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Economic Costs and Benefits of Imposing Short-Horizon Value-at-Risk Type Regulation

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Abstract

Regulators often set value-at-risk (VaR) constraints to limit the portfolio risk of institutional investors. For some investors, notably pension funds, the VaR constraint is enforced over a horizon which is significantly shorter than the investment horizon of the investor. Our paper aims to investigate the economic costs and benefits of this kind of regulation. Shorter regulatory constraint, on one hand, enables an institutional investor, like a pension fund, to avoid large losses when the investment environment worsens but, on the other hand, also limits the institutional investor's ability to benefit from an increase in stock prices. We show that the cost introduced by the short-term VaR constraints might over weight the benefits brought by such constraints.

Keywords: Portfolio Choice, Value-at-Risk, Pension Funds

JEL Classifications: G11, G23

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1 Introduction

This paper investigates the economic consequences of a difference in the planning horizon between an institutional investor pursuing long-term investment strategies and a regulator enforcing prudential standards and practices on a short-term basis. Such a misalignment of horizons between an institutional investor and a regulator are likely to exist in most developed financial markets and affect, for example, banks, insurance companies, and, notably, pension funds.

Consider the case of a bank operating under the Basel Accords. Under Basel II a bank will be required to hold credit risk capital determined by 1-year default probabilities and expected shortfalls. The regulatory horizon is likely to stand in sharp contrast with the horizon of long-term investment projects involving payoffs in a more distant future a bank has to evaluate with respect to his creditworthiness.

A second case is a pension fund, which faces long-term pension liabilities with typical maturities of 15 years or more under a regulatory framework which imposes short-term solvency constraints. In 2002, the Australian Prudential Regulation Authority (APRA) launched the Probability and Impact Rating System (PAIRS). Using this system, APRA first estimates each pension fund's probability that the institution may be unable to honor its financial promises to its members and then assigns supervisory stance accordingly. A more recent example can be observed in the Netherlands where a pension regulatory regime (Financieel Toetsings Kader, FTK) is effective as of January 2007. According to the Dutch regulation, pension funds should always keep the probability of underfunding one year ahead below 2.5%. Underfunding refers to the situation where the market value of a pension fund's assets falls below the market value of the pension fund's liabilities.

The existence of such funding constraints can be understood in light of the recent experience of a simultaneous decrease in pension assets due to a poor stock market performance and an increase in pension liabilities due to low interest rates. For the UK, KPMG estimated the aggregate funding deficit of the FTSE-100 companies reaches GBP 40 billion at the end of 2008. De Nederlandsche Bank reports that the average Dutch pension funding ratio dropped from 144% in 2007 to 99% in the third quarter of 2010. Of all Dutch pension funds, around 68% has a funding ratio below 105%. The situation in the US is equally alarming. The funding deficit in America's corporate pension funds is estimated to be 350bn USD (Jørgensen 2007).

The examples above demonstrate the particular importance of VaR constraints in regulatory practice despite the theoretical shortcomings of this risk measure (see Artzner et al.1999). For this reason, we focus on VaR-type regulation. This paper provides an analytical tool to assess the costs and benefits of a VaR-type regulation when the regulatory horizon is as long as the investment horizon and when the regulatory horizon is shorter than the investment horizon. In the latter case, within the investor's investment horizon, there are a number of subsequent and non-overlapping regulatory checks and the investment horizon is divided into a few equal-length sub-periods. In general, the investor has to insure his portfolio against the bad performance of the financial market to guarantee that (1) the current period VaR constraint is met and (2) there is enough wealth to fulfill the next periods VaR constraints. To do so, the investor has to hold more risk-free assets and less risky assets, thus, his ability to profit from the favorable financial market performance is limited. The economic benefits and costs are measured by the reduction in the expected portfolio wealth shortfall and the equivalent amount of wealth lost due to the VaR regulation respectively. We show that a short regulatory horizon can prevent portfolio wealth loss very effectively but at the same time also introduces a large opportunity cost by limiting the investor's ability to invest in risky assets and profit from the favorable stock market performance.

The strategic asset allocation problem has been studied extensively. For example, Kim and Omberg (1996) and Wachter (2002) study the optimal portfolio allocation where the price of risk is mean-reverting. Bajeux-Besnainou, Jordan and Portait (2003) and Sorensen (1999) solve the optimal investment problem when interest rates are stochastic. This paper is related to the literature studying the optimal portfolio trading strategy under constraints. Grossman and Vila (1992) provide explicit solutions to optimal portfolio problems containing leverage and minimum portfolio return constraints. Basak (1995) and Grossman and Zhou (1995) focus on the impact of a specific VaR constraint, the portfolio insurance[1], on asset price dynamics in a general equilibrium model. Van Binsbergen and Brandt (2006) assess the influence of ex ante (preventive) and ex post (punitive) risk constraints on dynamic portfolio trading strategies. Ex ante risk constraints include, among others, VaR and short sell constraints. Ex post risk constraints include the loss of the investment manager's personal compensation and reputation when the portfolio wealth turns out to be low. They found that ex ante risk constraints tend to decrease gain from dynamic investment while ex post risk constraints can be welfare improving.

Basak and Shapiro (2001) discuss the impact of the VaR type regulation on the institutional investors' portfolio wealth and trading strategies. Their results show that a VaR constraint keeps the portfolio value above or at the threshold value, e.g. the value of a

[1] Portfolio insurance is a special case of VaR constraint, which requires the probability that the portfolio wealth falls below a certain threshold value to be zero.

pension fund's liability, when the investment environment (states of the world) is favorable but generate a loss in unfavorable states of the world. The favorable (unfavorable) states are the ones in which it is cheap (expensive) for the investor to raise his portfolio wealth to the level of the threshold value. Ironically, the loss under a VaR constraint is even larger than the one without a VaR constraint in those unfavorable states. The unfavorable states of world occur with probability α. This probability is set by the regulator. The explanation is as follows. The VaR constrained investor is only concerned about the probability but neither the magnitude of the loss, nor in which (cheap or expensive) states this loss occurs. Therefore, it is optimal for him to incur losses in unfavorable states where it is most expensive to raise his portfolio wealth.

In Basak and Shapiro (2001), the VaR horizon is as long as the investment horizon and interest rates are deterministic. We extend the Basak and Shapiro (2001) paper by (1) embedding a number of subsequent and non-overlapping short-term VaR type regulations in the portfolio optimization problem and (2) allowing for a stochastic interest rate. We show that more frequent regulation can prevent the investor from generating a loss in unfavorable states due to the fact that there is a minimum amount of portfolio wealth required to fulfill future VaR constraints.

We also analyzed the portfolio choices of three types of investors, namely, an investor facing a number of short-term VaR constraints during his investment horizon (the multi-VaR investor), an investor facing one long-term VaR constraint during his investment horizon (the one-VaR investor), and an unconstrained investor (the benchmark investor). We find that the portfolio choice of the multi-VaR investor differs considerably from the one of the one-VaR investor and the benchmark investor. The deviation is most profound in the "transitional" states where there is a highest uncertainty regarding whether the future investment environment will turn out to be favorable or not. The one-VaR investor will have a large demand for risky assets and a low demand for riskless assets to finance a large portfolio wealth if the financial market turns out to be favorable, and generate a large amount of loss otherwise. The multi-VaR investor has a much smaller demand for risky assets and a much larger demand for riskless assets to compensate for the loss the risky assets might generate if the financial market turns out to be unfavorable and thus, guarantees that his portfolio wealth is sufficient to fulfill future VaR constraints in all circumstances.

Finally, we find that both the size and the probability of having a high or a low portfolio wealth level decrease as the regulatory frequency increases. More frequent VaR constraints does seem to be effective in reducing the size and the likelihood of a portfolio wealth loss but at the cost of losing the ability to benefit from the booming financial market.

Cuoco et al. (2008) considers the optimal trading strategy of institutional investors under short-horizon VaR constraints assuming that the portfolio allocation over the VaR horizon is constant and the interest rate is deterministic. We extend Cuoco et al. (2008) by allowing for optimal and time-varying portfolio allocations over the VaR horizon and having a stochastic interest rate. This enables us to quantify the costs and benefits of a VaR regulation given that the institutional investor behaves optimally and study the hedge behavior of the investor under short-horizon VaR constraints.

This paper is also related to the literature about dynamic trading strategies of pension funds. Sundaresan and Zapatero (1997) considers an optimal asset allocation with a power utility function in final surplus. Boulier et al. (2005) assume a constant investment opportunity set with a risky and a risk-free asset. In their paper, the pension plan sponsor aims to minimize the expected discounted value of future contributions over a given horizon. Inkmann and Blake (2008) proposes a new approach to the valuation of pension obligations taking into account the asset allocation strategy and the underfunding risk of a pension fund. This paper focuses on the optimal portfolio wealth of a pension fund when the regulatory horizon is shorter than its investment horizon and evaluates the economic costs and benefits of such a regulation. Advantages of having frequent short-term VaR constraints include, among others, smaller expected portfolio wealth losses.

The outline of the paper is as follows. Section 2 describes the investment environment. Section 3 studies the optimal portfolio wealth and trading strategies under VaR constraints and Section 4 discusses the economic costs and benefits of imposing short-term value-at-risk type of regulation. Section 5 concludes.

2 The Investment Environment

We consider a continuous-time stochastic economy on a finite horizon $[0, T]$ in a complete financial market. There are four assets in the market, namely, a zero-coupon bond maturing at time T, a cash account, a stock index and a constant maturity zero-coupon bond fund. The stock index (with dividend reinvested) is assumed to follow,

$$dS_t = (r_t + \Phi_S)\, S_t dt + \sigma_S S_t dZ_{S,t}, \tag{1}$$

where r_t is the short-term interest rate, Φ_S is the stock risk premium, σ_S is the instantaneous stock price volatility and $Z_{S,t}$ is a standard Brownian motion. The short-term interest rate r_t follows a Vasicek process,

$$dr_t = \kappa\left(\overline{r} - r_t\right)dt - \sigma_r dZ_{r,t}, \tag{2}$$

where κ determines the mean-reverting speed of the interest rate, $\overline{r}$ is the long-term interest rate, and σ_r is the instantaneous volatility of the interest rate. $Z_{r,t}$ is a standard Brownian motion. The two Brownian motions, $Z_{r,t}$ and $Z_{S,t}$, are correlated with correlation coefficient ρ_{sr}. Vasicek (1977) derived the price of a zero-coupon bond at time t with $T-t$ years to maturity and a face value of \$1, $P_t^{(T-t)}$,

$$P_t^{(T-t)} = \exp\left(-A\left(T-t\right) - B\left(T-t\right)r_t\right), \tag{3}$$

where

$$\begin{aligned} A\left(T-t\right) &= R_\infty\left(\left(T-t\right) - B\left(T-t\right)\right) + \frac{\sigma_r^2}{4\kappa}B\left(T-t\right)^2 \\ R_\infty &= \overline{r} + \frac{\Phi_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}, \\ B\left(T-t\right) &= \frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}, \end{aligned}$$

Φ_r describes the interest rate risk premium, and R_∞ describes the yield to maturity for a very long time-to-maturity bond (i.e., $T-t$ approaches to infinity). The duration of this bond D_t is defined as the elasticity of the bond price with respect to the short-term interest rate, i.e.,

$$D_t = -\frac{dP_t^{T-t}}{dr_t}\frac{1}{P_t^{T-t}}.$$

For this zero coupon bond, the duration is described by $B\left(T-t\right)$. Using the Ito-Doeblin Lemma, the dynamics of the zero-coupon bond prices are given by

$$\begin{aligned} \frac{dP_t^{(T-t)}}{P_t^{(T-t)}} &= \left(r_t + \Phi_r B\left(T-t\right)\right)dt + \sigma_r B\left(T-t\right)dZ_{r,t}, && (4) \\ &= \left(r_t + \Phi_P \sigma_{P_t^{(T-t)}}\right)dt + \sigma_{P_t^{(T-t)}} dZ_{r,t}, && (5) \end{aligned}$$

where the price of the bond risk, Φ_P, is Φ_r/σ_r, and the standard deviation of the bond $\sigma_{P_t^{(T-t)}}$ is $\sigma_r B\left(T-t\right)$.

Following (4), the price dynamics of a bond fund with M-year constant time to maturity[2], P_t^M, are

$$\frac{dP_t^M}{P_t^M} = \left(r_t + \Phi_r B\left(M\right)\right) dt + \sigma_r B\left(M\right) dZ_{r,t}.$$

All our bonds are not defautable.

Since the market is dynamically complete, there exists one unique pricing kernel. Let ζ_t describe the diffusion process of the pricing kernel,

$$\frac{d\zeta_t}{\zeta_t} = -r_t dt - \hat{\phi}_s dZ_{s,t} + \hat{\phi}_r dZ_{r,t}, \tag{6}$$

where

$$\begin{aligned} \hat{\phi}_s &= \frac{\sigma_r \Phi_S - \rho_{sr} \Phi_r \sigma_S}{\sigma_r \sigma_S \left(1 - \rho_{sr}^2\right)}, \\ \hat{\phi}_r &= \frac{\sigma_r \Phi_S \rho_{sr} - \Phi_r \sigma_S}{\sigma_r \sigma_S \left(1 - \rho_{sr}^2\right)}. \end{aligned}$$

Merton (1992) shows that the pricing kernel can be constructed as the inverse of the growth-optimum portfolio. Please See Appendix A for the derivation and the verification of ζ_t.

3 Optimal Portfolio Wealth and Trading Strategies

We consider the problem of an institutional investor who starts with an endowment W_0 and must select a portfolio $\pi \in \Pi$ so as to maximize the expected utility $E[u(W_T)]$ of the terminal value of the trading portfolio. Assume that the institutional investor has a power utility function with constant relative risk aversion (CRRA) parameter γ.

The regulator imposes a VaR constraint on the institutional investor: the probability that the portfolio wealth at time $t+\tau$ falls below $\underline{W}$ should not be larger than α, where α is usually a small number in the interval $[0,1]$. The VaR constraint can be formulated as

$$\Pr_t\left(W_{t+\tau} < \underline{W}\right) \leq \alpha, \quad t \in [0,T],$$

where τ, $\tau > 0$, is the regulatory horizon, $\alpha \in [0,1]$ and the "floor" $\underline{W}$ is specified exogenously. For a pension fund, the "floor" is the value of its liability at time $t+\tau$. In

[2] In a stochastic interest rate investment environment, if the zero-coupon bond matures at time T is the only bond in the portfolio, this bond then has two tasks, which are (1) to achieve the optimal interest rate risk exposure for speculative purpose and (2) to hedge the interest rate risk as it is a risk-free asset over the investment horizon. To uncouple these two functions we introduce a bond fund with H-year constant time to maturity, P_t^H, as in Bajeux-besnainou et al. (2003).

this paper, both τ and $\underline{W}$ are set by the regulator.

