, Li, 1998), or to satisfy buyers' demand for a varieties of goods (Shevichenko, 2004). While these are clearly sound reasons for the success of middlemen, the buyers' search is modeled as an undirected random matching process, thereby the middlemen's capacity cannot influence buyers' search decisions in these models.

---

[1]For most grocery items, electronics, toys, or video rentals, stockout is a major cause of customer dissatisfaction (Andersen 1996), even in the mail-order catalog transactions (Fitzsimons, 2000). In financial markets, when Stigler (1964) and Demsetz (1968) analyzed the role of dealers, they viewed dealers as the suppliers of immediacy to the market. To determine the costs to buyers and sellers of using the NYSE to contract with each other, Demsetz captured the compensation to the dealer by "the markup that is paid for predictable immediacy of exchange in organized markets; in other markets, it is the inventory markup of retailer or wholesaler" (p. 36).

This paper presents a model which allows us to study explicitly the dependence of both the buyers' search decision and the middlemen's price decision on their inventory capacity, based on a directed search approach[2]. This is an important issue that has never been addressed in the existing literature. In my framework, the middlemen's capacity enables them to serve many buyers at a time. As buyers value the immediacy under market frictions, the price charged by middlemen includes a premium even though the buyers and the products are homogeneous. Thus, the inventory maintained by middlemen can influence not only the matching efficiency, but also their market power.

More precisely, I consider retail markets that are operated by middlemen and sellers. In the retail markets, the only difference between sellers and middlemen is the selling capacity backed by inventories. Middlemen can stock their inventories in the competitive wholesale markets operated by sellers (i.e., producers).[3] Buyers have limited search time and their search activities are on an uncoordinated basis. Under these frictions, the middlemen's advantage in the selling capacity provides buyers with a lower likelihood of experiencing a stockout. Thus, the price difference between sellers and middlemen determines the retail premium charged for the more sure service rate, i.e. the immediacy service.

In this economy, the retail prices decrease with the number of middlemen because it leads to more competitive (i.e., less tight) retail markets, as is standard in the directed/competitive search literature. However, the middlemen's inventory capacity has non-trivial effects on prices. On one hand, a larger inventory makes it less likely that excess demand occurs at individual middlemen, generating a downward pressure on the middlemen's price. On the other hand, it influences the search decision of buyers: a larger capacity of a middleman can attract more buyers. This effect creates a tighter market that allows to charge a higher price, since the middleman knows that buyers receive zero payoff in the event of a stockout. These conflicting effects cause a non-monotonic response of the price to capacities: it goes up (down) when the initial total supply is scarce (abundant). This is because the stockout probability is initially high (low) in the former (latter) situation, thereby buyers are prepared to pay a higher premium for a larger inventory when the the initial scarcity of resources is higher.

---

[2]Unlike in traditional random matching models, directed search equilibria incorporate competition among sellers and buyers' choice of where to search. See, for example, Accemoglu and Shimer (1999), Albrecht, Gautier and Vroman (2006, 2010), Burdett, Shi and Wright (2001), Camera and Selcuk (2009), Coles and Eeckhout (2003), Faig and Jerez (2005), Galenianos and Kircher (2008), Guerrieri, Shimer and Wright (2010), Julien, Kennes, and King (2000), McAfee (1993), Menzio (2007), Moen (1997), Montogomery (1991), Peters (1991), Shi (2002ab), and Watanabe (2006, 2010).

[3]This assumption is for analytical tractability, and guarantees the middlemen's inventory to be deterministic and identical to individuals across all the periods. It is related to the one adopted in a monetary model of Lagos and Wright (2005) who establish a monetary equilibrium with a degenerate distribution of divisible money holdings in the presence of market frictions that could potentially lead to a complicated stochastic evolution of individual money balances.

This result may help us understand why popular items are sold at a larger premium by larger scaled intermediaries, as is the case of popular brands of used cars or newly-released rental videos, whereas everyday items are often sold at a larger discount by big supermarkets especially when they hold an excessive amount of inventories.[4]

The non-monotonicity of price has an interesting efficiency implication of the middlemen's market. The middlemen in this economy are efficiency enhancing, since their capacity can mitigate market frictions. Further, the concentration of middlemen's market, i.e., having fewer middleman, each with larger capacity, leads to a larger number of trades, which improves the matching efficiency. However, it does not necessarily accompany an increase in welfare. Indeed, the concentration of middlemen can deteriorate the welfare when the total supply is scarce, in which case the price increase is more than enough to offset the gain from the improved service rate. This result points to the tension between the matching efficiency and the economic welfare generated by the concentration of middlemen's market.

The rest of the paper is organized as follows. This introductory section closes with more detailed discussions on the related literature. Section 2 presents the basic setup and studies the steady state equilibrium allocations. Section 3 provides a characterization of the retail price differentials. Section 4 extends the analysis to allow for the free entry of middlemen. It is shown that the number of middlemen can be non-monotone in their capacity. Section 5 investigates the matching efficiency and the welfare. Section 6 concludes. All the proofs are in the Appendix.

Related literature: This study contributes to the literature of middlemen, initiated by Rubinstein and Wolinsky (1987), that emphasizes the middlemen's advantage over the original suppliers in the rate at which they meet buyers. In their model, it is assumed that: (i) the matching rates of agents are exogenous; (ii) the terms of trades are determined by Nash bargaining; (iii) middlemen can hold only one unit of a good as inventory. In contrast, my approach is based on a standard directed search equilibrium and allows me to study both the matching advantage of middlemen and its influence on their market power, because it incorporates: (i) buyers' choice of where to search so that the matching rate between buyers and suppliers is determined endogenously; (ii) competition among suppliers so

---

[4]Aguirregabiria (2005) finds that intermediaries' inventory is a critical variable to explain their pricing patterns in a supermarket chain, especially when customers trade-off the price against the service rate. Aguirregabiria (1999) identifies a negative effect of inventory ordering on the retail prices for groceries. A related evidence provided by Matsa (2009) shows that stockouts are negatively correlated with competition: supermarkets that face significantly high competition offer on average 5 percent lower stockout rates than otherwise similar stores.

that individual suppliers can influence the search-purchase behaviors of buyers through prices; (iii) middlemen's inventory holdings of more than one unit so that the dependence of both the buyers' search decision and the extent of competition on their inventory can be made explicit.

There are two papers in this literature that are closely related to the current study.[5] Shevichenko (2004) provides a random meeting model that allows for middlemen to hold a variety of goods as inventory with ex ante heterogeneity of products and preferences. The middlemen studied in his model are agents who can mitigate the severity of double-coincidence of wants problem. In his model, the price is determined by Nash bargaining, and the equilibrium price is dispersed with respect to the type of goods held by middlemen. By holding a larger variety of goods as inventory, middlemen can increase the chance that a random buyer finds his preferred good on their shelves, which may or may not improve their terms of trade. However, the inventory capacity does not affect the rate at which a random buyer meets with a middleman. In contrast to Shevichenko (2004), even though agents and goods are both homogeneous in my model, the inventory capacity of middlemen can affect their meeting rate with buyers, generating the buyers' tradeoff between the price and the matching rate. It is exactly this simple tradeoff that yields the differential equilibrium prices across sellers with different capacities and the differential degree of retail market competition depending on the capacity level.

In a companion paper, Watanabe (2010) presents a special case of the current model and studies the turnover behaviors of sellers to become middlemen under a simplifying assumption of infinite discounting. It is shown that turnover equilibrium can be multiple – one is stable and has many middlemen, each with few units and a high price, and the other is unstable and has few middlemen, each with many units and a low price. Assuming away the turnover issue, the current paper investigates the effect of middlemen's inventory capacities on the market outcomes. This issue is perhaps more relevant for the directed search approach to study the functioning of markets with middlemen.

This paper is also related to the recent literature on financial intermediaries pioneered by Duffie, Garleanu, and Pedersen (2005). They use a bargaining-based search model, together with time varying preference shocks, to formulate the trading frictions that are characteristic of over-the-counter markets.

---

[5]In the literature, Biglaiser (1993) and Li (1998) model middlemen as a guaranter of product qualities, who help to ameliorate lemons problems, and Masters (2007) and Watanabe (2010) endogenize the decision of agents to become a middleman. In another approach used in Gehrig (1993), Spulber (1996), Rust and Hall (2003), Caillaud and Jullien (2003), Hendershott and Zhang (2006) and Loertscher (2007) (see also the book by Spulber (1999)), price setting is emphasized as the middlemen's main role of market-makings, but the meeting rate is exogenous. For further issues, see, for instance, Galeotti and Moraga-Gonzalez (2009), Johri and Leach (2002), Moraga-Gonzalez and Wildenbeest (2012), Smith (2004), and Wong and Wright (2011).

On the methodological side, there are situations in which the notion of directed search might provide a better description of pricing and trading mechanisms than the standard model with bargaining and random search. For instance, the Securities and Exchange Commission Rule 11Ac1-5, later redesignated as Rule 605 of the National Market System Regulation, requires market centers to report various execution quality statistics (so called Dash 5 reports) in publicly traded securities. The information made available by each market center includes time of execution and trading costs (such as effective bid-ask spreads) on a stock-by-stock basis. Similarly, broker/dealers in over the counter secondary market have an obligation to report transactions in corporate bonds to the Trade Reporting and Compliance Engine. This type of information is consistent with the notion of directed search. Further, analyzing execution quality on Nasdaq and the NYSE using the Dash-5 data, Boehmer (2005) finds that high execution costs are systematically associated with fast execution speed, and low costs are associated with slow execution speed. This relationship holds both across markets and across order sizes. This is the key trade-off that is common in directed search equilibria.

On the positive side, although the details of the modeling setup are different, the current study is consistent with recent progress in this strand of literature. In particular, Lagos and Rocheteau (2007, 2009) generalize the framework by allowing market participants to hold an unrestricted amount of assets and show that the (average) bid-ask spread of dealers can be non-monotonic in their bargaining power or in the contact rate of investors.[6] In their model, a higher bargaining power or a lower contact rate has a positive effect on the spread, and changes the distribution of trade sizes, which can have an adverse effect on the spread. The latter effect can be considered as a general equilibrium effect that occurs through changes in the investors' hedging behaviors against the future preference shocks. My model shows there is non-monotonicity in the capacity of dealers. This result arises due to a rather simple tradeoff faced by traders between the price and the (endogenous) matching probability: traders are willing to pay a higher premium for a larger inventory of dealers when the total supply is scarce. Another difference is that in my model, agents are allowed to trade directly with each other and the premium for dealers is defined over this option, whereas the dealers are the only avenue of exchange in their model and the premium is defined with respect to no trade.

---

[6]Other papers in this literature include Afonso (2010), Gárleanu (2009), Lagos, Rocheteau, and Weill (2010), Miao (2006), Vayanos and Wang (2007), Vayanos and Weill (2008), and Weill (2007, 2008). In the finance literature, the formal models (without search frictions) emphasizing the relationship between the dealers' inventory holdings and the bid-ask spread are developed, for example, Stoll (1978), Amihud and Mendelson (1980), Ho and Stoll (1981), and Hendershott and Menkveld (2009).

Finally, in the directed/competitive search literature, Burdett, Shi, and Wright (2001)[7] and Shi (2001a) are the first to study the buyers' tradeoff between the price and the service probability for sellers with different capacities, assuming that the capacity level of high-capacity sellers is $k_m = 2$. Generalizing $k_m \geq 1$, this paper identifies a case in which the price differential in the retail market (i.e., retail premium) is not necessarily monotone in k_m. This leads to an interesting efficiency implication of the concentration of large firms: an economy with few firms, each with large capacity, can lead to a higher matching efficiency, but with a higher (lower) total welfare when the total supply is abundant (scarce), compared to having many firms, each with small capacity.[8] The essence of this welfare implication is neither about cost considerations, nor increasing returns to scale in production technologies, but about the way in which firms should mitigate market frictions.

2 Model

Consider an economy inhabited by a continuum of homogeneous buyers, sellers and middlemen, indexed b, s and m, respectively. The population of buyers is normalized to one, and the population of sellers and middlemen are denoted by S and M, respectively. The population of agents are constant over time. All agents are risk-neutral and infinitely lived. Time is discrete and each period is divided into two subperiods. During the first subperiod, a retail market is open for a homogeneous, indivisible good to buyers. The good is storable. This retail market is operated by sellers and middlemen, and is subject to search frictions as described in detail below. Each period, buyers have unit demand while each seller can sell $k_s = 1$ unit, and each middleman can sell $k_m \geq 1$ units of the good. The selling capacity of suppliers k_i is exogenously given, for both $i = s, m$. The consumption value of the good is normalized to unity. If a buyer successfully purchases in the retail market at a price p, then he obtains the per-period utility of one. Otherwise, he receives zero utility that period. A seller or a middleman

---

[7]In Coles and Eeckhout (2003), sellers can post a more general trading mechanism for a finite number of agents. They show that a continuum of equilibria exist including an equilibrium with a simple form of price posting (i.e., the one studied in Burdett, Shi and Wright), while sellers prefer an equilibrium with auction. With a continuum of agents, auction and price posting are practically equivalent, with sellers achieving the same revenue and guaranteeing buyers the same utility. A usual argument applies: relatively high transaction costs associated with establishing and implementing auction can make sellers prefer price posting. This makes sense in particular for the economy considered here where retail technologies are made explicit and play an important economic role for middlemen's profits.

[8]Moen (1997), Mortensen and Wright (2002), and Sattinger (2003) consider fictitious market makers (not middlemen), who replace the Warlasian auctioneer, to interpret the efficiency property of competitive search equilibrium: Given the market makers create different submarkets and announce a pair of price and service probability in each submarket, buyers and sellers are willing to trade in a submarket where the Hosios condition holds endogenously. The latter property holds true in the equilibrium with middlemen constructed in my model.

who sells z units at a price p obtains the revenue zp during the first subperiod.

Once the retail market is closed, another market opens during the second subperiod. This is a wholesale market operated by sellers, and middlemen can restock their units for the future retail markets.[9] There is no search friction in the wholesale market and the price is determined competitively. Sellers decide whether to produce for the current wholesale market and for the future retail market. The production cost is measured in terms of utility and is given by $c < 1$. Agents discount future payoffs at a rate $\beta \in [0, 1)$ across periods, but there is no discounting between the two sub-periods.

