Dynamic Correlation or Tail Dependence Hedging for Portfolio Selection

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Abstract

We solve for the optimal portfolio allocation in a setting where both conditional correlation and the clustering of extreme events are considered. We demonstrate that there is a substantial welfare loss in disregarding tail dependence, even when dynamic conditional correlation has been accounted for, and vice versa. Both effects have distinct portfolio implications and cannot substitute each other. We also isolate the hedging demands due to macroeconomic and market conditions that command important economic gains. Our results are robust to the sample period, the choice of the dependence structure, and both varying levels of average correlation and tail dependence coefficients.

JEL Classification: C15, C16, C51, G11

Keywords: correlation hedging, dynamic portfolio allocation, Monte Carlo simulation, tail dependence.

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It has become apparent in recent years that correlation between assets is time varying and peaks in periods of economic downturns[1]. Furthermore, there is ample evidence that the correlation measure is insufficient to capture asymmetric tail dependence between assets. Using Extreme Value Theory, Longin and Solnik (2001) find that international stock markets tend to be more highly correlated during extreme market downturns than during extreme market upturns, establishing a pattern of asymmetric tail dependence that linear measures of dependence cannot describe. Ang and Chen (2002) confirm this finding for the US market for correlations between stock returns and an aggregate market index. The analysis of risky asset dependencies has recently received a lot of attention simply because it is important for many finance related applications and forms the basis of portfolio diversification. The degree to which investors can reduce their risk by diversifying their portfolio depends on the dependence structure between asset returns. Modeling the exact form of dependence between the traded assets is thus crucial for any investment decision.

Motivated by the aforementioned evidence of time varying conditional correlations and asymmetric dependence of equity markets, we propose a continuous time process for asset prices that incorporates these findings in two distinct ways. First, we allow for both dependence between extreme realizations of asset returns as well as for dynamic conditional correlation. Second, we incorporate the impact of economic states and the market volatility level in driving assets' dependence structure. Our focus in this paper is centered on the implications of these realistic features of the data on the diversification benefits and agents' investment decisions overall[2]. More precisely, we wish to demonstrate the importance of considering dynamic correlation and the clustering of extreme events in solving for agents' portfolio problems. We also study the economic significance of considering asymmetric tail dependence beyond accounting for a time-varying conditional correlation and vice versa. In addition, we wish to shed light on the incremental benefit of considering the link between the state of the economy and the market on an agent's portfolio allocation. To this end, we aim at isolating explicitly the hedging demands for correlation risk due to stochastic changes in both economic and market variables.

In this study, we consider a comprehensive approach to model both asset co-movement asymmetries and their stochastic conditional correlation. Our aim is to stay as true as possible to stylized features of the

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[1]Using 150 years of equity data, Goetzmann et al. (2005) find that correlations between equity returns vary substantially over time and peak during periods characterized by highly integrated financial markets. Longin and Solnik (1995) study shifts in global equity markets correlation structure and reject the hypothesis of constant correlations among international stock markets. They find evidence that correlations increase during highly volatile periods. Other important work also connects the variability of stock return correlations to the business cycle. Ledoit et al. (2003) and Erb et al. (1994) show that correlations are time-varying and depend on the state of the economy, tending to be higher during periods of recession. Similar evidence is brought forward by Moskowitz (2003) who links time variation of volatilities and covariances to NBER recessions.

[2]It is important to note that these empirical characteristics of the data have found theoretical justification in the work of Ribeiro and Veronesi (2002). They show that in a Rational Expectations Equilibrium model, time variations in correlations are obtained endogenously as a result of changes in agents' uncertainty about the state of the economy. Further, by relating recessionary periods to a higher level of uncertainty, excess co-movements across international stock markets are obtained when the global economy slows down.

data while maintaining tractable solutions for the portfolio allocation. Asymmetric co-movement between the risky funds is achieved through the stationary distribution of the multivariate diffusion process of the state variables that drive the risky assets' prices. We introduce an asymmetric dependence structure of the distribution explicitly by using copula functions. This allows us to isolate the effect of the marginal distributions from that of the dependence structure itself. This construction of a multivariate diffusion with a pre-specified stationary distribution permits to obtain higher dependence when markets experience downturns than during upward periods. However, this approach by itself does not exploit the conditional correlation structure of the process, so we further propose a specification for modeling the correlation dynamics of risky asset returns that integrates the link to macroeconomic and market variables. In sum, our framework is able to consider all the aforementioned asset co-movement characteristics without reverting to an incomplete market setup[3]. For completeness, we also study the sensitivity of our results to the choice of the dependence function. We consider copula functions that incorporate dependence between extreme realizations of the state variables and copulas that imply no tail dependence. Then we examine the differences in the intertemporal hedging demands entailed by these two alternative data generating processes.

Being in a complete market setup while considering both asymmetric dependence and time varying correlations brings important advantages in solving for the agent's investment problem. We apply the standard portfolio solution methodology of Cox and Huang (1989), further developed by Ocone and Karatzas (1991) and Detemple et al. (2003)[4]. We obtain in closed form - up to a numerical integration - the optimal portfolio components in terms of mean-variance demand as well as intertemporal hedging demands. Our specification permits us to solve for the agent's investment problem under general utility preferences that are not constrained to the Constant Relative Risk Aversion (CRRA) case. Instead, we consider the more general Hyperbolic Absolute Risk Aversion (HARA) utility function, which has an additional parameter to the one that accounts for the level of risk aversion. This parameter imposes a lower boundary on wealth, so that this utility specification allows for intolerance towards wealth falling below this threshold. In sum, by implementing a realistic specification for the multivariate distribution of asset returns and investor's preferences, and solving for the portfolio holdings of an agent over typical investment horizons, this paper makes the following contributions:

1) We test whether the optimal portfolio hedging demands needed in the case when the data generating process allows for tail dependence through the stationary distribution are similar in magnitude

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[3]The inclusion of a multivariate jump process for example could also produce an asymetric tail dependence. However, this modeling choice would limit the tractibility of the model. In addition, it is not straightforward to isolate the hedging demands for correlation risk. Note also that introducing a jump factor would automatically cast the model into an incomplete market setup. In this case, more restrictive assumptions on asset price dynamics and agent's preferences should be put in place in order to obtain the optimal portfolio policy, as in Das and Uppal (2004).

[4]An alternative simulation-based methodology in a discrete-time setting is proposed by Brandt et al. (2005). It can handle a large scale problem with non-standard preferences without relying on distributional assumptions for the risky assets.

and direction to the ones obtained in a dynamic conditional correlation specification. To address this question, we consider an in-sample market timing exercise along realized paths of the state variables. We perform this exercise over a 20-year investment horizon and for two risky funds. We find that allowing for dynamic conditional correlation has a mixed effect on the intertemporal hedging demands depending on whether the investor incorporates tail dependence between the risky funds in her portfolio decision: it drives hedging demands up when no tail dependence is modeled, or down when the latter is taken into consideration. Results are more consistent when varying the assumptions in the stationary distribution: allowing for tail dependence in the stationary distribution always diminishes the intertemporal hedging demands. We also assess the effect of the investment horizon in driving our results. We change the investment period and find that these effects become more important when increasing the investment horizon.

2) To appreciate the economic importance of "considering vs. ignoring" either dynamic conditional correlation or tail dependence, we further translate our findings into the certainty equivalent cost. We show substantial utility loss due to disregarding either form of dependence. The loss increases dramatically with the investment horizon and for low levels of the agent's relative risk aversion. Subsequently, ignoring these features of the data would translate into suboptimal decisions for the investor. More importantly, we find substantial utility loss for disregarding dependence between extreme realizations, even when dynamic conditional correlation has already been accounted for, and vice versa (e.g. for an investment horizon that ranges from 1 to 5 years, the certainty equivalent cost rises from 1% to 12% of initial wealth for the most risk-averse investor we consider with a coefficient of relative risk aversion of 2 and for a data-generating process that is modeled to exclusively incorporate tail dependence). Thus, these findings suggest that it is economically crucial to consider both asymmetric tail dependence and dynamic conditional correlation in agents' investment decisions.

3) We also investigate the evolution of the assets' correlation hedging demands implied by the macroeconomic and market implication in driving assets' co-movement. Our specification permits us to isolate the correlation hedging demands required to hedge against fluctuations in these variables. We show that the total correlation hedging demands due to these variables are generally negative throughout the period we consider which leads toward a reduction in the total exposure to risky assets. The impact of the macroeconomic variable is more significant than the one due to the market influence, and directs the behavior of the hedging demands.

Two questions arise when examining the above results. First, to what extent are they driven by our sample period? Second, how robust are they to different levels of correlation and tail dependencies? We address these two issues separately.

First, we test whether the results are sensitive to the particular choice of the investment period. To do so, we consider two sub-periods that differ in both the level of stock market volatility and macroeconomic conditions?. We then consider an investor with investment horizon set at the end of each of these sub-periods. We show that for a relatively calm period with almost no extreme events, the impact of tail dependence disappears once we allow for a data generating process that incorporates dynamics in the conditional correlation behavior. However, for a hectic period with worsening macroeconomic conditions and a number of extreme events, especially towards its end, the importance of modeling tail dependence for the optimal hedging demand cannot be overwritten or substituted by allowing for dynamically varying correlations.

Second, we examine the sensitivity of the optimal hedging behavior for different levels of the average (or long run mean of) correlation. We document higher hedging demands for high correlation levels, especially when the impact of stochastic changes in conditional correlations on investor's utility is expected to be the highest. This finding is confirmed by the certainty equivalent cost of disregarding dynamic conditional correlation. The utility loss increases for higher levels of average correlation. Alternatively, we examine the impact of disregarding tail dependence for varying levels of tail dependence coefficients in the data generating process and find that there are far more significant costs of disregarding dependence between extreme realizations when its level increases, even when dynamic conditional correlation is already taken into account.

Related literature

The present study is closely related to the work of Buraschi et al. (2009) who solve for the optimal portfolio hedging behavior in the presence of correlation risk in a setting where both volatilities and correlations are stochastic. That gives rise to separate demands for volatility and correlation risk. They model covariance dynamics using the analytically tractable Wischart process and study the portfolio impact of stylized facts of asset returns such as volatility and correlation persistence and leverage effects. Our study confirms their results on volatility and correlation using a different setup. In addition, our model adds to this important work by considering both the incremental relevance of accounting for asymmetric tail dependence beyond dynamic conditional correlation as well as the importance of considering the effect of macroeconomic states and volatility levels in driving the dependence structure.

Our work also provides a novel specification compared to the model introduced in Liu (2007). While in Buraschi et al. (2009) the correlation between the risky assets is stochastic and is driven by its independent risk source, the model of Liu (2007) allows for stochastic correlations that are deterministic functions of return volatilities. His specification does not allow disentangling the portfolio effect of correlation from that of volatility. However, under some restrictive assumptions it is again possible to obtain explicit dynamic portfolio solutions for an investor with CRRA utility.

From a modeling perspective, our paper is inspired by the large literature on modeling asset co-movements. Popular choices for the time-varying correlation phenomenon are multivariate GARCH models (e.g. Bollerslev et al. (1988) or the principal component GARCH of Alexander (2002)), the parsimonious Dynamic Conditional Correlation model of Engle (2002), Engle and Sheppard (2001), and Tse and Tsui (2002), or alternatively the nonparametric model of range-based covariance obtained from intra-day returns as in Brandt and Diebold (2006), or the continuous time Wischart process, introduced by Bru (1991) that gives rise to an affine model and tractable portfolio allocation rules. In this paper, we allow for both asymmetric dependence and dynamic conditional correlation in a parsimonious way while keeping tractable the solution of the agent investment problem.

The present study is also closely related to the strand of literature that studies the implications of asset co-movement on dynamic portfolio choice. Ang and Bekaert (2002) consider a regime-switching model of asset returns that accounts for asymmetries in their dependence structure by including a 'bear' regime with low expected returns, coupled with high volatilities and correlations, and a 'normal' regime with high expected returns, low volatilities and correlations. They find that the asymmetric correlation structure between the two regimes becomes important for an international investor only when she is allowed to trade in the risk-free asset. Only in this case there are significant economic costs of disregarding regime switching. This paper is close in spirit to ours as a normal regime would correspond to a Gaussian dependence structure in our case, while a bear regime would be consistent with a dependence function that allows for extreme co-movements. However, they find statistically insignificant and economically negligible intertemporal hedging demands, even in the presence of a riskless asset, so that the welfare impact on the investor from behaving myopically is negligible. To the contrary, in our setup it is the hedging demands that mainly drive the differences between the portfolio policies under normal vs. tail dependent cases, or alternatively in constant vs. dynamic conditional correlation settings. They also appear to be a sizeable proportion of the optimal portfolio policy, even for a CRRA investor.

As mentioned above, another way to introduce tail dependence is to allow for jumps in the process of the state variables. Liu et al. (2003) model event-related jumps in prices and volatility in the double-jump framework, introduced by Duffie et al. (2000). The presence of event jumps renders the optimal portfolio holdings similar to those that can be obtained for an investor faced with short-selling and borrowing constraints. As well, event risk has a larger impact on the portfolio composition of investors with low levels of risk aversion. However, these results are obtained for a single risky asset portfolio.

Das and Uppal (2004) consider the impact of systemic risk on dynamic portfolio choice by introducing a jump component in asset prices that is common for all assets. They work in a constant investment opportunity set and find that investors who ignore systemic risk would have larger holdings of the risky assets. As well, there is higher cost associated to ignoring systemic risk for investors with low levels of risk aversion and levered portfolios. In this setting there are portfolio effects due to higher moments that

arise from the inclusion of jumps. However, jumps with constant jump intensity would have an effect through the tails of the conditional distribution, but not necessarily through the extremes of the invariant one. In our framework we have a clear distinction between the invariant distribution that matters in the long run and the conditional distributional features that have a local impact.

Alternatively, Cvitanic et al. (2008) develop optimal allocation rules under higher moments when risky assets are driven by a time-changed diffusion of the Variance Gamma type, and find that ignoring skewness and kurtosis leads to overinvestment in the risky assets and a substantial wealth loss, especially for high volatility levels.

