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Measuring Power and Satisfaction in Societies with Opinion Leaders: An Axiomatization

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Abstract

A well-known model in sociology and marketing is that of opinion leadership. Opinion leaders are actors who are able to affect the behavior of their followers. Hence, opinion leaders have some power over their followers, and they can exercise this power by influencing their followers choice of action. We study a two-action model for a society with opinion leaders. We assume that each member of the society has an inclination to choose one of these actions and that the collective choice is made by simple majority of the actions chosen by each member. For this model we axiomatize satisfaction and power scores, which allow us to investigate the effects of different opinion leader-follower structures.

Keywords: Collective choice, follower, opinion leader, power, satisfaction, axiomatization
JEL Classification: D71, D85

1 Introduction

The concept of opinion leadership received considerable attention in sociology and marketing. It rose out of the *two-step flow of communication* theory introduced by the 'Lazarsfeld group' (see, e.g., Katz and Lazarsfeld (1955), and Lazarsfeld, Berelson, and Gaudet (1968)). In its most rudimentary form it claims that 'ideas often flow *from* radio and print *to* the opinion leaders and *from* them to the less active sections of the population', see Lazarsfeld, Berelson, and Gaudet (1968). They investigated the influence of mass communication on the 1940 presidential election campaign in the US and found that the voters' choices were more influenced by actors which they called opinion leaders than by mass communication. Based on this observation they arrived at the conclusion that the communication process is not a one- but a two-step process. According to their model information distributed by mass media first reaches the so-called *opinion leaders*. These are actors who are specified as highly self-confident with strong opinions. In Lazarsfeld, Berelson, and Gaudet (1968) they act as intermediaries between the mass media and the recipients. In general, the latter actors are called *followers*. They feel attracted by the opinion leaders holding them in high esteem and are prepared to accept their opinion for their own behavior. Hence, a major characteristic of opinion leaders is their exercised power over their followers. After critiques of the model by the 'Lazarsfeld group' (see, e.g., Bostian (1970)), Troldahl (1966) introduced a modified version of their model called the *two-cycle flow of communication* model which corresponded to other results in the field (see, e.g., Deutschmann and Danielson (1960)). Troldahl's model distinguishes between two phases in the communication process. Phase one is a *flow of information* from the mass media to the members of the society which is assumed to be a *one-step process*, i.e., the information goes directly to all members of the society. Phase two is the *flow of influence* on beliefs and behavior which is assumed to be a *two-step process*. In a first step opinion leaders form their own opinion based on additional information provided by experts, such as academics, while in a subsequent second step they try to influence the behavior of their followers. Since Troldahl's contribution the literature on opinion leadership has provided a strong body of knowledge of how and why opinion leaders influence followers choices (see Hoyer and Stockburger-Sauer (2007)).

Opinion leaders form an attractive group for marketing and policy purposes (see, e.g., Hoyer and Stockburger-Sauer (2007)) as the existence (or non-existence) of opinion leaders in a society and their relations to their followers may have a considerable impact on market behavior (such as consumer or financial markets), and other social agglomerations being made up of individual actors choosing among a number of alternatives (open to them at a given time). Hence, it appears to be interesting to investigate the effect of different *opinion leader-follower structures* in markets or other collective decision-making situations in a society. This includes questions such as whether it would be worthwhile to establish a new opinion leader in a society or whether a change in an existing opinion leader-follower structure can be expected to make a difference to the society. However, to our best knowledge apart from our own recent work there exists no study which addresses such issues on bare theoretical grounds. In van den Brink, Rusinowska, and Steffen (2011), we laid the foundation to fill out this lacunae by introducing novel *power* and *satisfaction* scores for societies with opinion leaders and discussing some properties of both scores.[1] The power scores inform us about the power distribution among the members of the society with respect to their ability to affect the state of the society concerning a specific outcome, while satisfaction scores tell us to which degree members of the society can be expected to end up with an outcome that they like. Based upon these results, in this paper we provide a full axiomatization of the power and satisfaction scores for a specific opinion leader-follower collective choice situation.

For our analysis we consider the example of binary choice as it can also be found in Sinha and Raghavendra (2006) who study the effect of opinion leaders on market outcomes. It is assumed that an actor can choose among two alternatives. For instance, this can be a market in which the actors have to decide whether they should *buy* or *not buy* a joint product, or a voting situation in which the members of the society have to choose to vote either *yes* or *no* on a specific proposal. From now onwards we will refer to a voting situation only. However, all results presented in this paper also apply to markets. We assume that the actors in a society have to decide whether they would like to remain with the status quo or whether a specific *exogenous* proposal leading to a new state of the society should be adopted. We assume that the proposal has been distributed among all actors. Each actor has to form its own opinion on the proposal, i.e., without being influenced by any other actor. We will call this the actor's action inclination. The society is partitioned into *opinion leaders*, *followers*, and *independent actors*. In line with the inherent idea of opinion leadership we suppose that via informal discussions of the proposal the action inclinations of the opinion leaders are becoming public information prior to the real decision. Only after these discussions, all actors will choose their action which coincide with the actors action inclination if it is an opinion leader or independent actor. Concerning the followers we assume that for their choice of action they - independently of their own action inclination - adopt the action inclination of their opinion leaders if all of these have the same action inclination. Finally, based on the individual choices of all actors, a decision-making mechanism determines the collective choice, i.e., whether the proposal is

[1]Note that this research is in some respects also related to work on opinion leaders and the Condorcet Jury Theorem (see, e.g., Estlund (1994)), threshold models of collective behavior (see, e.g., Granovetter (1978), and Granovetter and Soong (1986)) and, in more general terms, to the literature on network externalities.

adopted or not. We assume that the collective choice is made by simple majority of the actions chosen by the actors.

In the literature we can find several scores and measures being introduced for analyzing collective decision-making situations with a possible influence between the actors.[2] For instance, some measures for arbitrary digraphs have been studied in van den Brink and Borm (2002) and van den Brink and Gilles (2000). Our analysis in the present paper and our earlier study (see van den Brink, Rusinowska, and Steffen (2011)) is related to these contributions as we represent opinion leader-follower structures by bipartite digraphs. Coming from a slightly different direction are the works presented in the voting power literature. One of the traditional measures is the Rae index (Rae (1969)) which measures the success of an actor in a voting situation. An actor is said to be successful if its vote coincides with the voting outcome. Such a successful actor can be additionally powerful. For the calculation of the voting power of an actor a number of measures have been suggested. The most prominent measures are the Banzhaf measure (Banzhaf (1965), see also Dubey and Shapley (1979), and Owen (1975)), and the Shapley-Shubik index (Shapley and Shubik (1954)). They ascribe power to an actor if its vote coincides with the voting outcome, but this outcome would have been different if the actor changed its vote.[3]

As we are concerned with measuring *satisfaction* and *power* distributions, our research is also related to the work on measurement of voting power. We use the notion of satisfaction in order to distinguish our approach from those in the standard voting power literature. We are aware of the fact that at least in Straffin (1978) and Straffin, Davis, and Brams (1982) the notion of satisfaction is used as a synonym of success. However, in our opinion in their framework referring to the relation between votes and the voting outcome, success appears to be the more appropriate notion, while in our context where we refer to the relation between action inclinations and the social outcome, the notion of satisfaction appears to be more natural.