In the single-constraint model, the regulatory horizon τ is as long as the investment horizon τ. At time 0, the regulator requires that the probability the portfolio wealth at time T falls below $\underline{W}$ should be smaller than α, say 2.5%,

$$\Pr_0 (W_T < \underline{W}) \leq \alpha.$$

In the two-constraint and the more general multi-constraint models, the investment horizon stays the same but the regulatory horizons become shorter and shorter. In the two-constraint model, the regulatory horizon equals half of the investment horizon, meaning that there are two subsequent and non-overlapping VaR constraints within the investment horizon, i.e.,

$$\begin{aligned} \Pr_0 \left(W_{\frac{T}{2}} < \underline{W} \right) &\leq \alpha, \\ \Pr_{\frac{T}{2}} (W_T < \underline{W}) &\leq \alpha. \end{aligned}$$

In the multi-constraint model, say m-constraint model, each VaR horizon is T/m and there are m non-overlapping and subsequent VaR constraints within the investment horizon.

3.1 Without Regulatory Constraints

When no VaR constraints are imposed, the investor's optimization problem is,

$$\max_{W_T} E_0 \frac{W_T^{1-\gamma}}{1-\gamma} \tag{7}$$

$$s.t. \;\; E_0 (\zeta_T W_T) \leq \zeta_0 W_0. \tag{8}$$

The solution to this problem is classical, but we provide a short recollection for expository reasons. Following the so-called Martingale method by Cox and Huang (1989), the optimal portfolio wealth at time T without a VaR constraint, W_T^u, is

$$\begin{aligned} W_T^u &= (y\zeta)^{-\frac{1}{\gamma}} \\ &= W_0 \exp \left[-\mu_{\zeta,t,T} \left(1 - \frac{1}{\gamma}\right) - \frac{1}{2} \left(1 - \frac{1}{\gamma}\right) \sigma^2_{\zeta,t,T} \right] \zeta_T^{-\frac{1}{\gamma}}, \end{aligned}$$

where the Lagrangian multiplier y equates the budget constraint (8), $\mu_{\zeta,t,T}$ and $\sigma^2_{\zeta,t,T}$ represent the mean and the variance of the log normally distributed pricing kernel at time t respectively. The values of $\mu_{\zeta,t,T}$ and $\sigma^2_{\zeta,t,T}$ are given in Appendix A.

Let t stand for any pre-horizon time. Let $\pi^u_{S_t}$, $\pi^u_{P^M_t}$, and $\pi^u_{P^{T-t}_t}$ stand for the percentages of portfolio wealth invested in the stock index, the bond fund and the zero-coupon bond at time t respectively. $1 - \pi^u_{S_t} - \pi^u_{P^M_t} - \pi^u_{P^{T-t}_t}$ percentage of portfolio wealth is invested in the cash account. The subscript u refers to the case when no VaR constraints are imposed. The optimal portfolio without a VaR constraint is then given by

$$\begin{bmatrix} \pi^u_{S_t} \\ \pi^u_{P^M_t} \\ \pi^u_{P^{T-t}_t} \end{bmatrix} = \underbrace{\frac{1}{\gamma}}_{\text{term 1}} \underbrace{\begin{bmatrix} \frac{1}{\sigma_S}\hat{\phi}_s \\ -\frac{1}{\sigma_r B(M)}\hat{\phi}_r \\ 0 \end{bmatrix}}_{\text{term 2}} + \underbrace{\left(1 - \frac{1}{\gamma}\right)}_{\text{term 3}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{term 4}}. \tag{9}$$

The optimal portfolio is a weighted average of two funds, a speculative fund (the second term on the right-hand side of (9)) and a hedge fund (the fourth term on the right-hand side of (9)). The portfolio strategy of the speculative fund depends on, among others, the risk premia of underlying assets. The portfolio strategy of the hedge fund is driven by the investor's desire to hedge interest rate risk. Therefore, the hedge fund consists only of the zero-coupon bond which matures at the end of the investment horizon. The portfolio weights assigned to these to funds are the relative risk tolerance $1/\gamma$ on the speculative fund and the complement $1 - 1/\gamma$ on the hedge fund. For an investor with a log utility ($\gamma = 1$), the hedge term vanishes. For an investor who is extremely risk averse ($\gamma = \infty$), this investor will invest his entire wealth in the hedge fund. Provided that the risk premia, Φ_S and Φ_r, are constant, portfolio weights on these four assets are constant over time as in Bajeux-Besnainou, et. al. (2003). Appendix B.1 provides a short derivation of (9).

The indirect utility of current wealth at time t, is the maximum expected utility conditional on the information available at time t. The maximum is obtained by the optimal wealth process. Without a VaR constraint, the indirect utility at time t takes the form,

$$J^u_t\left(W^u_t\right) = \underbrace{\frac{\left(W_t \exp\left[\left(A\left(T-t\right) + B\left(T-t\right) r_t\right)\right]\right)^{1-\gamma}}{1-\gamma}}_{\text{term 1}} \underbrace{\exp\left[\frac{1}{2}\left(\frac{1}{\gamma} - 1\right)\sigma^2_{\zeta,t,T}\right]}_{\text{term 2}} \tag{10}$$

Sorensen (2001) argues that the indirect utility of current wealth can be interpreted as the expected utility of future wealth as if current wealth were invested in a zero-coupon bond which matures at time T without default risk (the first term in (10)) and adjusted with the term $e^{\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\sigma^2_{\zeta,t,T}}$. As in (4), $\exp\left[\left(A\left(T-t\right) + B\left(T-t\right) r_t\right)\right]$ is the inverse of the price of a zero-coupon bond with $T - t$ years to maturity and a face value of \$1. The adjustment term reflects the investor's risk-return trade-off from holding risky securities. The indirect utility at time t is increasing with both portfolio wealth at time t, W_t, and the interest rate at time t, r_t. A high (low) interest rate will result in a high (low) yield of the zero-coupon bond and, therefore, a high indirect utility. Appendix B.2 provides a short derivation of (10).

3.2 The Single-Constraint Model

In our single-constraint model, the VaR horizon is as long as the investment horizon, the investor's optimization problem is

$$\max_{W_T} E_0 \frac{W_T^{1-\gamma}}{1-\gamma} \tag{11}$$

$$\text{subject to} \quad E_0\left[\zeta_T W_T\right] \leq \zeta_0 W_0, \tag{12}$$

$$\Pr_0\left(W_T < \underline{W}\right) \leq \alpha. \tag{13}$$

The problem (11) can be re-stated as

$$\mathbb{L}\left(W_T, \zeta_T\right) = E_0\left[\frac{W_T^{1-\gamma}}{1-\gamma} - y^{c1} W_T \zeta_T + y_2^{c1} \mathbb{I}_{W_T \geq \underline{W}}\right] + y \zeta_0 W_0 - y_2\left(1-\alpha\right), \tag{14}$$

where "$\mathbb{I}$" is an indicator function which takes the value of 1 if $W_T \geq \underline{W}$ holds, the Lagrangian multipliers y^{c1} and y_2^{c1} solve (12) and (13) respectively. The subscript $c1$ stands for one VaR constraint. Since the last two terms of (14) are constants, the optimal wealth in the single constraint model W_T^{c1} is the wealth which maximizes the function value within $E_0\left[\cdot\right]$ in (14).

Following Basak and Shapiro (2001), the optimal portfolio wealth is built pointwise on the realized value of the pricing kernel at time T (ζ_T). The pricing kernel, ζ_T, takes different values in different states of the world at time T. We use ω to indicate the states of the world, where $\omega \in \Omega$, and Ω refers to the sample space. For each given value of $\zeta_{T,\omega}$, $\omega \in \Omega$, the optimal portfolio wealth is the one which maximizes the value of $\mathbb{L}\left(W_T, \zeta_{T,\omega}\right)$. Basak and Shapiro (2001)'s result can be directly applied here even though in our case interest rates are stochastic. The optimal portfolio wealth at time T when the VaR horizon is as long as the investment horizon is

$$W_T^{c1} = \left(y^{c1}\zeta_T\right)^{-\frac{1}{\gamma}} \mathbb{I}_{\zeta_T \leq \min\left(\underline{\zeta}_T^{c1}, \overline{\zeta}_T^{c1}\right)} + \underline{W}\mathbb{I}_{\min\left(\underline{\zeta}_T^{c1}, \overline{\zeta}_T^{c1}\right) < \zeta_T \leq \overline{\zeta}_T^{c1}} + \left(y^{c1}\zeta_T\right)^{-\frac{1}{\gamma}} \mathbb{I}_{\zeta_T \geq \overline{\zeta}_T^{c1}}, \tag{15}$$

where $\underline{\zeta}_T^{c1}$ is defined by $\left(y^{c1}\underline{\zeta}_T^{c1}\right)^{-\frac{1}{\gamma}} \equiv \underline{W}$, $\overline{\zeta}_T^{c1}$ is defined by $\Pr_0\left(\zeta_T \geq \overline{\zeta}_T^{c1}\right) \equiv \alpha$ which means that at time 0 the probability that ζ_T will be larger than $\overline{\zeta}_T^{c1}$ is α. When the VaR constraint is binding, we have $\underline{\zeta}_T^{c1}$ smaller than $\overline{\zeta}_T^{c1}$. When the VaR constraint is not binding, we will have $\underline{\zeta}_T^{c1}$ larger than or equal to $\overline{\zeta}_T^{c1}$ and $W_T^{c1} = W_T^u$.

Figure 1 depicts the portfolio wealth at time T in the single constraint model, W_T^{c1}. In this figure, W_T^{c1} is compared with the portfolio wealth without a VaR constraint, W_T^u. The portfolio wealth without a VaR constraint W_T^u is a decreasing function of ζ_T. The portfolio wealth with a VaR constraint W_T^{c1} falls into three distinct regions in which the investor exhibits different investment behavior. For $\zeta_T \leq \underline{\zeta}_T^{c1}$ ("good" states), the investor behaves similar to the case where there are no VaR constraints. For $\overline{\zeta}_T^{c1} > \zeta_T > \underline{\zeta}_T^{c1}$ ("intermediate" states), the investor keeps his wealth at $\underline{W}$. For $\zeta_T \geq \overline{\zeta}_T^{c1}$ ("bad" states), the fund manager behaves again as if the VaR constraint is not imposed and incurs large losses. The probability that the investor will end up in the region where $\zeta_T \geq \overline{\zeta}_T^{c1}$ is α. Recall that the quantity $\zeta_{T,\omega}$ can be interpreted as the marginal cost at time 0 of obtaining one additional unit of wealth in state ω at time T. Thus, the investment environment worsens as the value of ζ_T increases. Under a VaR constraint, the investor is only concerned with the probability but not the magnitude of a loss, therefore, the investor chooses to raise the portfolio wealth to $\underline{W}$ in states when it is relatively cheap to do so. Thus, the wealth level in states where $\overline{\zeta}_T^{c1} > \zeta_T > \underline{\zeta}_T^{c1}$ is raised to $\underline{W}$ and there are large losses in the "bad" states where $\zeta_T \geq \overline{\zeta}_T^{c1}$ since the probability of ending up in these states is α and it is most expensive to raise the wealth level to $\underline{W}$ in these states. As a result, in the "bad" states, the institutional investor under a VaR constraint generates a larger loss than the one without a VaR constraint.

Basak and Shapiro (2001) shows that W_T^{c1} is equivalent to the sum of unconstrained portfolio wealth $(y^{c1}\zeta_t)^{\frac{1}{\gamma}}$ and a "corridor" option from which the investor will get $\underline{W} - (y^{c1}\zeta_T)^{-\frac{1}{\gamma}}$ when $\min\left(\underline{\zeta}_T^{c1}, \overline{\zeta}_T^{c1}\right) \leq \zeta_T \leq \overline{\zeta}_T^{c1}$ holds and nothing otherwise. That is,

$$W_T^{c1} = \left(y^{c1}\zeta_T\right)^{-\frac{1}{\gamma}} + \left(\underline{W} - \left(y^{c1}\zeta_T\right)^{-\frac{1}{\gamma}}\right) \mathbb{I}_{\min\left(\underline{\zeta}_T^{c1}, \overline{\zeta}_T^{c1}\right) \leq \zeta_T \leq \overline{\zeta}_T^{c1}}. \tag{16}$$

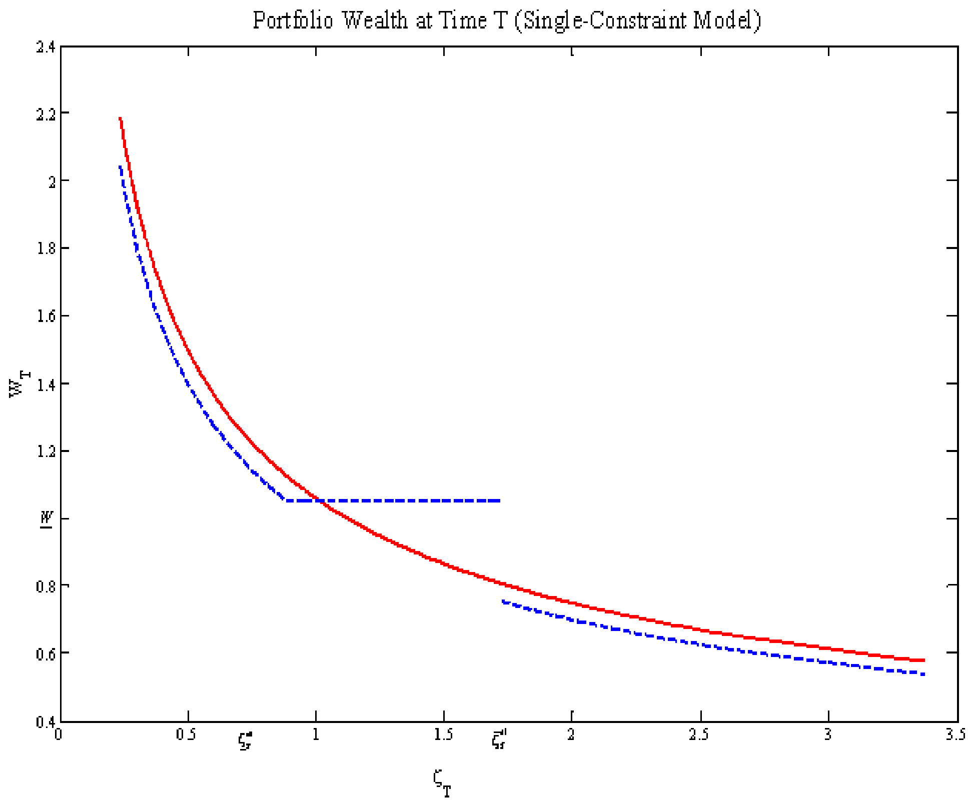

Figure 1: This figure depicts the portfolio wealth at time T in the single constraint model. The blue dashed line represents the optimal portfolio wealth at time T in the single-constraint model W_T^{c1} and the red solid line represnts the optimal portfolio wealth at time T without a VaR constraint W_T^u. $\underline{\zeta}_T^{c1}$ is defined as $\left(y^{c1}\underline{\zeta}_T^{c1}\right)^{-\frac{1}{\gamma}} \equiv \underline{W}$. $\overline{\zeta}_T^{c1}$ is defined as $\Pr_0\left(\zeta_T \geq \overline{\zeta}_T^{c1}\right) \equiv \alpha$. The parameter values are $W_0 = 1$, $\kappa = 0.15$, $\overline{r} = 0.05$, $\sigma_r = 0.015$, $\Phi_P = 0.05$, the maturity of the bond fund $M = 10$ years, $\sigma_s = 0.25$, the stock Sharpe Ratio $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 4\%$, $\gamma = 2$, and $\alpha = 0.025$.