Our findings contribute to this literature by documenting that there is substantial utility loss for disregarding dependence between extreme realizations, even when dynamic conditional correlation has already been accounted for, and vice versa. Thus, it is crucial to consider both asymmetric tail dependence and dynamic conditional correlation in any agent investment problem given non mean-variance or non-myopic preferences. Furthermore, our results show that it is important from the perspective of an allocation exercise to consider the implications of macroeconomic variables in driving asset co-movements.

The remainder of the paper is organized as follows. Section 1 describes the model, the solution to the portfolio choice problem, and the correlation hedging demands that appear due to changes in economic and market conditions driving asset dependence. In Section 2 we present numerical results on the importance of hedging demands that arise due to dynamic correlation or tail dependence. Section 3 concludes. Technical details are provided in the appendix.

1 The investment problem

This section describes the problem faced by the investor in allocating her wealth between a set of risky assets and a money market account in the presence of extreme asset dependencies. In what follows we introduce the model for the risky funds that accomodates these distributional feratures of the data. Further, we discuss the portfolio solution methodology under general von Neumann-Morgenstern preferences for which the non-linear dependence measure of tail dependence has a potential impact in addition to that of time-varying conditional correlations.

1.1 The economy

We define a filtered probability space $\left(\otimes, \left\{\mathcal{F}_t^Y\right\}_{t=0}^T, P^Y\right)$ over the investment horizon $[0, T]$ where $\mathcal{F}_t^Y$ is the filtration generated by state variables Y_t under the empirical probability measure P^Y. We consider a complete market setup with $d + 3$ state variables $Y_{it}, i = 1, ... d + 3$, where uncertainty is driven by a $(d + 1)$-dimensional Brownian motion $W' = \left(\left(W^X\right)', W^r\right)$. There are $d + 2$ securities available for investment: d stocks, a long term pure discount bond, and the risk-free asset. The state variable vector

Y_t consists of d state variables X_t, each one affecting its corresponding stock price process, two state variables F_t that proxy for macroeconomic states and market-wide volatility, and a state variable Y_t^r that governs the dynamics of the short rate r_t, that is $Y_t = (X_t, F_t, Y_t^r)^\mathsf{T}$.

The investor has at her disposal the following three asset categories. First, she can invest in a risk-free money market account and its value at time t is given by:

$$B_0(t) = \exp\left\{\int_0^t r(s, Y_s^r)\, ds\right\}. \tag{1.1}$$

Another tradeable asset in the portfolio is a default-free zero-coupon bond with a maturity τ. Its price $B(t, \tau)$ at time t can be expressed as a conditional expectation under the equivalent martingale measure Q:

$$B(t, \tau) = E^Q\left[\exp\left\{-\int_t^\tau r(s, Y_s^r)\, ds\right\} | \mathcal{F}_t^Y\right]. \tag{1.2}$$

The rest of the portfolio consists of a collection of stocks whose price processes are modeled using the d state variables X_t:

$$S_i(t) = \exp(X_{it} + \varphi(t)) \quad , i = 1, ..., d \tag{1.3}$$

where $\varphi(t)$ is a deterministic function of time. This specification was chosen in order to be as close as possible to the Geometric Brownian motion underlying the Black-Scholes formula for option pricing: if the process for X_{it} is given by $X_{it} = X_{i0} + \sigma_i \int_0^t dW_{it}^X$, then we are exactly in the Black-Scholes setting where all the assets are independent from each other; if alternatively we apply a stochastic time transformation to the Brownian motion and define the process for X_{it} as $X_{it} = X_{i0} + \int_0^t \sigma(t, X_{it})\, dW_{it}^X$, then we obtain a simple generalization of the Geometric Brownian motion that already departs from the normality assumption. As it is shown below, we also introduce a drift to the process for the state variables X_t which is consistent with a stationary distribution that allows for dependencies between tail realizations of the state variables. As well, correlations between the Brownians are allowed to be stochastic and be driven by macroeconomic and market-wide volatility state variables. The latter feature brings the model closer to the discrete-time alternative of a dynamic conditional correlation model, as the one introduced by Engle (2002). Thus, through the process of the state variables X_t we reproduce both increased tail dependence in bear markets and stochastic variations in their conditional correlation.

It is important to note that an investor with mean-variance preferences would not be influenced in her portfolio decision by any distributional assumptions beyond the first two moments. In such a case, the linear correlation coefficient will be the relevant dependence measure when solving for the optimal portfolio holdings and tail dependence features of the data will not matter for the investor. However, our analysis can be applied to a general utility specification under the standard assumptions of a strictly increasing

and concave utility function U that satisfies the Inada conditions. Thus, all distributional features of the data are potentially important for an investor with such preferences. For our particular application we choose a utility function with hyperbolic absolute risk aversion (HARA). It can be parametrized to model investor's intolerance towards wealth falling below a certain subsistence level for which relative risk aversion becomes infinite, or alternatively it can allow the latter to be strictly increasing and concave. It nests the standard constant relative risk aversion (CRRA) utility function used in literature.

We assume a one-factor affine model for the short rate, so that it is given by $r(t, Y_t^r) = \delta_0 + \delta_1 Y_t^r$. This modeling choice may be questionable as there is substantial empirical evidence concerning the shortcomings of affine models[5]. Also, using only one factor to capture the dynamics of the term structure may be too restrictive. But as the specification for the bond is marginal for our portfolio application, we proceed with this simple specification which ensures numerically tractable portfolio solutions. Also Y_t^r has the simple interpretation as a state variable that models the dynamics of the interest rate risk factor which further determine the hedging terms of the portfolio against changes in the stochastic interest rate. The specification of Y_t^r, as well as the resulting dynamics of the discount bond price are given in the appendix.

1.2 Modeling tail dependence and conditional correlation for the risky funds

In order to be able to disentangle the portfolio implications of unconditional tail dependencies of the risky funds from those ensuing from time-variations in conditional correlations, we need to specify a process for the state variables X so that both properties can be separated. Thus, we model unconditional tail dependence as a property of the multivariate stationary distribution of X, while a specification of the local correlation coefficient renders time variations in conditional correlation. Further, exploiting the Fokker-Planck equation, as in Chen et al. (2002), allows us to find a process for the state variables that is consistent with both the invariant distribution and conditional correlation. A similar approach is undertaken in Stefanova (2009), however it focuses exclusively on tail dependencies in the stationary density.

Following the overwhelming evidence that correlations between risky asset returns increase in volatile states, recessionary times and bear markets, we model conditional correlation as a function of macroeconomic and market variables. In order to account for the volatility effect, we choose the Chicago Board Options Exchange Volatility Index (VIX) which measures the implied volatility of S&P index options and thus incorporates market's expectations of near-term volatility. To incorporate the effect of the business cycle on the dynamics of correlation, we take the Chicago Fed National Activity Index (CFNAI) that synthesizes information on various macroeconomic factors in a single index[6]. This monthly index aggregates

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[5]Backus et al. (1998) show that term premiums generated by affine models are too low compared to the observed data; Duffee (2002) finds that this class of models is not flexible enough to replicate temporal patterns in interest rates.

[6]In order to appreciate the impact of both factors on the dynamics of asset correlation, we estimate a DCC model with

Figure 1. Evolution of the VIX index (upper panel) and of CFNAI index (bottom panel) for the period 1986 - 2006.

The VIX is quoted in terms of percentage points and the data is available at the daily frequency. The CFNAI is quoted monthly. A negative value of the CFNAI index indicates a below-average growth of the national economy, whereas a positive value of the index points towards an above-average growth. A zero value means that the economy grows at its historical average rate.
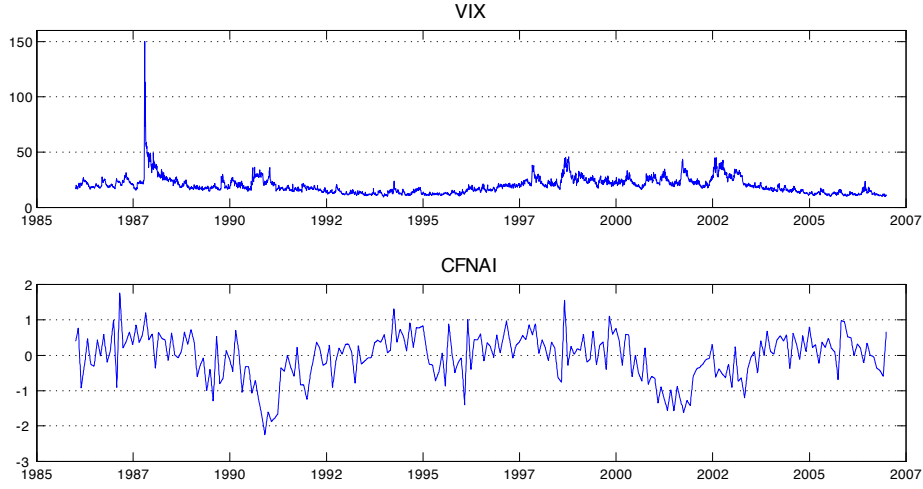
information on overall macroeconomic activity and inflation, as it is a weighted average of 85 indicators of national economic activity, ranging from production, employment, housing and consumption, income, sales, orders and inventories. The methodology behind the CFNAI is based on Stock and Watson (1999), who find a common factor behind various inflation indicators. The evolution of both variables is given in Figure 1.

1.2.1 The conditional correlation specification

Denoting the vector of volatility and macroeconomic variables as F_t, we consider the following general specification for the state variable process X_t:

$$dX_t = \mu\left(X_t, F_t\right) dt + \Lambda\left(X_t, F_t\right) dW_t^X \tag{1.4}$$

where Λ is a lower triangular matrix, defined as a function of the state variables X_t and F_t. The entries of the continuously differentiable positive definite matrix $\Sigma = \Lambda\Lambda'$ are given by $\nu_{ij}\left(X_t, F_t\right) = \Upsilon_{ij}\left(X_t, F_t\right) \sigma_i^X\left(X_t\right) \sigma_j^X\left(X_t\right)$. Borrowing the idea of Bibby and Sorensen (2003) for modeling the local

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exogenous factors on the return series of S&P 500 and NASDAQ - the two indices that we choose for the subsequent portfolio allocation exercise. Results that are not reported here for brevity indicate a significant positive (negative) relationship between correlation and the volatility index (the macroeconomic index).

volatility coefficient, we allow each $\sigma_i^X(X_t)$ to be a function of the state variables X_t:

$$\sigma_i^X(X_t) = \sigma_i \left[\widetilde{f^i}(x_i)\right]^{-\frac{1}{2}\kappa_i} \tag{1.5}$$

where $\widetilde{f^i}(x_i)$ is the non-normalized Normal Inverse Gaussian (NIG) density for X_i, and we have the following parameter restrictions: $\sigma_i > 0$ and $\kappa_i \in [0, 1]$. By expressing the volatility term as the inverse of a power function of the density $\widetilde{f}$ we obtain the usual U-shape for the volatility, typical for a stationary process. This specification is especially interesting, as it nests the constant conditional volatility as a special case, setting $\kappa_i = 0$.

The conditional correlation coefficient $\Upsilon_{ij}(X_t, F_t)$ is modeled as a function of the state variables X_t and the market and macroeconomic variables F_t:

$$\Upsilon_{ij}(X_t, F_t) = \mathcal{A}(h_{ij}(X_t, F_t)) = \frac{1 - \exp(-h_{ij}(X_t, F_t))}{1 + \exp(-h_{ij}(X_t, F_t))},$$
$$\text{where } h_{ij}(X_t, F_t) = \gamma_{ij,0} + \gamma_{ij,1}F_t^V + \gamma_{ij,2}\prod_{i=1}^{d}F(X_{it}) + \gamma_{ij,3}F_t^M \tag{1.6}$$
$$F_t^V = \log(VIX_t), \, F_t^M = CFNAI$$
$$\text{and } \mathcal{A}(h) = \frac{1 - \exp(-h)}{1 + \exp(-h)}.$$

The logistic transform $\mathcal{A}$ keeps the correlation coefficient in $[-1, 1]$. Apart the volatility level and macroeconomic conditions, as proxied by F_t^V and F_t^M, correlation also increases in extreme market downturns. As the economic cycle does not necessarily coincide with bear/bull financial markets, we also let the level of the state variables (expressed in terms of their probability integral transforms $F(X_i)$) to determine the dynamics of conditional correlations.

The Heston model (Heston (1993)) has been extensively studied and applied successfully in the context of modeling volatility. We follow this literature and consider a CIR process for F^V and a Vasicek process for the macroeconomic variable F^M:

$$dF_t^V = \kappa^V(\theta^V - F_t^V)dt + \sigma^V\sqrt{F_t^V}dW_t^X \tag{1.7}$$
$$dF_t^M = \kappa^M(\theta^M - F_t^M)dt + \sigma^M dW_t^X$$

These processes greatly facilitate the implementation of the portfolio allocation formula, as the Vasicek specification allows for a closed-form solution for the Malliavin derivative of the macroeconomic factor F^M, while the CIR diffusion term makes possible a variance-reduction technique for the Monte Carlo simulation of the Malliavin derivative of F^V.