In van den Brink, Rusinowska, and Steffen (2011) we measured satisfaction by the number of times the collective choice is the same as the action inclination of an actor.[4] We measured the power of an actor in a bipartite digraph by the number of times the actor has a *swing*, where a swing is defined as a situation where an actor by changing its action inclination, given the action inclinations of the others, enforces a change in the collective choice via a change in its action. In that paper we demonstrated that the power and satisfaction scores we introduced have some dictator and opinion leader properties in

[2] For the distinction between scores, measures, and indices, see Felsenthal and Machover (1998).

[3] Both measures can also be derived from a probabilistic framework, see, e.g., Straffin (1977, 1978) and, more recently, Laruelle and Valenciano (2005).

[4] Note that in van den Brink, Rusinowska, and Steffen (2011) we have just used the term inclination instead of action inclination.

common in case the followers, for their choice of action, adopt the action inclination of their opinion leaders independently of their own inclination if a certain fraction of these have the same action inclination. In the present paper we consider the specific, but quite usual, case that this fraction is equal to one, i.e., unanimity. In the current context this assumption appears to be in line with other findings in the literature. For instance, Asch's (1951, 1952, 1956) results imply that when a group takes a unanimous position, people may feel more pressure to conform. A very recent study underpinning this view comes from an experiment conducted by Verhulst and Levitan (2009). They found that participants were more likely to conform to the attitudes expressed by a unanimous group than by a non-unanimous group.

Assuming that a unanimity of opinion leaders is required for a follower to adopt their action inclination for its own choice of action implies that a follower only chooses an action which is not in line with its individual action inclination if all its opinion leaders have the (same) opposing action inclination. Based upon this assumption, in the present paper we show that the power and satisfaction scores satisfy even stronger opinion leader properties than those studied in van den Brink, Rusinowska, and Steffen (2011). Moreover, we introduce two different normalizations (i.e., units of measurement) and obtain full axiomatizations of both scores which differ in the normalization only.

The paper is structured as follows. In Sect. 2 we describe the model, and in Sect. 3 we define and illustrate the satisfaction and power scores for actors in societies with opinion leaders. In Sect. 4 we provide axiomatizations of the satisfaction and power scores differing only in the normalization that is applied. Finally, in Sect. 5 we draw some conclusions and discuss some possible extensions and applications of the model.

2 The model

Let $N = \{1, ..., n\}$ denote a society containing n actors which is partitioned into opinion leaders, followers, and independent actors. Adopting a distinction applied by Vanberg and Buchanan (1988) and Heckathorn (1987) we assume that each actor $k \in N$ has two types of inclinations: *constitutional* and *action inclinations*, where constitutional inclinations can be regarded to be on a 'higher level' than action inclinations as action inclinations have to be formed within the framework given by the constitutional inclinations. In our context constitutional inclinations are related to the organization of the society. They determine (i) whether an actor k is an opinion leader, follower, or independent actor, (ii) which opinion leader an actor k chooses if k is a follower, and (iii) the procedure for followers to follow their opinion leaders. Instead, action inclinations are related to the outcome of the collective choice to be made by the society, i.e., they state which actions an actor k would choose being ‘on its own’ and not being influenced by others.[5]

In this paper we assume that constitutional inclinations are exogenous. This implies that (i) we have a given partition of our society into opinion leaders, followers, and independent actors, and that (ii) our opinion leader-follower relationships are already fixed, i.e., it is given which actors might influence the choice of action of certain other actors by exercising some power over them.[6] Moreover, (iii) we assume a unanimity requirement applies for followers to follow their opinion leaders, i.e., a follower will only choose an action against its own action inclination if all its opinion leaders have an inclination different from its own. In this case the follower will adopt the action inclination of its opinion leaders. Action inclinations are left ‘unspecified’. As we assumed the constitutional inclinations to be exogenous, from now onwards we will just refer to *inclinations* instead of *action inclinations*.

Formally, we represent the structure of our society and the ‘opinion leader-follower’ relations by a *bipartite directed graph* (or *bipartite digraph*) (N, D) with a finite set of nodes N representing the actors, and $D \subset N \times N$ a binary relation on N such that each actor is either an opinion leader, a follower, or an independent actor. Since we take the set of actors N fixed, we represent a digraph (N, D) just by its binary relation D. Let $S_D(k)$ and $P_D(k)$ denote the set of successors and predecessors of actor k in digraph D, respectively, i.e., for each $k \in N$,

$$S_D(k) = \{j \in N : (k, j) \in D\}$$

and

$$P_D(k) = \{j \in N : (j, k) \in D\}.$$

As we assume that each actor is either an opinion leader, follower or independent actor, we consider digraphs D such that

$$|S_D(k)| \cdot |P_D(k)| = 0 \text{ for each } k \in N, \tag{2.1}$$

where $|X|$ denotes the cardinality of set X. Let $OL(D)$, $FOL(D)$, and $IND(D)$ denote the sets of all opinion leaders, followers, and independent actors in digraph D, respectively, i.e.,

$$OL(D) = \{k \in N : S_D(k) \neq \emptyset\}$$

[5]Note that action inclinations are the type of inclinations which were introduced by Hoede and Bakker (1982) for their power analysis of organizational structures.

[6]As a result of this *influence*, the ability of the followers to determine the outcome of the collective choice, i.e., their power to do something (with respect to the outcome of the collective choice) might be affected.

$$FOL(D) = \{k \in N : P_D(k) \neq \emptyset\}$$

and

$$IND(D) = N \setminus (OL(D) \cup FOL(D)).$$

Therefore, by assumption (2.1) we have that

$$OL(D) \cap FOL(D) = \emptyset,$$

and thus the sets $OL(D)$, $FOL(D)$ and $IND(D)$ form a partition of the set N. We denote the collection of all bipartite digraphs on N, represented by their binary relation, by $\mathcal{D}^N$.

Regarding the flow of information among the actors and their inclination formation we assume that (via the mass media) an exogenous proposal will be distributed among all actors $k \in N$. Having been informed about this proposal, each actor $k \in N$ will form its inclination on it. Concerning the nature of the proposal, following Sinha and Raghavendra (2006) we assume that an actor has a binary choice: actor k can have the inclination either to support the proposal in order to obtain a new state of the society (inclination to choose the *yes*-action denoted by 1), or to reject it in order to remain with the status quo (inclination to choose the *no*-action denoted by 0). The inclinations chosen by the members of our society are represented by an inclination vector $I = (I_1, ..., I_n) \in \{0,1\}^n$. This is a vector which k^{th} component, I_k, is 1 if actor k has the inclination to support the proposal, and 0 if it is inclined to reject it.

Now we can define an *opinion leader-follower collective choice situation* as a pair (I, D) with $I \in \{0,1\}^n$ and $D \in \mathcal{D}^N$ being a bipartite graph as described above.