The environment in the retail market each period is the same as in the standard competitive/directed search models (see, for example, Accemoglu and Shimer (1999), Burdett, Shi, and Wright (2001), Montgomery (1991), Peters (1991)). Any given first subperiod can be described as a simple two-stage game. In the first stage, sellers and middlemen simultaneously post a price which they are willing to sell at. Observing the prices, all buyers simultaneously decide which seller or middleman to visit in the second stage. Each buyer can visit one seller or one middleman. If more buyers visit a seller or middleman than its selling capacity, then the unit or units are allocated randomly. Assuming buyers cannot coordinate their actions over which seller or middleman to visit, a symmetric equilibrium is investigated where all buyers use the identical mixed strategy for any configuration of the announced prices. I focus my attention on a steady-state equilibrium where all sellers post the identical price p_s and all middlemen post the identical price p_m every period.

In any given period, each individual seller or middleman is characterized by an expected queue of buyers, denoted by x. The number of buyers visiting a given seller or middleman who has expected queue x is a random variable, denoted by n, which has the Poisson distribution, Prob.$(n = k) = \frac{e^{-x} x^k}{k!}$. In a symmetric equilibrium where x_i is the expected queue of buyers at i, each buyer visits some seller (and some middleman) with probability Sx_s (and Mx_m). They should satisfy the adding-up restriction,

$$Mx_m + Sx_s = 1, \tag{1}$$

requiring that the number of buyers visiting individual sellers and middlemen be summed up to the total population of buyers. The buyer's probability of being served by a supplier i depend on the queue x_i and the selling capacity k_i, and is denoted by $\eta(x_i, k_i)$.

---

[9]The timing of events is irrelevant here – the entire analysis remains unchanged with an alternative setting in which the retail market occurs in the second subperiod or production occurs in the retail market rather than in the restocking market. Also, one could alternatively assume that buyers own an income measured in terms of numeraire (or a compound good) and use it for buying the good in question. Here, I have selected the simplest possible setup.

In steady state, each seller holds $k_s = 1$ unit and each middleman holds $k_m \geq 1$ units at the start of every period. As the wholesale market is competitive, middlemen can restock at the sellers' reservation price c. That is, if the average quantity of sale by each individual middleman is $x_m \eta(x_m, k_m)$ in the retail market, then in total $M x_m \eta(x_m, k_m)$ units are supplied to the wholesale market, which clears at price c.[10] Simultaneously, sellers produce another unit for the next retail market, if the production cost is not too high, and if they have successfully sold in the retail market – holding more than one unit is never optimal since sellers only have the selling capacity of $k_s = 1$ in the retail market. This quantity amounts to $S x_s \eta(x_s, 1)$ units in total. Hence, in a steady state equilibrium, each seller holds $k_s = 1$ unit and each middleman holds $k_m \geq 1$ units, and the total units produced and consumed amount to

$$M x_m \eta(x_m, k_m) + S x_s \eta(x_s, 1)$$

each period.

Buyers' directed search   Assuming for the moment the existence of a symmetric equilibrium, the following lemma computes the buyer's probability of being served by a supplier i who has capacity k_i. The derivation is given in Watanabe (2006) (see also Watanabe (2010) for the finite agents version).

Lemma 1 *Given $x_i > 0$ and $k_i \geq 1$, the buyer's probability of being served by a supplier i that has capacity k_i is given by the following closed form expression.*

$$\eta(x_i, k_i) = \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} + \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)} \right)$$

where $\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$ and $\Gamma(k, x) = \int_x^\infty t^{k-1} e^{-t} dt$. $\eta(\cdot)$ is strictly decreasing (increasing) in x_i (in k_i) and satisfies $\eta(x_s, 1) = \frac{1 - e^{-x_s}}{x_s}$.

Given $\eta(\cdot)$ described above, I now characterize the expected queue of buyers. In any equilibrium where V_b is the value of a buyer, should a seller or a middleman deviate by setting price p in a period, the expected queue of buyers denoted by x satisfies[11]

$$V^b = \eta(x, k_i)(1 - p) + \beta V^b. \tag{2}$$

---

[10]Below, it is assumed that the capacity constraint of suppliers is binding for a middleman, so that he accumulates inventories up to the limit k_m. In financial markets, it can be justified by the reserve/capacity requirement. In other markets, one can assume a significantly high costs of maintaining an unfilled capacity or displaying unfilled shelves due to reputation concerns. To endogenize k_m is technically more involved and will be left for future research.

[11]See Peters (1991) for more issues on this large market property. Watanabe (2010) shows that the market equilibrium presented in this section is identical to the limiting solution in a finite setup counterpart as the population gets large.

A buyer choosing p is served with probability $\eta(x, k_i)$ in which case he obtains per-period utility $1 - p$. If not served by the seller or middleman, the buyer's payoff is zero that period. Irrespective of whether or not to be served, he enters the next period and his continuation value is βV^b. The situation is the same for all the other buyers. (2) is an implicit equation that determines $x = x(p, k_i \mid V^b) \in (0, \infty)$ as a strictly decreasing function of p given β, k_i and V^b.

Optimal pricing   Given the directed search of buyers described above, the next step is to characterize the equilibrium retail prices. In any equilibrium where V^b is the value of a buyer and where middlemen restock at the sellers' reservation price c in the Walrasian market, the optimal retail price of a supplier i who has a capacity k_i, denoted by $p_i(V^b)$, satisfies

$$p_i(V^b) = \mathrm{argmax}_p \left[x(p, k_i \mid V^b)\eta(x(p, k_i \mid V^b), k_i)(p - c) \right].$$

For $i = s$, a seller sells its good at price p with probability $x(p, 1\cdot)\eta(x(p, 1\cdot), 1)$, and produces a new unit with cost c for the next period. If unsuccessful in the retail market, then the seller's payoff is zero: if it sells the current unit to a middleman in the second subperiod, then it receives c from the middleman and produces a new unit for the next period with cost c; otherwise, the seller receives nothing and carries its unit into the next period. For $i = m$, a middleman's expected number of sales is $x(p, k_m\cdot)\eta(x(p, k_m\cdot), k_m)$, and it restocks at the competitive price c.

Substituting out p using (2), $p = 1 - \frac{1-\beta}{\eta(\cdot)}V^b$, the objective function of a supplier i, denoted by $\Pi_i(x)$, can be written as

$$\Pi_i(x) = x\eta(x, k_i)(1 - c) - x(1 - \beta)V^b$$

where $x = x(p, k_i \mid V^b)$ satisfies (2). The first-order condition is

$$\frac{\partial \Pi_i(x)}{\partial x} = \left(\eta(x, k_i) + x\frac{\partial \eta(x, k_i)}{\partial x} \right)(1 - c) - (1 - \beta)V^b = 0$$

for both $i = s, m$.[12] Rearranging the first order condition above using (2) and

$$\frac{\partial \eta(x, k_i)}{\partial x} = -\frac{k_i}{x^2}\left(1 - \frac{\Gamma(k_i + 1, x)}{\Gamma(k_i + 1)} \right),$$

---

[12] The second-order condition is satisfied as it holds that for both $i = s, m$

$$\frac{\partial^2 \Pi_i(x)}{\partial x^2} = -\frac{x^{k_i - 1}e^{-x}}{\Gamma(k_i)}(1 - c) < 0.$$

one can obtain the optimal price of the seller (if $i = s$) or the middleman (if $i = m$),

$$p_i(V^b) = c + \varphi^i(x, k_i)(1 - c) \tag{3}$$

where

$$\varphi^i(x, k_i) \equiv -\frac{\partial \eta(x, k_i)/\partial x}{\eta(x, k_i)/x}$$

is the elasticity of the matching rate of buyers.

Existence and uniqueness of steady-state equilibrium

Definition 1 *Given the population parameters S, M, the selling capacity k_i, $i = s, m$, the production cost c, and the discount factor β, a steady state equilibrium is a set of expected values V^j for $j = b, s, m$, and market outcomes x_i, p_i for $i = s, m$ such that:*

1. *Buyers' directed search satisfies (1) and (2);*

2. *Sellers' and middlemen's retail prices satisfy the first-order conditions (3) for $i = s, m$;*

3. *Middlemen restock their inventories in the wholesale market at the competitive price c. Each Middleman holds $k_m \geq 1$ units and each seller holds $k_s = 1$ unit in the retail market;*

4. *Agents have rational expectations. An agent $j = b, s, m$ receives $(1 - \beta)V^j$ each period.*

The analysis above has established the equilibrium prices $p_i(V^b)$ given V^b. Equilibrium implies buyers are indifferent between any of the individual suppliers $i = s, m$, leading to

$$(1 - \beta)V^b = \eta(x_s, 1)(1 - p_s) \tag{4}$$

$$= \eta(x_m, k_m)(1 - p_m), \tag{5}$$

where $x_i = x(p_i, k_i \mid V^b)$ is the equilibrium queue of buyers at $i = s, m$. Buyers successfully purchase the good from the seller or middleman with probability $\eta(x_i, k_i)$ each period. The value of sellers and middlemen are given by

$$(1 - \beta)V^s = x_s \eta(x_s, 1)(p_s - c) \tag{6}$$

$$(1 - \beta)V^m = x_m \eta(x_m, k_m)(p_m - c), \tag{7}$$

11

respectively. Sellers produce with cost c whenever needed, and middlemen restock at the competitive price c each period. To solve for the equilibrium, it is important to observe that indifference conditions (4) and (5) can be reduced to the following simple form: applying (3) for $i = s$ to (4) with a rearrangement,

$$\frac{(1-\beta)V^b}{1-c} = \eta(x_s, 1)(1 - \varphi^s(x_s, 1)) = e^{-x_s};$$

similarly, applying (3) for $i = m$ to (5) with a rearrangement,

$$\frac{(1-\beta)V^b}{1-c} = \eta(x_m, k_m)(1 - \varphi^m(x_m, k_m)) = \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)};$$

these two equations imply

$$\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} = e^{-x_s}. \tag{8}$$

The adding-up restriction (1) and the indifference condition (8) identify an equilibrium allocation $x_s, x_m > 0$.

Theorem 1 (Steady state equilibrium) *Given $c \in [0, \bar{c}]$, some $\bar{c} \in (0, \beta)$, a steady state equilibrium exists and is unique for all $\beta \in [0, 1)$, $S \in (0, \infty)$, $M \in (0, \infty)$ $k_m \geq 1$, satisfying $V^b \in (0, \frac{1-c}{1-\beta})$, $x_s \in (0, \frac{1}{S+M}]$, $x_m \in [\frac{1}{S+M}, \frac{1}{M})$, $p_i \in (c, 1)$, and $V^i \in (0, \frac{k_i(1-c)}{1-\beta})$, $i = s, m$.*

At the start of each period, all sellers hold one unit and all middlemen hold $k_m \geq 1$ units in the retail market. Sellers produce for the retail market given that production costs are not that high $c \leq \bar{c}$, and middlemen restock in the competitive wholesale market operated by sellers each period. The equilibrium allocation of buyers $x_s, x_m > 0$ is determined irrespective of the discount factor β and production cost c each period by (1) and (8). When $k_m = 1$, there is no difference between sellers and middlemen in the retail market and so all sellers and middlemen receive the identical number of buyers $x_s = x_m$ and post the identical price $p_s = p_m$. The indifference condition (8), combined with the adding up restriction, suggests that a supplier with a larger capacity should accommodate more buyers. Hence, an increase in the middlemen's capacity k_m induces more buyers to visit middlemen and fewer buyers to visit sellers, resulting in an increase in x_m and a decrease in x_s. An increase in the proportion of sellers S or middlemen M leads to a fewer number of buyers per each supplier, which decreases x_s, x_m.

3 Retail market prices

In this section, I study the behaviors of the retail prices. I begin by showing that the usual market-tightness effect leads to a lower price, as is standard in the directed/ competitive search literature.

Proposition 1 (Market tightness effect) *An increase in the population of sellers S or middlemen M (relative to that of buyers) leads to a lower retail market price p_i, $i = s, m$.*

As is consistent with the standard framework, the market-tightness effect implies that a larger supply makes the retail market less tight and more competitive, leading to a lower retail price.

I now investigate the comparative statistics results of middlemen's capacity k_m on the retail prices. In the current framework, the retail price differential, i.e., the price difference between sellers and middlemen in the retail market, is given by

$$p_m - p_s = [\varphi^m(x_m, k_m) - \varphi^s(x_s, 1)](1 - c),$$

where $\varphi^i(x_i, k_i)$ represents the supplier i's share of the net trading surplus $1 - c$ (see (3)).

Proposition 2 (Retail market premium) *For all $S, M \in (0, \infty)$, the retail market premium of middlemen is zero when $k_m = 1$ and is strictly positive when $1 < k_m < \infty$.*

When $k_m = 1$, there is no difference between sellers and middlemen in the retail market, and so the premium is zero. When $k_m > 1$, the price of middlemen is higher than that of sellers. The positive premium reflects the immediacy, or a relatively high rate of being served $\eta(x_m, k_m) > \eta(x_s, 1)$, that middlemen provide with its selling capacity $k_m > 1$. Below, I show that the behavior of p_m shapes critically that of the retail market premium.

The equilibrium price of middlemen is given by

$$p_m = c + \varphi^m(x, k_m)(1 - c)$$

where

$$\varphi^m(x_m, k_m) \quad = \quad \frac{1 - \frac{\Gamma(k_m + 1, x)}{\Gamma(k_m + 1)}}{x_m \eta(x_m, k_m)/k_m}.$$

Here, the denominator $x_m\eta(\cdot)/k_m$ represents the average probability of selling any given single unit, and the numerator,

$$\text{Prob.}(n > k_m) = \sum_{n=k_m+1}^{\infty} \frac{e^{-x_m}x_m^n}{n!} = 1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)},$$

represents the stock-out probability of an individual middleman – the probability that the number of buyers visiting the middleman n is strictly greater than its capacity k_m.[13] The stockout probability is decreasing in the capacity k_m and is increasing in the queue of buyers x_m.