1.2.2 The invariant density

With the choice of the stationary distribution we seek to answer several questions concerning the behavior of asset returns. Our major concern is the ability to allow assets to be dependent when they move towards the tails of the distribution, especially for the left tail. This would ensure our model the ability to replicate the empirical fact that asset returns are increasingly dependent as they jointly move towards the lower quantiles of their distribution, that is during market downturns. In order to allow our model the flexibility of different shapes of joint tail behaviour, while keeping the same specification for the marginal (fat) tails of $X_i, i = 1, ..., d$, we specify the invariant density q in terms of a copula function, following Sklar's representation theorem:

$$q\left(X_1, ..., X_d\right) \equiv \widetilde{c}\left(X_1, ..., X_d\right) \prod_{i=1}^{d} \widetilde{f^i}\left(X_i\right) \qquad (1.8)$$

where $\widetilde{c}\left(X_1, ..., X_d\right) = c\left(F^1\left(X_1\right), ..., F^d\left(X_d\right)\right)$ is a copula density defined over the univariate CDFs $F^i\left(X_i\right)$, and $\widetilde{f^i}\left(X_i\right)$ are the corresponding non-normalized univariate densities. We choose the Normal Inverse Gaussian (NIG) distribution[7] to model the univariate behavior via $\widetilde{f^i}\left(X_i\right)$ because of its proven ability to account for stylized facts of univariate asset return dynamics: autocorrelation of squared returns, semi-heavy tails, possibly asymmetric. Its tail behavior is richly parametrized, nesting tails that vary from an exponential to a power law. As well, NIG is one of the few members of the class of Generalized Hyperbolic (GH) distributions that is closed under convolution, that is if the distribution of log prices is modeled under a NIG law, then the distribution of the increments (asset returns) is also NIG. The univariate NIG diffusion is also an alternative to the widely used NIG Levy process (e.g. Eberlein and Keller 1995, Prause 1999) that allows for an infinite number of jumps in the price process. However, the latter also imposes independence of the increments, which is not the case for its diffusion counterpart that we consider.

The most important feature of the copula density representation (1.8) is that it allows us to separate the effect of the marginal behavior (e.g. fat tails) from the implications of the dependence structure (e.g. dependence in the tails) in our portfolio application. We hypothesize that the presence of tail dependence would cause a shift in the investor's holdings towards the riskless asset. The fact that we model dependence separately from the marginal behaviour implies that this shift is not motivated by the presence of fat tails per se, but by increased tail dependence, a property of the multivariate density only.

We define the coefficients of upper and lower tail dependence for couples of random variables X_1 and X_2. Upper (lower) tail dependence is the limit probability of the variable X_1 exceeding the upper quantile as it approaches it, conditional upon the fact that the second variable X_2 has exceeded (or is lower than) that same quantile. Both coefficients can be represented in terms of copula functions, so

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[7]See the appendix for details.

different copulas have different degrees of upper and lower tail dependence conditional on their parametric specification. Thus, in order to allow for different degrees of tail dependence, we consider the following copula specifications for C[8].

Case 1 *Gaussian copula C^{Ga}:* $\lambda_U = \lambda_L = 0$

In this case we do not allow for dependence between tail realizations of the state variables. The parameter that governs dependence is the correlation coefficient ρ.

Case 2 *A Gaussian - Symmetrized Joe-Clayton (SJC) mixture copula C^{Ga-SJC}:* $\lambda_U \neq \lambda_L$

The form of the mixture copula is given by:

$$C^{Ga-SJC} = \omega C^{SJC} + (1-\omega) C^{Ga}$$

where C^{Ga} stands for the Gaussian copula function and C^{SJC} - the Symmetrized Joe-Clayton copula, with a mixing parameter ω that determines the weights of each of the copulas. Proposed by Patton (2004), the symmetrized Joe-Clayton copula models separately upper and lower tail dependence and its form is particularly appealing, as the tail dependence coefficients are themselves the parameters of the copula function. We consider a mixture specification with this copula and the tail independent Gaussian one in order to answer the concerns raised in Poon et al. (2004) that a copula specification whose coefficients explicitly allow for tail dependence may overestimate the dependence in the tail regions.

1.2.3 A diffusion for X, consistent with unconditional tail dependence and conditional correlation

Having determined the invariant density and the local volatility coefficient, we can recover a diffusion for X that is consistent with them, using the Fokker-Planck equation, as in Chen et al. (2002):

$$\mu_j = \frac{1}{2} q^{-1} \sum_{i=1}^{d} \frac{\partial (\nu_{ij} q)}{\partial x_i} \tag{1.9}$$

where for ease of exposition we have replaced $\mu(X_t, F_t)$ by μ and $\nu_{ij}(X_t, F_t)$ by ν_{ij} (both will be used interchangeably with preference to the shorter notation when there is no ambiguity). [9]. Having recovered the diffusion for X_t and using the fact that the stock price is defined following (1.3), we can apply Itô's lemma in order to obtain the stock price process:

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[8]See the appendix for details on the alternative specifications of the copula functions used in the paper.

[9]We analyze the drift function implied by the Fokker-Planck equation. A plot of the function suggests the features of a mean-reverting process for asset returns. To save space, we do not report it in the analysis section. It is available upon request.

$$dS_{it} = S_{it}\mu_i^S(t, Y_t)\, dt + S_{it}\sum_{j=1}^{d}\Lambda_{ij}(t, Y_t)\, dW_{jt}^X \tag{1.10}$$

$$\text{where } \mu_i^S(t, Y_t) = \mu_i(Y_t) + \varphi'(t) + \frac{1}{2}\sum_{j=1}^{d}\overline{\sigma}_{ij}^2(Y_t).$$

Note that as we are in a complete market setting, the dynamics of the volatility and the macroeconomic variables F_t are governed by the same Brownian motions that drive the stock prices themselves. In this case we have an invertible matrix Λ, and we can subsequently define a market price of risk as $\Theta^S(t, Y_t) = \Lambda(t, Y_t)^{-1}\left(\mu^S(t, Y_t) - r(t, Y_t)\,\iota\right)$, where ι is a d-dimensional vector of ones. Let us stack the drift and diffusion terms for the bond and the stocks so that to obtain:

$$M(t, Y_t) = \begin{pmatrix} \mu_i^S(t, Y_t) \\ \mu^B(t, Y_t) \end{pmatrix}$$

$$\Xi(t, Y_t) = \begin{pmatrix} & & & 0 \\ \left[\Lambda^{(I)}(t, Y_t)\right] & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sigma^B(t, Y_t^r) \end{pmatrix}.$$

Then the market price of risk for all the tradeable assets is defined as:

$$\Theta(t, Y_t) = \Xi(t, Y_t)^{-1}\left(M(t, Y_t) - r(t, Y_t^r)\,\iota\right). \tag{1.11}$$

The market completeness implies the existence of a unique state price density ξ_t defined as

$$\xi_t \equiv B_0(t)^{-1}\eta_t \exp\left\{-\int_0^t r(s, Y_s^r)\, ds\right\} \times$$
$$\exp\left\{-\int_0^t \Theta(s, Y_s)^{\mathsf{T}}\, dW_s - \frac{1}{2}\int_0^t \Theta(s, Y_s)^{\mathsf{T}}\,\Theta(s, Y_s)\, ds\right\}$$

where η_t is the Radon-Nykodym derivative, $\mathcal{F}_t^Y$-adapted. We can also define the conditional state price density that converts cash flows at time $v \geq t$ into cash flows at time t:

$$\xi_{t,v}\equiv\xi_v/\xi_t=\exp\left\{ \begin{array}{c} -\int_t^v r(s, Y_s^r)\, ds - \int_t^v \Theta(s, Y_s)^{\mathsf{T}}\, dW_s \\ -\frac{1}{2}\int_t^v \Theta(s, Y_s)^{\mathsf{T}}\,\Theta(s, Y_s)\, ds \end{array} \right\}. \tag{1.12}$$

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1.3 The investor's objective function

We consider an investor who maximizes utility over terminal wealth, that we denote by $U(\omega_T)$, by choosing an optimal investment policy $\{\alpha_t\}_{t\in(0,T)}$ that belongs to an admissible set $\mathcal{A}$ for an investment horizon T:

$$\max_{\alpha\in\mathcal{A}} E\left[U(\omega_T)\right] \tag{1.13}$$

where the utility function U is strictly increasing, concave and differentiable, and satisfies the conditions $\lim_{x\to\infty} U'(\omega) = 0$ and $\lim_{x\to 0} U'(\omega) < \infty$. This standard utility specification includes the case of the Hyperbolic Relative Risk Aversion (HARA) utility function $U(\omega) = \frac{1}{1-\gamma}(\omega+b)^{1-\gamma}$ that we consider for this application. The coefficient of relative risk aversion, defined as $R(\omega) \equiv -\frac{U''(\omega)}{U'(\omega)}\omega$, is equal to $\gamma\frac{\omega}{\omega+b}$ for the HARA case, which boils down to a constant γ for the special case of CRRA utility. For $b < 0$ relative risk aversion is a convex and decreasing function of wealth, such that it becomes infinite when wealth gets near a subsistence boundary of $-b$. For nonnegative values of this coefficient, relative risk aversion is decreasing and concave.

The portfolio policy α is a $(d+1)$-dimensional progressively measurable process that is defined as the proportion of wealth allocated to the risky assets (d stocks and a long term pure discount bond). Thus, the amount invested in the risk-free asset (the money-market account) is $(\omega - \alpha^\intercal 1)$. The portfolio policy generates a wealth process ω whose dynamics are given by:

$$d\omega_t = \omega_t\left\{r_t dt + \alpha_t^\intercal\left[(M(t,Y_t) - r_t\iota)\,dt + \Xi(t,Y_t)\,dW_t\right]\right\}. \tag{1.14}$$

1.4 The complete market solution

The complete market setup that we have adopted allows us to solve for the optimal portfolio using the Martingale solution technique that restates the dynamic budget constraint (1.14) as a static one. We first solve for the optimal terminal wealth and then find the optimal portfolio policy that finances it. Following Cox and Huang (1989), the optimal terminal wealth is given by $\omega_T^* = I(y\xi_T)^+ = \max(I(y\xi_T), 0)$, where $I = [U']^{-1}$ denotes the inverse of the marginal utility function, and y satisfies the static budget constraint $E\left[\xi_T I(y\xi_T)^+\right] = \omega_0$, where ω_0 is the initial wealth.

Following Ocone and Karatzas (1991) and using the portfolio decomposition formula of Detemple et al. (2003), we get the following expression for the optimal portfolio policy that decomposes the portfolio holdings into a Mean Variance (MV) part α^{MV}, an Interest Rate (IR) Hedge α^{IRH} and a Market Price of Risk (MPR) hedge α^{MPRH}:

$$\alpha_t^* = \alpha_t^{MV} + \alpha_t^{IRH} + \alpha_t^{MPRH} \qquad \text{where} \qquad (1.15)$$

$$\alpha_t^{MV} = (\Lambda^\intercal(t, Y_t))^{-1} \frac{1}{R(\omega_T)} \Theta(t, Y_t) E_t \left[\xi_{t,T} \frac{\omega_T}{\omega_t} \frac{R(\omega_t)}{R(\omega_T)} 1_{\omega_T > 0} \right]$$

$$\left(\alpha_t^{IRH} \right)^\intercal = -(\Lambda^\intercal(t, Y_t))^{-1} E_t \left[\xi_{t,T} \frac{\omega_T}{\omega_t} \left(1 - R(\omega_T)^{-1} \right) I_{\omega_T > 0} H_{t,T}^r \right]$$

$$\left(\alpha^{MPRH} \right)^\intercal = -(\Lambda^\intercal(t, Y_t))^{-1} E_t \left[\xi_{t,T} \frac{\omega_T}{\omega_t} \left(1 - R(\omega_T)^{-1} \right) 1_{\omega_T > 0} H_{t,T}^\Theta \right]$$

The terms $H_{t,T}^r$ and $H_{t,T}^\Theta$ involve the sensitivities of the short rate and the market price of risk towards shocks in the Brownian motions that drive uncertainty in the model and are defined as follows:

$$H_{t,T}^r = \int_t^T \mathcal{D}_t r_s ds = \int_t^T \partial_2 r(s, Y_s) \mathcal{D}_t Y_s ds \qquad (1.16)$$

$$\begin{aligned} H_{t,T}^\Theta &= \int_t^T (dW_s + \Theta(s, Y_s) ds)^\intercal \mathcal{D}_t \Theta(s, Y_s) \qquad (1.17) \\ &= \int_t^T (dW_s + \Theta(s, Y_s) ds)^\intercal \partial_2 \Theta(s, Y_s) \mathcal{D}_t Y_s \end{aligned}$$

where the operator $\mathcal{D}$ is the Malliavin derivative, $\partial_2 f(t, x)$ refers to the derivative with respect to the second argument of $f(t, x)$. The second equality is obtained using the chain rule for Malliavin derivatives. Recall also that the state vector Y is given by $\left(X_1, ..., X_d, F^V, F^M, Y^r \right)^\intercal$.

Thus, the dynamic of the optimal portfolio behaviour is determined by two factors: the short rate and the market price of risk, as they completely characterize the stochastic discount factor used to price contingent claims in the economy. The two portfolio hedging components (α^{IRH} and α^{MPRH}) are motivated by the sensitivities of future short rates and market prices of risk to current shocks in the Brownians, which is reflected in their Malliavin derivatives. When risk aversion gets close to one, both dynamic hedging terms vanish and the investor's demand is entirely mean-variance motivated. When risk aversion becomes large, the optimal portfolio holdings are increasingly motivated by intertemporal hedging of the market price of risk and the interest rate.

In our setting the long term bond is the sole security in the portfolio that is used to hedge against changes in the short rate. To see this, consider the term $H_{t,T}^r$ that involves the sensitivity of the short rate to current shocks in the underlying sources of uncertainty. Recall that $r(s, Y_s) = \delta_0 + \delta_1 Y_t^r$, and that the $(d+3)$-dimensional state variable vector is defined as $Y \equiv \left(X_1, ..., X_d, F^V, F^M, Y^r \right)^\intercal$. Thus $\partial_2 r(s, Y_s) = (0, ..., 0, \delta_1)$, and using the fact that $\mathcal{D}_{d+1,t} Y_s = (0, ..., 0, \mathcal{D}_{d+1,t} Y_s^r)$, we have that:

$$H_{t,T}^r = \left(0, ..., 0, \int_t^T \delta_1 \mathcal{D}_{d+1,t} Y_s^r \right).$$

The implementation of the above portfolio decomposition formula follows Detemple et al. (2003) and relies on the fact that the Malliavin derivatives, as well as the state variables, follow stochastic differential equations that can be simulated using standard discretization techniques. Given the particular specification of some of the state variables, we can further apply the Doss transformation[10], reducing the stochastic differential equation of the given state variable to one with a constant diffusion term, which ensures that the Malliavin derivative does not involve a stochastic term. Specific solutions for the Malliavin derivative are given in the appendix.