With one VaR constraint, the intermediate wealth at time t is

$$
\begin{aligned}
W_t^{c1} &= \frac{1}{\zeta_t} E_t \zeta_T W_T^{c1} \\
&= \frac{e^{\Gamma_t}}{(y^{c1}\zeta_t)^{\frac{1}{\gamma}}} + \left[-\frac{e^{\Gamma_t}}{(y^{c1}\zeta_t)^{\frac{1}{\gamma}}} N\left(-d_{1,\min\left(\underline{\zeta}_T^{c1},\overline{\zeta}_T^{c1}\right)}\right) \right. \\
&\quad + \underline{W} e^{\left(\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}\right)} N\left(-d_{2,\min\left(\underline{\zeta}_T^{c1},\overline{\zeta}_T^{c1}\right)}\right) - \underline{W} e^{\left(\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}\right)} N\left(-d_{2,\overline{\zeta}_T^{c1}}\right) \\
&\quad \left. + \frac{e^{\Gamma_t}}{(y^{c1}\zeta_t)^{\frac{1}{\gamma}}} N\left(-d_{1,\overline{\zeta}_T^{c1}}\right) \right],
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
d_{2,x} &= \frac{\log\left(\frac{x}{\zeta_t}\right) - \left(\mu_{\zeta,t,T} + \sigma^2_{\zeta,t,T}\right)}{\sigma_{\zeta,t,T}}, \\
d_{1,x} &= d_{2,x} + \frac{1}{\gamma}\sigma_{\zeta,t,T}, \\
\Gamma_t &= \mu_{\zeta,t,T} + \frac{1}{2}\sigma^2_{\zeta,t,T} - \frac{1}{\gamma}\left(\boldsymbol{\mu_{\zeta,t,T}} + \sigma^2_{\zeta,t,T}\right) + \frac{1}{2}\frac{1}{\gamma^2}\sigma^2_{\zeta,t,T}.
\end{aligned}
$$

The first term in (17) derives from the unconstrained portfolio wealth and the remaining terms reflect the value of the "corridor" option at time t.

The percentages of wealth invested in the stock index, the bond fund and the bond are

$$
\begin{bmatrix} \pi^{c1}_{s,t} \\ \pi^{c1}_{P_t^M} \\ \pi^{c1}_{P_t^{T-t}} \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} \frac{\hat{\phi}_s}{\sigma_S} X^{c1}_{spec,t} \\ -\frac{\hat{\phi}_r}{\sigma_r B(M)} X^{c1}_{spec,t} \\ 0 \end{bmatrix} + \left(1 - \frac{1}{\gamma}\right) \begin{bmatrix} 0 \\ 0 \\ X^{c1}_{hedge,t} \end{bmatrix}, \tag{18}
$$

where

$$
\begin{aligned}
X^{c1}_{spec,t} &= -\frac{dW_t^{c1}}{d\zeta_t} \frac{\zeta_t}{W_t^{c1}} \gamma, && (19) \\
X^{c1}_{hedge,t} &= -\frac{dW_t^{c1}}{dr_t} \frac{1}{W_t^{c1} B(T-t)\left(1 - \frac{1}{\gamma}\right)}, && (20)
\end{aligned}
$$

$X^{c1}_{spec,t}$ is the demand of the speculative fund relative to demand in the unconstrained model. And $X^{c1}_{hedge,t}$ is the demand to the hedge fund relative to demand in the unconstrained model. Appendix C provides the derivations of $\pi^{c1}_{s,t}$, $\pi^{c1}_{P_t^M}$, $\pi^{c1}_{P_t^{T-t}}$, $X^{c1}_{spec,t}$ and $X^{c1}_{hedge,t}$.

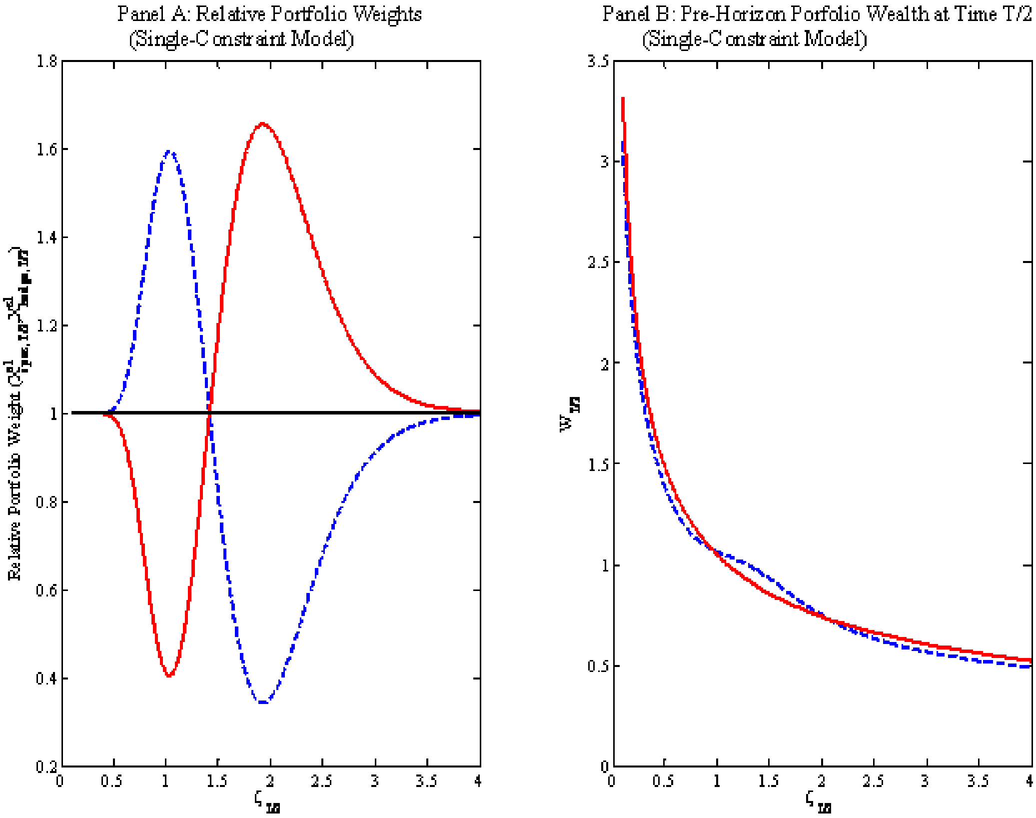

Figure 2: This figure shows the optimal portfolio allocation at time $t = T/2$ in Panel A and optimal portfolio wealth at time t in Panel B with and without a VaR constraint. In this figure, the interest rate at time t is fixed at 4%. In Panel A, the red solid line represents the portfolio weight allocated to the speculative fund relative to the unconstrained portfolio weight $X^{c1}_{spec,t}$ and the blue dashed line represents the portfolio weight allocated to the hedge fund relative to the unconstrained weight $X^{c1}_{hedge,t}$. In Panel B, the red solid line represents the portfolio wealth at time t without a VaR constraint W^u_t and the blue dashed line represents the portfolio wealth at time t in the single-constraint model W^{c1}_t. The parameter values are $W_0 = 1$, $\kappa = 0.15$, $\overline{r} = 0.05$, $\sigma_r = 0.015$, $\Phi_P = 0.05$, $M = 10$ years, $\sigma_s = 0.25$, $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 4\%$, and $\alpha = 0.025$.

Both the optimal portfolio allocation and portfolio wealth depend on the interest rate and the pricing kernel value at time $T/2$. Figure 2 shows the optimal portfolio allocation at time $T/2$ in the left panel (Panel A) and portfolio wealth at time $T/2$ in the right panel (Panel B) with and without a VaR constraint when the interest rate at time t is fixed at 4%. Figure 3 shows the portfolio choice at time t with different interest rates.

In Figure 2 Panel A, the red solid line represents the portfolio weight allocated to the speculative fund relative to the unconstrained portfolio weight $X^{c1}_{spec,t}$ and the blue dashed line represents the portfolio weight allocated to the hedge fund relative to the unconstrained portfolio weight $X^{c1}_{hedge,t}$. In Panel B, the red solid line represents the portfolio wealth at time t without a VaR constraint W^u_t and the blue dashed line represents the portfolio wealth at time t in the single-constraint model W^{c1}_t. In the Panel A, as the investment environment becomes worse, the VaR investor allocates less (more) to the speculative (hedge) fund to fulfill the VaR constraint at time T. As the investment environment deteriorates further but the chances of ending up in the "intermediate" states are still very high, the VaR investor starts to increase (decrease) allocation to the speculative (hedge) fund. At that time, the investor is gambling. If the financial market at time T ends up in "intermediate" states, he can finance his portfolio wealth at $\underline{W}$ by holding more risky assets. If the financial market at time T ends up in "bad" states, his portfolio wealth will suffer a loss which is larger than the one without a VaR constraint. The VaR constrained investor is only concerned about the probability but not the magnitude of the loss. Therefore, the investor is willing to incur losses in compliance with the VaR constraint. As the investment environment becomes even worse, the investor is sure that he will end up in the "bad" states at time T where the VaR constraint is not binding. Thus, he behaves as if the VaR constraint is not imposed. The investment behavior depicted in Panel A leads to a hump shaped portfolio wealth at time $T/2$ shown in Panel B.

In figure 3 Panel A, the red (grey) solid line and the red dashed line represent the relative portfolio weights in the speculative fund (X^{c1}_{spec}) and the hedge fund (X^{c1}_{hedge}) respectively when $r_t = 10\%$ (0%). In general, a higher interest rate is related to a higher yield on the default-risk-free zero-coupon bond P^{T-t}_t which makes the VaR constraint easier to be fulfilled. Thus, a higher interest rate shifts the portfolio allocation curve rightward. For example, when the interest rate is 0 and the value of the pricing kernel is 1, the investor considers to increase his allocation to the risky assets to finance his portfolio wealth at time T to the level of $\underline{W}$. However, when the interest rate is 10% and the value of the pricing kernel is 1, the investor will not consider to increase his allocation to the risky assets since the high yield from the riskless bond is sufficient to finance his

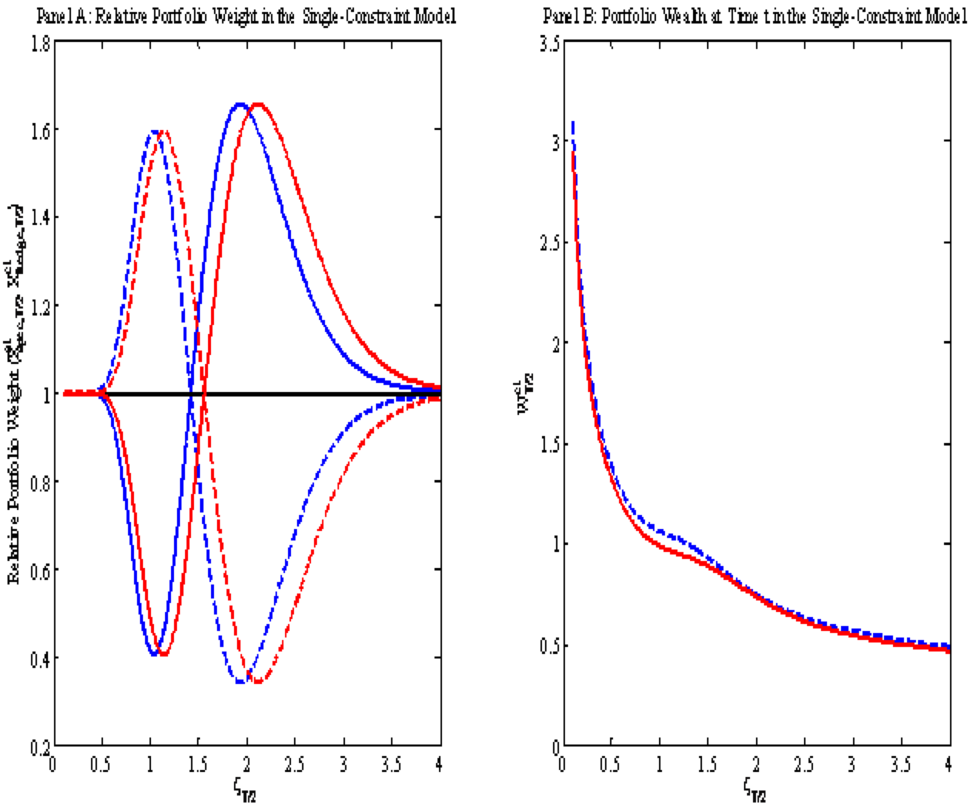

Figure 3: This figure shows the relative portfolio weights (Panel A) and the portfolio wealth at time t ($t = T/2$) with different interest rate in the one-constraint model. In panel A, the red (blue) solid line and the red dashed line represent the relative portfolio weights in the speculative and the hedge fund respectively when $r_t = 10\%$ (0%). In panel B, the red (blue) line and the red dashed line represent the portfolio wealth with one-VaR constraint when $r_t = 10\%$ (0%). The parameter values are $W_0 = 1$, $\kappa = 0.15$, $\overline{r} = 0.05$, $\sigma_r = 0.015$, $\Phi_P = 0.05$, $M = 10$ years, $\sigma_s = 0.25$, $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 0.04$, and $\alpha = 0.025$.

portfolio wealth at time T to the level of $\underline{W}$. Panel B shows the portfolio wealth with one-VaR constraint W_t^{c1} is decreasing as the interest rate r_t increases.