There are two important effects of an increase in the capacity of middlemen k_m on their price p_m. On the one hand, a larger capacity of a middleman implies a smaller likelihood of excess demand and a smaller stockout probability of the middleman, given x_m. On the other hand, an increase in k_m implies an increase in the number of buyers to visit middlemen, rather than sellers. The latter effect makes the middlemen's market tighter and increases the stockout probability of individual middlemen. Suppose that the latter effect is large enough to make the price of middlemen increase with their capacity. Then, it means that middlemen can extract a larger share of trading surplus from buyers, since they know that buyers receive zero payoff in the event of stockouts. Conversely, suppose that the increase in x_m is relatively small, so that a middleman has a smaller likelihood of successfully selling out its entire units and its stockout becomes less likely. Then, buyers can receive a larger surplus share per unit, since buyers know that the middleman receives zero payoff from unsold units. Hereafter, I normalize $S = 1$, to simplify the analysis. Denote by

$$X \equiv \frac{1}{Mk_m+1} < 1$$

the per-period ratio of the total demand to the total supply in the retail market.

Proposition 3 (Retail market prices/differential)   *1. The retail price of middlemen p_m is increasing in sufficiently low k_m, if and only if $X > X^* \in (0,1)$, and is decreasing in sufficiently large k_m for any given $X \in (0,1)$.*

   2. The retail price of sellers p_s is decreasing in all k_m, for any given $X \in (0,1)$.

   3. The retail market differential $p_m - p_s \geq 0$ is increasing in sufficiently low k_m and is decreasing in sufficiently large k_m for any given $X \in (0,1)$.

---

[13]The second equation follows from the series definition of cumulative gamma function, $\sum_{n=0}^{k} \frac{e^{-x}x^n}{n!} = \frac{\Gamma(k+1,x)}{\Gamma(k+1)}$.
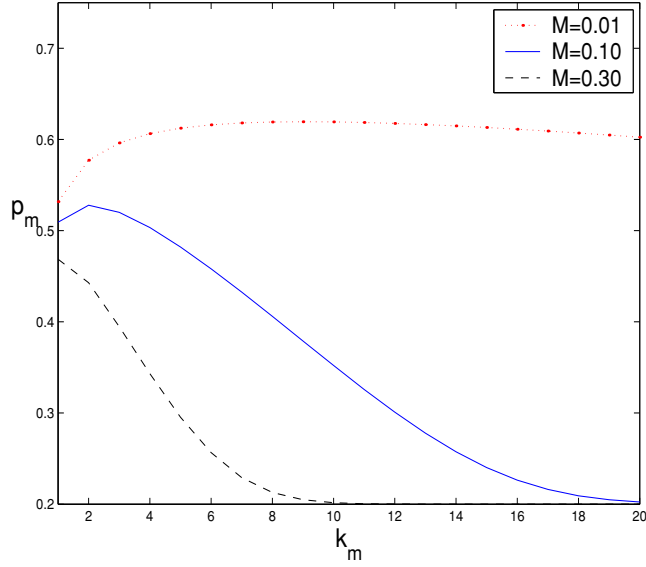
Figure 1: Retail price of middlemen

Figure 1 plots the behaviors of the price p_m and Figure 2 the behaviors of the price differential $p_m - p_s$ in response to changes in the middlemen's capacity k_m, for given values of M (and hence X). The non-monotonicity occurs because more buyers visit middlemen as their capacity increases. For relatively small k_m, the stockout probability is relatively high given values of x_m. Since the increase in the number of buyers visiting middlemen x_m is sufficiently large when the total demand X is high, an increase in k_m can make a tighter middlemen's market and result in a higher price p_m, if k_m (X) is initially low (high). For relatively large k_m, the total demand ratio is relatively low and the stockout probability is already low. In this situation, a larger k_m leads to an increase in the total supply that makes it less likely that an excess demand occurs at individual middlemen, resulting in a lower price. Unlike p_m, the sellers's price p_s is monotone decreasing in k_m, because the sellers' market gets less tight as k_m increases. This implies, the non-monotonicity of the retail market differential, $p_m - p_s$, is driven by that of the middlemen's price: there is a larger (smaller) premium for a larger selling capacity of middlemen when the total demand ratio X is initially high (low).

Notice that in the above analysis, changes in k_m affect the total supply. To abstract it from the effects of changes in the total demand-supply ratio, I examine the same comparative statistics exercise but, this time, fixing the middlemen's total supply denoted by $G = Mk_m$.
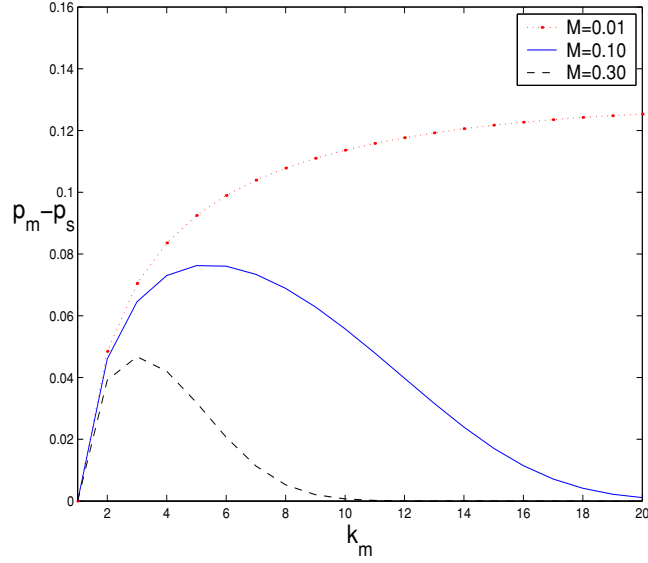
15

Figure 2: Retail price differentials

Proposition 4 (Fixed supply in middlemen's market) *Fix the total supply by middlemen* $G = Mk_m$.

1. *The retail price of middlemen* p_m *is increasing in sufficiently low* k_m, *if* $X = \frac{1}{G+1} > X^* \in (0,1)$, *and the retail market differential* $p_m - p_s \geq 0$ *is increasing in sufficiently low* k_m *for any given* $X \in (0,1)$

2. *The retail price of middlemen and the retail price differential satisfy* $p_m \to c$ *and* $p_m - p_s \to 0$, *respectively, when* $X < X^{**} \in (0,1)$, *and* $p_m > c$ *and* $p_m - p_s > 0$ *when* $X > X^{**} \in (0,1)$, *for sufficiently large* k_m.

The fixed total supply $G = Mk_m$ generates two margins: one is the intensive margin (as already seen above) and the other is the extensive margin, where M decreases with k_m. As the extensive margin implies a price increase, the item 1 in Proposition 4 shows that the condition of price increase with low k_m is less stringent than before. The item 2 shows that with fixed total supply in the middlemen's market, the price of middlemen p_m can be above the marginal cost c and the price differential can be positive $p_m - p_s > 0$ even for large k_m: when the total demand ratio $X = \frac{1}{G+1}$ is relatively high, the number of buyers to middlemen x_m increases large enough to make the middlemen's market tighter.

16

Figure 3 (a) plots the behavior of p_m and Figure 3 (b) that of price differential $p_m - p_s$, with different values of X. With fixed supply, the retail price/premium of middlemen can be substantially high for large k_m, especially when the total demand X is relatively high.
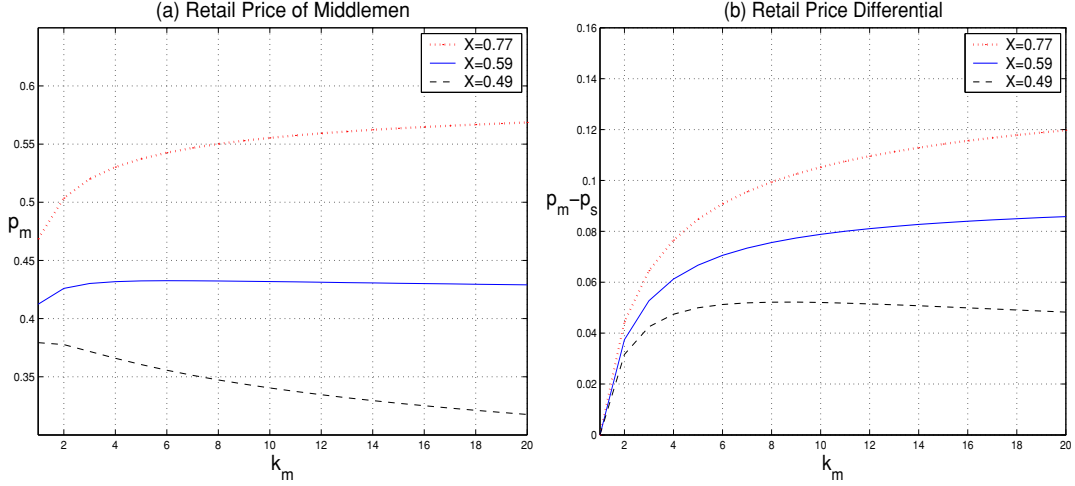

Figure 3: Concentration of middlemen's market

4   Free entry equilibrium

In this section, I allow for the number of middlemen to be determined endogenously by free entry. So far, I have implicity assumed that middlemen hold the initial endowment k_m at the start of their lifetime period. Suppose now that the initial endowment can be obtained by paying $ck_m > 0$ from the sellers' market, where as before $c > 0$ represents the wholesale price. Middlemen possess the inventory management technologies that enable them to operate with a relatively large selling capacity in the retail market. Denote by c_k the latter cost of inventory management per unit payed period by period.[14] An agent chooses to be a middlemen if the value of being a middlemen is non-negative, $-(c + \frac{c_k}{1-\beta})k_m + V^m \geq 0$, given values of $k_m \geq 1$ and $M > 0$. A symmetric free entry equilibrium is a steady state equilibrium described in Theorem 1 where entry and exit occur until the middlemen operating in the markets earn zero expected net profits, just to cover the cost. The equilibrium number

---

[14]All the results presented below go through with non linear management costs, as long as the marginal cost is not highly nonlinear in k_m.

of middlemen $M > 0$ is determined by the free entry condition, $V^m = (c + \frac{c_k}{1-\beta})k_m$, or

$$\left(1 - \frac{\Gamma(k_m + 1, x_m)}{\Gamma(k_m + 1)}\right) \frac{1 - c}{1 - \beta} = c + \frac{c_k}{1 - \beta}, \tag{9}$$

where the L.H.S. represents the per-unit profit of middlemen (apply the price in (3), $i = m$, to the value V^m in (7)) and the R.H.S. the lifetime per-unit cost. Define the total per-unit cost per period as $C \equiv \frac{c(1-\beta)+c_k}{1-c}$ and its upper bound as $\bar{C} \equiv \lim_{M \to 0} 1 - \frac{\Gamma(k_m+1,x_m)}{\Gamma(k_m+1)} \in (0, 1)$.
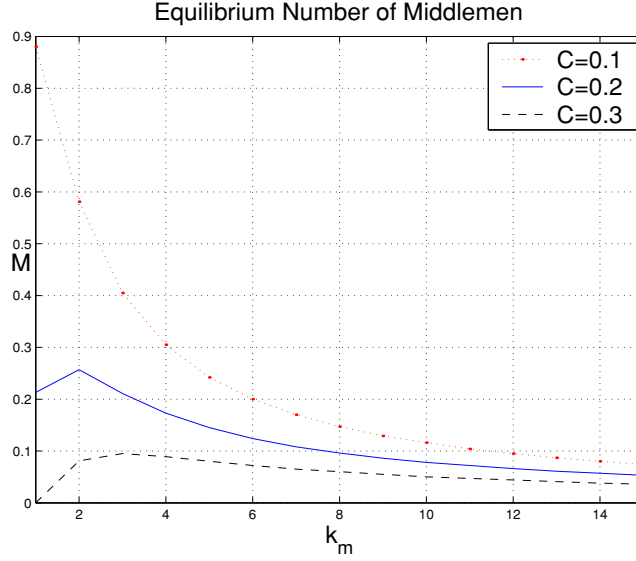

Figure 4: Free entry equilibrium

Proposition 5 (Free entry equilibrium) *Given values of $C \in (0, \bar{C})$, a free entry equilibrium exists and is unique. The equilibrium number of middlemen $M \in (0, \infty)$ is: monotone decreasing in C; increasing in low k_m if and only if $C > C^* \in (0, \bar{C})$; decreasing in large k_m for any $C \in (0, \bar{C})$.*

The per-unit profit (i.e., L.H.S. of (9)) is decreasing in the number of middlemen, thereby a larger cost leads to fewer middlemen given values of k_m. Figure 4 plots the number of middlemen M and k_m, for different values of C. The equilibrium number of middlemen can be non monotone in k_m when C is relatively high.[15] Notice that the per-unit profit is proportional to the stockout probability, $1 - \frac{\Gamma(k_m+1,x_m)}{\Gamma(k_m+1)}$, thereby a similar logic to the one stated before applies to the non-monotonicity of M:

---

[15]Through these effects, the price of middlemen can be non-monotonic in k_m. The analysis is available upon request.

when the total per-unit cost C is high and the initial units k_m are low, the number of middlemen is initially small and the total demand-supply ratio X is initially high. In this situation, a larger k_m can generate a tighter market of middlemen and increase the profitability of operating as a middleman. Hence, the number of middlemen increases with low k_m when C is high. The opposite happens when C is low or k_m is high, since in such a case, a larger k_m makes it more likely that middlemen remain their units unsold, which reduces their profits and the number of operating middlemen.

5 Efficiency and welfare

I now study the implications of middlemen's inventory holdings on the matching efficiency and the welfare. The per-period total number of trades in this economy, denoted by T, is given by

$$T = M x_m \eta(x_m, k_m) + x_s \eta(x_s, 1),$$

and the total welfare net of middlemen's profits, denoted by W, is given by

$$W = V^b + V^s = \frac{e^{-x_s}(1-c)}{1-\beta} + \frac{(1 - e^{x_s} - x_s e^{-x_s})(1-c)}{1-\beta} = \frac{(1 - x_s e^{-x_s})(1-c)}{1-\beta}.$$

To show first that the middlemen in this economy are efficiency/welfare enhancing, consider an alternative economy in which there are no middlemen, and buyers and sellers can trade only in a private market each period. Let $\underline{S} = M k_m + 1$ be the population of sellers. Then, the total number of trades and the net welfare in this alternative economy are

$$\underline{T} = \underline{S}\, \underline{x}_s \eta(\underline{x}_s, 1), \qquad \underline{W} = \frac{(1 - \underline{x}_s e^{-\underline{x}_s})(1-c)}{1-\beta},$$

respectively, in each period, where $\underline{x}_s = 1/\underline{S}$ is the queue of buyers at individual sellers. Comparing the these two economies, which have the same total supply ($= M k_m + 1$) and total demand (normalized to one) per period, the following proposition shows $T \geq \underline{T}$ and $W \geq \underline{W}$ with strict inequality for $k_m > 1$.