1.5 Hedging for correlation risk and tail dependence

Tail dependencies and conditional correlations impact the dynamic hedging terms in the optimal portfolio through the market price of risk function defined in 1.11.Its dynamics are determined by the state vector Y and thus reflect the assumptions concerning tail dependence as a property of the stationary distribution of the state variables X, as well as properties of their conditional distribution in terms of conditional correlation coefficients. In Stefanova (2009) it is shown that different degrees of tail dependence command significantly different intertemporal hedging demands for a non-mean-variance investor, the effect being stronger for a longer horizon. Disregarding dependence in the tails also gives rise to substantial economic losses in terms of certainty equivalent costs.

In addition, in this paper we model conditional correlations as a function of market-wide volatility and a macroeconomic index to proxy for the state of the economy. Thus, the sensitivities of those variables to shocks in the underlying Brownian motions give rise to hedging demands that can be related to correlation hedging. [11]

By considering both the hedging demands that arise due to tail dependencies and the hedges for correlation risk, we seek to determine whether they substitute each other or whether they are complementary. As tail dependence is modeled as a property of the stationary density, it is important in the long run, while the impact of conditional correlations is local.

Further, our specification allows us to explicitly isolate the correlation hedging demands in the portfolio that arise from stochastic changes in the volatility and macroeconomic variables that drive conditional correlation. Consider the second term $H_{t,T}^{\Theta}$ in the portfolio decomposition formula that handles the sensitivity of the market price of risk towards shocks in the underlying state variables. Let us introduce the

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[10]See Detemple et al. (2003) for further details.

[11]In our setup the Brownian driving the risk-free rate process is independent from those underlying the risky assets. Thus, neither tail dependence nor dynamic correlation impact its dynamics and the hedging demands that it gives rise to. This setting is easily extendable to a more complete model that considers correlation between the risk-free and the risky assets. We leave that for a further extension of this study.

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following vector of state variables: $Y_t^* = (X_t, Y_t^r)^\intercal$ and denote by $\partial_2 \Theta(t, Y_t^*, F_t^V, F_t^M)$ the first derivative of the market price of risk with respect to Y_t^*, by $\partial_3 \Theta(t, Y_t^*, F_t^V, F_t^M)$ its derivative with respect to the market volatility factor F_t^V, and by $\partial_4 \Theta(t, Y_t^*, F_t^V, F_t^M)$ the derivative with respect to the market factor F_t^M. Then the term $H_{t,T}^\Theta$ in the market price of risk hedge can be expressed as:

$$H_{t,T}^\Theta = \int_t^T (dW_s + \Theta(s, Y_s)ds)^\intercal \, \partial_2 \Theta(t, Y_t^*, F_t^V, F_t^M) \mathcal{D}_t Y_s^*$$
$$+ V_{t,T,i}^\Theta + M_{t,T,i}^\Theta$$

where

$$V_{t,T,i}^\Theta = \int_t^T (dW_s + \Theta(s, Y_s)ds)^\intercal \, \partial_3 \Theta(t, Y_t^*, F_t^V, F_t^M) \mathcal{D}_t Y_s^V$$

and

$$M_{t,T,i}^\Theta = \int_t^T (dW_s + \Theta(s, Y_s)ds)^\intercal \, \partial_4 \Theta(t, Y_t^*, F_t^V, F_t^M) \mathcal{D}_t Y_s^M$$

The last two terms $V_{t,T,i}^\Theta$ and $M_{t,T,i}^\Theta$ involve the Malliavin derivatives of the volatility and macroeconomic proxies with respect to the Brownian shocks. As the latter are solely responsible for describing the dynamics of conditional correlation in the process for asset returns, then the component in the Market Price of Risk (MPR) hedging term that arises due to the necessity to hedge changes in these two variables that drive conditional correlation and hence the market price of risk can be expressed as:

$$\left(\alpha^{CORR}\right)^\intercal = -\left(\Lambda^\intercal(t, Y_t)\right)^{-1} E_t \left[\xi_{t,T} \frac{\omega_T}{\omega_t} \left(1 - R(\omega_T)^{-1}\right) 1_{\omega_T > 0} C_{t,T}^\Theta \right] \tag{1.18}$$

where $C_{t,T,i}^\Theta = V_{t,T,i}^\Theta + M_{t,T,i}^\Theta$, $i = 1, ..., d$. This defines the explicitly identifiable correlation hedging demand in our setting.

Since we have defined the conditional correlation dynamics in (1.6) as being driven as well by the state variables X which introduces market level effects through the product of their CDFs, this also commands correlation hedging demands on top of those motivated by stochastic changes in the two macroeconomic and volatility variables. However, they cannot be isolated from the total market price of risk hedging terms in the same straightforward manner. We can only jusdge their effect implicitly by comparing the intertemporal MPR hedges for two data generating processes that differ in their conditional correlation specifications: one with constant and one with dynamic conditional correlations.

Given the fact that the optimal portfolio policy is formed so that it can finance terminal wealth, we can alternatively restate the above result in terms of the sensitivity of the cost of optimal terminal wealth to current shocks in the volatility and macroeconomic variables driving the conditional correlation dynamics. Recall that optimal wealth at time t is given by $\omega_t^* = E_t [\xi_{t,T} \omega_T^*]$, where $\xi_{t,T} \omega_T^* = \xi_{t,T} I (y \xi_t \xi_{t,T})^+$ represents its cost. Then for a nonnegative $I(y\xi_T)$ its sensitivity with respect to fluctuations in F is

given by:

$$
\left[I\left(y\xi_t\xi_{t,T} \right) + y\xi_t\xi_{t,T}I'\left(y\xi_t\xi_{t,T} \right) \right]\left(-\xi_{t,T} \right) \times
$$
$$
\int_t^T \left(dW_s + \Theta(s, Y_s)ds \right)^{\mathsf{T}} \partial_2\Theta(s, Y_s)D_tF_s
$$

where we have used (1.12) and the fact that $I'(y) = \left(u''\left(I(y) \right) \right)^{-1}$ which follows from the definition of $I(y)$ as the inverse of the marginal utility. The portfolio terms that are responsible for the sensitivity of the cost of optimal terminal wealth to fluctuations in the market-wide volatility and macroeconomic variables are the correlation hedging demands defined in (1.18).

2 Results

In order to appreciate the impact of the correlation hedging demands on the optimal portfolio composition in a realistic setting and compare them to the intertemporal hedges that arise due to incorporating tail dependence, we offer an application based on real data. We consider a portfolio, formed by a 10-year pure discount bond, as well as two risky funds, represented by old and new economy stocks: S&P 500 and NASDAQ. An application with this choice of a dataset can be found in Detemple et al. (2003). Data is observed at the daily frequency (except for the CFNAI factor, which is observed monthly) and covers the period 1986-2007.

Without loss of generality, we assume that the coefficients in the short rate specification are given by $\delta_0 = 0$ and $\delta_1 = 1$, so that for the short rate we have $r\left(t, Y_t^r \right) = Y_t^r$. As well, we consider the CIR case for $\alpha^r = 0$ and $\beta^r = 1$. In this case there are no closed-form solutions for the interest rate hedging term. Nevertheless, for the simulations of its Malliavin derivatives we can apply a variance stabilization technique following the Doss transformation that renders constant the diffusion term of the process for Y^r, as explained in the appendix.

Before discussing the estimation results for the various diffusion specifications that we have chosen for the state variables X, let us first look at the data itself in order to verify whether the stylized facts that we aim to reproduce are indeed present. In non-reported results on the estimation of a DCC model with market-wide volatility and macroeconomic variables driving conditional correlation[12] we can distinguish periods of relatively high or low correlation that can be attributed to the influence of the aforementioned variables. Motivated by this evidence, we split the estimation period into two subsamples, one characterized by decreasing and low volatility and improving macroeconomic conditions (1988-1996), and the other characterized by high volatility and declining and relatively low CFNAI index,

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[12]See also footnote (5). Results on the estimation of a DCC model with factors related to market-wide volatility and macroeconomic conditions are available upon request.

Figure 2. Quantile dependence plots

Plots of quantile dependence for the de-trended log-prices of S&P 500 vs. NASDAQ for the 1988-1996 and 1996-2004 subperiods.
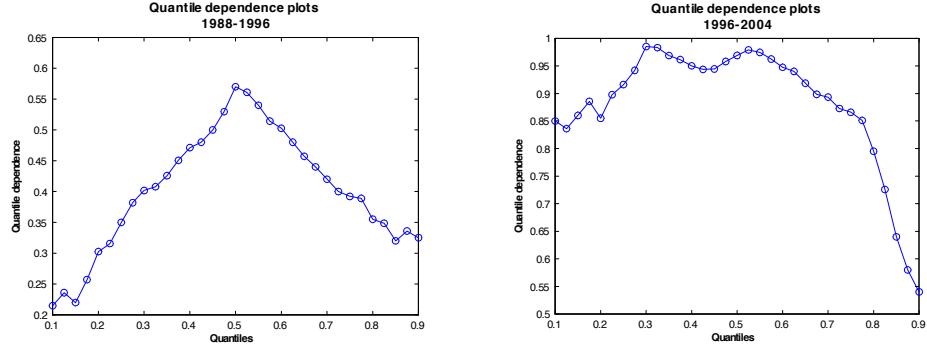
pointing towards a declining economy (1996-2004). We then construct quantile dependence plots for the de-trended log-prices of both indices for the corresponding subsamples.

As we can see in Figure 2, during the first relatively calm period dependence in the extreme quantiles of the joint distribution decreases substantially, even though it does not disappear completely, as one would expect under a Gaussian distributional assumption. As well, the test of tail dependence symmetry introduced in Hong et al. (2003) does not fail to reject symmetric tails for this particular period[13]. On the other hand, the period of (1996-2004) brings about extremely high dependence in the tail quantiles, especially in the left tail, and the dependence symmetry test rejects symmetric tails for the period. Thus, the unconditional distribution of the two risky funds that we have chosen has the features that we try to assess - increased dependence when markets experience extreme downturns.

The processes for the state variables are estimated using Markov Chain Monte Carlo and the Simulation Filter of Golightly and Wilkinson (2006). This estimation methodology is particularly convenient for highly nonlinear multivariate diffusions, as in our case. It also allows us to filter out unobservable data points, as is the case of the CFNAI factor, which is observed monthly, whereas the two indices and the VIX factor are observed at the daily frequency. Parameter estimates along with their corresponding batch-mean Monte Carlo standard errors and simulated inefficiency factors (SIF)[14] for the observable variables are given in Table I.

Let us now turn to the estimation results for the whole sample period, as well as the two subsamples for the two conditional correlation specifications (DCC and CCC) and the two alternative stationary distribution assumptions (the tail independent Gaussian and the asymmetric tail dependent Gaussian-SJC diffusions). As in this application we aim to determine the impact of the stationary distribution

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[13]Results not reported for brevity and available upon request.

[14]The SIF is estimated as the variance of the sample mean divided by the variance of the posterior distribution.

Table I. Parameter estimates for the observable factors

Estimated parameters for the observable factors VIX and CFNAI that have the following specifications:

$$dF_t^V = \kappa^V \left(\theta^V - F_t^V\right) dt + \sigma^V \sqrt{F_t^V} dW_t^X$$
$$dF_t^M = \kappa^M \left(\theta^M - F_t^M\right) dt + \sigma^M dW_t^X$$

where $i = \{V, M\}$.

parameter	$CFNAI$	MC s.e.	SIF	VIX	MC s.e.	SIF
κ^i	2.2521	0.0027	0.8153	1.2094	0.0021	0.8002
θ^i	-0.0457	0.0018	1.7702	2.7800	0.0007	0.8863
$\left(\sigma^i\right)^2$	2.9383	0.0005	0.8631	0.1230	0.0000	1.9260

and hence tail dependence on the optimal portfolio holdings regardless of the univariate marginals, we do not proceed to a full-scale optimization of all model parameters, as would be otherwise preferred. However, we rather undertake a two-step estimation procedure. In a first step, we impose that the two price processes are independent from each other (i.e. independent (or product) copula for their stationary distribution as well as zero conditional correlation). Thus, we are able to estimate them separately and further use the same marginal distribution parameters for all alternative processes that we consider. In this manner, differences in portfolio demands between the alternative specifications will not depend on the particular parameter choice of the univariate marginals. Parameter estimates are reported in Table II. The trend parameters k_i for each of the state variables X_i are estimated separately as a linear trend. Their values are 0.1014 for S&P 500 and 0.1100 for NASDAQ. In a second step, we assume the marginal parameters as known and we proceed to the estimation of the multivariate processes by assuming all the alternative specifications for the stationary distribution of the conditional correlation. Results are reported in Table III.

Note that the conditional correlation parameters that pertain to volatility (γ_1) are generally positive, pointing towards an increase in conditional correlation when there is rise in market-wide volatility. An exception to this is the 1996-2004 period during which the VIX coefficient is negatively estimated for all stationary distributional assumptions. On the other hand, the parameter pertaining to the macroeconomic factor (γ_3) is always negatively estimated, pointing towards a decrease in conditional correlation when there is an improvement in macroeconomic conditions, and vice versa.