In line with the inherent idea of an opinion leader, we assume that via informal (public) discussions of the proposal the inclinations of opinion leaders are becoming public information prior to the formal decision on the proposal, i.e., each follower is aware of the inclination of its opinion leader(s).[7] Only after these informal discussion actors will secretly (or simultaneously) choose their action. We assume that the actors in $k \in OL(D) \cup IND(D)$ make their simultaneous choice of action according to their own inclinations. For each follower we assume that, independently from its own inclination, it will choose the action which corresponds to the inclination of its opinion leaders if all of them have the same inclination. Otherwise, if the inclinations of its opinion leaders are not all the same, the follower will choose the action which corresponds to its own inclination.

Let $V = V(I, D) \in \{0,1\}^n$ denote the choice vector, that is, a vector which k^{th} component, V_k, is 1 if actor k has chosen to support the proposal, and 0 if k has chosen to reject it. Thus, the choice vector $V = V(I, D) \in \{0,1\}^n$ is given by:

$$V_k = I_k \text{ if } k \in OL(D) \cup IND(D),$$

[7]Note that while our model allows that the public discussions may change the inclinations of some actors, we suppose that when choosing their action, and after that, inclinations do not alter.

and for $k \in FOL(D)$:

$$V_k = \begin{cases} x & \text{if } I_j = x \text{ for all } j \in P_D(k) \\ I_k & \text{otherwise,} \end{cases} \tag{2.2}$$

where $x \in \{0,1\}$.

After all actors have chosen their actions, a collective choice is resulting according to the decision-making mechanism in use. The decision-making mechanism is given by the *collective decision function* $C\colon \{0,1\}^n \times \mathcal{D}^N \rightarrow \{0,1\}$ which assigns an outcome to every pair $(I,D) \in \{0,1\}^n \times \mathcal{D}^N$, that is, the value 0 if the collective decision is *no*, and the value 1 if the collective decision is *yes*. Usually, one only considers collective decision functions C that are neutral[8] and anonymous[9]. In this paper, we define the collective decision function by simple majority voting. Let for an action $x \in \{0,1\}$ and choice vector $V = V(I,D) \in \{0,1\}^n$ the number of actors choosing the action x be denoted by

$$n_x(V(I,D)) = |\{k \in N : V_k = x\}|.$$

Restricting our analysis to situations in which the number of actors is odd, the collective decision function is defined, for each (I,D), as follows:

$$C(I,D) = \begin{cases} 1 & \text{if } n_1(V(I,D)) > n_0(V(I,D)) \\ 0 & \text{if } n_0(V(I,D)) > n_1(V(I,D)). \end{cases} \tag{2.3}$$

Before we present our satisfaction and power scores in the next section, it appears to be appropriate to add a remark on the common measurement of success and power in voting games. Both usually relate the vote(s) of each actor to the voting outcome. An actor is said to be successful in a voting game if the actor's vote coincides with the voting outcome, i.e., the collective choice. In addition, if an actor is successful then power is ascribed to such an actor if, given the votes of the others, by changing its own vote the actor changes the voting outcome. It is said that the actor has a swing. Hence, roughly speaking, power in this context refers to an ability of an actor, i.e., what the actor is able to do (by changing its vote) against some resistance of others (represented by those given votes of the others which are not in line with the 'new' vote of the actor in question) irrespective of the actual occurrence of this resistance (see van den Brink and Steffen (2008) referring to Braham (2008)). These definitions of success and power are sufficient for many applications. However, being applied to the measurement of success and power in voting games they come along with a number of implicit simplifying assumptions (see Morriss 1987/2002:154-156).

[8] A collective decision function C is *neutral* if $[C(I,D) = 1$ if and only if $C(I^c,D) = 0]$, where $I_k^c = 1$ if and only if $I_k = 0$.

[9] A collective decision function C is *anonymous* if for every permutation $\pi\colon N \rightarrow N$, $C(I,D) = C(\pi(I),\pi(D))$ with $\pi(I)_k = I_{\pi(k)}$ and $(\pi(k),\pi(j)) \in \pi(D)$ if and only if $(k,j) \in D$.

Among others they assume (i) that actors vote in line with their inclination, and (ii) that each actor's choice of a vote is not influenced by another actor. The implication of (i) is that for measurement purposes it is sufficient to consider the votes of the actors and the resulting voting outcome, and to ignore the fact that inclinations are usually part of any definition of power defined as an ability. Following Morriss (1987/2002:26), 'abilities are things that we can do *when we want*'. As our opinion leader-follower collective choice situations allow for situations under which both assumptions above are violated which, in fact, is the usual case in our context, we have to relax both. This implies that for our purposes we have to include the actors' inclinations into our measurement.

3 Measuring satisfaction and power

In van den Brink, Rusinowska, and Steffen (2011) satisfaction and power scores for bipartite digraphs which represent opinion leader-follower collective choice situations as described in the previous section, are introduced. In general, a score for bipartite digraphs is a function $f\colon \mathcal{D}^N \to \mathbb{R}^n$ which assigns an n-dimensional real vector to every bipartite digraph on N.

Since in our model actors will actually choose either the *yes*- or *no*-action if they have the corresponding inclination to do so, or if they are influenced by their opinion leaders to make that choice, we propose to ascribe *satisfaction* to an actor if the actor's *inclination* prior to its actual choice coincides with the collective choice and to measure the satisfaction of an actor in an opinion leader-follower collective choice situation by the actor's likelihood to be satisfied.

Formally, first, in order to ascribe satisfaction to an actor we define a satisfaction score of an actor under a given inclination vector, i.e., for each $(I, D) \in \{0,1\}^n \times \mathcal{D}^N$ and $k \in N$

$$\overline{SAT}_k(I, D) = \begin{cases} 1 & \text{if } C(I, D) = I_k \\ 0 & \text{otherwise.} \end{cases}$$

Next, based on the satisfaction of an actor under each inclination vector, we define the *satisfaction* score in a bipartite digraph, $SAT\colon \mathcal{D}^N \to \mathbb{R}^n$, given by

$$SAT_k(D) = \sum_{I \in \{0,1\}^n} \overline{SAT}_k(I, D) \quad \text{for each } k \in N. \tag{3.4}$$

In a similar way we ascribe power to an actor if the actor, by changing its inclination, is able to alter the collective choice and measure the power of the actor in an opinion leader-follower collective choice situation by the actor's likelihood to be powerful.

Hence, formally, actor $k \in N$ has a *swing* in (I, D) according to collective decision function C if $C(I, D) \neq C(I', D)$ with $I'_k \neq I_k$ and $I'_j = I_j$ for all $j \in N \setminus \{k\}$. In order to ascribe power to actor k we define a power score of actor k under a given inclination vector:

$$\overline{POW}_k(I,D) = \begin{cases} 1 & \text{if } k \text{ has a swing in } (I,D) \\ 0 & \text{otherwise.} \end{cases}$$

Then, the *power* score $POW\colon \mathcal{D}^N \to \mathbb{R}^n$ is given by

$$POW_k(D) = \sum_{I \in \{0,1\}^n} \overline{POW}_k(I,D) \quad \text{for each } k \in N. \tag{3.5}$$

To illustrate the satisfaction and power scores we present two examples, one with three actors and one with five actors.