Proposition 6 (Efficiency of middlemen) *The middlemen in this economy are efficiency enhancing in terms of the total number of trades and the net welfare.*

In our economy, even with fixed total supply, the composition of middlemen's markets, in terms of the individual scale k_m and the number M of middlemen, can affect the number of trades and the

net welfare. The following proposition shows that the concentration of middlemen's markets have a positive impact on the number of trades, but not necessarily on the economic welfare.

Proposition 7 (Efficiency implication of middlemen's markets) *Fix the total supply $G = Mk_m$.*

1. *The total number of trades T is increasing in both small and large values of k_m, for all $X \in (0, 1)$.*

2. *The net welfare W is increasing in low k_m for all $X \in (0, 1)$, and satisfies $W \to \frac{1-c}{1-\beta}$ when $X < X^{**} \in (0, 1)$, and $W < \frac{1-c}{1-\beta}$ when $X > X^{**} \in (0, 1)$, for sufficiently large k_m.*
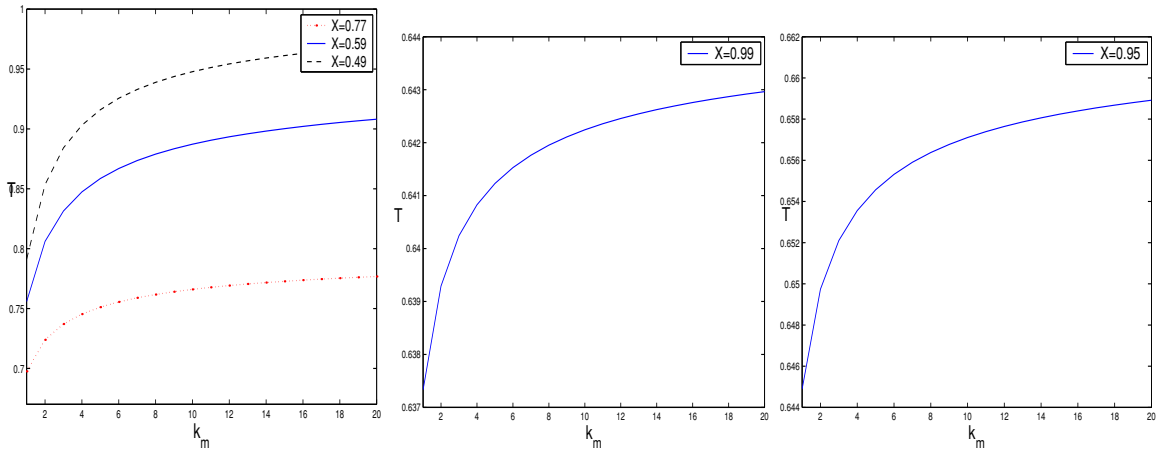

Figure 5: Matching efficiency

The item 1 in the proposition shows that the concentration of middlemen's market can mitigate the search frictions in the retail market: few middlemen, each with large scale, lead to a larger total number of trades than many middlemen, each with small scale. Figure 5 plots the relationship between the total matching rate T and the selling capacity of middlemen k_m, for given fixed values of $G = Mk_m$. However, the item 2 suggests that the concentration of middlemen's markets does not necessarily maximize the net welfare: W increases with k_m if the concentration of middlemen's market is initially low, but otherwise can deteriorate when the total demand-supply ratio X is relatively high. Figure 6 plots the relationship between W and k_m for given values of X. The reason of the welfare decrease is that when X is high, the demand effect of a larger capacity to increase x_m is large, hence the size of

20

price increase is large enough to offset the buyers' benefit of the improved service rate. All in all, the result points to the tension between the matching efficiency (measured in terms of the total number of trades T) and the welfare, generated by the concentration of middlemen's market.
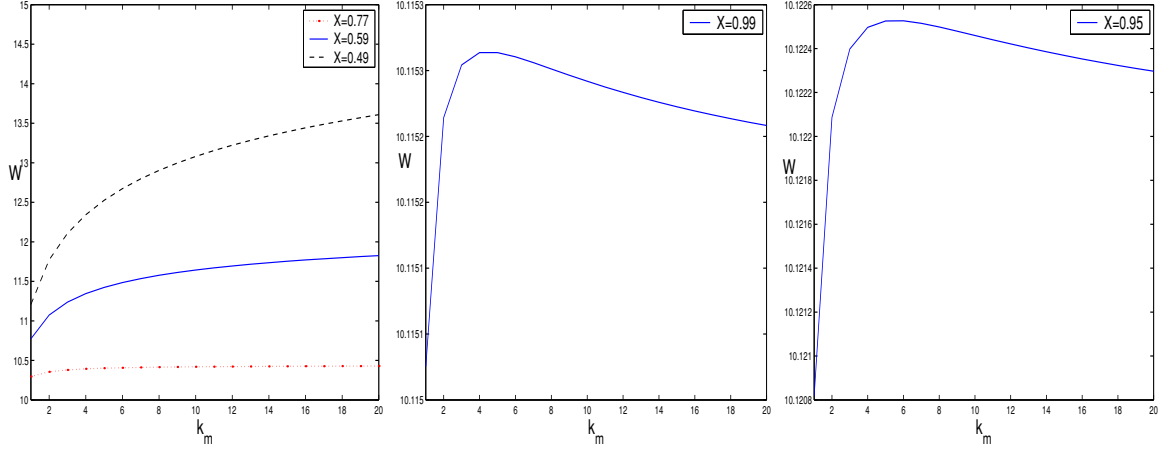

Figure 6: Total welfare

6   Conclusion

This paper proposed a simple theory of middlemen using a standard directed search approach. It offers wide applicability and economic insights into many empirically relevant forms of middlemen. Middlemen's inventories can provide buyers with immediacy service under market frictions, thereby the retail price of middlemen includes a premium to buyers. The model generates two important effects of middlemen's inventories that serve as the critical determinant of the retail premium. On the one hand, middlemen can attract more buyers with a larger selling capacity, which allows them to charge a higher price. On the other hand, it puts downward pressure on their retail price. These conflicting effects cause non-monotonic responses of the retail price/premium to changes in their inventories. The middlemen in this economy are efficiency enhancing. Interestingly, the concentration of middlemen's market introduces the tension between the matching efficiency and the welfare, especially when the total supply is relatively scarce.

Appendix

Proof of Theorem 1

The analysis in the main text has established that (1), (3), (4), (5), (6) and (7) describe necessary and sufficient conditions for an equilibrium, given that suppliers i holds k_i units at the start of each period. All that remains here is to establish a solution to these conditions, $x_s, x_m, p_s, p_m, V^b, V^s, V^m > 0$, exists and is unique. The proof takes 3 steps. Step 1 establishes a unique solution $x_s, x_m > 0$ for all $k_m \geq 1$, $S \in (0, \infty)$ and $M \in (0, \infty)$, using (1), (3), (4) and (5). With a slight abuse of notation, let $x_i(k_m, S, M)$ denote this solution for $i = s, m$. Given this solution, Step 2 then identifies a unique solution $V^j \in (0, \frac{1-c}{1-\beta})$ to (4), (5) and (6) for $j = b, s$. The rest of the equilibrium values are identified immediately: given x_i, (3) determines a unique $p_i \in (c, 1)$ for $i = s, m$; given x_m and p_m, (7) determines a unique $V^m \in (0, \frac{k_m(1-c)}{1-\beta})$. Hence, given the initial units k_i, $i = s, m$, for all $\beta \in [0, 1)$, $k_m \geq 1$, $S \in (0, \infty)$, $M \in (0, \infty)$, $c \in [0, \bar{c}]$, this solution then satisfies (1), (3), (4), (5), (6), and (7). Finally, Step 3 shows that in any period, sellers produce a unit for the retail market given $c \leq \bar{c}$, some $\bar{c} \in (0, \beta)$. As middlemen restock their units in the Walrasian market each period, this implies all sellers hold $k_s = 1$ unit and all middlemen hold $k_m \geq 1$ units at the start of each period, and so the established solution is indeed a steady state equilibrium.

Step 1 For any $k_m \geq 1$, $S \in (0, \infty)$ and $M \in (0, \infty)$, a solution $x_i = x_i(k_m, S, M)$ to (1), (3), (4) and (5) exists and is unique for $i = s, m$ that is: continuous in $S, M, k_m \in \mathbf{R}_+$; strictly decreasing in S, M; strictly increasing (or decreasing) in k_m if $i = m$ (or if $i = s$) satisfying $x_s(1, \cdot) = x_m(1, \cdot) = 1/(S+M)$, $x_s(k_m, \cdot) \to 0$ and $x_m(k_m, \cdot) \to 1/M$ as $k_m \to \infty$.

Proof of Step 1. In the main text, it has been shown that (3), (4) and (5) imply (8). Substituting out x_m in (8) by using (1),

$$\frac{\Gamma\left(k_m, \frac{1-Sx_s}{M}\right)}{\Gamma(k_m)} = e^{-x_s} \tag{10}$$

where $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$ and $\Gamma(k, x) = \int_x^\infty t^{k-1}e^{-t}dt$. The L.H.S. of this equation, denoted by $\Phi(x_s, k_m, S, M)$, is continuous and strictly increasing in x_s and $k_m \in \mathbf{R}_+$, satisfying:

$$\Phi(x_s, \cdot) \to \frac{\Gamma\left(k_m, \frac{1}{M}\right)}{\Gamma(k_m)} < 1 \text{ as } x_s \to 0; \quad \Phi\left(\frac{1}{S+M}, \cdot\right) = \frac{\Gamma\left(k_m, \frac{1}{S+M}\right)}{\Gamma(k_m)} \geq e^{-\frac{1}{S+M}}$$

with equality only when $k_m = 1$;

$$\Phi(x_s, 1, \cdot) = e^{-\frac{1-Sx_s}{M}}; \quad \Phi(x_s, k_m, \cdot) \to 1 \text{ as } k_m \to \infty.$$

Similarly, $\Phi(\cdot)$ is continuous and strictly increasing in S, M for any $x_s \in (0, \frac{1}{S+M})$ and $k_m \geq 1$. It follows therefore that a unique solution $x_s = x_s(k_m, S, M) \in (0, \frac{1}{S+M}]$ exists that is: continuous and strictly decreasing in $k_m \in [1, \infty) \subset \mathbf{R}_+$ satisfying $x_s(1, \cdot) = \frac{1}{S+M}$ and $x_s(k_m, \cdot) \to 0$ as $k_m \to \infty$; continuous and strictly decreasing in S, M.

Applying this solution to (1), one can obtain a unique solution $x_m = x_m(k_m, S, M) \in [\frac{1}{S+M}, \frac{1}{M})$ that is: continuous and strictly decreasing in S and M; continuous and strictly increasing in $k_m \in [1, \infty) \subset \mathbf{R}_+$ satisfying $x_m(1, \cdot) = \frac{1}{S+M}$ and $x_m(k_m, \cdot) \to \frac{1}{M}$ as $k_m \to \infty$. This completes the proof of Step 1.

Step 2 Given $x_s \in (0, 1/(S+M)]$ established in Step 1, there exists a unique solution $V^j \in (0, 1)$, $j = b, s$, to (3), (4), and (6).

Proof of Step 2. (3), (4), and (6) imply V^b satisfies

$$V^b = \frac{e^{-x_s}(1-c)}{1-\beta}.$$

The R.H.S of this equation, denoted by $\Upsilon_b(x_s)$, is strictly decreasing in $x^s \in (0, \infty)$ and satisfies: $\Upsilon_b(\cdot) \to \frac{1-c}{1-\beta}$ as $x_s \to 0$; $\Upsilon(\cdot) \to 0$ as $x_s \to \infty$. As equilibrium implies $x_s \in (0, 1/(S+M)]$, there exists a unique $V^b \in (0, \frac{1-c}{1-\beta})$ that satisfies $V^b = \Upsilon_b(\cdot)$. (3), (4), and (6) also imply

$$V^s = \frac{(1 - e^{-x_s} - x_s e^{-x_s})(1-c)}{1-\beta}$$

and this time, the R.H.S. of this equation, denoted by $\Upsilon_s(x_s)$, is strictly increasing in $x^s \in (0, \infty)$ and satisfies: $\Upsilon_s(\cdot) \to 0$ as $x_s \to 0$; $\Upsilon_s(\cdot) \to \frac{1-c}{1-\beta}$ as $x_s \to \infty$, thereby there exists a unique solution $V^s \in (0, \frac{1-c}{1-\beta})$. This completes the proof of Step 2.

Step 3 Sellers produce a unit for the retail market if $c \leq \bar{c}$, some $\bar{c} \in (0, \beta)$.

Proof of Step 3. Observe that in any given second sub-period, sellers produce a unit for future sale if and only if

$$c \leq \beta \left(x_s \eta(x_s, 1)p_s + (1 - x_s \eta(x_s, 1))c \right)$$

where the R.H.S. represents the expected discounted value of production: the first term in the parenthesis is the expected revenue and the second term is the net value of the produced unit in the next second sub-period – it can be sold to a middleman, which generates c or can be used for saving the production cost c for the next retail sale. Hence, sellers produce if and only if

$$c \leq \frac{1 - e^{-x_s} - x_s e^{-x_s}}{1 - \beta e^{-x_s}(1 + x_s)} \equiv \bar{c} \in (0, \beta).$$

This completes the proof of Step 3. ∎

Proof of Proposition 1

Differentiation yields

$$\frac{dp_i}{dS} = \frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i} \frac{dx_i}{dS} (1-c),$$

for $i = s, m$. Remember that $\frac{dx_i}{dS} < 0$, $i = s, m$ (see the proof of Step 2 in Theorem 1). Below, I show that $\frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i} > 0$ for all the possible values of x_i. There are three cases.