2.1 Hedging for correlation risk and tail dependence

Our first objective is to test whether the data generating processes for the risky assets that we consider command for substantial portfolio hedging demands and whether the latter are different in magnitude

Table II. Univariate parameter estimates

Parameter estimates from the univariate Normal Inverse Gaussian (NIG) diffusions with density $f_{NIG}(x;\theta)$, where $\theta = (\alpha, \beta, \delta, \mu)$ is the vector of NIG parameters that satisfy the restrictions, given in the Appendix. The diffusion for each of the state variables X_{it} has the following specification:

$$
\begin{aligned}
dX_{it} &= b\left(X_{it};\theta_i\right)dt + v\left(X_{it};\theta_i\right)dW_{it} \\
\text{where } b\left(x;\theta\right) &= \frac{1}{2}v\left(x;\theta\right)\frac{d}{dx}\ln\left[v\left(x;\theta\right)f_{NIG}\left(x;\theta\right)\right] \\
v\left(x;\theta\right) &= \sigma^2 f_{NIG}\left(x;\theta\right)^{-\kappa}, \quad \sigma^2 > 0, \kappa \in [0,1]
\end{aligned}
$$

Monte Carlo standard errors, obtained using the batch-mean approach (multiplied by a factor of 1000) and the simulation inefficiency factor (SIF) are reported for each parameter estimate.

parameter	X_1 $(S\&P500)$	MC s.e.	SIF	X_2 $(NASDAQ)$	MC s.e.	SIF
α	5.6431	0.0601	1.0262	4.2938	0.2138	0.8070
β	-0.6272	0.3091	1.1979	-0.7072	0.4151	0.6343
δ^2	0.0471	0.0016	0.7755	0.0549	0.0026	0.8782
μ	4.6342	0.0083	1.0129	5.1191	0.0146	0.6724
σ^2	0.0268	0.0006	0.8375	0.0222	0.0003	0.2821
κ	0.5776	0.0128	1.0339	0.5349	0.0356	1.2291

and direction when allowing for tail dependence or conditional correlation. To address this issue, we first conduct a market timing exercise over the estimation horizon. We solve for the optimal investment policy along the observed or filtered values of the state variables throughout the period[15]. We also filter out the market prices of risk for the risky funds, implied by the data. It is through the market price of risk process that the distributional features of the risky assets (tail dependence or conditional correlation) are reflected in the optimal portfolio composition. We then simulate ahead the Malliavin derivatives of the state variables and the state price density process in order to evaluate the conditional expectations needed to obtain the portfolio terms that involve hedging against changes in the interest rate (1.16) and the market price of risk (1.17).

First, we obtain the optimal portfolio terms for the whole period between 1986-2007 for an investor with a constant, moving-window horizon of 4 years in order to abstract from horizon effects when studying the implications of tail dependence and conditional correlation. Next, we consider an investor who keeps her investment horizon fixed at the end of the period, thus investigating the horizon effect on the optimal portfolio shares.

During the 20 year investment period, one can distinguish different regimes. Some recessionary periods are associated with high volatility and negative CFNAI implying a rise in conditional correlation.

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[15] As the CFNAI index is observed at a monthly frequency, we filter the unobservable data points at the daily frequency using the MCMC sequential filter.

Table III. Parameter estimates from the multivariate diffusion specifications (1986-2006)

Estimates for the parameters of the stationary density, defined in terms of copula functions, and the parameters governing the correlation dynamics for a bivariate diffusion, defined as:

$$dX_t = \mu\left(X_t\right)dt + \Lambda\left(X_t\right)dW_t^X$$

$$\text{where } \Lambda = \begin{bmatrix} \sigma_1\left[\widetilde{f}^1\left(x1\right)\right]^{-\frac{1}{2}\kappa_1} & 0 \\ \Upsilon_{12}\left(X_t\right)\sigma_2\left[\widetilde{f}^2\left(x_2\right)\right]^{-\frac{1}{2}\kappa_2} & \sqrt{1-\Upsilon_{12}^2\left(X_t\right)}\sigma_2\left[\widetilde{f}^2\left(x_2\right)\right]^{-\frac{1}{2}\kappa_2} \end{bmatrix}$$

$$\mu_j = \frac{1}{2}q^{-1}\sum_{i=1}^{2}\frac{\partial\left(\nu_{ij}q\right)}{\partial x_i}, \quad j=1,2$$

$$\text{and } q\left(x_1,...,x_d\right) \equiv \widetilde{c}\left(x_1,...,x_d\right)\prod_{i=1}^{d}\widetilde{f}^i\left(x_i\right)$$

where ν_{ij} are entries of the matrix $\Sigma = \Lambda\Lambda^\intercal$, and $q\left(x_1,...,x_d\right)$ is the stationary density of the diffusion, defined in terms of a copula function $\widetilde{c}$ and the NIG marginal densities $\widetilde{f}^i$. Parameter estimates are given for two cases of copulas: Ga refers to the Gaussian copula, and $Ga-SJC$ - to the mixture Gaussian-Symmetrized Joe-Clayton copula. The copula parameters are as follows: ρ is the Gaussian correlation parameter, τ_U and τ_L are the upper and lower tail dependence parameters of the Symmetrized Joe-Clayton copula, and ω is the weighting parameter in the Symmetrized Joe-Clayton copula. The parameters that describe the correlation dynamics are $\gamma_i, i=0,...,3$, consistent with the specification in (1.6). The Constant Conditional Correlation model in Panel 2 assumes that all correlation parameters are zero but γ_0.

Panel 1. Dynamic conditional correlation						
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF
ρ	0.4036	0.3654	0.9608	0.4596	0.7086	1.6656
τ_U	-	-	-	0.4669	0.3453	0.6012
τ_L	-	-	-	0.5178	0.3165	1.1565
ω	-	-	-	0.5513	0.7156	0.7900
γ_0	1.7273	0.0166	0.6051	1.7401	0.0252	0.6578
γ_1	0.0060	0.0126	0.9784	0.0034	0.0062	0.3958
γ_2	-0.2873	0.0642	0.9762	-0.2745	0.0484	0.6097
γ_3	-0.3086	0.0263	1.0807	-0.3487	0.0240	0.9340

Panel 2. Constant conditional correlation						
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF
ρ	0.4565	0.2337	1.2678	0.4918	0.3299	1.1136
τ_U	-	-	-	0.5012	0.6331	2.3965
τ_L	-	-	-	0.5801	0.4020	1.6656
ω	-	-	-	0.3816	0.6329	1.4994
γ_0	1.9955	0.0139	1.8733	2.0374	0.0128	0.9472

Table III (A). Parameter estimates from the multivariate diffusion specifications (1988-1996)

Panel 1. Dynamic conditional correlation						
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF
ρ	0.4011	0.3787	0.4744	0.3705	0.8203	1.3778
τ_U	-	-	-	0.5159	0.9509	0.9491
τ_L	-	-	-	0.5466	0.7998	0.7297
ω	-	-	-	0.5258	1.4451	1.5272
γ_0	2.1724	0.0426	0.4523	2.1661	0.0716	1.0785
γ_1	0.0102	0.0207	1.1832	0.0079	0.0185	0.5042
γ_2	-0.7282	0.3580	1.2605	-0.9716	0.2619	0.5328
γ_3	-0.2691	0.1471	1.2750	-0.2887	0.1229	0.7116
Panel 2. Constant conditional correlation						
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF
ρ	0.3348	0.5310	0.7963	0.4497	0.5185	0.7457
τ_U	-	-	-	0.5447	1.0077	1.2661
τ_L	-	-	-	0.5016	1.1308	1.7278
ω	-	-	-	0.5765	0.8678	0.9065
γ_0	1.7174	0.0585	1.8229	1.6532	0.0460	0.9813

Table III (B). Parameter estimates from the multivariate diffusion specifications (1996-2004)

Panel 1. Dynamic conditional correlation						
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF
ρ	0.5380	0.6157	0.5776	0.5383	1.0569	0.6704
τ_U	-	-	-	0.5093	0.4792	0.2987
τ_L	-	-	-	0.5322	0.9009	1.5219
ω	-	-	-	0.5023	0.7294	1.1890
γ_0	1.9191	0.1318	2.0517	1.9198	0.0576	0.4783
γ_1	-0.0134	0.0221	0.5537	-0.0034	0.0157	0.4258
γ_2	-0.7266	0.1758	0.8608	-0.7292	0.2284	1.6010
γ_3	-0.0825	0.0983	0.5140	-0.1403	0.0834	0.6450
Panel 2. Constant conditional correlation						
param	Ga	MC s.e.	SIF	Ga-SJC	MC s.e.	SIF
ρ	0.3533	0.5154	0.7736	0.3853	1.4276	0.6995
τ_U	-	-	-	0.5242	0.7559	0.9003
τ_L	-	-	-	0.5091	0.7778	0.7893
ω	-	-	-	0.5142	0.9299	0.7847
γ_0	1.1262	0.0668	1.1726	1.1751	0.0473	0.6244

Other periods are relatively calm and characterized with low volatility, mostly positive levels of the CFNAI index and hence low conditional correlation. Therefore we proceed to a second market timing experiment, considering instead two subperiods of 8 years. The first one spans between 1988 and 1996 and is characterized by increased volatility and a recession in the US economy in the beginning of the period (between July 1990 and March 1991, as determined by NBER), followed by improving macroeconomic conditions (positive and rising CFNAI), as well as relatively low and declining volatility, accompanied by falling dynamic conditional correlation. On the other hand, the second period spanning between 1996 and 2004 is characterized by increased volatility for the whole period, a recession towards the end of the period (March 2001 marks the end of a 10-year expansion period, according to NBER, and there is a trough in business activity in November 2001), as well as a rising trend in the dynamic conditional correlations for the period. For both subperiods we consider an investor who has a fixed investment horizon at the end of each period.

### 2.1.1 Hedging for correlation risk and tail dependence for the whole estimation horizon

To get an impression of the magnitude and the variability of the hedging demands for the risky assets in the portfolio, let us first consider the results displayed in Figure 3 for a HARA investor with varying degrees of relative risk aversion[16]. The intertemporal hedging demands are a sizeable component of the total portfolio, and they are responsible for a larger portion of the portfolio demands if we increase the level of relative risk aversion of the investor. As well, the hedging demands are larger for longer horizons: an investor with a horizon fixed at the end of the 20-year sample period (left-hand figures) has higher hedging demands at each period of time than an investor who has a short rolling-window horizon (right-hand figures). Also the fixed horizon would cause the hedging demands to shrink as we approach it (it is visible during the last 4 years on the left column of Figure 3), so that the Mean-Variance component becomes increasingly more important in the total portfolio holdings. These results are based on a data generating process with tail dependence (Gaussian-SJC diffusion) and dynamic correlation.

In our setup the intertemporal hedging demands that arise to hedge against stochastic changes in the state variables underlying the market price of risk processes for the risky funds can be attributed to market-wide volatility or macroeconomic effects (via the VIX and the CFNAI variables) and to market-level effects (via the idiosyncratic state variables X). These variables also drive the dynamics of conditional correlation between the risky asset returns. Thus, we can assess their effect on the part of the MPR hedges that is due to correlation hedging. The hedging demands due to the volatility and macro-economic variables can be isolated explicitly (see 1.18). The market level component in the correlation

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[16]In our setup the long term bond is the only security in the investor's portfolio that is responsible for hedging interest rate risk. As well the Brownian motion driving the short rate is independent of the Brownian motions underlying the dynamics of the rest of the state variables. Also, the short rate does not enter the stock price dynamics. Thus, the hedging terms for the risky assets consist solely of market price of risk hedges. Due to the chosen specification of the market price of risk of the long term bond, it has a negative market price of risk hedging term, and a positive interest rate hedge.

Figure 3. Total portfolio holdings and intertemporal hedging demands for the two risky stocks over the entire sample

The figure displays the holdings of the two risky stocks in the portfolio for the entire sample period 1986-2007. The total holdings are contrasted with the intertemporal hedging demands, which for the stocks are entirely given by the market price of risk hedges. The figure on the left represents the portfolio holdings for a fixed investment horizon at the end of the 20-year sample. The figure on the right represents the holdings for a moving-window 4-year horizon. The two top figures concern a HARA investor with relative risk aversion of 5, $b = -0.01$, and initial wealth normalized to one, while the bottom two - a HARA investor with relative risk aversion of 10. The data generating process is a Gaussian-SJC diffusion with dynamic correlation.
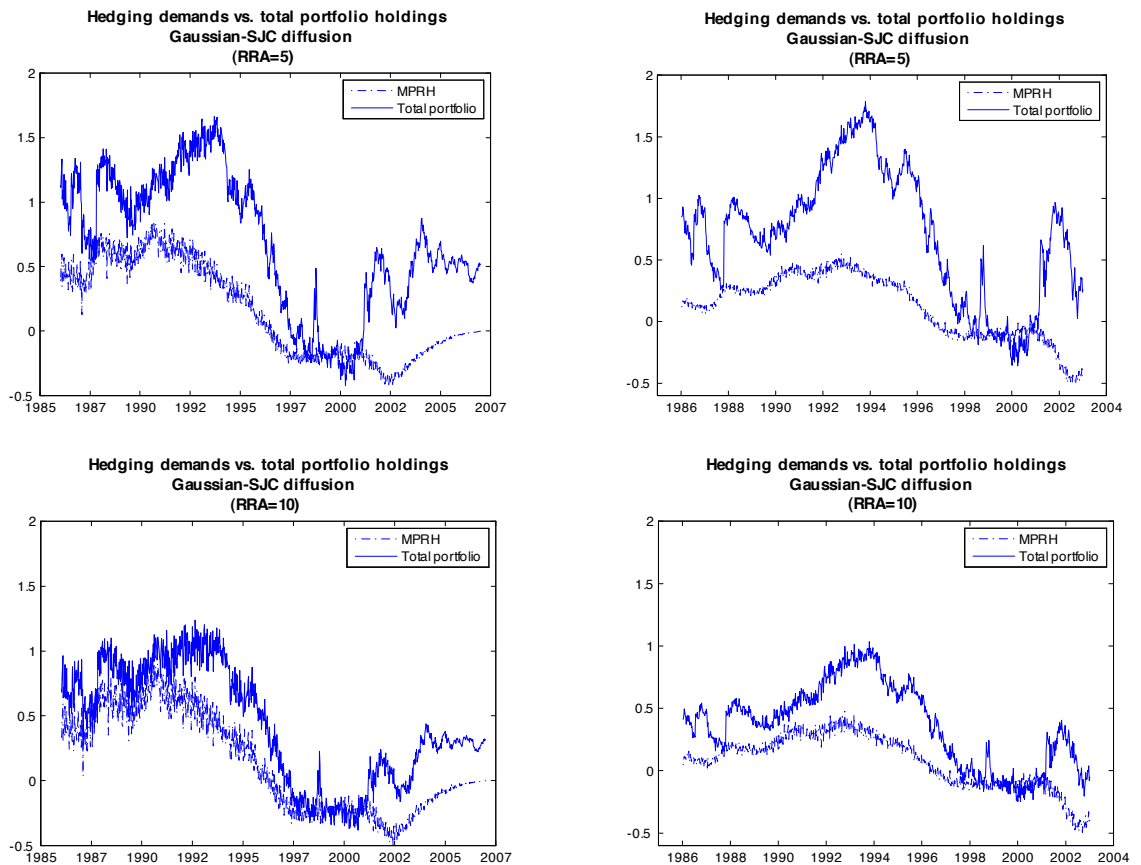
hedges however can only be assessed indirectly by contrasting the MPR hedging parts for a process with dynamic vs. constant conditional correlation. The MPR hedges for the two conditional correlation specifications, as well as their correlation hedging components are plotted on Figure 4 for both tail dependent and independent processes. The investment horizon is fixed at the end of the sample period.