3.3 Two- and Multi-Constraint Models

The optimal portfolio wealth under two VaR constraints, W_T^{c2}, solves the problem

$$\begin{aligned} & \max E_0 \frac{W_T^{1-\gamma}}{1-\gamma} \\ s.t. \quad & E_0 W_T \zeta_T = \zeta_0 W_0, \\ & \Pr_0\left(W_{\frac{T}{2}} \leq \underline{W}\right) \leq \alpha, \\ & \Pr_{\frac{T}{2}}\left(W_T \leq \underline{W}\right) \leq \alpha. \end{aligned} \tag{21}$$

Our model directly embeds two VaR type constraints. We are going to use the backward iterative solution procedure to solve (21). First, we solve the maximization problem in the second period, that is, $[T/2, T]$. This second period problem is identical to the one-constraint model. We assume that at time $T/2$, the investor starts with wealth $W_{\frac{T}{2}}$. Following the same solution method as the one in the one-constraint model, we find the optimal wealth at time T, W_T^{c2}, and the indirect utility function at time $T/2$ $J_{\frac{T}{2}}\left(W_{\frac{T}{2}}^{c2}\right)$. Second, we solve the maximization problem in the first period, that is, $[0, T/2]$. The difference between the maximization problem in the second and first period is that in the second period, the objective function is $\max E_{\frac{T}{2}} \frac{W_T^{1-\gamma}}{1-\gamma}$ while in the first period the objective function is the indirect utility of the problem, namely, $\max E_0 J_{\frac{T}{2}}^{c2}\left(W_{\frac{T}{2}}^{c2}\right)$.

We discuss these two steps in more details. The maximization problem for the second period is

$$\begin{aligned} & \max_{W_T} E_{\frac{T}{2}} \frac{W_T^{1-\gamma}}{1-\gamma} \\ s.t. \quad & E_{\frac{T}{2}}\left(\zeta_T W_T\right) = \zeta_{\frac{T}{2}} W_{\frac{T}{2}} \\ & \Pr_{\frac{T}{2}}\left(W_T < \underline{W}\right) \leq \alpha. \end{aligned} \tag{22}$$

The optimal portfolio wealth at time T under two-VaR constraints is

$$W_T^{c2} = \left(y_{\frac{T}{2}} \zeta_T\right)^{-\frac{1}{\gamma}} \mathbb{I}_{\zeta_T \leq \min\left(\underline{\zeta}_T^{c2}, \overline{\zeta}_T^{v2}\right)} + \underline{W} \mathbb{I}_{\min\left(\underline{\zeta}_T^{c2}, \overline{\zeta}_T^{v2}\right) < \zeta_T \leq \overline{\zeta}_T^{v2}} + \left(y_{\frac{T}{2}} \zeta_T\right)^{-\frac{1}{\gamma}} \mathbb{I}_{\zeta_T > \overline{\zeta}_T^{v2}},$$

where $\mathbb{I}$ denotes an indicator function, $\underline{\zeta}_T^{c2}$ is defined as $\left(y_{\frac{T}{2}} \underline{\zeta}_T^{c2}\right)^{-\frac{1}{\gamma}} \equiv \underline{W}$, $y_{\frac{T}{2}}$ solves the budget constraint function and $\overline{\zeta}_T^{v2}$ is defined as $\Pr_{\frac{T}{2}}\left(\zeta_T \geq \overline{\zeta}_T^{v2}\right) = \alpha$.

The indirect utility at time T/2, $J_{\frac{T}{2}}^{c2}\left(W_{\frac{T}{2}}^{c2}\right)$, takes the form

$$
\begin{aligned}
J_{\frac{T}{2}}^{c2}\left(W_{\frac{T}{2}}^{c2}, r_{\frac{T}{2}}\right) &= E_{\frac{T}{2}} \frac{\left(W_T^{c2}\right)^{1-\gamma}}{1-\gamma} \\
&= \underbrace{\frac{1}{1-\gamma} e^{\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\sigma^2_{\zeta,T/2,T}} \left[\left(y^{c2}\zeta_t\right)^{-\frac{1}{\gamma}} e^{A\left(\frac{T}{2}\right)+B\left(\frac{T}{2}\right)r_{T/2}}\right]^{1-\gamma}}_{\text{term 1}} \\
&\quad \times \underbrace{\left[N\left(D_{1,\min\left(\underline{\zeta}_T^{c2},\overline{\zeta}_T^{c2}\right)}\right) + N\left(-D_{1,\overline{\zeta}_T^{c2}}\right)\right]}_{\text{term 2}} \\
&\quad + \underbrace{\frac{\underline{W}^{1-\gamma}}{1-\gamma}}_{\text{term 3}} \times \underbrace{\left(N\left(-D_{2,\min\left(\underline{\zeta}_T^{c1},\overline{\zeta}_T^{c2}\right)}\right) - N\left(-D_{2,\overline{\zeta}_T^{c1}}\right)\right)}_{\text{term 4}}
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
D_{2,x} &= \frac{\log\left(\frac{x}{\zeta_{\frac{T}{2}}}\right) - \mu_{\zeta,\frac{T}{2},T}}{\sigma_{\zeta,\frac{T}{2},T}}, \\
D_{1,x} &= D_{2,x} - \sigma_{\zeta,\frac{T}{2},T}\left(1 - \frac{1}{\gamma}\right).
\end{aligned}
$$

The first two terms of (23) derive from the utility over final wealth in the "good" ($\zeta_T \leq \min\left(\underline{\zeta}_T^{c2}, \overline{\zeta}_T^{c2}\right)$) and "bad" states ($\zeta_T \geq \overline{\zeta}$) and the probabilities of ending up in these states. The third and fourth terms derive from the utility over portfolio wealth in the "intermediate" states ($\min\left(\underline{\zeta}_T^{c2}, \overline{\zeta}_T^{c2}\right) \leq \zeta_T \leq \overline{\zeta}_T^{c2}$) and the probabilities of ending up in these states.

The indirect utility (23) depends on both the interest rate at time $T/2$ and the portfolio wealth at time $T/2$. Figure 4 shows the indirect function at time T/2 for $r_{T/2}$ equals to 0%, 5% and 10% respectively. For small portfolio wealth, the indirect utility under a VaR constraint is smaller than the one without a VaR constraint. As the wealth increases, the two indirect utilities converge as the VaR constraint becomes less binding. For any given portfolio wealth at time $T/2$, the indirect utility $J_{\frac{T}{2}}\left(W_{\frac{T}{2}}^{c2}, r_{\frac{T}{2}}\right)$ increases with the interest rate. This result is driven by two facts. First, a part of the indirect utility is expected utility over portfolio wealth in the "good" and "bad" states at time T. In these states, the investor under a VaR constraint behaves as if the VaR constraint has not been imposed. As discussed earlier, the indirect utility at time $T/2$ without a VaR constraint

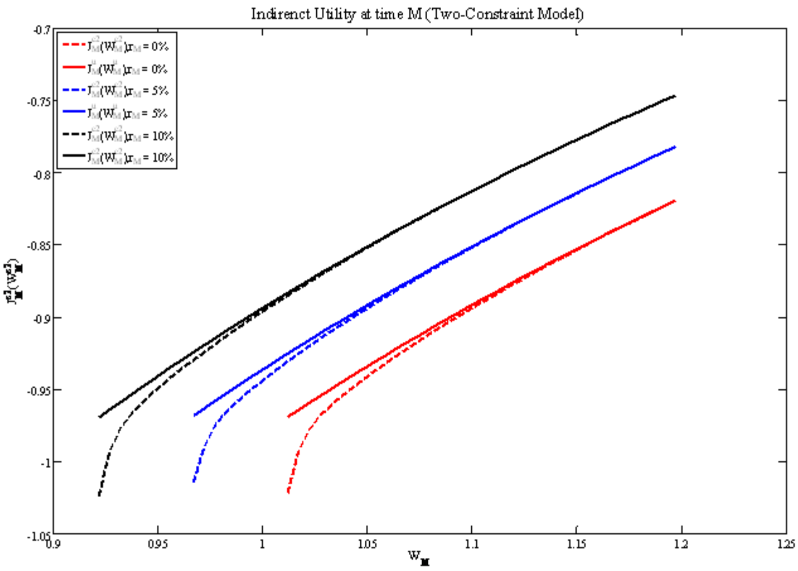

Figure 4: This figure shows that the indirect function at time M for r_M equals to 0%, 5% and 10% respectively. The indirect utility is defined in (23). The parameter values are $W_0 = 1$, $\kappa = 0.15$, $\overline{r} = 0.05$, $\sigma_r = 0.015$, $\Phi_P = 0.05$, $M = 10$ years, $\sigma_s = 0.25$, $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 0.04$, and $\alpha = 0.025$.

is increasing with the interest rate at time $T/2$. Second, high interest rate is coupled with higher zero coupon bond yield which makes the VaR constraint easier to be fulfilled.

The optimal trading strategy in the period from time 0 to time $T/2$ is a solution to the problem

$$\begin{aligned} & \max E_0 J^{c2}_{\frac{T}{2}}\left(W^{c2}_{\frac{T}{2}}\right) && (24) \\ s.t. \quad & E_0 W^{c2}_{\frac{T}{2}} \zeta_{\frac{T}{2}} = \zeta_0 W_0, && \\ & \Pr_0\left(W^{c2}_{\frac{T}{2}} \leq \underline{W}\right) \leq \alpha, && (25) \\ & \Pr_0\left(W^{c2}_{\frac{T}{2}} < W_{\frac{T}{2},\min}\right) = 0, && (26) \end{aligned}$$

where $W_{\frac{T}{2},\min}$ is the minimal portfolio wealth required to fulfill the next period's VaR constraint. The maximization problem in the first-period has one extra constraint (26). The minimum wealth at time T which makes it possible to fulfill the VaR constraint in (22) is

$$W_{T,\min} = \begin{cases} \underline{W} & \text{if } \zeta_T < \overline{\zeta}^{c2}_T \\ 0 & \text{if } \zeta_T \geq \overline{\zeta}^{c2}_T \end{cases},$$

i.e., keeping the portfolio wealth at time T at the level of $\underline{W}$ in the "good" and "intermediate" states and leaves the portfolio wealth at 0 in the "bad" states. Therefore, the minimum wealth at time $T/2$, $W_{\frac{T}{2},\min}$, equals $\frac{1}{\zeta_{\frac{T}{2}}}E_{\frac{T}{2}}\underline{W}\zeta_T\mathbb{I}_{\zeta_T<\bar{\zeta}_T^{v2}}$. If the wealth at time $T/2$ is smaller than $W_{\frac{T}{2},\min}$, it is not possible to fulfill the VaR constraint in the next period. The minimum wealth at time $T/2$ is negatively related to the interest rate at time $T/2$ since the higher the interest rate the easier it is to fulfill the VaR constraint at time T.

The Lagrangian for the constrained optimization problem (24) is given by

$$\begin{aligned} L &= E_0\left[J_{\frac{T}{2}}\left(W_{\frac{T}{2}}^{c2}\right) - y_0^{c2}\zeta_{\frac{T}{2}}W_{\frac{T}{2}}^{c2} + y_2^{c2}\mathbb{I}_{W_{\frac{T}{2}}^{c2}\geq\underline{W}} - y_3^{c2}\mathbb{I}_{W_{\frac{T}{2}}^{c2}<W_{\frac{T}{2},\min}}\right] \\ &\quad +y_0^{c2}\zeta_0W_0 - y_2^{c2}\left(1-\alpha\right), \end{aligned} \tag{27}$$

where y_0^{c2}, y_2^{c2} and y_3^{c2} are Lagrangian multipliers solving $\zeta_0W_0 - E_0\left(\zeta_{\frac{T}{2}}W_{\frac{T}{2}}^{c2}\right) = 0$, $E_0\mathbb{I}_{W_{\frac{T}{2}}^{c2}\geq\underline{W}} = 1-\alpha$ and $E_0\mathbb{I}_{W_{\frac{T}{2}}^{c2}<W_{\frac{T}{2},\min}} = 0$ respectively, with $y_0^{c2}\geq 0$, $y_2^{c2}\geq 0$ and $y_3^{c2}=\infty$. The Lagrangian multiplier y_2^{c2} guarantees that the VaR constraint in (21) is satisfied and the the Lagrangian multiplier y_3^{c2} makes sure that the portfolio wealth at time $T/2$ is sufficient to fulfill the VaR constraint next period. The last two terms of (27) are constants. Thus, finding a $W_{\frac{T}{2}}^{v2}$ which maximizes the value of (27) is equivalent to finding a portfolio wealth which maximizes the value of the function within $E_0\left[\cdot\right]$ in (27).

Again following the martingale method, the optimal portfolio wealth at time $T/2$ can be found using a pointwise optimization. That is, for each pair of interest rate at time $T/2$ and the pricing kernel value at time $T/2$, the optimal portfolio wealth is the one which maximizes the function $L\left(r_{\frac{T}{2}},\zeta_{\frac{T}{2}},W_{\frac{T}{2}}^{c2}\right)$.