Example 1 Let $N = \{1,2,3\}$. For this set of actors the first column in Table 1 contains all feasible inclination vectors. As a bench mark case let us begin with the empty digraph $D_\emptyset = \emptyset$, i.e., all actors are independent. Hence, the choice vector, defined in (2.2), is the same as the inclination vector, i.e., $V(I, D_\emptyset) = I$. Having defined the collective decision function $C(I,D)$ by simple majority voting, the resulting outcomes of the choices given by $V(I, D_\emptyset) = I$ are displayed in the second column. This results in $SAT(D_\emptyset) = (6,6,6)$ and $POW(D_\emptyset) = (4,4,4)$.

Table 1: Three actors

I	$C(I,D_\emptyset)$	$V(I,D_1)$	$C(I,V(I,D_1))$	$V(I,D_2)$	$C(I,V(I,D_2))$	$V(I,D_3)$	$C(I,V(I,D_3))$
(0,0,0)	0	-	-	-	-	-	-
(1,0,0)	0	(1,1,0)	1	(1,1,1)	1	-	-
(0,1,0)	0	(0,0,0)	-	(0,0,0)	-	(0,0,0)	-
(0,0,1)	0	-	-	(0,0,0)	-	-	-
(1,1,0)	1	-	-	(1,1,1)	1	-	
(1,0,1)	1	(1,1,1)	-	(1,1,1)	-	(1,1,1)	-
(0,1,1)	1	(0,0,1)	0	(0,0,0)	0	-	-
(1,1,1)	1	-	-	-	-	-	-

Next, let us consider the digraph $D_1 = \{(1,2)\}$, i.e., the case of actor 1 being the opinion leader of actor 2 (implying that actor 2 is the follower of actor 1), while actor 3 remains independent. Column 3 of Table 1 displays the choice vectors $V(I,D_1)$ and column 4 the resulting outcomes of the choices for this case under simple majority voting. By a '-' in column 3 we indicate that the choice is identical to the inclination. By a '-' in column 4 we indicate that the outcome is the same as with the empty digraph, i.e., when all actors choose according to their own inclination. Obviously, when the choices do not change then also the outcome does not change, but the choices may change without affecting the outcome.

Based on Table 1 we can now compute the satisfaction and power scores for digraph $D_1 = \{(1,2)\}$: $SAT(D_1) = (8,4,4)$ and $POW(D_1) = (8,0,0)$. Note that these scores are the same in case actor 1 is a 'real' dictator, i.e., if we consider the digraph $D_2 = \{(1,2),(1,3)\}$, although the choices are different for all inclinations except $(0,0,0)$ and $(1,1,1)$. Note also that the scores for $D_3 = \{(1,2),(3,2)\}$ are the same as for the empty digraph, although the choices for inclinations $(0,1,0)$ and $(1,0,1)$ are different. So, in this three actor example satisfaction and power are distributed either according to the situation that there are only independent actors, or there is a dictator.

□

Some more interesting cases are obtained when we consider five actors. We take the following opinion leader-follower structures from van den Brink, Rusinowska, and Steffen (2011), but discuss the unanimity model of the underlying paper.

Example 2 Consider $N = \{1,2,3,4,5\}$. Like in Table 1 of the previous example, also the first two columns of Table 2 display all feasible inclinations and the resulting outcomes under simple majority voting for the empty digraph, i.e., for the case that all actors behave independently. By the following columns we provide the corresponding choice vectors and the outcomes for the following five digraphs as also represented by Figure 1:

$D_\emptyset = \emptyset$
$D_1 = \{(1,2)\}$
$D_2 = \{(1,2),(3,2)\}$
$D_3 = \{(1,2),(3,2),(4,2)\}$, and
$D_4 = \{(1,2),(3,4)\}$.

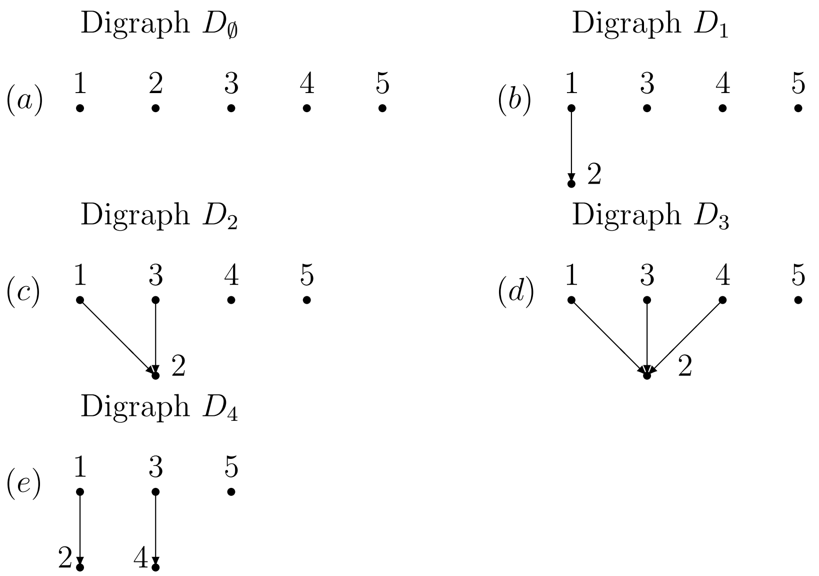

Fig. 1: Digraphs of Example 2

Table 2: Five actors (Notation: $C(I, D_m)$ denotes here $C(I, V(I, D_m))$ for $m = 1, 2, 3, 4$)