The effect of dynamically varying conditional correlations on the total hedging demands is mixed. When tail dependence is not taken into account (i.e. in the case of a Gaussian diffusion) it leads to a marginal increase in the hedging demands. On the other hand, for the case when tail dependence matters for the investor (i.e. in the case of a Gaussian-SJC diffusion), dynamic conditional correlation leads to a slight increase in the total hedging demands only over the first half of the investment horizon when conditional correlations are declining and there is no evidence of tail dependence. Otherwise it leads to a decrease in the hedging demands, as it is the case over the second half of the period characterized by rising conditional correlations and increased tail dependence. Decomposing these correlation hedges further into demands driven by the market level (through X) and by the volatility and macroeconomic variables shows that this difference in the impact of conditional correlations can be explained mainly by market level effects that are related to the presence of tail dependence: an investor who incorporates clustering of extreme events in her portfolio decision has negative or positive correlation hedging demands depending on whether the risky assets show evidence of tail dependence or not over the investment horizon. On the other hand, an investor who disregards tail dependence has generally positive correlation hedging demands due to market level effects, i.e. accounting for conditional correlation leads to an overall increase in the hedging demand for the risky funds. The estimates of the correlation hedges due to market level effects contain more noise than the explicitly isolated ones due to observed variables, as they are obtained indirectly by subtracting the MPR hedging parts for a process with dynamic vs. constant conditional correlation. The correlation hedges due to the VIX and the CFNAI variables are smaller in magnitude and generally lead to a decrease in the demand for risky funds, which is more pronounced in the tail dependence case. We can also distinguish periods with peaks in the absolute value of these correlation hedging demands that can be attributable to some market events (e.g. the market crashes in 1987, 1990-1992, 2001).

Alternatively, we can assess the importance of the correlation hedging component by taking into account the relative holdings of the two risky assets, rather than the combined risky shares (5, Panel A). Conditional correlation leads to a decrease in the spread between the holdings of the two risky funds. Correlation hedging demands (as determined by the difference between the market price of risk hedging terms implied by a dynamic vs. a constant conditional correlation model) are generally positive for NASDAQ, and close to zero or negative for S&P 500. The latter clearly prevail during the bear market in the second part of the investment period for an investor who incorporates tail dependence in her portfolio decision, thus leading to a reduction in the total hedging demands. We gather some additional

Figure 4. Hedging demands along realized paths for the risky stocks for the 20-year fixed investment horizon

Plotted are the intertemporal MPR hedging demands along realized paths of the state variables for the whole sample period for the risky stocks for a fixed investment horizon at the end of the period The upper row figures plot the MPR hedging demands obtained under a DCC specification vs. those under CCC for a tail-dependant Gaussian-SJC diffusion (left figure) and for a tail-independant Gaussian diffusion (right figure). Below are plotted the correlation hedging demands due to market level effects, as well as those due to stochastic changes in the market-wide volatility and macroeconomic variables. The left column results correspond to a tail-dependent data generating process (Gaussian-SJC diffusion), while those to the right have an underlying tail independent data generating process (Gaussian diffusion).
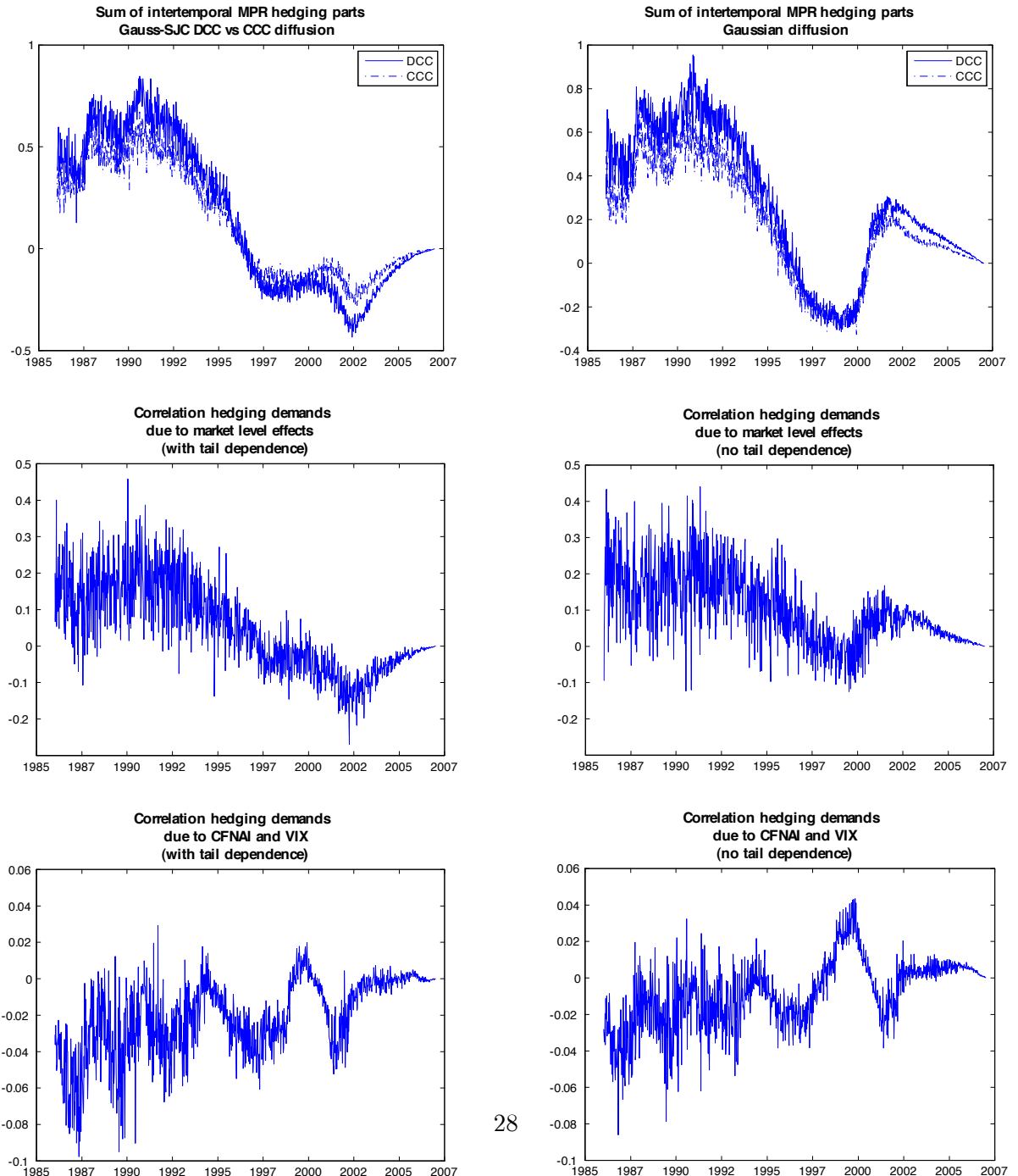
Figure 5. Panel A. Individual hedging demands for the SP500 and NASDAQ along real-ized paths of the state variables for the 20-year fixed investment horizon

Plotted are the intertemporal demands along realized paths of the state variables for the whole sample period for the individual risky funds. The investment horizon is fixed at the end of the period The lefthand side figures plot the intertemporal hedging demands obtained under a DCC specification vs. those under CCC for a tail-dependant Gaussian-SJC diffusion, and the righthand side figures - for a tail-independant Gaussian diffusion.
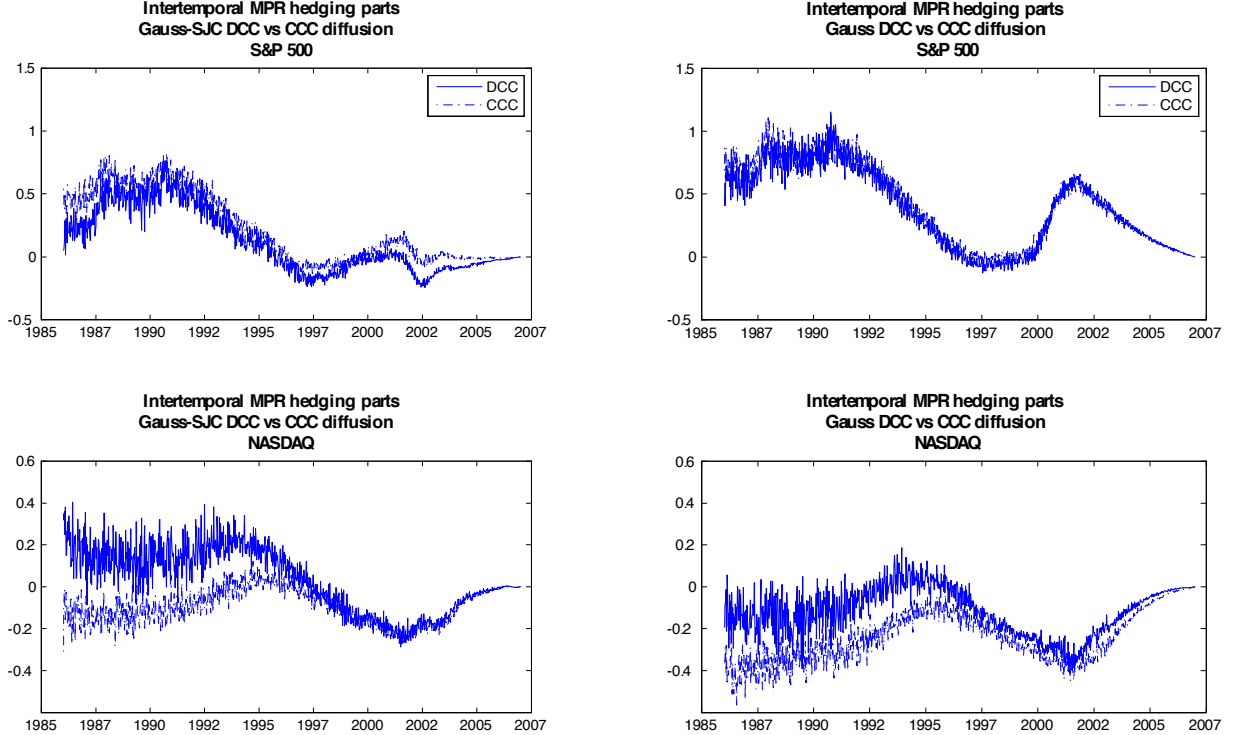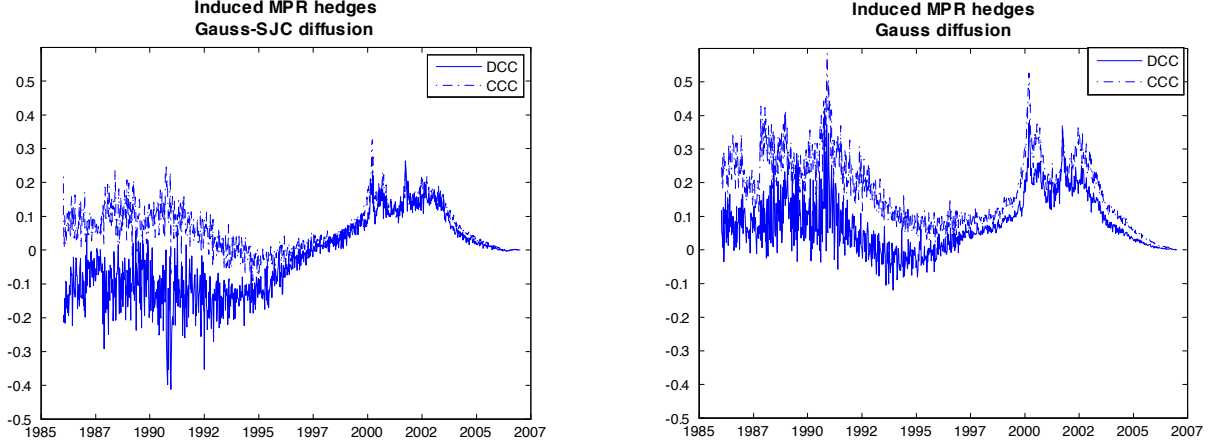

insight into the impact of conditional correlation by considering the induced portfolio holdings of S&P 500 from the position in NASDAQ (Figure 5, Panel B). Regardless of whether we allow or not for extreme dependencies, dynamic correlation leads to a decrease in the induced correlation hedging demands. As well, accounting for both clustering of extreme events and conditional correlation leads to a significant reduction in the induced hedges.

Thus, we can conclude that the effect on the optimal hedging behaviour of the investor from allowing for dynamic conditional correlations depends on whether or not she takes clustering of extreme events into consideration. Correlation hedging demands cause a distinct change in portfolio composition in both cases by reducing the MPR hedging demands for the risky assets in absolute terms. The effect on the overall risky fund holdings translates into a shift to risky assets for a tail-independent data generating process, while on the other hand the riskless holdings increase when the investor accounts for tail dependence,

Figure 5. Panel B. Induced hedging demands

Induced MPR hedging demands for S&P 500 for a HARA investor with a 20-year horizon and a relative risk aversion coefficient of 5. Data generating processes are DCC and CCC with Gaussian or Gaussian-SJC stationary distribution.

provided the investment period is charactedized by clustered tail events. This difference in the impact of correlation hedging demands comes primarily from the market level effect in the dynamics of conditional correlations modeled via the state variables X. When they are jointly low (and this is more likely for a tail dependent process), conditional correlation is higher with a portfolio effect of reduced risky holdings.