⊙ **Case 1.** $x_i < k_i$: Observe that

$$\frac{\partial \varphi^i(x_m, \cdot)}{\partial x_i} = -\frac{\frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i+1, x_i)}{\Gamma(k_i+1)}\right)}{\eta(\cdot)^2} \frac{\partial \eta(\cdot)}{\partial x_i} + \frac{\frac{\partial}{\partial x_i}\left[\frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i+1, x_i)}{\Gamma(k_i+1)}\right)\right]}{\eta(\cdot)}. \tag{11}$$

The first term in the R.H.S. of (11) is positive. The numerator of the second term in (11) is

$$\frac{\partial}{\partial x_i}\left[\frac{k_i}{x_i}\left(1 - \frac{\Gamma(k_i+1, x_i)}{\Gamma(k_i+1)}\right)\right] = \frac{\partial}{\partial x_i}\left[\sum_{j=k_i}^{\infty} \frac{x_i^j e^{-x_i}}{j!} \frac{k_i}{j+1}\right] = \sum_{j=k_i}^{\infty} \frac{x_i^{j-1} e^{-x_i}(j - x_i)}{j!} \frac{k_i}{j+1} > 0$$

if $x_i < k_i$.

⊙ **Case 2.** $x_i \geq k_i$ **and** $\frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} \leq \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)}$**:**   Rewrite (11) as

$$x_i \eta(\cdot)^2 \frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i} = \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)} \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} + \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)}\right) \left(\frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)} - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}\right).$$

If $\frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} \leq \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)}$, then the second term above is positive and so $\frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i} > 0$.

⊙ **Case 3.** $x_i \geq k_i$ **and** $\frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} > \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)}$**:**   Using $\frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)} = \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} - \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i + 1)}$, (11) can be further rearranged to $x_i \eta(\cdot)^2 \frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i}$

$$= \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)} \eta(x_i, k_i) - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}\right) + \frac{x_i^{k_i - 1} e^{-x_i}}{\Gamma(k_i)} \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}$$

$$= \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)} \left[\left(1 + \frac{1}{x_i}\right) \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} + \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)}\right)\right] - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}\right)$$

$$> \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)} - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} \left(1 - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}\right), \tag{12}$$

where the last inequality is because it holds that

$$\frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} \left(1 - \frac{k_i}{x_i}\right) \left(1 - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}\right) > \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)} \left[1 - \left(1 + \frac{1}{x_i}\right) \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} - \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)}\right)\right]$$

$$\Longleftarrow \quad \left(1 - \frac{k_i}{x_i}\right) \left(1 - \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)}\right) > 1 - \left(1 + \frac{1}{x_i}\right) \frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} - \frac{k_i}{x_i} \left(1 - \frac{\Gamma(k_i + 1, x_i)}{\Gamma(k_i + 1)}\right)$$

$$\Longleftrightarrow \quad \frac{1}{x_i} \left(\frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} - \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)}\right) > 0$$

for $\frac{\Gamma(k_i, x_i)}{\Gamma(k_i)} > \frac{x_i^{k_i} e^{-x_i}}{\Gamma(k_i)}$. Now, define

$$\Phi_g(x, k) \equiv \frac{x^k e^{-x}}{\Gamma(k)} - \frac{\Gamma(k, x)}{\Gamma(k)} \left(1 - \frac{\Gamma(k, x)}{\Gamma(k)}\right)$$

for $x \geq k \in [1, \infty) \subseteq \mathbf{R}_+$. Observe that $\lim_{x \to \infty} \Phi_g(x, k) = 0$, and

$$\frac{\partial \Phi_g(x, k)}{\partial x} = \frac{x^{k-1} e^{-x}}{\Gamma(k)} \left(k + 1 - x - 2 \frac{\Gamma(k, x)}{\Gamma(k)}\right) \gtreqless 0 \iff x \lesseqgtr x^+$$

where $x^+ \in (k, k + 1)$ is a unique solution to $x^+ = k + 1 - 2 \frac{\Gamma(k, x^+)}{\Gamma(k)}$, hence $\frac{\partial \Phi_g(x, k)}{\partial x} > 0$ at $x = k$. Therefore, if $\Phi_g(k, k) > 0$ then $\Phi_g(x, k) > 0$ for all $x \in [k, \infty)$. To show this corner condition $\Phi_g(k, k) > 0$ holds true, notice first that

$$\Phi_g(k, k) > \frac{k^k e^{-k}}{\Gamma(k)} - \frac{1}{4}$$

holds true for any $k \in [1, \infty)$. Now, observe that

$$\frac{d}{dk} \ln\left(\frac{k^k e^{-k}}{\Gamma(k)}\right) = \ln(k) - \psi(k),$$

24

where $\psi(k) = \frac{d \ln \Gamma(k)}{dk}$ is the Psi (or digamma) function, which has the definite-integral representation that leads to

$$\psi(k) = \int_0^\infty \left(e^{-t} - \frac{1}{(1+t)^k} \right) \frac{dt}{t}$$

$$= \ln k - \frac{1}{2k} - 2 \int_0^\infty \frac{t \, dt}{(t^2 + k^2)(e^{2\pi t} - 1)}$$

(see, for example, Abramowitz and Stegun (1964) p.259). The last expression leads to

$$\frac{d}{dk} \ln \left(\frac{k^k e^{-k}}{\Gamma(k)} \right) = \frac{1}{2k} + 2 \int_0^\infty \frac{t \, dt}{(t^2 + k^2)(e^{2\pi t} - 1)} > 0,$$

for all $k \in [1, \infty)$. Since $\frac{k^k e^{-k}}{\Gamma(k)} = e^{-1}$ ($\simeq 0.37 > \frac{1}{4}$) when $k = 1$, this implies that the term $\frac{k^k e^{-k}}{\Gamma(k)}$ is greater than $\frac{1}{4}$. This further implies $\Phi_g(k, k) > 0$ for all $k \in [1, \infty)$ and $\Phi_g(x, k) > 0$ for all $x \in [k, \infty)$. This shows that the R.H.S. of (12) is positive and so $\frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i} > 0$.

The above covers all the possible cases and, therefore, it has been shown that $\frac{\partial \varphi^i(x_i, \cdot)}{\partial x_i} > 0$, for all $x_i \in (0, \infty)$, $i = s, m$. The result on parameter M follows from exactly the same procedure. ∎

Proof of Proposition 2

As given in the test, the retail price differential is

$$(1-c)^{-1}(p_m - p_s) = \varphi^m(\cdot) - \varphi^s(\cdot) = \frac{\frac{k_m}{x_m}\left(1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}\right)}{\eta(x_m, k_m)} - \frac{\frac{1 - e^{-x_s} - x_s e^{-x_s}}{x_s}}{\eta(x_s, 1)}.$$

From this, it follows that $\eta(x_m, k_m)\eta(x_s, 1)(\varphi^m - \varphi^s)$

$$= \frac{1 - e^{-x_s}}{x_s} \frac{k_m}{x_m}\left(1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}\right) - \left\{ \frac{\Gamma(k_m), x_m}{\Gamma(k_m)} + \frac{k_m}{x_m}\left(1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}\right) \right\} \frac{1 - e^{-x_s} - x_s e^{-x_s}}{x_s}$$

$$= \left[\frac{k_m}{x_m}\left(1 - \frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}\right) - \frac{1 - e^{-x_s} - x_s e^{-x_s}}{x_s} \right] e^{-x_s}$$

$$= \left[\frac{(1 - e^{-x_s})(k_m x_s - x_m)}{x_m x_s} + \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} - \frac{x_m^{k_m-1} e^{-x_m}}{\Gamma(k_m)} \right] e^{-x_s}$$

$$= \left[(1 - e^{-x_s})((Mk_m + S)x_s - 1) + \frac{\Gamma(k_m - 1, \frac{1-Sx_s}{M})}{\Gamma(k_m - 1)} x_s(1 - Sx_s) \right] \frac{e^{-x_s}}{(1 - Sx_s)x_s}$$

where I have used (8) for the second equality, (8) and $\frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)} = \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} + \frac{x_m^{k_m} e^{-x_m}}{\Gamma(k_m+1)}$ for the third equality, and (1) and $\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} = \frac{\Gamma(k_m-1, x_m)}{\Gamma(k_m-1)} + \frac{x_m^{k_m-1} e^{-x_m}}{\Gamma(k_m)}$ for the last equality. Define $\Lambda_x(x_s)$ as the parenthesis terms in the last expression above for $x_s \in (0, \frac{1}{S})$ and $k_m > 1$. Then, it satisfies $\Lambda_x(x_s) \to 0$ as $x_s \to 0$, $\Lambda_x(x_s) \to (1 - e^{-\frac{1}{S}})\frac{Mk_m}{S} > 0$ as $x_s \to \frac{1}{S}$, and

$$\frac{d\Lambda_x(x_s)}{dx_s} = (Mk_m + S)(1 - e^{-x_s} + x_s e^{-x_s}) - \frac{\Gamma(k_m - 1, \frac{1-Sx_s}{M})}{\Gamma(k_m - 1)}(2Sx_s - 1) + \frac{x_m^{k_m-2} e^{-x_m}}{\Gamma(k-1)} \frac{Sx_s(1 - Sx_s)}{M}$$

$$= Mk_m(1 - e^{-x_s} + x_s e^{-x_s}) + S(1 - e^{-x_s} + kx_s e^{-x_s}) - \frac{\Gamma(k_m - 1, \frac{1-Sx_s}{M})}{\Gamma(k_m - 1)}(Sx_s(k_m + 1) - 1)$$

$$> Mk_m(1 - e^{-x_s} + x_s e^{-x_s}) + S(1 - e^{-x_s} + kx_s e^{-x_s}) - e^{-x_s}(Sx_s(k_m + 1) - 1)$$

$$= Mk_m(1 - e^{-x_s} + x_s e^{-x_s}) + S(1 - e^{-x_s} - x_s e^{-x_s}) + e^{-x_s}$$

$$> 0,$$

where I have used $\frac{\Gamma(k_m-1,x_m)}{\Gamma(k_m-1)} < \frac{\Gamma(k_m,x_m)}{\Gamma(k_m)}$ $(= e^{-x_s}$ by (8)) for the second inequality. This implies $\Lambda_x(x_s) > 0$ for all $x \in (0, \frac{1}{S})$ given $k_m > 1$. Hence, $\varphi^m - \varphi^s > 0$ and so $p_m - p_s > 0$ for all $k_m > 1$ and $S, M \in (0, \infty)$. Since $p_m - p_s = 0$ when $k_m = 1$, this proves the claims in the proposition. ∎

Proof of Proposition 3

⊙ **Retail price of middlemen p_m:**   For the expositional ease, let

$$\nabla_1 \equiv \frac{x_m}{k_m} \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)}; \quad \nabla_2 \equiv 1 - \frac{\Gamma(k_m + 1, x_m)}{\Gamma(k_m + 1)}.$$

Differentiating p_m $(= \varphi^m(x_m, k_m))$ with respect to $k_m \in [1, \infty) \subset \mathbf{R}_+$,

$$(\nabla_1 + \nabla_2)^2 \frac{d\varphi^m(x_m, k_m)}{dk_m}$$

$$= (\nabla_1 + \nabla_2)^2 \frac{d}{dk_m} \left(\frac{\nabla_2}{\nabla_1 + \nabla_2} \right)$$

$$= -\nabla_1 \frac{\partial \frac{\Gamma(k_m+1,x_m)}{\Gamma(k_m+1)}}{\partial k_m} + \nabla_2 \left(\frac{\nabla_1}{k_m} - \frac{x_m}{k_m} \frac{\partial \frac{\Gamma(k_m,x_m)}{\Gamma(k_m)}}{\partial k_m} \right) + \frac{dx_m}{dk_m} \left((\nabla_1 + \nabla_2) \frac{x_m^{k_m} e^{-x_m}}{\Gamma(k_m+1)} - \frac{\nabla_1 \nabla_2}{x_m} \right).$$

In Step 1 in the proof of Theorem 1, it has been shown that

$$\frac{dx_m}{dk_m} = \frac{\frac{\partial(\Gamma(k_m,x_m)/\Gamma(k_m))}{\partial k_m}}{\frac{x_m^{k_m-1} e^{-x_m}}{\Gamma(k_m)} + Me^{-x_s}},$$

where, as already mentioned in the text, I used here the normalization $S = 1$.