The numerical procedures to find the optimal portfolio wealth $W_{\frac{T}{2}}^{c2}$ are as follows. First, we simulate N scenarios of interest rates, $r_{\frac{T}{2},i}$, and pricing kernel values at time $\frac{T}{2}$, $\zeta_{\frac{T}{2},i}$, with $i=1,2,...N$. Second, we create a vector with H different portfolio wealth in a very broad range, $W_{\frac{T}{2},j}$ with $j=1,2,...H$. Third, since the indirect utility (23) depends on both the interest rate at time $T/2$ and the portfolio wealth at time $T/2$, for each interest rate $r_{\frac{T}{2},i}$, we evaluate the value of $J_{\frac{T}{2}}\left(r_{\frac{T}{2},i},W_{\frac{T}{2},j}\right)$ for all $W'_{\frac{T}{2},j}s$[3]. Fourth, for each scenario of interest rate and pricing kernel value, i.e., $r_{\frac{T}{2},i},\zeta_{\frac{T}{2},i}$ with $i=1,2,...N$, we evaluate the function value $L\left(r_{\frac{T}{2},i},\zeta_{\frac{T}{2},i},W_{\frac{T}{2},j}\right)$ for all $W'_{\frac{T}{2},j}s$ with $j=1,2,...H$. Finally,

[3] To speed up the numerical process, we could first evaluate the indirect utility function value $J_M\left(\cdot\right)$ for a small sample of interest rates. As we see in figure 4, for each given portfolio wealth the value function value is almost linearly increasing with interest rates. This relationship enable us to use linear interpolation to evaluate the function value $J_M\left(r_{M,i},W_{M,j}\right)$ for $i\in[1,N]$.

for each pair of $r_{\frac{T}{2},i}$ and $\zeta_{\frac{T}{2},i}$, the optimal portfolio wealth is the one which maximizes the value of $L(\cdot)$.

The optimal portfolio wealth $W^{c2}_{\frac{T}{2}}$ depends on both $\zeta_{\frac{T}{2}}$ and $r_{\frac{T}{2}}$. Panel A of Figure 5 compares the optimal portfolio wealth in the two-constraint ($W^{c2}_{\frac{T}{2}}$), single-constraint ($W^{c1}_{\frac{T}{2}}$) and unconstrained models ($W^{u}_{\frac{T}{2}}$) at time $T/2$ when the interest rate $r_{T/2}$ is 4%. Panel B depicts the optimal portfolio wealth under two VaR constraints when interest rates at time $T/2$ are 4% and 10% respectively.

Panel A shows that, in "good" states (low pricing kernel values), the portfolio wealth without any VaR constraints $W^{u}_{\frac{T}{2}}$ is the largest, followed by $W^{c1}_{\frac{T}{2}}$ and $W^{c2}_{\frac{T}{2}}$. When the investment environment deteriorates, the investor with two VaR constraints keeps his wealth level at $\underline{W}$. When the investment environment deteriorates further, the investor with two VaR constraints keeps his portfolio wealth at $W_{\frac{T}{2},\min}$ so that he has enough wealth to fulfill the next period's VaR constraint. Thus, at time $\frac{T}{2}$, the optimal portfolio wealth under two VaR constraints in "bad" states are much larger than both the one under one-VaR constraint and the one without any VaR constraints. However, in "good" states the portfolio wealth under two VaR constraints is much smaller than both the one under one VaR constraint and the one without any VaR constraints.

From Panel B, we can conclude (1) in "good" states, the optimal portfolio wealth decreases as interest rates increase, (2) the minimum wealth at time $T/2$ ($W_{\frac{T}{2},\min}$) decreases as the interest rate increases since a high interest rate leads to a high zero-coupon bond yield and thus reduces the minimum amount of wealth necessary to fulfill the next period's VaR constraint, and (3) the states in which the investor keeps his portfolio wealth at $\underline{W}$ no longer depend only on the pricing kernel values. For example, in Panel B, the investor choose to keep his portfolio wealth at $\underline{W}$ in the state where $\zeta_t = \zeta_1$ and $r_t = 0\%$ while leave the portfolio wealth at $W_{\frac{T}{2},\min}$ in the state where $\zeta_t = \zeta_2$ and $r_t = 10\%$ even though ζ_1 is larger than $\overline{\zeta}^{c2}_{\frac{T}{2}}$ and ζ_2 is smaller than $\overline{\zeta}^{c2}_{\frac{T}{2}}$. The investor decides in which states he keeps his portfolio wealth at $\underline{\mathrm{W}}$ not only on the value of the pricing kernel but also on the interest rates. For each state at time t, $t \in [0, T]$, the cost of raising the portfolio wealth from the unconstrained portfolio wealth W^u_t to $\underline{W}$ equals $\zeta_t (\underline{W} - W^u_t)$. At time T, as shown in Figure 1, the unconstrained portfolio wealth W^u_T monotonically decreases as ζ_T increases. Therefore, at time T, it is always cheaper to raise the wealth level to $\underline{W}$ in states where $\zeta_T < \overline{\zeta}^{c2}_{T}$. While at time $T/2$, the unconstraint portfolio wealth depends on both the pricing kernel value and the interest rate. For any given value of $\zeta_{\frac{T}{2}}$, the unconstrained portfolio wealth $W^u_{\frac{T}{2}}$ decreases when the interest rate increases. Thus,

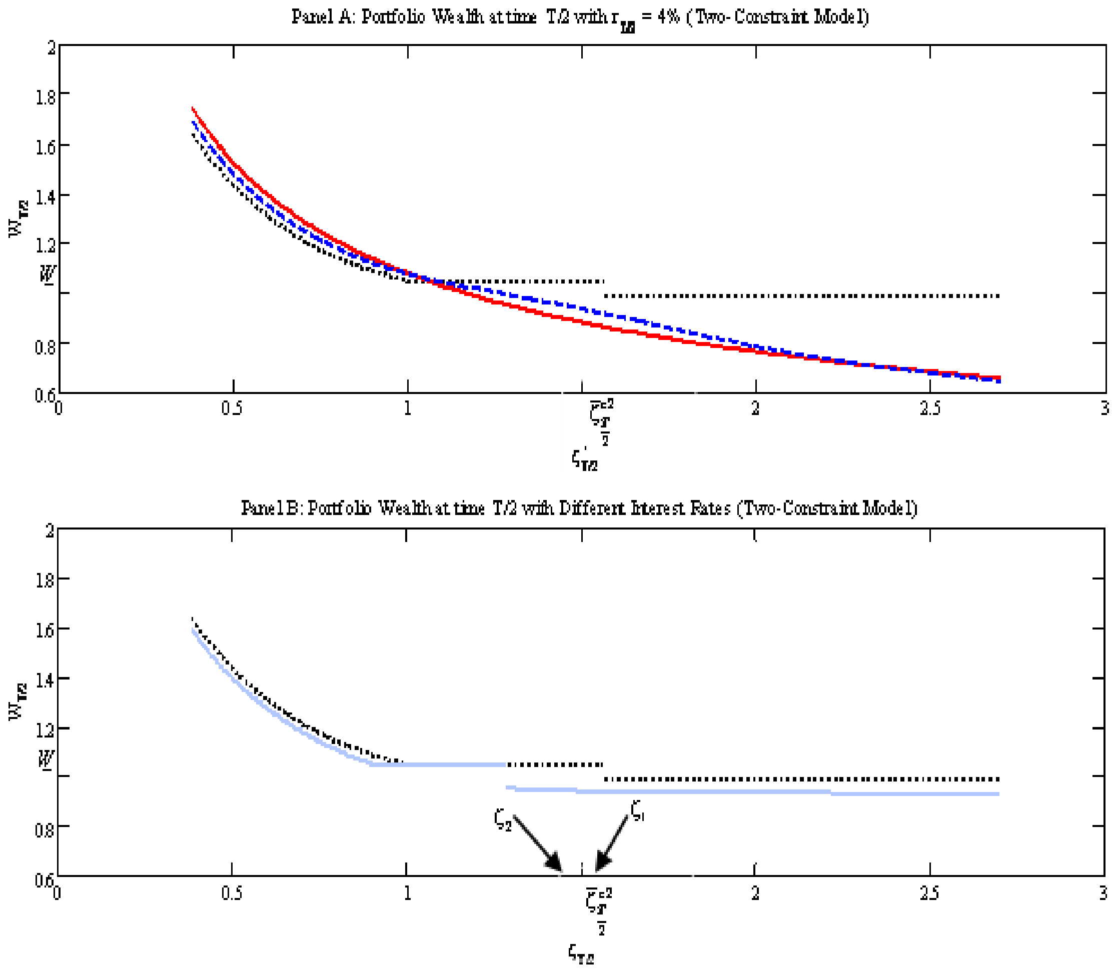

Figure 5: Panel A compares the optimal portfolio wealth in the two-constraint model $W^{c2}_{\frac{T}{2}}$ (the black dotted line), single-constraint model $W^{c1}_{\frac{T}{2}}$ (the blue solid line) and unconstrained model $W^{u}_{\frac{T}{2}}$ (the red solid line) at time $T/2$. The interest rate is 0.04. Panel B shows the optimal portfolio wealth at time T/2 under two VaR constraints when interest rates are 4% (the black dotted line) and 10% (the grey solid line) respectively. The parameter values are $W_0 = 1.05$, $\kappa = 0.15$, $\overline{r} = 5\%$, $\sigma_r = 1.5\%$, $\Phi_P = 0.05$, $M = 10$ years, $\sigma_s = 0.25$, $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 4\%$, and $\alpha = 0.025$.

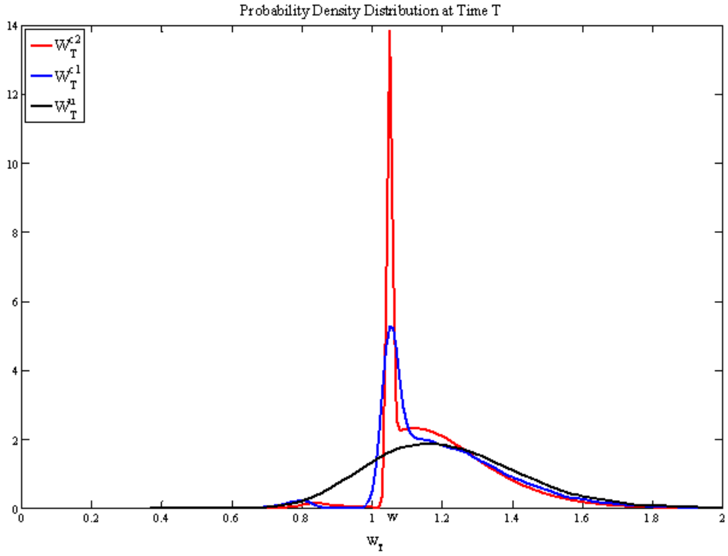

Figure 6: This figure depicts the probability density functions of portfolio wealth at time T under one-VaR constraint (W_T^{c1}), two-VaR constraints (W_T^{c2}) respectively. The probability density functions of W_T^{c1} and W_T^{c2} are then compared with the probability density function of portfolio wealth at time T without a VaR constraint (W_T^u).

the cost of raising the portfolio wealth to $\underline{W}$ in states with large pricing kernel value, i.e., $\zeta_{\frac{T}{2}} > \overline{\zeta}^{c2}_{\frac{T}{2}}$ and low interest rates might be cheaper than the one in states with small pricing kernel value, i.e., $\zeta_{\frac{T}{2}} \leq \overline{\zeta}^{c2}_{\frac{T}{2}}$, and high interest rates.

Figure 5 shows that two subsequent and non-overlapping VaR constraints can reduce the portfolio wealth loss at time $T/2$ very effectively but at the cost of lowering the portfolio wealth when the financial market is booming. Since the institutional investor is interested in the final portfolio wealth, Figure 6 depicts the probability density functions of portfolio wealth at time T under one VaR constraint (W_T^{c1}), two VaR constraints (W_T^{c2})[4] respectively. The probability density functions are then compared with the one without a VaR constraint (W_T^u).

For both W_T^{c1} and W_T^{c2}, there is a probability mass build-up at the wealth level $\underline{W}$. The probability mass for W_T^{c2} is much larger than the one for W_T^{c1}. Both probability distribution exhibit a discontinuity. For the two-constraint model, no wealth falls between $\underline{W}$ and $W_{\frac{T}{2},\min}$ at time T/2 as shown in Figure 6. For the one-constraint model, there is

[4]The final wealth under two VaR constraints are estimated as follows. First, we estimated the indirect utility at time $T/2$. Second, we estimated the optimal portfolio wealth $W^{c2}_{\frac{T}{2}}$. $W^{c2}_{\frac{T}{2}}$ is related to the interest rate at time $T/2$ and the pricing kernel value at time $T/2$. We simulated 2500 scenarios of $r_{T/2}$ and $\zeta_{T/2}$. Third, we estimated the optimal portfolio wealth at time T for each simulated scenario at time $T/2$.

no wealth falls between $\underline{W}$ and $\left(y_0^{c1}\overline{\zeta}_T^{c1}\right)^{-\frac{1}{\gamma}}$ at time T as shown in Figure 6.

The probability mass build-up for large wealth is smaller as the regulatory frequency increases. However, the probability mass build-up for small wealth is also smaller as the regulatory frequency increases, meaning both the size and the probability of portfolio wealth loss decreases as the regulatory frequency increases. Overall, more frequent VaR constraints does seem to be effective in reducing portfolio wealth loss but at the cost of losing the ability to gain wealth when the financial market is booming.