I	$C(I, D_\emptyset)$	$V(I, D_1)$	$C(I, D_1)$	$V(I, D_2)$	$C(I, D_2)$	$V(I, D_3)$	$C(I, D_3)$	$V(I, D_4)$	$C(I, D_4)$
(0,0,0,0,0)	0	-	-	-	-	-	-	-	-
(1,0,0,0,0)	0	(1,1,0,0,0)	-	-	-	-	-	(1,1,0,0,0)	-
(0,1,0,0,0)	0	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-	(0,0,0,0,0)	-
(0,0,1,0,0)	0	-	-	-	-	-	-	(0,0,1,1,0)	-
(0,0,0,1,0)	0	-	-	-	-	-	-	(0,0,0,0,0)	-
(0,0,0,0,1)	0	-	-	-	-	-	-	-	-
(1,1,0,0,0)	0	-	-	-	-	-	-	-	-
(1,0,1,0,0)	0	(1,1,1,0,0)	1	(1,1,1,0,0)	1	-	-	(1,1,1,1,0)	1
(1,0,0,1,0)	0	(1,1,0,1,0)	1	-	-	-	-	(1,1,0,0,0)	-
(1,0,0,0,1)	0	(1,1,0,0,1)	1	-	-	-	-	(1,1,0,0,1)	1
(0,1,1,0,0)	0	(0,0,1,0,0)	-	-	-	-	-	(0,0,1,1,0)	-
(0,1,0,1,0)	0	(0,0,0,1,0)	-	(0,0,0,1,0)	-	-	-	(0,0,0,0,0)	-
(0,1,0,0,1)	0	(0,0,0,0,1)	-	(0,0,0,0,1)	-	(0,0,0,0,1)	-	(0,0,0,0,1)	-
(0,0,1,1,0)	0	-	-	-	-	-	-	-	-
(0,0,1,0,1)	0	-	-	-	-	-	-	(0,0,1,1,1)	1
(0,0,0,1,1)	0	-	-	-	-	-	-	(0,0,0,0,1)	-
(1,1,1,0,0)	1	-	-	-	-	-	-	(1,1,1,1,0)	-
(1,1,0,1,0)	1	-	-	-	-	-	-	(1,1,0,0,0)	0
(1,1,0,0,1)	1	-	-	-	-	-	-	-	-
(1,0,1,1,0)	1	(1,1,1,1,0)	-	(1,1,1,1,0)	-	(1,1,1,1,0)	-	(1,1,1,1,0)	-
(1,0,1,0,1)	1	(1,1,1,0,1)	-	(1,1,1,0,1)	-	-	-	(1,1,1,1,1)	-
(1,0,0,1,1)	1	(1,1,0,1,1)	-	-	-	-	-	(1,1,0,0,1)	-
(0,1,1,1,0)	1	(0,0,1,1,0)	0	-	-	-	-	(0,0,1,1,0)	0
(0,1,1,0,1)	1	(0,0,1,0,1)	0	-	-	-	-	(0,0,1,1,1)	-
(0,1,0,1,1)	1	(0,0,0,1,1)	0	(0,0,0,1,1)	0	-	-	(0,0,0,0,1)	0
(0,0,1,1,1)	1	-	-	-	-	-	-	-	-
(1,1,1,1,0)	1	-	-	-	-	-	-	-	-
(1,1,1,0,1)	1	-	-	-	-	-	-	(1,1,1,1,1)	-
(1,1,0,1,1)	1	-	-	-	-	-	-	(1,1,0,0,1)	-
(1,0,1,1,1)	1	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-	(1,1,1,1,1)	-
(0,1,1,1,1)	1	(0,0,1,1,1)	-	-	-	-	-	(0,0,1,1,1)	-
(1,1,1,1,1)	1	-	-	-	-	-	-	-	-

For the empty digraph we have $SAT(D_\emptyset) = (22, 22, 22, 22, 22)$ and $POW(D_\emptyset) = (12, 12, 12, 12, 12)$. In case actor 1 is a dictator, i.e., $D = \{(1, 2), (1, 3), (1, 4), (1, 5)\}$ we obtain the scores $SAT(D) = (32, 16, 16, 16, 16)$ and $POW(D) = (32, 0, 0, 0, 0)$. (The same scores we obtain if one actor is the unique opinion leader of at least half of the other actors, although the choices may be different.) For the other digraphs we obtain the following scores:

$SAT(D_1) = (28, 16, 20, 20, 20)$, $POW(D_1) = (24, 0, 8, 8, 8)$
$SAT(D_2) = (24, 20, 24, 20, 20)$, $POW(D_2) = (16, 8, 16, 8, 8)$
$SAT(D_3) = (22, 22, 22, 22, 22)$, $POW(D_3) = (12, 12, 12, 12, 12)$
$SAT(D_4) = (24, 16, 24, 16, 24)$, $POW(D_4) = (16, 0, 16, 0, 16)$.

□

4 Axiomatizations

In this section we provide full axiomatizations of the satisfaction score SAT and power score POW. In van den Brink, Rusinowska, and Steffen (2011) it is shown that both scores satisfy the first four properties discussed below. Symmetry states that the value of a score for actors with a symmetric position in the bipartite digraph is the same.

Symmetry If $S_D(k) = S_D(j)$ and $P_D(k) = P_D(j)$ then $f_k(D) = f_j(D)$.

Next, we consider two dictator properties. Clearly, a dictator, i.e., a unique opinion leader who is followed by all other actors, has the power to change the outcome for any voting profile by changing its own inclination and, if the dictator votes according to its inclination, then the outcome will be the inclination of the dictator. Therefore the *dictator property* states that, if there is a dictator, then the score of the dictator is equal to the total number of possible inclination vectors. Note, that since we assume that no actor can be at the same time a follower and an opinion leader, the dictator as defined above cannot be a follower.

Dictator property If $D \in \mathcal{D}^N$ and $h \in N$ is such that $S_D(h) = N \backslash \{h\}$, then $f_h(D) = 2^n$.

Secondly, since a follower who has only one opinion leader has always to follow this opinion leader, *dictated independence* states that the score of a follower with one opinion leader does not change as long as this follower is dictated by a sole opinion leader.

Dictated independence If $D, D' \in \mathcal{D}^N$ and $k \in N$ are such that $|P_D(k)| = |P_{D'}(k)| = 1$, then $f_k(D) = f_k(D')$.

Next we present two opinion leader properties saying something about the changes in score for different actors when the opinion leader-follower structure changes, in particular when an actor gets a new opinion leader. The properties that we consider are inspired by similar properties for solutions in cooperative game theory, where studying these kinds of properties has a longer history.

In the context of cooperative TU-games, Lehrer (1988) and Haller (1994) introduced properties that consider collusion of players. In particular, Haller (1994) considers different types of collusion neutrality properties requiring that the sum of the payoffs of two colluding players does not change, see also Malawski (2004). In van den Brink (2010) these properties are stated in terms of games in which the players belong to some hierarchical structure, the so-called *games with a permission structure*. There deleting a domination link between a successor and a predecessor is interpreted as a collusion between this predecessor and another predecessor with respect to the influence over the successor. In that context power neutrality states that the sum of payoffs of the two colluding predecessors should not change. We are now going to apply this idea to define two axioms for opinion leader-follower collective choice situations.

Suppose that an independent actor gets an opinion leader. The *opposite gain property* states that, if an actor becomes a sole opinion leader of another actor who was previously independent, then the sum of the scores of these two actors does not change. This implies that in case the opinion leader gains this goes fully at the cost of the follower.

Opposite gain property Let $D, D' \in \mathcal{D}^N$, $j \in IND(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') - f_h(D) = f_j(D) - f_j(D')$.

Horizontal neutrality states that, if a follower gets one more opinion leader, then the sum of scores of the 'new' and an 'old' opinion leader does not change. In other words, the change for the new opinion leader is opposite but in absolute value equal to the change for an 'old' opinion leader.[10]

Horizontal neutrality Let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $g \in P_D(j)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') - f_h(D) = f_g(D) - f_g(D')$.