It is interesting to consider alternatively the effect of tail dependence on optimal hedging demands which we model through the stationary distribution of the state variables (see Figure 4). Regardless of whether dynamics in conditional correlation are taken into account, the risk of clustered tail events invariably lowers the exposure to risky assets when the investment period is characterized by increased tail dependencies. Thus, for portfolio allocation, the impact of tail dependence through the unconditional distribution cannot be swept away by allowing conditional correlation to vary through time, rising in down markets. However, there is virtually no tail effect over the first part of the investment horizon for both conditional correlation specifications. It appears that for different subperiods of this relatively long sample hedging demands may have qualitatively different behavior. In order to gather more insight into the reasons behind the differences in those demands, we next consider the two 8-year subperiods that we introduced earlier.

2.1.2 Hedging for correlation risk and tail dependence for the two subperiods

In Figure 7 we have plotted the hedging demands of a HARA investor with a relative risk aversion coefficient of 5 who models the stock price process using different assumptions on the risky asset co-movements, while keeping conditional correlation constant (first row). We contrast the intertemporal

Figure 6. Hedging demands induced by tail dependence along realized paths for the risky stocks for the 20-year fixed investment horizon

. Hedging demands due to tail dependence for the risky stocks for the 20-year fixed investment horizon for a DCC diffusion (left column) and a CCC diffusion (right column). The upper figures plot the total MPR hedging demands for a tail dependent (Gaussian-SJC) and a tail independent (Gaussian) diffusion, while the bottom figures plot the hedging terms due to tail dependence, as obtained by subtracting the MPR hedges obtained without tail dependence from those with tail dependence.
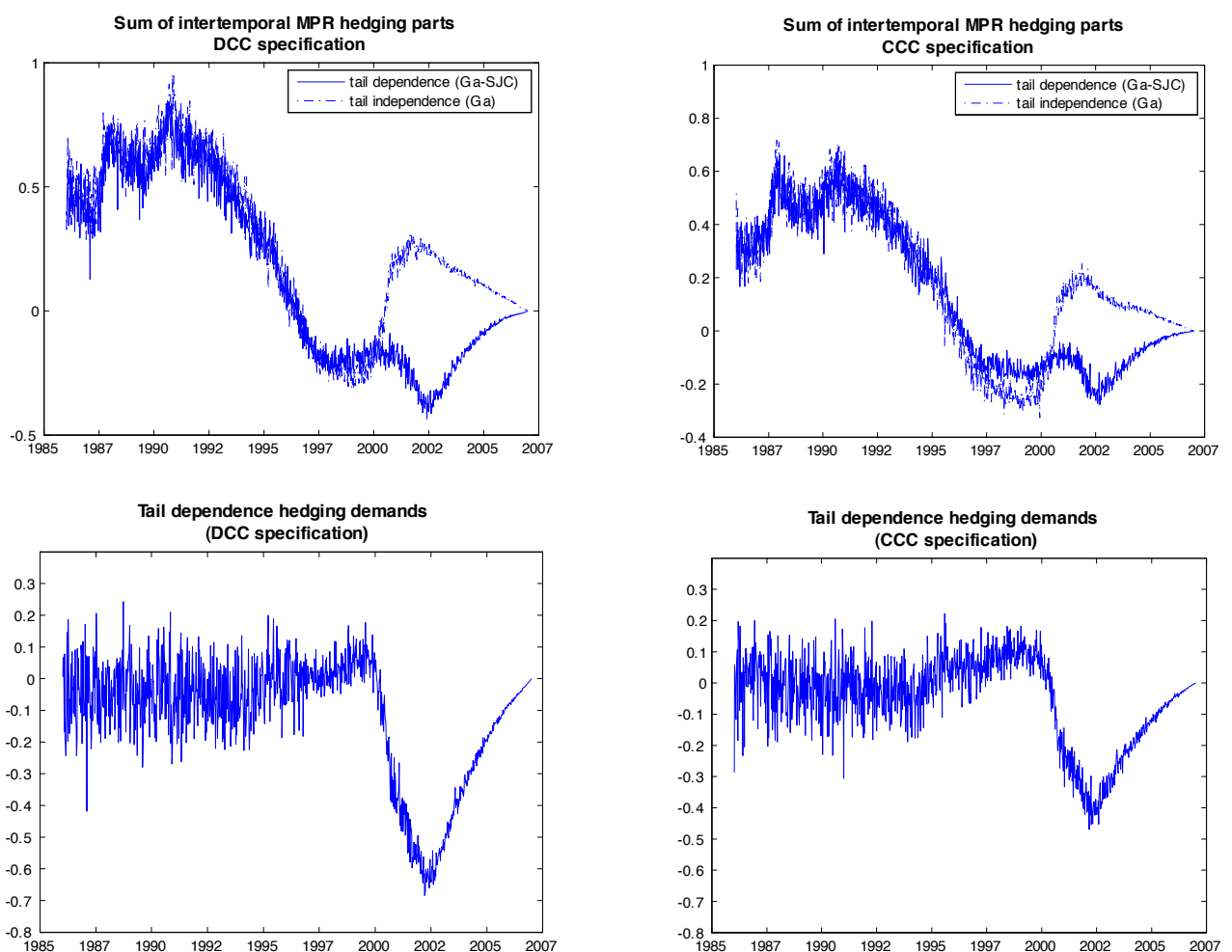
hedging terms implied by a tail-independent Gaussian vs. an asymmetrically tail dependent Gaussian-SJC diffusion for the state variables X in order to gauge the effect of disregarding risky asset extreme co-movements. This is done in the context of a market timing exercise over the two 8-year subperiods. It is repeated for an investor whose data generating process allows for dynamic movements in conditional correlation, caused by market level or volatility effects, or macroeconomic conditions (second row in Figure 7). With this we aim at answering the following questions. Is there a correlation hedging effect on the optimal portfolio composition above the one implied by extreme dependencies between risky funds? Do the economic and market conditions, whose influence we incorporate in the data generating process for the assets in the portfolio through their conditional correlations, have an impact on portfolio decisions. Does the portfolio allocation to risky assets differ across periods marked by different states of the economy?

Comparing the intertemporal hedging demands for each one of the two subperiods, regardless of the assumptions we have made on the conditional correlation or the treatment of extremes, we see that those demands are generally positive throughout the first relatively calm period of economy on the rise and generally negative for the second hectic period of slowing down economy. There is just one notable exception to this rule - the hedging demands turn positive towards the second half of the 1996-2004 period for the Gaussian diffusion for both constant and dynamic specifications for the conditional correlation. Thus, failing to account for tail dependence increases the demand for the two risky funds and the fact that we allow for dynamically varying conditional correlation does not change this effect. It appears, following this preliminary observation, that unconditional (asymmetric) dependence has a portfolio impact beyond the one induced by correlation hedging. On the other hand, allowing for dynamic conditional correlation generally leads to higher portfolio hedging demands across all specifications. This brings evidence in support of the existence of correlation hedging demands above the effect of extreme asset dependencies.

As expected, for all cases in which the data generating process implies constant conditional correlation, accounting for tail dependence leads to smaller hedging demands in absolute value for both risky funds, which in turn reduces the total intertemporal demands for the risky assets. Those differences are more pronounced during the 1996-2004 period when the investor completely disregards tail dependence by assuming a Gaussian diffusion.

However, when we allow for dynamically varying correlations, some interesting results follow. In this case the large difference between the alternative unconditional distribution assumptions seems to vanish for the first subperiod. Allowing for tail dependence have virtually no impact on the hedging demands. So, for this relatively calm period of improving economic conditions the presence of tail dependence does not lead to any significant change in the portfolio composition beyond the impact of correlation hedging. The picture for the second highly volatile period is quite different. Accounting for tail dependence still leads to a decrease in absolute terms of the hedging components for both risky funds which generally leads to a decrease in the total hedging demand. Thus, for a volatile period of deteriorating economic

conditions tail dependence has a significant impact on the portfolio composition, even when dynamic conditional correlation has been accounted for.

2.2 Investment horizon effects

Having examined the distinct ways that dynamic conditional correlation or tail dependence influence the optimal portfolio decisions for a particular period and for realized paths of the state variables, we now turn to a simulations experiment that determines optimal portfolio shares for varying investment horizons while simulating ahead all the state variables involved. With this we aim to determine whether, for the set of estimated parameters of the corresponding processes, the relative importance of conditional and unconditional dependence on portfolio hedging demands remains qualitatively the same as with the historical data considered.

Thus, we set up a first simulations exercise that aims at determining the importance of correlation hedging demands for a HARA investor who also incorporates in her portfolio decision the fact that the process underlying stock prices has asymmetric tail dependence, implied by the Gaussian-SJC diffusion. Then we alternate the way to model conditional correlation by letting it be either constant or dynamic. In this way we can analyze the correlation hedging demands that arise beyond those that could be attributed to tail dependence through the unconditional distribution. We use the parameters estimated from a Gaussian-SJC process with DCC for the whole estimation period as a benchmark. Then, in order to obtain a CCC model, we set all parameters driving conditional correlation to zero, except for γ_0. We calibrate this parameter in order to reflect the same average correlation throughout the estimation period as the one implied by the benchmark process. In order to gauge the relative importance of adding either the volatility or the macroeconomic index to the dynamic correlation specification, we alternatively set either γ_1 (the VIX coefficient) or γ_3 (the CFNAI coefficient) to its corresponding value from the benchmark process, while setting all the other parameters to zero except γ_0 that is again calibrated in order to reflect the same average correlation. We then simulate ahead all the state variables involved in each of the four alternative processes, as well as their Malliavin derivatives, in order to obtain the Monte Carlo estimates of their conditional expectations in (1.15) and thus the intertemporal hedging demands. Results for investment horizons of 1 and 5 years are reported in Table IV, Panels A through C and Panel E.

The major conclusion that we draw from these results is that for all investment horizons considered, as well as for all degrees of relative risk aversion, the market price of risk hedge for the DCC model is the lowest. If we add only the macroeconomic index to render conditional correlation dynamic, we get results that are quite close to the benchmark model. So for this application the macroeconomic factor seems to be the major driving force to determine the optimal portfolio composition. However, adding only the VIX index does not change in any substantial way the portfolio holdings and they remain virtually

Figure 7. Market price of risk hedging terms for the two subperiods

The figure displays the intertemporal hedging demands of a HARA investor with RRA coefficient of 5 and an investment horizon fixed at the end of the period. The results of a Gaussian-SJC diffusion are plotted against those coming from a Gaussian diffusion. Two alternatives are considered: one with constant conditional correlations (first row), and one with dynamic conditional correlations (second row). The two left columns plot the results for the 1988-1996 period, the two right ones - the results for the 1996-2004 period.
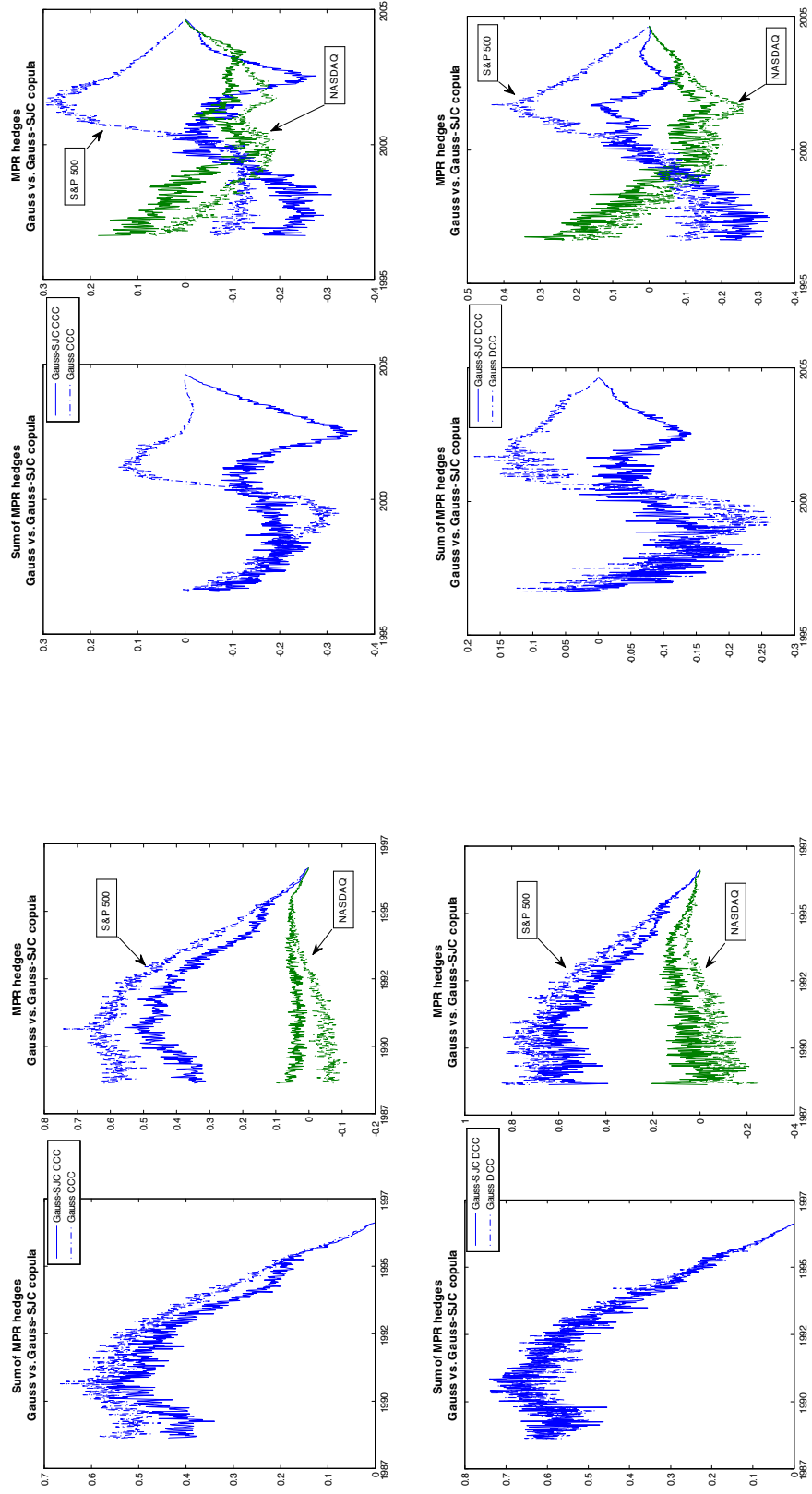
Table IV. Portfolio hedging terms through simulation

Intertemporal hedging demands for the 2 funds with different specifications for the conditional dependence through the conditional correlation (Gaussian-SJC diffusion with DCC. For each investment horizon of 1 and 5 years the table displays the sum of market price of risk hedges (MPRH), the separate demands for S&P 500 and NASDAQ, as well as the two correlation hedges (CorrH), corresponding to the macroeconomic factor F^M and the volatility factor F^V (the latter multiplied by 100).