At time t, $0 \leq t \leq \frac{T}{2}$, the portfolio wealth is

$$W_t^{v2}=\frac{1}{\zeta_t}E_t\left(\zeta_{\frac{T}{2}}W_{\frac{T}{2}}^{v2}\right),$$

and the optimal portfolio allocation is

$$\begin{bmatrix} \pi_{s,t}^{c2} \\ \pi_{P_t^{\frac{T}{2}}}^{c2} \\ \pi_{P_t^{T-t}}^{c2} \end{bmatrix} = \frac{1}{\gamma}\begin{bmatrix} \frac{\hat{\phi}_s}{\sigma_S}X_{\zeta_t}^{c2} \\ -\frac{\hat{\phi}_r}{\sigma_r B\left(\frac{T}{2}\right)}X_{\zeta_t}^{c2} \\ 0 \end{bmatrix} + \left(1-\frac{1}{\gamma}\right)\begin{bmatrix} 0 \\ 0 \\ X_{r_t}^{c2} \end{bmatrix}, \tag{28}$$

where

$$X_{spec,t}^{c2} = -\frac{dW_t^{c2}}{d\zeta_t}\frac{\zeta_t}{W_t^{c2}}\gamma, \tag{29}$$

$$X_{hedge,t}^{c2} = -\frac{dW_t^{c2}}{dr_t}\frac{1}{W_t^{c2}B\left(T-t\right)\left(1-\frac{1}{\gamma}\right)}, \tag{30}$$

X_{spec}^{c2} is the demand of the speculative fund relative to demand in the unconstrained model, and X_{hedge}^{c1} is the demand to the hedge fund relative to demand in the unconstrained model. The first order derivatives are approximated as follows,

$$\frac{dW_t^{c2}}{d\zeta_t} \approx \frac{W_{t,\zeta_t+\Delta\zeta_t}^{c2}-W_{t,\zeta_t-\Delta\zeta_t}^{c2}}{2\times\Delta\zeta_t},$$

$$\frac{dW_t^{c2}}{dr_t} \approx \frac{W_{t,r_t+\Delta r_t}^{c2}-W_{t,r_t-\Delta r_t}^{c2}}{2\times\Delta r_t},$$

where $W_{t,\zeta_t+\Delta\zeta_t}^{c2}$ ($W_{t,\zeta_t-\Delta\zeta_t}^{c2}$) refers to the portfolio wealth at time t in the two-constraint model when the pricing kernel takes the value of $\zeta_t+\Delta\zeta_t$ ($\zeta_t-\Delta\zeta_t$) while other parameter values keep unchanged and $W_{t,r_t+\Delta r_t}^{c2}$ ($W_{t,r_t-\Delta r_t}^{c2}$) refers to the portfolio wealth at time t in the two-constraint model when the interest rate takes the value of $r_t+\Delta r_t$ ($r_t-\Delta r_t$).

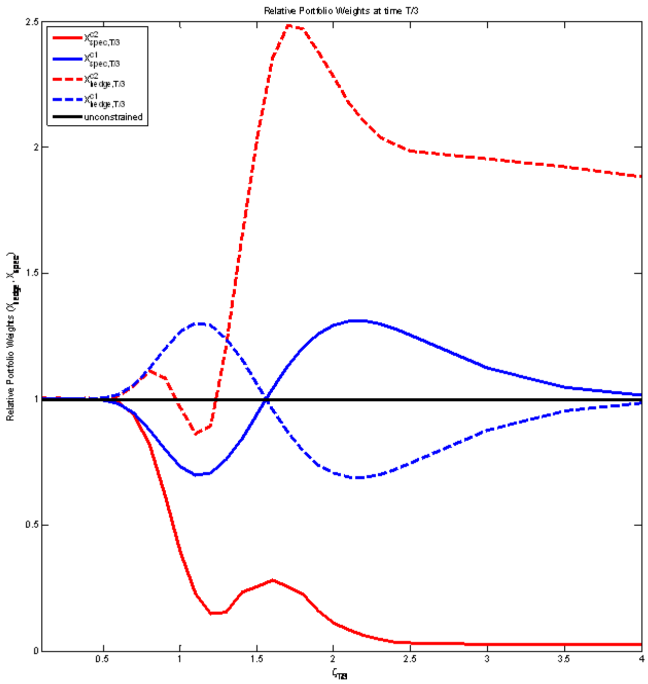

Figure 7: This figure shows the relative portfolio weights in two-constraint, i.e.., X^{c2}_{hedge} and X^{c2}_{spec}, and one-constraint model, i.e.., X^{c1}_{hedge} and X^{c1}_{spec}, where X^{c1}_{hedge}, X^{c1}_{spec}, X^{c2}_{hedge}, and X^{c2}_{spec} are defined in (19), (20), (29), and (30) respectively. $X^{'}_{spec}$ s are the demand of the speculative fund relative to demand in the unconstrained model, and $X'_{hedge}s$ are the demand to the hedge fund relative to demand in the unconstrained model. The parameter values are $W_0 = 1.05$, $\kappa = 0.15$, $\overline{r} = 0.05$, $\sigma_r = 1.5\%$, $\Phi_P = 5\%$, $M = 10$ years, $\sigma_s = 25\%$, $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 4\%$, and $\alpha = 0.025$.

Figure 7 shows portfolio weights in two-constraint and single-constraint models relative to the one in the unconstrained model at time $T/3$ when the interest rate $r_{T/3}$ is 4%. When the financial market performs extremely well, both the investor under two VaR constraints and the investor under one VaR constraint behaves as if no VaR constraints have been imposed. As the market deteriorates, both investors increase their holdings in the hedge fund and decrease their holdings in the speculative fund. As the financial market deteriorates further, both investors increase their holding in the speculative fund. At that time, even though the financial market does not perform well but it is still very likely that the investors might end up in the "bad" states at time T and T/2. Therefore, both investors are gambling. At the same time, however, unlike the investor under one VaR constraint who decreases his holdings in the hedge fund, the investor under two VaR constraints *increases* his holdings in the hedge fund to compensate for the possible loss generated by the speculative fund. By doing so, the investor under two VaR constraints guarantees that his portfolio wealth is large enough to fulfill next period's VaR constraint in all circumstances. As the financial market deteriorates still further, the investor under two VaR constraints decreases his holdings in the speculative fund to keep his portfolio wealth at $W_{\frac{T}{2},\min}$ at time $T/2$. At the same time, he decreases his holdings in the hedge fund since it is no longer necessary to invest in the hedge fund to compensate for the possible loss generated by the speculative fund. But his holdings in the hedge fund remains at a very high level to keep the portfolio wealth at time $T/2$ large enough to fulfill next period's VaR constraint.

The analysis above can easily be extended to more than three constraints. For example, if there are m subsequent and non-overlapping VaR constraints within the investment horizon, we start by solving the optimal portfolio wealth in the last period and then proceed backwards by repeating the numerical procedures developed for finding the first period's optimal portfolio wealth in the two-constraint model.

4 Benefit and Cost Analysis

In this section, we consider a pension fund with 15-year investment horizon as an example to analyze the cost and the benefit of VaR-type prudential regulation. The regulatory horizon considered here is 1 year, meaning that in the 15-year investment horizion, there are 15 non-overlapping VaR constraints.

The economic cost is measured by the certainty equivalent loss ce relative to the unconstrained portfolio allocation problem. The certainty equivalent loss ce is defined as the equivalent amount of wealth lost due to the VaR regulation, i.e.,

$$J_0^u (W_0 - ce) = J_0^{cm} (W_0),$$

where $J_0^u (\cdot)$ stands for the indirect utility at time 0 without a VaR constraint, and $J_0^{cm} (W_0)$ is the indirect utility at time 0 with m VaR constraints. The economic benefit is measured by a reduction in the expected shortfall at time 0. The expected shortfall, SF_0^{cm}, is defined as

$$SF_0^{cm} = E_0 \max (\underline{W} - W_T, 0).$$

We assume that $\kappa = 1.5\%$, $\overline{r} = 5\%$, $\sigma_r = 1.5\%$, $\Phi_P = 5\%$, $M = 10$ years, $\sigma_S = 25\%$, $\rho_{sr} = 20\%$, $\underline{W} = 1.05$, $r_0 = 2\%$, and the stock Sharpe ratio $\lambda_s = 25\%$. These set of parameters are close to those obtained by empirical studies, for example, Chan et al. (1992). In particular, ρ_{sr} is chosen to be positive so that the correlation between interest rate and stock price is negative which is suggested by Campbell (1987). For a pension fund, the natural choice of the "floor" $\underline{\text{W}}$ is its liability. In this paper, we assumed that the value of the pension fund's liability is constant over time but it can be easily extended to the case when the liability value is stochastic as long as the liability value is exogenously determined.

Figure 8 shows the certainty equivalent loss and the expected portfolio wealth shortfall of a pension fund with $\gamma = 2$ and $\alpha = 2.5\%$.

We find that the fifteen VaR constraints can significantly reduce the portfolio wealth shortfall. It is almost guaranteed that at the end of the investment horizon the pension fund's portfolio wealth will be above $\underline{W}$. For example, when the initial portfolio wealth is 1.1, which corresponds to a funding ratio of about 1.05, the expected shortfall is about 0.35% when no VaR constraints are imposed. The expected shortfall decreases to almost 0 as the regulatory frequency increases. However, the certainty equivalent loss is 4.3% when there are 15 VaR constraints and 0.05% when there are two VaR constraints. In this numerical example, the cost introduced by the 15 VaR constraints over weights the benefit these VaR constraints bring.

5 Conclusions

The value-at-risk type constraint is often adopted by regulators to limit the portfolio risk of institutional investors. However, the regulatory horizon is usually much shorter than the institutional investors' investment horizon. We find, e.g, that constrained investor, as

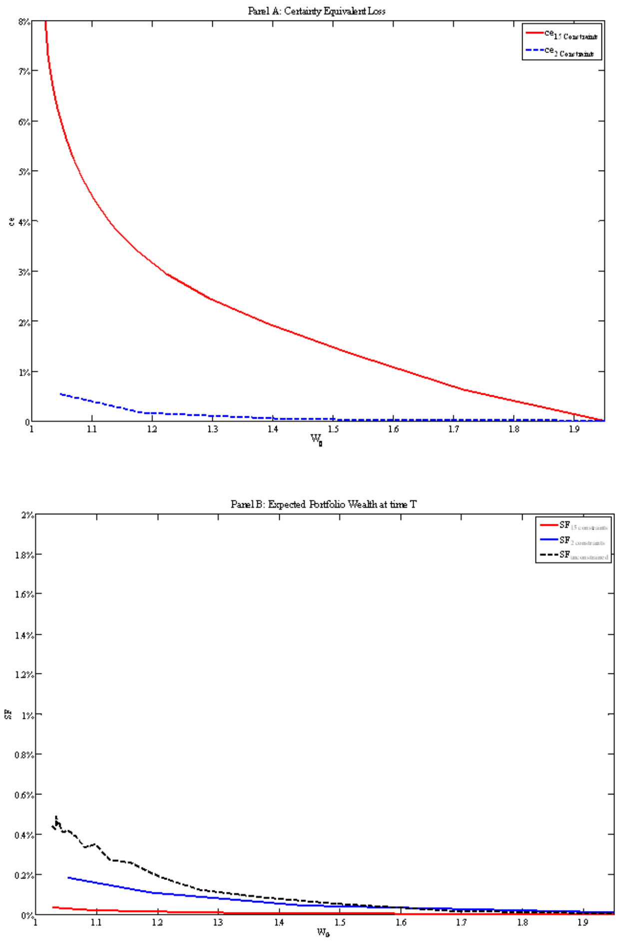

Figure 8: This figure shows the certainty equivalent loss (Panel A) and the expected shortfall (Panel B) of a pension fund with a 15-year investment horizion. The parameter values are $\kappa = 0.15$, $\overline{r} = 0.05$, $\sigma_r = 0.015$, $\Phi_P = 0.05$, $M = 10$ years, $\sigma_s = 0.25$, $\lambda_s = 0.25$, $\rho_{sr} = 0.2$, $\underline{W} = 1.05$, $r_0 = 2\%$, and $\alpha = 0.025$.

expected, often invests more in the risk-free asset than unconstrained investors. However, unintendedly, constraints may under certain market conditions also lead to gambling behavior in order to be able to meet future regulatory constraints. Also shorter regulatory horizon, on the one hand, enables an institutional investor like a pension fund to avoid large losses when the investment environment worsens but, on the other hand, also limits the institutional investor's ability to benefit from an increase in stock prices.

Appendix A: The Derivation of the Pricing Kernel and the Bond Price

A.1 The Derivation of the Pricing Kernel

Assume that the investor will invest π_s, π_p^{T-t} and $1-\pi_s-\pi_p^{T-t}$ percent of his wealth in the stock index, the zero coupon bond with $T-t$ years to maturity and the cash account respectively. Let W_t be the portfolio wealth,

$$\frac{dW_t}{W_t} = \left(r_t + \Pi^\top \Phi\right) dt + \Pi^\top \Sigma dZ_t, \tag{31}$$

where

$$\begin{aligned}
\Pi &= \begin{bmatrix} \pi_s \\ \pi_p^{T-t} \end{bmatrix}, \\
\Phi &= \begin{bmatrix} \Phi_S \\ \Phi_r B\left(T-t\right) \end{bmatrix}, \\
\Sigma &= \begin{bmatrix} \sigma_S & 0 \\ 0 & \sigma_r B\left(T-t\right) \end{bmatrix}, \\
dZ_t &= \begin{bmatrix} dZ_{s,t} \\ dZ_{r,t} \end{bmatrix}.
\end{aligned}$$

Merton (1992) shows that the pricing kernel is the inverse of a growth-optimum portfolio. Under the following three assumptions, (1) the stochastic structure and the information flow is represented by a probability space (Ω, F, P) where the filtration $F = (F_t)$, $t \in [0, T]$, satisfies the usual conditions (see, for instance, Duffie (1992)), (2) markets are free of arbitrage, frictionless, and continuously open, and (3) the prices of the risky assets and the zero-coupon bond follow semi-martingales with finite expectation and variance, the growth-optimum portfolio is the optimal portfolio with the objective of maximizing

$E_0 \log(W_T)$ and the pricing kernel is the inverse of the growth-optimum portfolio (see Bajeux-Besnainou and Portait 1998).