One of the most widely studied and applied properties in the context of cooperative games in which the players belong to some binary communication structure, is *fairness* introduced by Myerson (1977). It states that deleting a communication link between two players changes their individual payoffs by the same amount. In van den Brink (1997) this type of equal gain/loss property is stated in terms of the above mentioned games with a hierarchical permission structure. For the opinion leader-follower collective choice situation of the underlying paper, suppose that a follower gets one more opinion leader. In van den Brink, Rusinowska, and Steffen (2011) it is shown that in case a follower follows a qualified majority of its opinion leaders, both their satisfaction and power score satisfy the *equal absolute change property* stating that the changes in scores of this follower and of its new opinion leader are either the same or are opposite, but the same in absolute values. It turns out that in the unanimity case considered in this paper (i.e., a follower chooses against its own inclination only if all its opinion leaders have the same inclination that is different from its own) both the satisfaction score *SAT* and power score *POW* even satisfy the stronger *equal gain property* stating that the changes in scores of this follower and of its new opinion leader are the same.

Equal gain property Let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h, j)\}$. Then $f_h(D') - f_h(D) = f_j(D') - f_j(D)$.

We illustrate these axioms with the digraphs presented in Figure 1.

Example 3 Consider the five actor opinion leader-follower collective choice situation of Example 2. We see that the satisfaction and power scores of a dictated actor (i.e., an actor that is subordinate to exactly one opinion leader) stay the same as long as the actor is

[10]Note that this is a stronger version of the *power neutrality for two opinion leaders* used in van den Brink, Rusinowska, and Steffen (2011).

dictated by this opinion leader (see, for example, the scores for actor 2 in D_1 and D_4), illustrating dictated independence. Further, if we let an independent actor become the follower of one opinion leader, then both the gain in power and the gain in satisfaction of the opinion leader go at the cost of the follower. For example, by going from the empty digraph $D_\emptyset$ to D_1 actor 1 becomes opinion leader for actor 2, and the sum of their satisfaction and power (44, respectively, 24) does not change, illustrating the opposite gain property. If also actor 3 becomes an opinion leader for actor 2 (i.e., we go from D_1 to D_2), then the gain in satisfaction and power of actors 2 and 3 change the same (an increase of 4, respectively, 8), illustrating the equal gain property, and is in absolute value equal to the loss of actor 1, illustrating horizontal neutrality.

□

As we will prove below, both, the satisfaction score *SAT* and the power score *POW* satisfy the above six properties. Obviously, since satisfaction and power are related but different concepts (see, e.g., Dowding (1996)) each will satisfy a different normalization. As normalization of satisfaction we take that the sum of all scores is equal to the total number of individual satisfactions, i.e., for each inclination vector we count how many actors have an inclination that coincides with the social outcome, and we add all these satisfactions over all inclination vectors.

Satisfaction normalization For every $D \in \mathcal{D}^N$ it holds that

$$\sum_{k\in N} f_k(D) = \sum_{I\in\{0,1\}^n} |\{k \in N : I_k = C(I, D)\}|.$$

As normalization of power we take that the sum of all scores is equal to the total number of individual swings, i.e., for each inclination vector we count how many actors have a swing, and we add all these swings over all inclination vectors.

Power normalization For every $D \in \mathcal{D}^N$ it holds that

$$\sum_{k\in N} f_k(D) = \sum_{I\in\{0,1\}^n} |\{k \in N : k \text{ has a swing in } (I, D)\}|.$$

It turns out that adding satisfaction normalization to the first six axioms characterizes the satisfaction score.

Theorem 1 *Let the choice vector V be defined by (2.2). A score $f\colon \mathcal{D}^N \to \mathbb{R}^n$ is the satisfaction score $SAT\colon \mathcal{D}^N \to \mathbb{R}^n$ if and only if it satisfies symmetry, the dictator property, dictated independence, the opposite gain property, horizontal neutrality, the equal gain property, and satisfaction normalization.*

Proof

SAT satisfying symmetry, the dictator property, dictated independence, and the opposite gain property is shown in van den Brink, Rusinowska, and Steffen (2011).[11] It is obvious that *SAT* satisfies satisfaction normalization.

To show the equal gain property and horizontal neutrality, let $D, D' \in \mathcal{D}^N$, $j \in FOL(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h,j)\}$. If $C(I,D) \neq C(I,D')$ then it must hold that actor j initially had to vote against its inclination and now can vote according to its inclination because its new opinion leader h has the same inclination. Moreover, all 'other' opinion leaders of j must have the opposite inclination. Thus, for $g \in P_D(j)$ we have $C(I,D) = I_g \neq I_j = I_h$ and $C(I,D') = I_j = I_h \neq I_g$. So, $\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D) = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_g(I,D) - \overline{SAT}_g(I,D') = 1$.

Obviously, if $C(I,D) = C(I,D')$, then $\overline{SAT}_j(I,D') - \overline{SAT}_j(I,D) = \overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_g(I,D) - \overline{SAT}_g(I,D') = 0$. Thus, with (3.4) we have $SAT_j(D) - SAT_j(D') = SAT_h(D) - SAT_h(D') = SAT_g(D') - SAT_g(D)$, showing that *SAT* satisfies the equal gain property and horizontal neutrality.

To prove uniqueness, suppose that the score $f\colon \mathcal{D}^N \to \mathbb{R}^n$ satisfies the seven axioms, and let $D \in \mathcal{D}^N$. We prove that f must be equal to *SAT* in several steps. We first prove uniqueness in case there is at most one actor that is an opinion leader by induction on its number of followers.

First, suppose that the opinion leader is a dictator, i.e., there is an $h \in N$ such that $S_D(h) = N \setminus \{h\}$. Then the dictator property implies that $f_h(D) = 2^n$. Since each actor $k \in N \setminus \{h\}$ has the same inclination as h in half of the inclination vectors in $\{0,1\}^n$, satisfaction normalization implies that $\sum_{k \in N \setminus \{h\}} f_k(D) = \sum_{k \in N \setminus \{h\}} 2^{n-1} = (n-1)2^{n-1}$. Symmetry then implies that $f_k(D) = 2^{n-1}$ is determined for all $k \in N \setminus \{h\}$.

Proceeding by induction, suppose that $f(\widehat{D})$ is uniquely determined whenever $|S_{\widehat{D}}(h)| > |S_D(h)|$. Take a $j \in N \setminus (\{h\} \cup S_D(h))$. Note that j is an independent actor since h is the only actor with successors. Consider $D' \in \mathcal{D}^N$ given by $D' = D \cup \{(h,j)\}$. Dictated independence and the induction hypothesis imply that $f_k(D)$ is uniquely determined for all $k \in S_D(h)$. Symmetry implies that there is a constant $c \in \mathbb{R}$ such that

$$f_k(D) = c \text{ for all } k \in N \setminus (\{h\} \cup S_D(h)). \tag{4.6}$$

The opposite gain property implies that

$$f_h(D') - f_h(D) = f_j(D) - f_j(D'), \tag{4.7}$$

where $f_h(D')$ and $f_j(D')$ are given by the induction hypothesis. Then, with satisfaction normalization, (4.6) and (4.7) yield $(n - 1 - |S_D(h)|) + 1 + 1 = n - |S_D(h)| + 1$ linearly

[11] To make the paper self-contained we also put these proofs in the appendix of this paper.

independent equations with the $n - |S_D(h)| + 1$ unknowns, c and $f_i(D),\ k \in N \setminus S_D(h)$. Thus $f(D)$ is uniquely determined.