I now evaluate the above derivatives at $k_m = 1$. Let $x \equiv x_m = x_s = 1/(M+1) \in (0,1)$ at $k_m = 1$. Observe that

$$\frac{\partial(\Gamma(k_m,x)/\Gamma(k_m))}{\partial k_m}\Big|_{k_m=1} = \frac{\partial\Gamma(k_m,x)/\partial k_m}{\Gamma(k_m)}\Big|_{k_m=1} - \frac{\Gamma(k_m,x)}{\Gamma(k_m)} \frac{\partial\Gamma(k_m)/\partial k_m}{\Gamma(k_m)}\Big|_{k_m=1}$$

$$= e^{-x}\ln x + E_1(x) + e^{-x}\gamma,$$

where in the second equality I have used:

$$\frac{\partial\Gamma(k_m,x)/\partial k_m}{\Gamma(k_m)}\Big|_{k_m=1} = \frac{\partial\Gamma(k_m,x)}{\partial k_m}\Big|_{k_m=1} = e^{-x}\ln x + E_1(x); \quad \frac{\partial\Gamma(k_m)/\partial k_m}{\Gamma(k_m)}\Big|_{k_m=1} = -\gamma$$

(see Geddes, Glasser, Moore, and Scott (1990) for the former, and Abramowitz and Stegun (1964) p.228 for the latter, for example), where

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$$

is the exponential integral and γ $(= 0.5772..)$ is the Euler-Mascheroni constant. Similarly, observe that

$$\frac{\partial(\Gamma(k_m+1,x)/\Gamma(k_m+1))}{\partial k_m}\Big|_{k_m=1} = \frac{\partial}{\partial k_m} \left(\frac{\Gamma(k_m,x)}{\Gamma(k_m)} + \frac{x^{k_m} e^{-x}}{\Gamma(k_m+1)} \right)\Big|_{k_m=1}$$

$$= e^{-x}(1+x)(\ln x + \gamma) - xe^{-x} + E_1(x).$$

26

Applying these derivative expressions, and noting $\nabla_1 = xe^{-x}$ and $\nabla_2 = 1 - e^{-x} - xe^{-x}$ when $k_m = 1$, one obtains

$$(\nabla_1 + \nabla_2)^2 \frac{d\varphi^m(x, k_m)}{dk_m}\Big|_{k_m = 1}$$

$$= -xe^{-x}\left(E_1(x)(e^x - x) - 1 + e^{-x} + \ln x + \gamma\right) + \frac{x - 1 + e^{-x}}{1 + M}\left(E_1(x) + e^{-x}(\ln x + \gamma)\right).$$

In the above expression, the terms in the first parenthesis, denoted by $\Theta_1(x) \equiv E_1(x)(e^x - x) - 1 + e^{-x} + \ln x + \gamma$, satisfy:

$$\lim_{x \to 0} \Theta_1(x) = \lim_{x \to 0}(E_1(x) + \ln x) + \gamma = \lim_{x \to 0} E_{in}(x) = 0,$$

where I used $\lim_{x \to 0} E_1(x)x = \lim_{x \to 0} xe^{-x} = 0$ (by the l'Hospital rule) in the first equality, and $E_1(x) = -\gamma - \ln x + E_{in}(x)$ in the second equality, where

$$E_{in}(x) = \int_0^x (1 - e^{-t})\frac{dt}{t}$$

is the entire function (see footnote 3, p.228 in Abramowitz and Stegun (1964));

$$\frac{d\Theta_1(x)}{dx} = E_1(x)(e^x - 1) > 0.$$

Hence, $\Theta_1 > 0$ for all $x \in (0, 1]$. The terms in the second bracket, denoted by $\Theta_2(x) \equiv E_1(x) + e^{-x}(\ln x + \gamma)$, satisfy:

$$\lim_{x \to 0} \Theta_2(x) = \lim_{x \to 0}(E_1(x) + \ln x) + \gamma = \lim_{x \to 0} E_{in}(x) = 0; \quad \Theta_2(1) = E_1(1) + e^{-1}\gamma > 1;$$

$$\frac{d\Theta_2(x)}{dx} = -e^{-x}(\ln x + \gamma) \gtreqless 0 \Longleftrightarrow x \lesseqgtr e^{-\gamma}; \quad \Theta_2(e^{-\gamma}) = E_1(e^{-\gamma}) > 0.$$

Hence, $\Theta_2(x)$ achieves the unique minimum at $x = 0$ within $x \in [0, 1]$, which equals to zero, thereby $\Theta_2(x) > 0$ for all $x \in (0, 1]$.

Now, since $\Theta_1(x) > 0$, $\Theta_2(x) > 0$ for all $x \in (0, 1]$, the condition of price increase is given by $\frac{d\varphi^m(x, k_m)}{dk_m}\Big|_{k_m = 1} > 0 \Longleftrightarrow$

$$M < \frac{(x - 1 + e^{-x})\Theta_2(x) - xe^{-x}\Theta_1(x)}{xe^{-x}\Theta_1(x)}. \tag{13}$$

In what follows, I identify the values of x ($= 1/(M + 1) \in (0, 1)$) (and hence $M \in (0, \infty)$) that satisfy the condition of price increase (13). For this purpose, define

$$\Omega(x) \equiv (x - 1 + e^{-x})\Theta_2(x) - e^{-x}\Theta_1(x).$$

Note the inequality (13) holds true if and only if $\Omega(x) > 0$. $\Omega(\cdot)$ satisfies: $\lim_{x \to 0} \Omega(x) = 0$;

$$\Omega(1) = e^{-1}(\Theta_2(1) - \Theta_1(1)) = e^{-1}\left[-E_1(1)(e^1 - 2) + (1 - e^{-1})(1 - \gamma)\right] > 0$$

since $E_1(1)(e^1 - 2) \simeq 0.22 * 0.72 \simeq 0.16 < 0.27 \simeq (1 - e^{-1})(1 - \gamma)$;

$$\frac{d\Omega(x)}{dx} = e^{-x}\Theta_1(x) + \left(2(1 - e^{-x}) - x\right)e^{-x}(\ln x + \gamma).$$

From the last expression, it follows that $\lim_{x \to 0} \frac{d\Omega(x)}{dx} = \lim_{x \to 0}(2(1 - e^{-x}) - x)\ln x = 0$ (by using the l'Hospital's rule twice) and $\frac{d\Omega(x)}{dx} > 0$ for $x > e^{-\gamma}$. To identify the sign of the derivative for $x \le e^{-\gamma}$,

suppose that $\Omega(x) \geq 0$ for $x \in (0, e^{-\gamma}]$. Then, it has to hold that $e^{-x}\Theta_1(x) \leq (x - 1 + e^{-x})\Theta_2(x)$, which further implies

$$
\begin{aligned}
\frac{d\Omega(x)}{dx} &\leq (x - 1 + e^{-x})\Theta_2(x) + \left(2(1 - e^{-x}) - x\right)e^{-x}(\ln x + \gamma) \\
&= (x - 1 + e^{-x})E_1(x) + (1 - e^{-x})e^{-x}(\ln x + \gamma) \equiv \Upsilon(x)
\end{aligned}
$$

for $x \in (0, e^{-\gamma}]$. Observe that $\lim_{x \to 0} \Upsilon(x) = 0$ (by using the l'Hospital's rule thrice on the first term and twice on the second term) and $\Upsilon(e^{-\gamma}) > 0$. Further,

$$
\frac{d\Upsilon(x)}{dx} = (1 - e^{-x})\Theta_2(x) - (2 - 3e^{-x})e^{-x}(\ln x + \gamma) + \frac{e^{-x}(2(1 - e^{-x}) - x)}{x} \rightarrow -\infty < 0
$$

as $x \to 0$. This implies there exists some $x' \in (0, e^{-\gamma})$ such that $\Upsilon(x') = 0$ and $\Upsilon(x) < 0$ for $x < x'$. The latter further implies $\frac{d\Omega(x)}{dx} < 0$ for $x < x'$, a contradiction to $\frac{d\Omega(x)}{dx} \geq 0$ (which is implied by $\Omega(x) \geq 0$ and $\lim_{x \to 0} \Omega(x) = 0$ for an interval of x close to 0). Hence, we must have $\Omega(x) < 0$ for an interval x close to zero. As $\Omega(x)$ is continuous in $x \in (0,1)$ and $\Omega(1) > 0$, this implies that there exists some $x^* \in (0,1)$ such that $\Omega(x^*) = 0$ and $\Omega(x) < 0$ for $x \in (0, x^*)$.

Observe now that $\Omega(e^{-\gamma}) = -(2 - x - e^{-x} - xe^{-x})E_1(x) + e^{-x}(1 - e^{-x}) \mid_{x = e^{-\gamma} \simeq 0.56} \simeq -0.55 * 0.49 + 0.25 < 0$. This implies, since $\Omega(x)$ is increasing in $x \in (e^{-\gamma}, 1)$, it has to be that $x^* \in (e^{-\gamma}, 1)$. This further implies that $\Omega(x)$ must cross the horizontal axis (of $\Omega(\cdot) = 0$) from below and only once at $x^* \in (e^{-\gamma}, 1)$. As $\lim_{x \to 0} \Omega(x) = 0 < \Omega(1)$, it should hold that

$$
\Omega(x) \leq 0 \quad \text{for} \quad x \leq x^* \in (0,1) \quad \text{and} \quad \Omega(x) > 0 \quad \text{for} \quad x > x^*.
$$

Therefore, the condition of price increase (13) holds true if and only if $x \in (x^*, 1)$, and since $x = X$ when $k_m = 1$, this proves the first claim in the proposition with $x^* = X^* \in (0,1)$.

To prove the second claim, it is sufficient to observe that since $x_m \to 1/M$, $x_m\eta(x_m, k_m) \to 1/M$, $k_m\nabla_2 \to 0$ as $k_m \to \infty$, it holds that $\varphi^m(x_m, k_m) \to 0$ as $k_m \to \infty$. ∎

⊙ **Retail price of sellers p_s:** It is sufficient to observe that $x_s(\cdot)$ is strictly decreasing in all $k_m \geq 1$ (as shown in Step 1 in the proof of Theorem 1) and $p_s (= \varphi^s(\cdot))$ is strictly increasing in all $x_s \in (0,1)$ (as shown in the proof of Proposition 1). In the limit as $k_m \to \infty$, we have $x_s \to 0$ and so $\varphi^s(x_s, 1) \to 0$. ∎

⊙ **Retail market premium $p_m - p_s$:** The above analysis shows that $p_m \to c$ as $k_m \to \infty$ and $p_s \to c$ as $k_m \to \infty$. Hence, $p_m - p_s \to 0$ as $k_m \to \infty$. Since $p_m = p_s$ when $k_m = 1$ and $p_m > p_s$ when $1 < k_m < \infty$, this implies the price differential must be increasing (decreasing) in low (high) k_m. ∎

Proof of Proposition 4

⊙ **Retail price of middlemen p_m:** With the fixed total supply of middlemen $G = Mk_m$, the only modification appears in the adding-up restriction (1), which now becomes (with normalization $S = 1$)

$$
\frac{G}{k_m}x_m + x_s = 1.
$$

This affects the analysis in Step 1 in the proof of Theorem 1, so that now I have

$$
\frac{dx_m}{dk_m} = \frac{\frac{\partial(\Gamma(k_m, x_m)/\Gamma(k_m))}{\partial k_m} + \frac{Gx_m}{k_m^2}e^{-x_s}}{\frac{x_m^{k_m-1}e^{-x_m}}{\Gamma(k_m)} + \frac{G}{k_m}e^{-x_s}}.
$$

Observe that there is an additional, positive term in the numerator of this expression. This modification further affects the following parts of the analysis: the derivative in question becomes

$$(\nabla_1 + \nabla_2)^2 \frac{d\varphi^m(x, k_m)}{dk_m}\Big|_{k_m=1 \ \& \ G=Mk_m}$$

$$= -xe^{-x}\left(E_1(x)(e^x - x) - 1 + e^{-x} + \ln x + \gamma\right) + \frac{x - 1 + e^{-x}}{1 + G}\left(E_1(x) + e^{-x}(\ln x + \gamma) + Gxe^{-x}\right),$$

where a positive term is added inside the second bracket; the condition for price increase (13) is then modified to $\frac{d\varphi^m(x, k_m)}{dk_m}\Big|_{k_m=1 \ \& \ G=Mk_m} > 0 \iff$

$$G\left(1 - \frac{x - (1 - e^{-x})}{\Theta_1(x)}\right) < \frac{(x - 1 + e^{-x})\Theta_2(x) - xe^{-x}\Theta_1(x)}{xe^{-x}\Theta_1(x)}.$$

Observe here that the R.H.S. remains the same as before, while the L.H.S. is now multiplies by a new term which is less than one. As $G = M$ when $k_m = 1$, this implies that the above inequality holds for all $x = X > x^* = X^* \in (0, 1)$ (see the proof of Proposition 3) and so $\frac{d\varphi^m(x, k_m)}{dk_m}\Big|_{k_m=1 \ \& \ G=Mk_m} > 0$ for all $x \in (x^*, 1)$. This proves the first claim for p_m in the proposition.

The second claim can be shown by using the following property (see Temme (1996) p.285):

$$\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to D \quad \text{as} \quad k_m \to \infty \tag{14}$$

where $D \in [0, 1]$ satisfies: $D = 1$ if and only if $x_m < k_m$; $D = 0$ if and only if $x_m > k_m$.

Throughout the proof given below, keep in mind that with the fixed total supply $G = Mk_m \in (0, \infty)$, it has to be that $M = G/k_m \to 0$ as $k_m \to \infty$, thus $x_m \to \infty$ as $k_m \to \infty$. There are three cases. Consider first the case $G < 1$. Suppose $x_m > k_m$ as $k_m \to \infty$. This leads to $\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to 0$ as $k_m \to \infty$ by (14) and so $x_s \to \infty$ as $k_m \to \infty$ by (8). However, this contradicts to (1) which requires $x_s \in [0, 1]$. Suppose $x_m < k_m$ as $k_m \to \infty$. Then, $\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to 1$ as $k_m \to \infty$ by (14) and so $x_s \to 0$ as $k_m \to \infty$ by (8). However, this contradicts to (1) and $G < 1$, or

$$M(x_m - k_m) + x_s = 1 - G > 0$$

which requires $x_s > 0$, if $x_m < k_m$. Therefore, the only possible solution when $G < 1$ is $x_m = k_m$ as $k_m \to \infty$, which in turn leads to $x_s = 1 - G$ by (1), as is consistent with (14), requiring $\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} = e^{-x_s} \in (0, 1)$ as $k_m \to \infty$ and $x_m = k_m$. In this solution, it holds that:

$$\eta(x_m, k_m) = \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} + \frac{k_m}{x_m}\left(1 - \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)}\right) - \frac{x^{k_m-1}e^{-x}}{\Gamma(k_m)} \to 1 \quad \text{as} \quad k_m \to \infty$$

because $\frac{x^{k_m-1}e^{-x}}{\Gamma(k_m)} \to 0$ as $k_m \to \infty$ for any $\frac{x_m}{k_m} \in (0, \infty)$;

$$\varphi^m(x_m, k_m) \to 1 - e^{-(1-G)} \quad \text{as} \quad k_m \to \infty.$$

Consider next the case $G = 1$. Suppose $x_m < k_m$ as $k_m \to \infty$. Then, $\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to 1$ as $k_m \to \infty$ by (14) and so $x_s \to 0$ as $k_m \to \infty$ by (8). However, this contradicts to (1) and $G = 1$, or

$$M(x_m - k_m) + x_s = 1 - G = 0$$

which requires $x_s > 0$, if $x_m < k_m$. Similarly, $x_m \geq k_m$ as $k_m \to \infty$ cannot be the solution. Therefore, there is no limiting solution as $k_m \to \infty$ with the fixed total supply $G = Mk_m$ when $G = 1$.