The growth-optimum portfolio is given by

$$\begin{aligned}
\Pi &= \left(\Sigma\rho\Sigma^{\top}\right)^{-1}\Phi, \\
&= \left[\begin{array}{c} \frac{\Phi_S}{\sigma_S^2(1-\rho_{sr}^2)} - \frac{\rho_{sr}\Phi_r}{\sigma_S\sigma_r(1-\rho_{sr}^2)} \\ \frac{-\Phi_S\rho_{sr}}{\sigma_S\sigma_r B(t,T)(1-\rho_{sr}^2)} + \frac{\Phi_r}{\sigma_r^2 B(t,T)(1-\rho_{sr}^2)} \end{array}\right],
\end{aligned} \tag{32}$$

where

$$\rho = \left[\begin{array}{cc} 1 & \rho_{sr} \\ \rho_{sr} & 1 \end{array}\right],$$

and $dZ_{s,t} \times dZ_{r,t} = \rho_{sr} dt$. Insert (32) into (31), we get

$$\begin{aligned}
\frac{dW_t}{W_t} &= \left(r_t + \Pi^{\top}\Phi\right) dt + \Pi^{\top}\Sigma dZ \\
&= \left(r_t + \hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right) dt + \hat{\phi}_s dZ_{s,t} - \hat{\phi}_r dZ_{r,t},
\end{aligned} \tag{33}$$

where

$$\hat{\phi}_s = \frac{\sigma_r\Phi_S - \rho_{sr}\Phi_r\sigma_S}{\sigma_r\sigma_S\left(1-\rho_{sr}^2\right)}, \tag{34}$$

$$\hat{\phi}_r = \frac{\sigma_r\Phi_S\rho_{sr} - \Phi_r\sigma_S}{\sigma_r\sigma_S\left(1-\rho_{sr}^2\right)}. \tag{35}$$

Therefore, the growth-optimum portfolio wealth at time t is

$$\begin{aligned}
W_t &= W_0 \exp\left(\int_0^t r_u du + \frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right) t + \hat{\phi}_s\left(Z_{s,t} - Z_{s,0}\right)\right. \\
&\quad \left. -\hat{\phi}_r\left(Z_{r,t} - Z_{r,0}\right)\right).
\end{aligned} \tag{36}$$

The pricing kernel, which is the inverse of the growth optimum portfolio, is

$$\begin{aligned}
\zeta_t &= \zeta_0 \exp\left[-\int_0^t r_u du - \frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right) t\right. \\
&\quad \left. -\hat{\phi}_s\left(Z_{s,t} - Z_{s,0}\right) + \hat{\phi}_r\left(Z_{r,t} - Z_{r,0}\right)\right].
\end{aligned} \tag{37}$$

Applying Ito-Doeblin lemma to (37), we have

$$\frac{d\zeta_t}{\zeta_t} = -r_t dt - \hat{\phi}_s dZ_{s,t} + \hat{\phi}_r dZ_{r,t}.$$

If the pricing kernel derived in (37) is correct, we must have

$$S_0 = \frac{1}{\zeta_0} E_0 \zeta_t S_t, \tag{38}$$

where S_t is a stock price. The diffusion process of the stock price is described in (1). It can be verified that the stock price is

$$S_t = S_0 \exp\left(\int_0^t r_u du - \frac{1}{2}\sigma_s^2 t + \Phi_S t + \sigma_S \left(Z_{s,t} - Z_{s,0}\right)\right). \tag{39}$$

Thus,

$$\begin{aligned}
E_0 \zeta_t S_t &= \zeta_0 S_0 E_0 \left[\exp\left(-\int r_u du - \frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right) t \right.\right. \\
&\quad \left. -\hat{\phi}_s \left(Z_{s,t} - Z_{s,0}\right) + \hat{\phi}_r \left(Z_{r,t} - Z_{r,0}\right)\right) \\
&\quad \left. \times \exp\left(\int r_u du - \frac{1}{2}\sigma_s^2 t + \Phi_S t + \sigma_S \left(Z_{s,t} - Z_{s,0}\right)\right)\right] \qquad (40) \\
&= \zeta_0 S_0 E_0 \exp\left[\left(-\frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right) + \Phi_S - \frac{1}{2}\sigma_s^2\right) t \right. \\
&\quad \left. \times \left(-\left(\hat{\phi}_s - \sigma_S\right)\left(Z_{s,t} - Z_{s,0}\right) + \hat{\phi}_r \left(Z_{r,t} - Z_{r,0}\right)\right)\right] \\
&= \zeta_0 S_0 \exp\left(\Phi_S + \hat{\phi}_s \sigma_S - \hat{\phi}_r \sigma_S \rho\right) t \qquad (41)
\end{aligned}$$

Substituting (34) and (35) into (41), we get

$$\Phi_S + \hat{\phi}_s \sigma_S - \hat{\phi}_r \sigma_S \rho_{sr} = 0,$$

and therefore,

$$S_0 = \frac{1}{\zeta_0} E_0 \zeta_t S_t$$

holds. Thus, the pricing kernel derived in (37) is correct.

Mamon (2004) showed that $-\int_0^t r_u du$ is log-normally distributed with mean

$$E\left[-\int_0^t r_u du\right] = -\frac{r_t - \overline{r}}{\kappa}\left(1 - e^{-\kappa(t-0)}\right) - \overline{r}\left(t - 0\right),$$

and variance,

$$Var\left[-\int_0^t r_u du\right] = -\frac{\sigma_r^2}{2\kappa}B(t-0)^2 - \frac{\sigma_r^2}{\kappa^2}\left(B(t-0)-(t-0)\right).$$

Therefore, $log\left(\zeta_t/\zeta_0\right)$ is normally distributed with mean

$$\mu_{\zeta,0,t} = \left(\overline{r}-r_0\right)B(t-0) - \overline{r}(t-0) - \frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right)(t-0), \tag{42}$$

and variance

$$\begin{aligned}\sigma_{\zeta,0,t}^2 &= -\frac{\sigma_r^2}{2\kappa}B(t-0)^2 + \left(2\frac{\Phi_r}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right)\left(B(t-0)-(t-0)\right) \\ &\quad +\hat{\phi}_s^2(t-0) + \hat{\phi}_r^2(t-0) - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r(t-0).\end{aligned} \tag{43}$$

A.2 The Derivation of the Bond Price

The price of a bond that pays \$1 at time T without a default risk is

$$P_t^{(T-t)} = \frac{1}{\zeta_t}E_t\left(\zeta_T \times \$1\right). \tag{44}$$

Inserting (37) into (44), we get

$$\begin{aligned}P_t^{(T-t)} &= E_t \exp\left(-\int_t^T r_u du - \frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right)\tau\right. \\ &\qquad \left. -\hat{\phi}_s\left(Z_{s,T}-Z_{s,t}\right) + \hat{\phi}_r\left(Z_{r,T}-Z_{r,t}\right)\right).\end{aligned} \tag{45}$$

Since $log\frac{\zeta_T}{\zeta_t}$ is normally distributed with mean $\mu_{\zeta,t,T}$ and variance $\sigma_{\zeta,t,T}^2$, we have

$$\begin{aligned}P_t^{(T-t)} &= \exp\left[\mu_{\zeta,t,T} + \frac{1}{2}\sigma_{\zeta,t,T}^2\right] \\ &= \exp\left[\left(\left(\overline{r}-r_t\right)B(\tau) - \overline{r}\tau - \frac{1}{2}\left(\hat{\phi}_s^2 - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r + \hat{\phi}_r^2\right)\tau\right)\right. \\ &\quad \left. +\frac{1}{2}\left(-\frac{\sigma_r^2}{2\kappa}B(\tau)^2 + \left(2\frac{\phi_r}{\kappa} - \frac{\sigma_r^2}{\kappa^2}\right)\left(B(\tau)-\tau\right) + \hat{\phi}_s^2\tau + \hat{\phi}_r^2\tau - 2\rho_{sr}\hat{\phi}_s\hat{\phi}_r\tau\right)\right] \\ &= \exp\left(R_\infty\left(B(\tau)-\tau\right) - r_t B(\tau) - \frac{\sigma_r^2}{4\kappa}B(\tau)^2\right),\end{aligned} \tag{46}$$

where

$$\begin{aligned}\tau &= T-t,\\ R_\infty &= \overline{r}+\frac{\Phi_r}{\kappa}-\frac{\sigma_r^2}{2\kappa^2},\\ B\left(T-t\right) &= \frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}.\end{aligned}$$

Applying Ito-Doeblin Lemma to (46), we have

$$dP_t=\frac{dP_t}{dt}+\frac{dP_t}{dr_t}dr_t+\frac{1}{2}\frac{dP_t^2}{d^2r_t}\left(dr_t\right)^2, \tag{47}$$

where

$$\begin{aligned}\frac{dP_t}{dt} &= P_t\left(\frac{dB\left(\tau\right)}{dt}\left(R_\infty-r_t\right)+R_\infty-\frac{\sigma_r^2}{2\kappa}B\left(\tau\right)\frac{dB\left(\tau\right)}{dt}\right)dt && (48)\\ \frac{dP_t}{dr_t}dr_t &= P_t\left(-B\left(\tau\right)\right)\left(\left(\kappa\left(\overline{r}-r_t\right)\right)dt-\sigma dZ_{r,t}\right) && (49)\\ \frac{dP_t^2}{d^2r_t}\left(dr_t\right)^2 &= P_tB^2\left(\tau\right)\sigma^2dt && (50)\\ \frac{dB\left(\tau\right)}{dt} &= -e^{-\kappa(T-t)} && (51)\end{aligned}$$

Substituting (48)~(51) into (47), we have

$$\begin{aligned}\frac{dP_t}{P_t} &= \left[\frac{\Phi_r}{\kappa}\left(1-e^{-\kappa(\tau)}\right)+r_t\right]dt+\sigma_rB\left(\tau\right)dZ_{r,t} && (52)\\ &= \left[r_t+\Phi_P\sigma_{P_t}\right]dt+\sigma_PdZ_{r,t},\end{aligned}$$

where

$$\begin{aligned}\Phi_P &= \frac{\Phi_r}{\sigma_r},\\ \sigma_{P_t} &= \sigma_rB\left(\tau\right).\end{aligned}$$

Appendix B: Optimal Portfolio Wealth and Indirect Utility without a VaR Constraint

B.1 The Optimal Portfolio Allocation

The pre-horizon portfolio wealth without a VaR constraint is

$$\begin{aligned} W_t^u &= \frac{1}{\zeta_t} E_t \zeta_T W_T^u \\ &= (y\zeta_t)^{-\frac{1}{\gamma}} \exp\left[\mu_{\zeta,t,T}\left(1-\frac{1}{\gamma}\right) + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma_{\zeta,t,T}^2\right]. \end{aligned} \tag{53}$$

Applying Ito-Doeblin lemma to (53), we have

$$dW_t^u = [...]\, dt + \frac{1}{\gamma} W_t \hat{\phi}_s dZ_{s,t} - \frac{1}{\gamma} W_t \hat{\phi}_r dZ_{r,t} + W_t \left(1-\frac{1}{\gamma}\right) B\left(T-t\right) \sigma_r dz_{r,t}. \tag{54}$$

Assume that the investor will invest π_s, π_p^M and $1-\pi_s-\pi_p^M$ percent of his wealth in the stock index, the bond fund with M-year constant time to maturity and the cash account respectively. Let W_t be the portfolio wealth,

$$\frac{dW_t}{W_t} = \left(r_t + \pi_s \Phi_S + \pi_p^M \Phi_r B\left(M\right)\right) dt + \pi_s \sigma_S dZ_s + \pi_p^M \sigma_r B\left(M\right) dZ_r. \tag{55}$$

Equating the coefficients of dz_s and dz_r in (54) and (55), we get

$$\begin{bmatrix} \pi_s \\ \pi_p^M \\ \pi_c \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} \frac{1}{\sigma_S}\hat{\phi}_s \\ -\frac{1}{\sigma_r B(M)}\hat{\phi}_r \\ 1-\frac{1}{\sigma_S}\hat{\phi}_s + \frac{1}{\sigma_r B(M)}\hat{\phi}_r \end{bmatrix} + \left(1-\frac{1}{\gamma}\right) \begin{bmatrix} 0 \\ \frac{\sigma_r B(T-t)}{\sigma_r B(M)} \\ 1-\frac{\sigma_r B(T-t)}{\sigma_r B(M)} \end{bmatrix}, \tag{56}$$

where π_c stands for the allocation to cash.

As can be seen in (56), the bond with constant maturity is held for both the speculative and the hedge purposes. To uncouple these two tasks, we introduce a zero-coupon bond that matures at time T.

If we invest in cash, stock index, bond fund and bond maturing at time T. There is one redundant asset. Investment \$1 in P_t^{T-t} is equivalent to investment $\$\frac{\sigma_r B(T-t)}{\sigma_r B(M)}$ in P_t^M and $\$\left(1-\frac{\sigma_r B(T-t)}{\sigma_r B(M)}\right)$ in cash, since the cash flow from investing $\$\frac{\sigma_r B(T-t)}{\sigma_r B(M)}$ in P_t^M and

\$ $\left(1-\frac{\sigma_r B(T-t)}{\sigma_r B(M)}\right)$ in cash is

$$\begin{aligned}
& \frac{\sigma_r B(T-t)}{\sigma_r B(M)} \frac{dP_t^M}{P_t^M} + \left(1-\frac{\sigma_r B(T-t)}{\sigma_r B(M)}\right) r_t dt \\
= & \frac{\sigma_r B(T-t)}{\sigma_r B(M)} \left[(r_t + \Phi_r B(M))\, dt + \sigma_r B(M)\, dZ_{r,t}\right] \\
& + \left(1-\frac{\sigma_r B(T-t)}{\sigma_r B(M)}\right) r_t dt \\
= & \left[\Phi_r \frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa} + r_t\right] dt + \sigma_r B(T-t)\, dZ_{r,t},
\end{aligned}$$

which is the same as the diffusion process of the risk-free bond price with $T-t$ years to maturity.