Next, we prove that $f(D)$ is uniquely determined for all $D \in \mathcal{D}^N$ by induction on $|D|$. From above it follows that $f(D)$ is uniquely determined if $D = \emptyset$.[12]
Proceeding by induction, assume that $f(\widehat{D})$ is uniquely determined whenever $|\widehat{D}| < |D|$. We distinguish the following cases with respect to actor $k \in N$ (of which at least one must occur):

1. If $|P_D(k)| = 1$ then dictated independence and the case with a dictator considered before imply that $f_k(D) = 2^{n-1}$ is uniquely determined.

2. If there is a $j \in S_D(k)$ with $|P_D(j)| = 1$ then the opposite gain property implies that

$$f_k(D) + f_j(D) = f_k(D \setminus \{(k,j)\}) + f_j(D \setminus \{(k,j)\}). \tag{4.8}$$

   Since actor j is as in case 1, we determined $f_j(D)$. With the induction hypothesis $f_k(D \setminus \{(k,j)\})$ and $f_j(D \setminus \{(k,j)\})$ are determined. Thus, with (4.8), $f_k(D) = f_k(D \setminus \{(k,j)\}) + f_j(D \setminus \{(k,j)\}) - f_j(D)$ is uniquely determined.

3. If there is a $j \in S_D(k)$ with $|P_D(j)| \geq 2$ then take $h \in P_D(j) \setminus \{k\}$. The equal gain property implies that

$$f_k(D) - f_k(D \setminus \{(k,j)\}) = f_j(D) - f_j(D \setminus \{(k,j)\}) \tag{4.9}$$

   and

$$f_h(D) - f_h(D \setminus \{(h,j)\}) = f_j(D) - f_j(D \setminus \{(h,j)\}). \tag{4.10}$$

   Horizontal neutrality implies that

$$f_k(D) - f_k(D \setminus \{(k,j)\}) = f_h(D \setminus \{(k,j)\}) - f_h(D). \tag{4.11}$$

   With the induction hypothesis the values in graphs $D \setminus \{(k,j)\}$ and $D \setminus \{(h,j)\}$ are uniquely determined. Thus, with the three linearly independent equations (4.9), (4.10) and (4.11), the payoffs $f_k(D),\ f_j(D)$ and $f_h(D)$ are uniquely determined.

4. If $|P_D(k)| \geq 2$ then $f_k(D)$ is uniquely determined as in the previous case (with the roles for k and j reversed).

[12]Note that this also follows from symmetry and satisfaction normalization.

5. Finally, symmetry implies that there is a constant $c \in \mathbb{R}$ such that $f_k(D) = c$ for all $k \in \mathit{IND}(D)$. Since above we determined all $f_j(D)$ for $j \in \mathit{OL}(D) \cup \mathit{FOL}(D)$, satisfaction normalization determines c.

Thus, all $f_k(D),\ k \in N$, are uniquely determined. □

As mentioned earlier, the power score *POW* satisfies all properties except satisfaction normalization. Replacing satisfaction normalization by power normalization characterizes *POW*.

Theorem 2 *Let the choice vector V be defined by (2.2). A score $f\colon \mathcal{D}^N \to \mathbb{R}^n$ is the power score $\mathit{POW}\colon \mathcal{D}^N \to \mathbb{R}^n$ if and only if it satisfies symmetry, the dictator property, dictated independence, the opposite gain property, horizontal neutrality, the equal gain property, and power normalization.*

Proof

POW satisfying symmetry, the dictator property, dictated independence, and the opposite gain property is shown in van den Brink, Rusinowska, and Steffen (2011).[13] It is obvious that *POW* satisfies power normalization.

To show the equal gain property and horizontal neutrality[14], let $D, D' \in \mathcal{D}^N$, $j \in \mathit{FOL}(D)$, $h \in \mathit{OL}(D) \cup \mathit{IND}(D)$, and $D' = D \cup \{(h, j)\}$. Note that $\overline{\mathit{POW}}_k(I, D) = 1$ implies that (i) $\overline{\mathit{POW}}_k(I, D') = 1$ for $k \in \{h, j\}$, and (ii) $\overline{\mathit{POW}}_k(I, D) = 0$ implies that $\overline{\mathit{POW}}_k(I, D') = 0$ for all $k \in P_D(j)$. Since $[\overline{\mathit{POW}}_h(I, D) = 0$ and $\overline{\mathit{POW}}_h(I, D') = 1]$ if and only if $[I_j = I_h \neq I_k$ for all $k \in P_D(j)$ and $C(I', D) \neq C(I, D)$ for $I'_j = I'_h \neq I_h$ and $I'_k = I_k$ for all $k \in N \setminus \{h, j\}]$ if and only if $[\overline{\mathit{POW}}_j(I, D) = 0$ and $\overline{\mathit{POW}}_j(I, D') = 1]$ if and only if $[\overline{\mathit{POW}}_k(I, D) = 1$ and $\overline{\mathit{POW}}_k(I, D') = 0$ for all $k \in P_D(j)]$, *POW* satisfies the equal gain property and horizontal neutrality.

Uniqueness follows similar as the uniqueness part of the proof of Theorem 1, but using the alternative power normalization. □

Note that Theorems 1 and 2 characterize the satisfaction score *SAT* and the power score *POW* by the same axioms except the normalization axiom. Thus, we have two comparable axiomatizations which exactly illustrate the difference between satisfaction and power

[13]Again, to make the paper self-contained we put these proofs in the appendix of this paper.

[14]In van den Brink, Rusinowska, and Steffen (2011) it is already shown that horizontal neutrality holds if the follower has exactly one opinion leader.

which lies in the normalization being applied.[15] This expresses the basic difference in measuring satisfaction or power.

Furthermore, note that satisfaction normalization and the dictator property imply that the satisfaction score *SAT* also satisfies the property that in case there is a dictator, the satisfaction score of a dictated actor is half the satisfaction score of the dictator. On the other hand, power normalization and the dictator property imply that the power score *POW* satisfies the property that the power score of actors that are subordinate to a dictator is equal to zero.[16]

Proposition 3 *(i)* **(Dictator domination satisfaction property)** *If* $D \in \mathcal{D}^N$, $OL(D) = \{h\}$, *and* $FOL(D) = N \setminus \{h\}$, *then* $SAT_k(D) = \frac{1}{2} SAT_h(D)$ *for all* $k \in N \setminus \{h\}$.

(ii) **(Dictator domination power property)** *If* $D \in \mathcal{D}^N$, $OL(D) = \{h\}$, *and* $FOL(D) = N \setminus \{h\}$, *then* $POW_k(D) = 0$ *for all* $k \in N \setminus \{h\}$.

This expresses another difference between measuring satisfaction and power. If there is a dictator then the other actors cannot influence the outcome of the voting process since they have to follow the dictator. Therefore their power is equal to zero. However, also an actor that is subordinate to a dictator might end up with a social outcome that coincides with its inclination. Ex ante, a subordinate of a dictator will have its inclination coincide with that of the dictator in half of the cases. Since a dictator always dictates the outcome we arrive at a satisfaction score of the subordinate that is half the satisfaction score of the dictator.