29

Consider finally the case $G > 1$. Then, by (1),

$$M(x_m - k_m) + x_s = 1 - G < 0,$$

implying that $x_m < k_m$ as $k_m \to \infty$, leading to $\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to 1$ and $x_s \to 0$ by (8) and (14), is the only solution. Therefore, $1 - \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} \to 0$ as $k_m \to \infty$, which implies $\varphi^m(\cdot) \to 0$ as $k_m \to \infty$ and thus $\varphi^m(\cdot) > \lim_{k_m \to \infty} \varphi^m(\cdot)$ for all $k_m \geq 1$.

Therefore, it has to hold that $p_m > c$ as $k_m \to \infty$ if and only if $G < 1$ or $X = \frac{1}{G+1} > X^{**} = \frac{1}{2}$. ∎

⊙ **Retail market premium $p_m - p_s$:** With fixed $G = Mk_m$, we have

$$\frac{dx_s}{dk_m} = -\frac{\frac{\partial(\Gamma(k_m, x_m)/\Gamma(k_m))}{\partial k_m} - \frac{x_m^{k_m - 1} e^{-x_m}}{\Gamma(k_m)} \frac{1 - x_s}{G}}{\frac{x_m^{k_m - 1} e^{-x_m}}{\Gamma(k_m)} \frac{k_m}{G} + e^{-x_s}} = -\frac{E_1(x) + e^{-x}(\ln x + \gamma) - e^{-x} \frac{1-x}{M}}{e^x(1 + \frac{1}{M})} \Big|_{k_m = 1 \ \& \ G = Mk_m},$$

and

$$(\nabla_1 + \nabla_2)^2 \left(\frac{d\varphi^m(x, k_m)}{dk_m} - \frac{d\varphi^s(x, 1)}{dk_m} \right)_{\Big|_{k_m = 1 \ \& \ G = Mk_m}} = -xe^{-x}\Theta_1(x) + (x - 1 + e^{-x})\Theta_2(x) - (1 - x)xe^{-x}$$

where, as before, $\Theta_1(x) \equiv E_1(x)(e^x - x) - 1 + e^{-x} + \ln x + \gamma > 0$ and $\Theta_2(x) \equiv E_1(x) + e^{-x}(\ln x + \gamma) > 0$. Define the L.H.S. of the above derivative as

$$\Omega_f(x) \equiv E_1(x)(-(1 - e^{-x}) + x^2 e^{-x}) + xe^{-x}(x - e^{-x}) - (1 - e^{-x})e^{-x}(\ln x + \gamma).$$

$\Omega_f(\cdot)$ satisfies: $\lim_{x \to 0} \Omega_f(x) = 0$; $\Omega_f(1) = -E_1(1)(1 - 2e^{-1}) + e^{-1}(1 - e^{-1})(1 - \gamma) = -0.22 * 0.26 + 0.23 * 0.43 > 0$;

$$\frac{d\Omega_f(x)}{dx} = -E_1(x)e^{-x}(1 - x)^2 + e^{-x}(x(2 - x + e^{-x}) - e^{-x}) + (1 - 2e^{-x})e^{-x}(\ln x + \gamma).$$

Note $\frac{d\Omega_f(x)}{dx} = e^{-1} + (1 - 2e^{-1})\gamma > 0$ at $x = 1$.

Now, suppose that there exists some interval of x in the neighborhood of $x = 0$ such that $\Omega_f(x) \leq 0$. Then, within that interval of x, we must have

$$e^{-x}(\ln x + \gamma) \geq \frac{1}{1 - e^x} \left[xe^{-x} - \frac{1 - e^{-x} - x^2 e - x}{1 - e^{-x}} E_1(x) \right].$$

Applying this inequality to the expression of $\frac{d\Omega_f(x)}{dx}$, we have

$$\frac{d\Omega_f(x)}{dx} \geq \frac{1 - e^{-x} - xe^{-x}}{1 - e^{-x}} \left[e^{-x}(x - e^{-x}) + \frac{xe^{-x}(1 - e^{-x})}{1 - e^{-x} - xe^{-x}} - (1 - e^{-x} - xe^{-x})E_1(x) \right]$$

$$\equiv \frac{1 - e^{-x} - xe^{-x}}{1 - e^{-x}} \Upsilon_f(x).$$

Observe that $\Upsilon_f(x) \to 1 > 0$ as $x \to 0$. This implies that $\frac{d\Omega_f(x)}{dx} \geq 0$ for an interval close to zero, if $\Omega_f(x) \leq 0$. However, this is impossible since $\Omega_f(x) \to 0$ as $x \to 0$. Hence, we must have $\Omega_f(x) > 0$ in the interval close to $x = 0$.

This result further implies that if $\Omega_f(x) < 0$ in some interval of $x \in (0, 1)$ then, since $\Omega_f(1) > 0$, there must exist at least two points, denoted $x_- > x_+ \in (0, 1)$, such that $\frac{d\Omega_f(x)}{dx} = 0$ at $x = x_-, x_+$ and $\Omega_f(x_-) < 0 < \Omega_f(x_+)$. Keeping this in mind, suppose that there exists some $x_* \in (0, 1)$ such that $\frac{d\Omega_f(x)}{dx} = 0$ at $x = x_*$ (if not, then the claim holds true automatically). Then, we must have

$$\Omega_f(x_*) = \frac{1 - e^{-x_*} - x_* e^{-x_*}}{1 - 2e^{-x_*}} \left[-(1 - e^{-x_*} - x_* e^{-x_*})E_1(x_*) + e^{-x_*}(x_* - e^{-x_*} + \frac{x_*(1 - e^{-x_*})}{1 - e^{-x_*} - x_* e^{-x_*}}) \right].$$

Here, the terms in the parenthesis satisfy

$$
\begin{aligned}
f_f(x) &\equiv -(1 - e^{-x} - xe^{-x})E_1(x) + e^{-x}\left(x - e^{-x} + \frac{x(1 - e^{-x})}{1 - e^{-x} - xe^{-x}}\right) \\
&> -(1 - e^{-x} - xe^{-x})E_1(x) + e^{-x}\left(1 - e^{-x}\right) \equiv f_r(x) > 0
\end{aligned}
$$

for all $x \in (0, 1]$, where $f_r(x) > 0$ was introduced in the proof of Proposition 3. Hence, for all $x \in (0, 1]$, it holds that $f_f(x) > 0$, and the result obtained there applies: since $1 - e^{-x} - xe^{-x} > 0$ for all $x \in (0, 1]$ and $1 - 2e^{-x} > 0$ if and only if $x > \ln(2) \in (0, 1)$, we must have $\Omega_f(x_{*-}) < 0 < \Omega_f(x_{*+})$ for some $x_{*-} < \ln(2) < x_{*+}$ if $\frac{d\Omega_f(x)}{dx} = 0$ at $x = x_{*-}, x_{*+}$, and so it is impossible to have $x_- > x_+ \in (0, 1)$ satisfying $\frac{d\Omega_f(x)}{dx} = 0$ at $x = x_-, x_+$ and $\Omega_f(x_-) < 0 < \Omega_f(x_+)$.

All in all, the above covers all the possibilities of $\Omega_f(x) \leq 0$ for $x \in (0, 1]$, which turns out to be impossible, and so we must have $\Omega_f(x) > 0$ for all $x \in (0, 1]$. This proves the first claim.

The second claim follows from the result obtained in the proof of p_m above: when $G < 1$, we have

$$
\varphi^m(x_m, k_m) - \varphi^s(x_s, 1) \to \frac{e^{-x_s}(x_s - 1 + e^{-x_s})}{1 - e^{x_s}} > 0
$$

where $x_s \to 1 - G > 0$, as $k_m \to \infty$; when $G > 1$,

$$
\varphi^m(x_m, k_m) - \varphi^s(x_s, 1) \to 0.
$$

as $k_m \to \infty$. Therefore, $p_m - p_s > 0$ as $k_m \to \infty$ if and only if $G < 1$ or $X = \frac{1}{G+1} > X^{**} = \frac{1}{2}$. $\blacksquare$

Proof of Proposition 5

From the free entry condition (9), the fixed point condition for the equilibrium number of middlemen $M \in (0, \infty)$ is given by

$$
\Phi_m(M, \cdot) \equiv 1 - \frac{\Gamma(k_m + 1, x_m)}{\Gamma(k_m + 1)} = \frac{c(1 - \beta) + c_k}{1 - c} \equiv C \tag{15}
$$

where $x_m = x_m(M)$ is strictly decreasing in M and satisfies $x_m \to 0$ as $M \to \infty$, as shown in Step 1 in the proof of Proposition 1. It then follows that $\Phi_m = \Phi_m(M, \cdot)$ is continuous and strictly decreasing in $M \in (0, \infty)$ and satisfies $\Phi_m \to 0 < C$ as $M \to \infty$. Therefore, with $\bar{C} \equiv \lim_{M \to 0} \Phi_m \in (0, 1)$, there exists a unique $M \in (0, \infty)$ that satisfies (15) given $C \in (0, \bar{C})$. The comparative statistics of C is immediate: the equilibrium M is strictly decreasing in C satisfying $M \to \infty$ as $C \to 0$ and $M \to 0$ as $C \to \bar{C}$.

For the comparative statics of k_m, observe that

$$
\frac{d\Phi_m}{dk_m} = -\frac{\partial}{\partial k_m}\frac{\Gamma(k_m + 1, x_m)}{\Gamma(k_m + 1)} + \frac{x_m^{k_m} e^{-x_m}}{\Gamma(k_m + 1)}\frac{dx_m}{dk_m}.
$$

Evaluating this derivative at $k_m = 1$ using the expression of $\frac{\partial}{\partial k_m}\frac{\Gamma(k_m+1, x_m)}{\Gamma(k_m+1)}\big|_{k_m=1}$ and $\frac{dx_m}{dk_m}\big|_{k_m=1}$ derived in the proof of Proposition 3, we have

$$
\frac{d\Phi_m}{dk_m}\Big|_{k_m=1} = -(1 - x^2)E_1(x) + xe^{-x} - (1 + x - x^2)e^{-x}(\ln x + \gamma) \equiv \Omega_m(x).
$$

Observe that: $\lim_{x \to 0}\Omega_m(x) = -\lim_{x \to 0}(E_1(x) + e^{-x}(\ln(x) + \gamma)) = 0$; $\Omega_m(1) = e^{-1}(1 - \gamma) > 0$;

$$
\frac{d\Omega_m(x)}{dx} = x\left[2E_1(x) - e^{-x} + (3 - x)e^{-x}(\ln x + \gamma)\right] \equiv xf_m(x).
$$

31

Here, the terms in the parenthesis satisfy $f_m(x) \to -\infty < 0 < 2E_1(1) + e^{-1}(2\gamma - 1) = f_m(1)$, implying there exists some $x_{.*} \in (0,1)$ such that $f_m(x_{.*}) \geq 0$ if and only if $x \geq x_{.*}$. Since $\Omega_m(0) < 0 < \Omega_m(1)$, this implies $\Omega_m(x) \geq 0$ if and only if $x \geq x_{.*}$. As $\Phi_m(M)$ is monotone decreasing in M, the last result implies that M is decreasing in low k_m if $x = \frac{1}{M+1} < x_{.*}$ and is increasing in low k_m if $x = \frac{1}{M+1} > x_{.*}$. As M is strictly decreasing in c, c_k, this proves the claim in the proposition.

As for large k_m, notice that the L.H.S of the fixed point condition (15) should be a positive number less than one (given $C \in (0, \bar{C}) < 1$). This is the case if and only if $x_m = k_m$ and $x_s = 1 - Mk_m > 0$ as $k_m \to \infty$ (see the property (14) in the proof of Proposition 4). This is possible only when $M \to 0$ as $k_m \to \infty$. ∎

Proof of Proposition 6

Observe that the total number of trades satisfies

$$
\begin{aligned}
T - \underline{T} &= Mx_m\eta(x_m, k_m) + x_s\eta(x_x, 1) - \underline{S}\,\underline{x}_s\eta(\underline{x}_s, 1) \\
&= Mx_m(\eta(x_m, k_m) - \eta(x_s, 1)) + (\eta(x_s, 1) - \eta(\underline{x}_s, 1)),
\end{aligned}
$$

where the second equality is by (1) and $\underline{S}\,\underline{x}_s = \frac{1}{Mk_m+1}(Mk_m + 1) = 1$. For $k_m = 1$, it holds that $x_m = x_s = \underline{x}_s = \frac{1}{M+1}$, implying $\eta(x_m, k_m) = \eta(x_s, 1)$ and $\eta(x_s, 1) = \eta(\underline{x}_s, 1)$, thereby $T = \underline{T}$. For $k_m > 1$, $\eta(x_m, k_m) > \eta(x_s, 1)$ and $\eta(x_s, 1) > \eta(\underline{x}_s, 1)$, since $\eta(x_s, 1)$ is decreasing in x_s and $x_s < \underline{x}_s$ for $k_m > 1$, thereby $T > \underline{T}$.

Notice that the net welfare is strictly deceasing in all $x_s \in (0,1)$. Since $x_s = \underline{x}_s$ when $k_m = 1$ and $x_s < \underline{x}_s$ when $k_m > 1$, we must have $W = \underline{W}$ when $k_m = 1$ and $W > \underline{W}$ when $k_m > 1$. ∎

Proof of Proposition 7

To examine the effect of small k_m's on T given fixed supply $G = Mk_m$, observe that

$$
\frac{dT}{dk_m}_{|G=Mk_m} = Mx_m\left[\frac{\partial}{\partial k_m}\frac{\Gamma(k_m, x_m)}{\Gamma(k_m)} - \frac{k_m}{x_m}\frac{\partial}{\partial k_m}\frac{\Gamma(k_m + 1, x_m)}{\Gamma(k_m + 1)}\right].
$$

Evaluating this derivative at $k_m = 1$,

$$
\frac{dT}{dk_m}_{|G=Mk_m\ \&\ k_m=1} = M\left[e^{-x}(x - \ln x - \gamma) - (1 - x)E_1(x)\right].
$$

where $x \equiv x_m = \frac{1}{G+1} \in (0,1)$ at $k_m = 1$. Define the parenthesis terms above as $\Psi_T(x)$ for $x \in (0,1)$. It satisfies: $\lim_{x \to 0}\Psi_T(x) = \lim_{x \to 0} -(E_1(x) + \ln x + \gamma) = \lim_{x \to 0} -E_{in}(x) = 0$ (see the proof of Proposition 3); $\lim_{x \to 1}\Psi_T(x) = e^{-1}(1 - \gamma) > 0$;

$$
\frac{d\Psi_T(x)}{dx} = -e^{-x}(x - \ln x - \gamma) + E_1(x).
$$

In the last expression, observe that $\frac{d\Psi_T}{dx} \to 0$ as $x \to 0$ and $\frac{d\Psi_T}{dx} \to e^{-1}(\gamma - 1) + E_1(1) \simeq 0.37 * (0.58 - 1) + 0.22 \simeq 0.064 > 0$, and that $\frac{d^2\Psi_T}{dx^2} = -e^{-x}(1 - x + \ln x + \gamma)$, satisfying $\frac{d^2\Psi_T}{dx^2} \to +\infty > 0$ as $x \to 0$, $\frac{d^2\Psi_T}{dx^2} \to -e^{-1}\gamma < 0$ as $x \to 1$ and $\frac{d^2\Psi_T}{dx^2} = 0$ at some $\hat{x} \in (0,1)$ such that $\hat{x} = 1 + \ln\hat{x} + \gamma$. Then, $\frac{d\Psi_T}{dx} = -e^{-\hat{x}} + E_1(\hat{x}) > 0$ implies $\frac{d\Psi_T}{dx} > 0$ for all $x \in (0,1)$, which further implies $\Psi_T > 0$ for all $x \in (0,1)$ and so $\frac{dT}{dk_m}_{|G=Mk_m\ \&\ k_m=1} > 0$ for all $G = Mk_m \in (0, \infty)$.