If the investor will include the risk-free bond $P_t^{(T-t)}$ in his portfolio, the optimal portfolio choice is

$$\begin{bmatrix} \pi_s \\ \pi_p^M \\ \pi_c \\ \pi_p^{T-t} \end{bmatrix} = \frac{1}{\gamma} \begin{bmatrix} \frac{1}{\sigma_S}\hat{\phi}_s \\ -\frac{1}{\sigma_M}\hat{\phi}_r \\ 1-\frac{1}{\sigma_S}\hat{\phi}_s + \frac{1}{\sigma_M}\hat{\phi}_r \\ 0 \end{bmatrix} + \left(1-\frac{1}{\gamma}\right) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{57}$$

B.2 The Indirect Utility

The indirect utility of current wealth at time t, is the maximum expected utility conditional on the information available at time t. The maximum is obtained by the optimal wealth process. That is,

$$\begin{aligned}
& \max_{W_T} E_t \frac{W_T^{1-\gamma}}{1-\gamma}, \\
& s.t.,\ \zeta_t W_t = E_t \zeta_T W_T.
\end{aligned} \tag{58}$$

The optimal wealth solving (58) is

$$W_T^u = (k\zeta_T)^{-\frac{1}{\gamma}}, \tag{59}$$

where the Lagrangian multiplier $k = \left[\zeta_t^{\frac{1}{\gamma}} W_t \exp\left(-\mu_{\zeta,t,T}\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma^2_{\zeta,t,T}\right)\right]^{-\gamma}$.
Inserting (59) into (58), the indirect utility at time t $J_t^u\left(W_t^u\right)$ is

$$\begin{aligned} J_t^u\left(W_t^u\right) &= \frac{1}{1-\gamma}\left(W_t^u\right)^{1-\gamma} \exp\left(-\mu_{\zeta,t,T}\left(1-\frac{1}{\gamma}\right)(1-\gamma) - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma^2_{\zeta,t,T}(1-\gamma)\right) \\ &\quad \times E_t \frac{\zeta_T^{1-\frac{1}{\gamma}}}{\zeta_t^{1-\frac{1}{\gamma}}}. \end{aligned} \tag{60}$$

Since

$$E_t \frac{\zeta_T^{1-\frac{1}{\gamma}}}{\zeta_t^{1-\frac{1}{\gamma}}} = \exp\left(\mu_{\zeta,t,T}\left(1-\frac{1}{\gamma}\right) + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma^2_{\zeta,t,T}\right), \tag{61}$$

we have

$$\begin{aligned} J_t^u\left(W_t^u\right) &= \frac{1}{1-\gamma}\left(W_t^u\right)^{1-\gamma} \exp\left(-\mu_{\zeta,t,T}(1-\gamma) + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \sigma^2_{\zeta,t,T}\gamma\right) \\ &= \frac{1}{1-\gamma}\left(W_t^u\right)^{1-\gamma} \exp\left[\left(-\mu_{\zeta,t,T} - \frac{1}{2}\sigma^2_{\zeta,t,T}\right)(1-\gamma)\right] \\ &\quad \times \exp\left[\frac{1}{2}\sigma^2_{\zeta,t,T}\left(\frac{1}{\gamma}-1\right)\right]. \end{aligned} \tag{62}$$

Appendix C: The Intermediate portfolio wealth and The Optimal portfolio allocation in The Single-Constraint Model

C.1 The intermediate Portfolio Wealth

At time t, the portfolio wealth in the single-constraint model is

$$\begin{aligned}
W_t^{c1} &= \frac{1}{\zeta_t} E_t W_T^{c1} \qquad (63)\\
&= \underbrace{\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}} N\left(d_1\left(\min\left(\underline{\zeta}_T^{v1}, \overline{\zeta}_T^{v1}\right)\right)\right)}_{\text{term 1}}\\
&\quad \underbrace{+\underline{W} e^{\left(\mu_{\zeta,t,T}+\frac{1}{2}\sigma^2_{\zeta,t,T}\right)}\left[N\left(-d_2\left(\min\left(\underline{\zeta}_T^{v1}, \overline{\zeta}_T^{v1}\right)\right)\right) - N\left(-d_2\left(\overline{\zeta}_T^{v1}\right)\right)\right]}_{\text{term 2}}\\
&\quad \underbrace{+\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}} N\left(-d_1\left(\overline{\zeta}_T^{v1}\right)\right)}_{\text{term 3}},
\end{aligned}$$

where

$$\begin{aligned}
d_2(x) &= \frac{\log\left(\frac{x}{\zeta_t}\right) - \left(\mu_{\zeta,t,T} + \sigma^2_{\zeta,t,T}\right)}{\sigma_{\zeta,t,T}},\\
d_1(x) &= d_2(x) + \frac{1}{\gamma}\sigma_{\zeta,t,T},\\
\Gamma_t &= \mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}} + \frac{1}{2}\sigma^2_{\zeta,t,T} - \frac{1}{\gamma}\left(\mu_{\zeta,t,T} + \sigma^2_{\zeta,t,T}\right) + \frac{1}{2}\frac{1}{\gamma^2}\sigma^2_{\zeta,t,T}.\\
&= \left(1 - \frac{1}{\gamma}\right)\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}} + \left(\frac{1}{2} - \frac{1}{\gamma} + \frac{1}{2\gamma^2}\right)\sigma^2_{\zeta,t,T}.
\end{aligned}$$

Y_{T-t} is defined as $\log(\zeta_T/\zeta_t)$. Now, Y_{T-t} is normally distributed with mean $\mu_{\zeta,t,T}$ and variance $\sigma^2_{\zeta,t,T}$. First Term of (63) comes from:

$$\begin{aligned}
& E_t\left(\frac{\zeta_T}{\zeta_t}(y\zeta_T)^{-\frac{1}{\gamma}} 1_{\left\{\zeta_T \le \underline{\zeta}_T^{c1}\right\}}\right)\\
&= E_t\left(e^{Y_{T-t}}(y\zeta_T)^{-\frac{1}{\gamma}} 1_{\left\{Y_{T-t} \le \log\left(\frac{\underline{\zeta}}{\zeta_t}\right)\right\}}\right)\\
&= \int_{-\infty}^{\log\left(\frac{\underline{\zeta}}{\zeta_t}\right)} e^{Y_{T-t}}(y\zeta_T)^{-\frac{1}{\gamma}} \frac{1}{\sqrt{2\pi\sigma^2_{\zeta,t,T}}} e^{-\frac{1}{2\sigma^2_{\zeta,t,T}}\left(Y_{T-t}-\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}\right)^2} dY_{T-t}\\
&= y^{-\frac{1}{\gamma}}\zeta_t^{-\frac{1}{\gamma}} \exp\left(\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}} + \frac{1}{2}\sigma^2_{\zeta,t,T} - \frac{1}{\gamma}\left(\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}} + \sigma^2_{\zeta,t,T}\right) + \frac{1}{2}\frac{1}{\gamma^2}\sigma^2_{\zeta,t,T}\right) N\left(d_1\left(\underline{\zeta}\right)\right).
\end{aligned}$$

C.2 The Optimal Portfolio Allocation in the One-Constraint Model

The optimal portfolio with cash, stock and a bond fund with constant maturity

Applying Ito-Doeblin lemma to (63), the diffusion process of the pre-horizon wealth at time t W_t^{v1} is

$$\frac{dW_t^{c1}}{W_t^{c1}} = [...]\, dt + \frac{1}{W_t^{c1}}\left(-\frac{dW_t}{dr_t}\sigma_r + \frac{dW_t}{d\zeta_t}\hat{\phi}_r\zeta_t\right) dZ_{r,t} - \frac{1}{W_t^{c1}}\hat{\phi}_s\zeta_t\frac{dW_t}{d\zeta_t}dZ_{s,t.} \tag{64}$$

Assume that the investor will invest π_s, π_p^M and $1-\pi_s-\pi_p^M$ percent of his wealth in the stock index, the bond fund with M-year constant time to maturity and the cash account respectively. Let W_t be the portfolio wealth,

$$\frac{dW_t}{W_t} = \left(r_t + \pi_s\Phi_S + \pi_p^M\Phi_r B\left(M\right)\right) dt + \pi_s\sigma_S dz_s + \pi_p^M\sigma_r B\left(M\right) dz_r. \tag{65}$$

Equating the coefficients of (64) and (65), the optimal portfolio wealth without investing in the risk-free bond P_t^{T-t} is

$$\begin{aligned}\begin{bmatrix}\pi_s\\ \pi_p^M\\ \pi_c\end{bmatrix} &= \frac{1}{\gamma}\begin{bmatrix}\frac{\hat{\phi}_s}{\sigma_S}X_{spec,t}^{c1}\\ -\frac{\hat{\phi}_r}{\sigma_r B(M)}X_{spec,t}^{c1}\\ 1-\frac{\hat{\phi}_s}{\sigma_S}X_{spec,t}^{c1}+\frac{\hat{\phi}_r}{\sigma_r B(M)}X_{spec,t}^{c1}\end{bmatrix}\\ &\quad+\left(1-\frac{1}{\gamma}\right)\begin{bmatrix}0\\ \frac{1}{B(M)}\frac{\mathbf{1-e^{-\kappa(T-t)}}}{\boldsymbol{\kappa}}X_{hedge,t}^{c1}\\ 1-\frac{1}{B(M)}\frac{\mathbf{1-e^{-\kappa(T-t)}}}{\boldsymbol{\kappa}}X_{hedge,t}^{c1}\end{bmatrix},\end{aligned} \tag{66}$$

where

$$\begin{aligned}X_{spec,t}^{c1} &= -\frac{dW_t}{d\zeta_t}\frac{\zeta_t}{W_t}\gamma,\\ X_{hedge,t}^{c1} &= -\frac{1}{W_t\frac{\mathbf{1-e^{-\kappa(T-t)}}}{\boldsymbol{\kappa}}\left(1-\frac{1}{\gamma}\right)}\frac{dW_t}{dr_t},\end{aligned}$$

$$
\begin{aligned}
\frac{d\mathbf{W}_t}{dr_t} &= -\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}}\mathbf{N}\left(d_1\left(\underline{\zeta}\right)\right)\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}\left(1-\frac{1}{\gamma}\right) \\
&+\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}}\boldsymbol{\phi}\left(d_1\left(\underline{\zeta}\right)\right)\frac{\left(\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}\right)}{\boldsymbol{\kappa}\sigma_{\zeta,t,T}} \\
&-\underline{W}e^{\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N\left(-d_2\left(\underline{\zeta}\right)\right)\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}} \\
&-\underline{W}e^{\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}+\frac{1}{2}\sigma^2_{\zeta,t,T}}\phi\left(-d_2\left(\underline{\zeta}\right)\right)\frac{\left(\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}\right)}{\boldsymbol{\kappa}\sigma_{\zeta,t,T}} \\
&+\underline{W}e^{\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}+\frac{1}{2}\sigma^2_{\zeta,t,T}}N\left(-d_2\left(\overline{\zeta}\right)\right)\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}} \\
&+\underline{W}e^{\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}+\frac{1}{2}\sigma^2_{\zeta,t,T}}\phi\left(-d_2\left(\overline{\zeta}\right)\right)\frac{\left(\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}\right)}{\boldsymbol{\kappa}\sigma_{\zeta,t,T}} \\
&-\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}}\mathbf{N}\left(-d_1\left(\overline{\zeta}\right)\right)\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}\left(1-\frac{1}{\gamma}\right) \\
&-\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}}\boldsymbol{\phi}\left(-d_1\left(\overline{\zeta}\right)\right)\frac{\left(\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}\right)}{\boldsymbol{\kappa}\sigma_{\zeta,t,T}},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\mathbf{W}_t}{d\zeta_t} &= \left(-\frac{1}{\gamma}\right)\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}\zeta_t}\mathbf{N}\left(d_1\left(\underline{\zeta}\right)\right) \\
&+\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}}\boldsymbol{\phi}\left(d_1\left(\underline{\zeta}\right)\right)\left(\frac{-\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\right) \\
&+\underline{W}\mathbf{e}^{\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}+\frac{1}{2}\sigma^2_{\zeta,t,T}}\phi\left(-d_2\left(\underline{\zeta}\right)\right)\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}} \\
&-\underline{W}\mathbf{e}^{\mu_{\boldsymbol{\zeta},\mathbf{t},\mathbf{T}}+\frac{1}{2}\sigma^2_{\zeta,t,T}}\phi\left(-d_2\left(\overline{\zeta}\right)\right)\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}} \\
&+\left(-\frac{1}{\gamma}\right)\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}\zeta_t}\mathbf{N}\left(-d_1\left(\overline{\zeta}\right)\right) \\
&+\frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}}\boldsymbol{\phi}\left(-d_1\left(\overline{\zeta}\right)\right)\left(\frac{\zeta_t^{-1}}{\sigma_{\zeta,t,T}}\right).
\end{aligned}
$$

The optimal portfolio with cash, stock, a bond fund with constant maturity and a zero-coupon bond

The cash flow of investing $\frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}$ in the bond fund P_t^M and $1-\frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}$ in cash is

$$\begin{aligned}
& \frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}\frac{dP_t^M}{P_t^M}+\left[1-\frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}\right]r_t dt \\
= & \frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}\left[\left(r_t+\Phi_r B\left(M\right)\right)dt-\sigma_r B\left(M\right)dZ_{r,t}\right] \\
& +\left[1-\frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}\right]r_t dt \\
= & \left[r_t+\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}\Phi_r\right]dt-\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}\sigma_r dZ_{r,t}.
\end{aligned}$$

The cash flow of investing $1-X_{r_t}$ in cash and X_{r_t} in the riskfree bond P_t^{T-t} is

$$\begin{aligned}
& \left[1-X_{r_t}\right]r_t dt+X_{r_t}\frac{dP_t^{T-t}}{P_t^{T-t}} \\
= & \left[1-X_{r_t}\right]r_t dt+X_{r_t}\left[\left(r_t+\Phi_r B\left(T-t\right)\right)dt-\sigma_r B\left(T-t\right)dZ_{r,t}\right] \\
= & \left[r_t+X_{r_t}\Phi_r B\left(T-t\right)\right]dt-X_{r_t}\sigma_r B\left(T-t\right)dZ_{r,t},
\end{aligned}$$

where

$$B\left(T-t\right)=\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}.$$

Thus, investing $\frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}$ in the bond fund P_t^M and $1-\frac{1}{B_M}\frac{\mathbf{1}-\mathbf{e}^{-\boldsymbol{\kappa}(T-t)}}{\boldsymbol{\kappa}}X_{r_t}$ in cash generate the same cash flow as investing $1-X_{r_t}$ in cash and X_{r_t} in the riskfree bond $P_t^{(T-t)}$. Therefore, the optimal portfolio choice with a riskfree bond maturing at time T is

$$\begin{aligned}
\begin{bmatrix}\pi_s\\ \pi_p^M\\ \pi_c\\ \pi_p^{T-t}\end{bmatrix} = & \frac{1}{\gamma}\begin{bmatrix}\frac{\hat{\phi}_s}{\sigma_S}X_{spec,t}^{c1}\\ -\frac{\hat{\phi}_r}{\sigma_r B_M}X_{spec,t}^{c1}\\ 1-\frac{\hat{\phi}_s}{\sigma_S}X_{spec,t}^{c1}+\frac{\hat{\phi}_r}{\sigma_r B_M}X_{spec,t}^{c1}\\ 0\end{bmatrix} \\
& +\left(1-\frac{1}{\gamma}\right)\begin{bmatrix}0\\ 0\\ 1-X_{hedge,t}^{c1}\\ X_{hedge,t}^{c1}\end{bmatrix}.
\end{aligned} \tag{67}$$

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