5 Future Research

The existence of opinion leaders and their influence over other actors can be seen in every day life situations: in small as well as in large societies, be it in politics or business. Both satisfaction and power are the very natural measures of actors' *strength* or *status* in such situations. Since both are different concepts, it is worth to analyze what the common and different properties of the scores for both concepts display. Although, as mentioned in the introduction, there exist several related theoretical studies in the literature on voting models and on networks, the approach which we use in this paper, i.e., the analysis of *opinion leader-follower* structures, and the properties of the scores in question has brought

[15] A similar difference is shown by van den Brink and Gilles (2000) for the outdegree measure and the β-measure as scoring methods for directed graphs, highlighting that a normalization is not always so innocent as it might appear.

[16] The power score *POW* even satisfies a stronger property which states that if a follower has a unique opinion leader, then its power is equal to zero, i.e., if $D \in \mathcal{D}^N$, $j \in FOL(D)$ and $h \in OL(D)$ such that $P_D(j) = \{h\}$, then $f_j(D) = 0$.

up several innovative elements and can also be regarded to contribute to knowledge, in particular, in marketing.

Some generalizations that we will consider in the future are the following. First, one could allow for an actor to be an opinion leader and follower at the same time, i.e., to have a society with more than two 'layers'. In terms of Troldahl's (1966) *two-cycle flow of communication* model this would allow us to include the experts as an additional group of actors into our analysis acting as opinion leaders of the opinion leaders. Secondly, one could enlarge the set of possible actions, i.e., instead of allowing for just two actions one could follow some works on abstention (see, e.g., Braham and Steffen (2002), Felsenthal and Machover (1997, 1998, 2001), Tchantcho, Diffo Lambo, Pongou, and Mbama Ebgoulou (2008)), and multi-choice games (see, e.g., Grabisch and Rusinowska (2010), and Hsiao and Raghavan (1993)). Related models are also games with r alternatives, where the alternatives are not ordered; see Bolger (1986, 1993, 2000, 2002). Also in Freixas (2005a, 2005b) and Freixas and Zwicker (2003), the authors consider decision-making situations, i.e., voting systems, with several levels of approval in the input and output, where those levels are qualitatively ordered. They introduce (j, k) simple games, in which each actor expresses one of j possible levels of input support, and the output consists of one of k possible levels of collective support. Thirdly, one could endogenize the 'higher level' constitutional inclinations.

In this paper the choice vector is determined by unanimity. A further item for future research is to axiomatize the satisfaction and power score for other methods determining the choice vector, such as those defined by qualified majority. Moreover, one could apply the formal approach in the present paper to related problems in organizational hierarchies where an organizational choice is to be made, see, e.g., Hammond and Thomas (1990). Finally, one could apply our framework to concrete real life situations. Here, in particular, in the light of the financial crisis of 2007-2010 the herding behavior in financial markets (see, e.g., Devenow and Welch (1996)) appears to be a promissing field for an application.

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Appendix

To make the paper self-contained, this appendix contains proofs of parts of Theorems 1 and 2 that are also given in van den Brink, Rusinowska, and Steffen (2011).

Proposition A.1 *Let the choice vector V be defined by (2.2). The satisfaction score SAT satisfies symmetry, the dictator property, dictated independence, and the opposite gain property.*

Proof
It is straightforward that SAT satisfies symmetry.

The dictator property follows straightforward since a dictator is followed in all 2^n inclination vectors in $\{0,1\}^n$, i.e., if $S_D(h) = N \setminus \{h\}$, then $C(I,D) = I_h$ for all $I \in \{0,1\}^n$.

To show dictated independence, note that actor k always chooses an action according to j's inclination if $P_D(k) = \{j\}$. That means that the collective choice is independent of actor k's inclination, i.e., $C(I,D) = C(I',D)$ if $I_h = I'_h$ for all $h \in N \setminus \{k\}$. Hence, in half of the inclination vectors $C(I,D) = I_k$ and in the other half $C(I,D) \neq I_k$. So, SAT satisfies dictated independence.

To show the opposite gain property, let $D, D' \in \mathcal{D}^N$, $j \in IND(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h,j)\}$. Suppose that $C(I,D) \neq C(I,D')$. Then it must hold that actor j had to deviate from its inclination and follow h, and this must result in a change of collective choice from I_j to I_h, with $I_j \neq I_h$. So, $C(I,D) = I_j \neq I_h$ and $C(I,D') = I_h \neq I_j$. Then, $\overline{SAT}_j(I,D') = \overline{SAT}_j(I,D) - 1$ and $\overline{SAT}_h(I,D') = \overline{SAT}_h(I,D) + 1$. So, $\overline{SAT}_h(I,D') - \overline{SAT}_h(I,D) = \overline{SAT}_j(I,D) - \overline{SAT}_j(I,D')$.
Obviously, this last equality also holds if $C(I,D) = C(I,D')$. Thus, we have $SAT_h(D') - SAT_h(D) = SAT_j(D) - SAT_j(D')$, showing that SAT satisfies the opposite gain property.

□

Proposition A.2 *Let the choice vector V be defined by (2.2). The power score POW satisfies symmetry, the dictator property, dictated independence, and the opposite gain property.*

PROOF
It is straightforward that POW satisfies symmetry.

Since a dictator has a swing in every inclination vector, POW satisfies the dictator property.

Since an actor with a unique opinion leader never has a swing, POW satisfies dictated independence.

To show the opposite gain property, let $D, D' \in \mathcal{D}^N$, $j \in IND(D)$, $h \in OL(D) \cup IND(D)$, and $D' = D \cup \{(h,j)\}$. Since in D' voter j has to choose an action according to its unique opinion leader h, j never has a swing in D', i.e., $POW_j(D') = 0$. So, we have to show that $POW_h(D') - POW_h(D) = POW_j(D)$.
We distinguish the following three cases. (i) If h does not have a swing in (I,D) but j has a swing in (I,D) then h has a swing in (I,D'), i.e., if $\overline{POW}_h(I,D) = 0$ and $\overline{POW}_j(I,D) = 1$ then $\overline{POW}_h(I,D') = 1$.
(ii) If h has a swing in (I,D) then h has a swing in (I,D'), i.e., if $\overline{POW}_h(I,D) = 1$ then $\overline{POW}_h(I,D') = 1$. If, moreover, also j has a swing in (I,D) then h also has a swing

in (I', D') with $I'_j = I'_h \neq I_h = I_j$, i.e., if $\overline{POW}_h(I, D) = 1$ and $\overline{POW}_j(I, D) = 1$ then $\overline{POW}_h(I', D') = 1$.

(iii) Finally, if h does not have a swing in (I, D) and j has a swing in (I, D) then the only possibility for h to have a swing in (I, D') is as described in the last case before. So, $POW_h(D') = \sum_{I \in \{0,1\}^n} \overline{POW}_h(I, D') = \sum_{I \in \{0,1\}^n} (\overline{POW}_h(I, D) + \overline{POW}_j(I, D))$, showing that POW satisfies the opposite gain property. □