Next, there are two cases for the effect of large values of k_m, given fixed supply. Suppose $G > 1$. Then, $x_s \to 0$ as $k_m \to \infty$ (see the proof of Proposition 4), and so T approaches to one, the highest possible value, as $k_m \to \infty$. Suppose next $G < 1$. Then, $x_s \to 1 - G > 0$ and $M(x_m - k_m) \to 0$ as $k_m \to \infty$. The latter implies $\eta(x_m, k_m) \to 1$ and so $Mx_m\eta(x_m, k_m) \to Mk_m = G$ as $k_m \to \infty$. This further means that the middlemen's market approaches to the market clearing, which leads to the maximum possible T, as $k_m \to \infty$. Therefore, in any case, T must be strictly increasing in sufficiently large k_m.

Now, consider the net welfare W. Given values of fixed supply $G = Mk_m$, observe that

$$\frac{dx_s}{dk_m}_{|G=Mk_m} = \frac{\frac{1-x_s}{G} \frac{x_m^{k_m-1} e^{-x_m}}{\Gamma(k_m)} - \frac{\partial}{\partial k_m} \frac{\Gamma(k_m, x_m)}{\Gamma(k_m)}}{\frac{k_m}{G} \frac{x_m^{k_m-1} e^{-x_m}}{\Gamma(k_m)} + e^{-x_s}}$$

(see the proof of Proposition 4). The denominator of the above expression is clearly positive. To examine the sign of the denominator terms evaluated at $k_m = 1$, define $\Theta_W(x) \equiv e^{-x}(-x + \ln x + \gamma) + E_1(x)$ for $x \in (0, 1)$. Note that $\frac{dx_s}{dk_m} < 0$ at $k_m = 1$ if and only if $\Theta_W(x) > 0$. Observe that: $\lim_{x \to 0} \Theta_W(x) = \lim_{x \to 0} E_1(x) + \ln x + \gamma = \lim_{x \to 0} E_{in}(x) = 0$ (see the proof of Proposition 3); $\lim_{x \to 1} \Theta_W(x) = e^{-1}(\gamma - 1) + E_1(1) \simeq 0.37 * (0.58 - 1) + 0.22 \simeq 0.064 > 0$;

$$\frac{d\Theta_W}{dx} = -e^{-x}(\ln x + \gamma - x + 1).$$

The parenthesis terms in the last derivative are monotone increasing in $x \in (0, 1)$, being negative as $x \to 0$ and positive as $x \to 1$. Therefore, $\Theta_W(x) > 0$ for all $x \in (0, 1)$, implying $\frac{dx_s}{dk_m} < 0$ at $k_m = 1$.

As for large values of k_m, there are two cases. Consider first the case $G > 1$, where $x_s \to 0$ as $k_m \to \infty$. Then, $W \to \frac{1-c}{1-\beta}$, the highest possible value, as $k_m \to \infty$. Consider next the case $G < 1$ where $x_s \to 1 - G > 0$ as $k_m \to \infty$. In this case, we have $W \to \frac{(1-c)(1-(1-G)e^{-(1-G)})}{1-\beta} < \frac{1-c}{1-\beta}$ for all $G \in (0, 1)$ as $k_m \to \infty$. Therefore, $W < \frac{1-c}{1-\beta}$ as $k_m \to \infty$ if and only if $G < 1$ or $X = \frac{1}{G+1} > X^{**} = \frac{1}{2}$. ∎

References

[1] Abramowitz, M., and I.A. Stegun, eds. (1965), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover.

[2] Acemoglu, D., and R. Shimer (1999), Holdups and Efficiency with Search Frictions, *International Economic Review*, 40, 827-849.

[3] Afonso, G.M. (2010), Liquidity and Congestion, *Journal of Financial Intermediation*, forthcoming.

[4] Aguirregabiria, V. (1999), The Dynamics of Markups and Inventories in Retailing Firms, *Review of Economic Studies*,, 66, 275-308.

[5] ——————— (2003), Strategic Stockouts in Supermarkets, Boston University manuscript.

[6] Albrecht,J., P. Gautier, and S. Vroman (2006), Equilibrium Directed Search with Multiple Applications, *Review of Economic Studies*, 73, 869-91.

[7] ——————————— (2010), Directed Search in the Housing Market, mimeo.

[8] Amihud, Y. and H. Mendelson (1980), Dealership Market: Marketmaking with Inventory, *Journal of Financial Economics*, 8, 31-53.

[9] Andersen Consulting (1996), *Where to Look for Incremental Sales Gains: The Retail Problem of Out-of-Stock Merchandise* (Coca-Cola Retailing Research Council).

[10] Biglaiser, G. (1993), Middlemen as Experts, *RAND Journal of Economics*, 24, 212-223.

[11] Boehmer, E. (2005), Dimensions of Execution Quality: Recent Evidence for US Equity Markets, *Journal of Financial Economics*, 78, 553-582.

[12] Burdett, K., S. Shi, and R. Wright (2001), Pricing and Matching with Frictions, *Journal of Political Economy*, 109, 1060-85.

[13] Caillaud, B., B. Jullien (2003), Chicken and Egg: Competition among Intermediation Service Providers, *RAND Journal of Economics*, 34, 521-552.

[14] Camera, G., and C. Selcuk (2009), Price Dispersion with Directed Search, *Journal of the European Economic Association*, Vol. 7, No. 6: 1193-1224.

[15] Coles, M., and J. Eeckhout (2003), Indeterminancy and Directed Search, *Journal of Economic Theory* 111, 265-276.

[16] Dana, J. D., and K. E. Spier (2001), Revenue Sharing and Vertical Control in the Video Rental Industry, *Journal of Industrial Economics*, 49(3), 223-245.

[17] Demsetz, H. (1968), The Cost of Transacting, *Quarterly Journal of Economics*, 82(1), 33-53.

[18] Duffie, D., N. Garleanu, and L. H. Pedersen (2005), Over-the-Counter Markets, *Econometrica*, 73, 1815.1847.

[19] Faig, M., and B. Jerez (2005), A Theory of Commerce, *Journal of Economic Theory*, 22, 60-99.

[20] Fitzsimons, G.J. (2000), Consumer Response to Stockouts, *Journal of Consumer Research*, 27, 249-266.

[21] Galenianos, M., and P. Kircher (2008), Directed Search with Multiple Job Applications, *Journal of Economic Theory*, 114, 445-471.

[22] Galeotti, A. and J.L. Moraga-Gonzalez (2009), Platform Intermediation in a Market for Differentiated Products, *European Economic Review*, 54, 417-28.

[23] Gàrleanu, N. (2009), Portfolio Choice and Pricing in Illiquid Markets, *Journal of Economic Theory*, 144(2), 53264.

[24] Geddes, K.O., M.L. Glasser, R.A. Moore and T.C. Scott (1990), Evaluation of Classes of Definite Integrals Involving Elementary Functions via Differentiation of Special Functions, *Applicable Algebra in Engineering, Communication and Computing*, vol. 1, 149-165.

[25] Gehrig, T. (1993), Intermediation in Search Markets, *Journal of Economics and Management Strategy* 2-1, 97-120.

[26] Guerrieri, V., R. Shimer and R. Wright (2010), Adverse Selection in Competitive Search Equilibrium, forthcoming in *Econometrica*.

[27] Hendershott, T. and J. Zhang (2006), A Model of Direct and Intermediated Sales, *Jurnal of Economics and Management Strategy*, 15, 279-316.

[28] Hendershott, T. and A.J. Menkveld (2009), Price Pressures, Working Paper, UC Berkeley and VU University Amsterdam.

[29] Ho, T. and H.R. Stoll (1983), The Dynamics of Dealer Markets under Competition, *Journal of Finance*, 38, 1053-1074.

[30] Johri, A. and J. Leach (2002), Middlemen and the Allocation of Heterogeneous Goods, *International Economic Review* 43, 347-361.

[31] Julien, B., J. Kennes and I. King (2000), Bidding for Labor, *Review of Economic Dynamics*, 3, 619-649.

[32] Lagos, R., and G. Rocheteau (2007), Search in Assest Markets: Market Structure, Liquidity, and Welfare, *American Economic Review, Papers and Preceedings*, 97-2, 198-202.

[33] ——————————————— (2009), Liquidity in Asset Markets With Search Frictions, *Econometrica*, 77, 403-426.

[34] Lagos, R., G. Rocheteau, and P. Weill (2010), Crashes and Recoveries in Illiquid Markets, *Journal of Economic Theory*, forthcoming.

[35] Lagos, R. and R. Wright (2005), A Unified Framework for Monetary Theory and Policy Analysis, *Journal of Political Economy*, 113, 463-484.

[36] Li, Y. (1998), Middlemen and Private Information, *Journal of Monetary Economics*, 42, 131-159.

[37] Loertscher, S. (2007), Horizontally Differentiated Market Makers. *Journal of Economics and Management Strategy*, 16, 793825.

[38] Manheim Consulting (2007), Used Car Market Report.

[39] Masters, A. (2007), Middlemen in Search Equilibrium, *International Economic Review*, 48, 343-62.

[40] Matsa, D.A. (2009), Competition and Product Quality in the Supermarket Industry, *Quarterly Journal of Economics*, forthcoming.

[41] McAfee, R.P. (1993), Mechanism Design by Competing Sellers, *Econometrica*, 61, 1281-1312.

[42] Menzio, G. (2007), A Theory of Partially Directed Search, *Journal of Political Economy*, 115, 748-769.

[43] Miao, J. (2006), A Search Model of Centralized and Decentralized Trade, *Review of Economic Dynamics* 9, 68-92.

[44] Moen, E. (1997), Competitive Search Equilibrium, *Journal of Political Economy*, 105, 385-411.

35

[45] Montgomery, J. (1991) Equilibrium Wage Dispersion and Interindustry Wage Differentials, *Quarterly Journal of Economics*, 106, 163-179.

[46] Moraga-Gonzalez, J.L., and M. Wildenbeest (2012), Comparison Sites, in Martin Peitz and Joel Waldfogel (editors), *Handbook of the Digital Economy*, Oxford University Press.

[47] Mortensen, D., and R. Wright (2002), Competitive Pricing and Efficiency in Search Equilibrium, *International Economic Review*, 43, 1-20.

[48] Peters, M. (1991), Ex-Ante Price Offers in Matching Games : Non-Steady States, *Econometrica*, 59, 1425-1454.

[49] Rubinstein, A., and A. Wolinsky. (1987), Middlemen, *Quarterly Journal of Economics* 102, 581-593.

[50] Rust, J., and R., Hall. (2003), Middlemen versus Market Makers: A Theory of Competitive Exchange, *Journal of Political Economy,* 111, 353-403.

[51] Sattinger, M. (2003), Brokers and the Equilibrium Price Function, SUNY Albany manuscript.

[52] Shevichenko, A. (2004), Middlemen, *International Economic Review* 45, 1-24.

[53] Shi, S. (2002a), Product Market and the Size-Wage Differential, *International Economic Review,* 43, 21-54.

[54] Shi, S. (2002b), A Directed-Search Model of Inequality with Heterogeneous Skills and Skill-Biased Technology, *Review of Economic Studies,* 69, 467-491.

[55] Smith, E. (2004), Intermediated Search, *Economica* 71, 619-636.

[56] Spulber, D.F. (1996), Market Making by Price Setting Firms, *Review of Economic Studies* 63, 559-80.

[57] Spulber, D.F. (1999), *Market Microstructure*, Cambridge.

[58] Stoll, H.R. (1978), The Supply of Dealer Services in Securities Markets, *Journal of Finance*, 33, 1133-1151.

[59] Stigler, G.J. (1964), Public Regulation of the Securities Markets, *Journal of Business*, 117-42.

[60] Temme, N.M. (1996), *Special Functions: An Introduction to the Classical Functions of Mathematical Physics.* Wiley, New York.

[61] Vayanos, D. and T. Wang (2007) Search and Endogenous Concentration of Liquidity in Asset Markets, *Journal of Economic Theory*, 136(1), 66-104.

[62] Vayanos, D. and P. Weill (2008), A Search-Based Theory of the On-the-Run Phenomenon, *Journal of Finance*, 63(3), 1361-98.

[63] Watanabe, M. (2006), Middlemen: The Visible Market Makers, University of Carlos III Working paper.

[64] ————————— (2010), A Model of Merchants, *Journal of Economic Theory*, 145, 1865-1889.

[65] Weill, P. (2007), Leaning against the Wind, *Review of Economic Studies*, 74(4), 1329-54.

[66] ———— (2008), Liquidity Premia in Dynamic Bargaining Markets, *Journal of Economic Theory*, 140(1), 66-96.

[67] Wong, Y. and R. Wright (2011), Buyers, Sellers and Middlemen: Variations on Search-Theoretic Themes, University of Winsoncine Working paper.