Panel A. Gaussian-SJC CCC diffusion ($\gamma_1 = \gamma_2 = \gamma_3 = 0$)

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, $\gamma=5$	0.1056	0.0614	0.0442	-	-	0.8623	0.6588	0.2035	-	-
CRRA, $\gamma=10$	0.0643	0.0370	0.0273	-	-	0.7341	0.5807	0.1534	-	-
HARA, $\gamma=5,b=-0.2$	0.0873	0.0505	0.0368	-	-	0.8201	0.6315	0.1885	-	-
HARA, $\gamma=10,b=-0.2$	0.0539	0.0307	0.0232	-	-	0.7113	0.5661	0.1451	-	-

Panel B. Gaussian-SJC DCC diffusion with only VIX driving conditional correlation ($\gamma_2 = \gamma_3 = 0$)

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, $\gamma=5$	0.1060	0.0613	0.0446	-	0.4368	0.8617	0.6575	0.2041	-	0.4103
CRRA, $\gamma=10$	0.0646	0.0370	0.0276	-	0.2815	0.7348	0.5809	0.1539	-	0.1492
HARA, $\gamma=5,b=-0.2$	0.0878	0.0505	0.0372	-	0.3693	0.8203	0.6314	0.1890	-	0.3392
HARA, $\gamma=10,b=-0.2$	0.0541	0.0307	0.0234	-	0.2429	0.7112	0.5657	0.1455	-	0.1078

Panel C. Gaussian-SJC DCC diffusion with only CFNAI driving conditional correlation ($\gamma_1 = \gamma_2 = 0$)

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, $\gamma=5$	0.0353	0.0620	-0.0267	-0.0673	-	0.8030	0.7259	0.0771	-0.0521	-
CRRA, $\gamma=10$	0.0204	0.0379	-0.0174	-0.0403	-	0.7356	0.6523	0.0833	0.0024	-
HARA, $\gamma=5,b=-0.2$	0.0285	0.0512	-0.0227	-0.0558	-	0.7798	0.7021	0.0777	-0.0373	-
HARA, $\gamma=10,b=-0.2$	0.0160	0.0313	-0.0153	-0.0338	-	0.7215	0.6371	0.0844	0.0101	-

Table IV (cont.). Intertemporal hedging demands for the 2 stocks with different specifications for the unconditional dependence (Gaussian diffusion with no tail dependence, Student's t diffusion with symmetric tail dependence, and Gaussian-SJC diffusion with asymmetric tail dependence). Dynamic conditional correlation with volatility and macroeconomic variables. For each investment horizon of 1 and 5 years the table displays the sum of market price of risk hedges (MPRH), the separate demands for S&P 500 and NASDAQ, as well as the two correlation hedges (CorrH), corresponding to the macroeconomic factor F^M and the volatility factor F^V (the latter multiplied by 100).

Panel D. Gaussian DCC diffusion

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0557	0.0652	-0.0094	-0.0469	0.5440	0.9605	0.7419	0.2187	-0.0127	0.1512
CRRA, γ=10	0.0374	0.0431	-0.0057	-0.0277	0.3209	0.8905	0.6694	0.2211	0.0237	-0.2819
HARA, γ=5,b=-0.2	0.0476	0.0555	-0.0078	-0.0387	0.4491	0.9353	0.7167	0.2186	-0.0026	0.0358
HARA, γ=10,b=-0.2	0.0321	0.0372	-0.0052	-0.0232	0.2673	0.8752	0.6559	0.2193	0.0292	-0.3400

Panel E. Gaussian-SJC DCC diffusion

Horizon	1 year					5 years				
	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100	MPRH Sum	MPRH S&P500	MPRH NASDAQ	CorrH F^M	CorrH F^V*100
CRRA, γ=5	0.0268	0.0594	-0.0326	-0.0652	0.3782	0.8419	0.7265	0.1154	-0.0245	0.1429
CRRA, γ=10	0.0140	0.0356	-0.0216	-0.0386	0.2243	0.7708	0.6500	0.1208	0.0226	-0.1317
HARA, γ=5,b=-0.2	0.0206	0.0484	-0.0277	-0.0538	0.3129	0.8166	0.7007	0.1159	-0.0110	0.0664
HARA, γ=10,b=-0.2	0.0107	0.0297	-0.0190	-0.0323	0.1885	0.7552	0.6350	0.1202	0.0303	-0.1761

unchanged with respect to the CCC model. As in the portfolio allocation example along realized paths of the state variables, here we also observe a larger spread between the holdings of S&P 500 and NASDAQ for the DCC case with respect to CCC. These results are confirmed for a CRRA as well as a HARA investor. They are valid for all investment horizons considered as well as different levels of risk aversion. Increasing the level of risk aversion invariably leads to a decrease in the intertemporal hedging demands in absolute terms. This also happens for a HARA investor with a certain subsistence level b below which she is unwilling to fall as compared to a CRRA investor.

A second simulations experiment aims at determining the importance of the stationary distribution and hence tail dependence for an investor who has already accounted for dynamically varying conditional correlation. We pick again the Gaussian-SJC diffusion with DCC as the benchmark case and compare its implied hedging demands with those from a Gaussian alternative. Results are presented in Panels D and E of Table IV. As in the portfolio example over realized paths of the state variables, the stationary distribution still plays a role in determining the hedging demands, rendering them smaller in the presence of tail dependence. For shorter horizons its effect is smaller than the effect of disregarding conditional correlation, but at the 5-year horizon the Gaussian diffusion renders the highest hedging demands, even higher than the CCC case, which confirms our findings of the market timing exercise.

The above results may be sensitive to the level of conditional correlation that we impose. Thus, we repeat the simulations experiment with a Gaussian-SJC diffusion and DCC for varying values of the γ_0 parameter for the conditional correlation. We consider levels of γ_0 of 1, 2 and 3 which correspond to conditional correlation levels (averaged over the estimation period) of 0.45, 0.75 and 0.90. We find the appropriate CCC calibration for the conditional correlation parameters, keeping the same average correlation levels as in the DCC case. Results are plotted in Figure 8.

Regardless of the investment horizon, for relatively low correlation levels (0.45) the DCC model implies significantly lower intertemporal hedging demands compared to a CCC specification, even after tail dependence has been accounted for through the Gaussian-SJC stationary distribution. For extremely high correlation levels (the case of $\gamma_0 = 3$) the roles of DCC and CCC change and now it is the latter that implies lower hedging demands (as illustrated by the comparison between the lefthand and the righthand panel of Figure 8). Depending on the investment horizon, we may have higher or lower hedge levels for a mean conditional correlation of 0.75. This behavior can thus explain the higher hedging demands implied by the DCC specification over a realized path of the state variables that we encountered earlier.

2.3 Certainty equivalent cost of ignoring hedging demands due to correlation risk and tail dependence

We follow the common approach in literature on portfolio choice and study the effect of ignoring correlation hedging on the wealth of the investor using the utility loss, or the certainty equivalent cost (see Liu

Figure 8. Dynamic correlation-induced portfolio hedging terms through simulation: the influence of correlation level

Intertemporal hedging demands for a benchmark Gaussian-SJC diffusion with DCC vs. a CCC specification with parameter calibrated to match the mean conditional correlation of the DCC model. Varying average values of conditional correlation through the parameter γ_o. HARA investor with $b = -0.2$ and varying degrees of relative risk aversion, and investment horizon of 1 and 5 years.
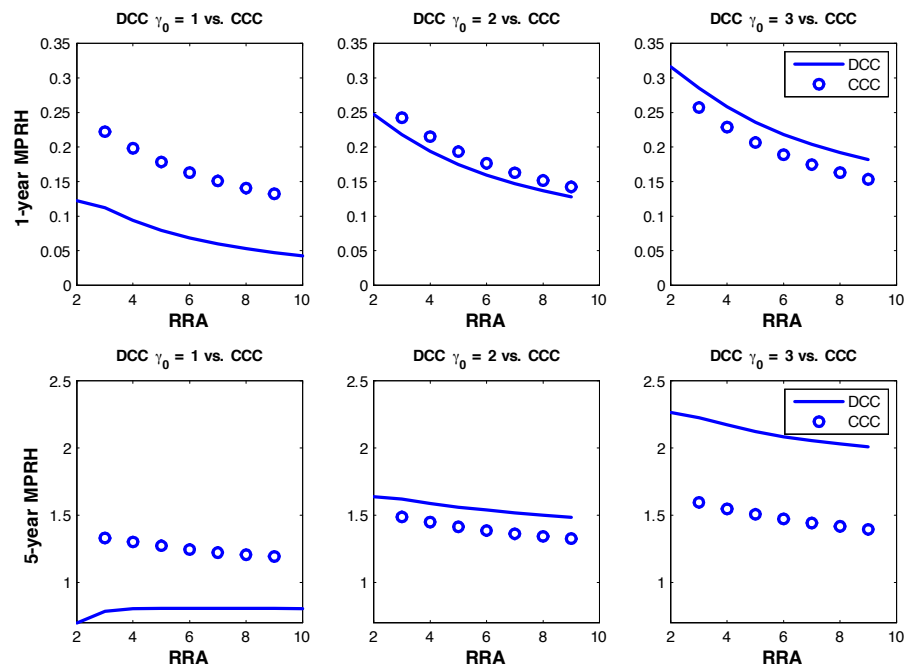
et al. 2003). The approach consists of computing the additional amount of wealth that would be needed for an investor to consider a suboptimal allocation strategy (that results from ignoring correlation and tail dependence hedging) instead of the optimal one (that takes into account the dynamics of conditional correlation and the clustering of extreme realizations), in order to achieve the same expected utility of terminal wealth. In other words, we are looking to determine the amount *ceq* such that:

$$E\left[U\left(\omega_T^* \mid \omega_0 = 1\right)\right] = E\left[U\left(\omega_T \mid \omega_0 = 1 + ceq\right)\right]$$

where ω_T^* is the terminal wealth achieved under the optimal investment strategy and ω_T is the terminal wealth under the suboptimal one.

The first question that we address, in accordance with the simulation exercise above, is whether the investor would lose anything if she disregards the dynamics of conditional correlation, modeled using volatility and macroeconomic indices, given that tail dependence in the unconditional distribution has already been accounted for. Thus, we choose as a benchmark process the Gaussian-SJC diffusion with DCC. Then we consider an alternative CCC model by setting all conditional correlation parameters to zero except for γ_0. The γ_0 parameter is calibrated to reflect the same average correlation as the DCC benchmark over the estimation horizon. We consider again a HARA investor with varying degrees of relative risk aversion and a parameter b in the utility function equal to -0.2, 0 or 0.2. The case of $b = 0$ corresponds to a CRRA investor, while if $b < 0$ relative risk aversion is decreasing and convex in wealth, in which case the investor is intolerant towards wealth falling below a certain subsistence level $-b$. Alternatively, if $b > 0$, then relative risk aversion is increasing and concave. Table V summarizes the results on the certainty equivalent cost in each case, calculated in cents per dollar. In summary, the cost of disregarding the dynamics of conditional correlation decreases with rising levels of the risk aversion coefficient and is highest for a HARA investor with relative risk aversion that is increasing and concave in wealth.

We next address the alternative problem of finding the utility cost for an investor who disregards that extreme realizations of the assets in her portfolio may be dependent, as modeled through the stationary distribution of X. Results are summarized in Table VI, where we take as a benchmark process either the DCC Gaussian-SJC diffusion (left column), or the CCC one (right column) against the tail independent Gaussian counterpart. To isolate only the impact of the tail dependence through the stationary distribution, conditional correlation parameters for all processes are taken from the Gaussian-SJC type with DCC.

The main conclusion that we can draw from comparing the wealth loss across the alternative specifications is that the investor loses more from disregarding tail dependence if she has not taken into account the dynamics in conditional correlation. It is an anticipated result, as both ways of modeling dependence through the dynamics of the conditional correlation or through the stationary distribution

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Table V. Certainty equivalent cost of ignoring dynamic conditional correlation, modeled with volatility and macroeconomic indices

The benchmark process is a Gaussian-SJC diffusion with DCC. The conditional correlation parameters of the alternative CCC processes are calibrated in order to reflect the same mean conditional correlation as the benchmark process. The Certainty Equivalent Cost is given in cents per dollar. Investment horizon is 5 years.

	The cost of disregarding DCC (CCC alternative)		
	HARA (b = -0.2)	CRRA (b = 0)	HARA (b = 0.2)
$\gamma = 2$	2.3054	2.4039	2.5024
$\gamma = 4$	1.8987	1.9369	1.9751
$\gamma = 6$	1.7983	1.8216	1.8449
$\gamma = 8$	1.7538	1.7706	1.7873
$\gamma = 10$	1.7289	1.7419	1.7549

**Table VI. Certainty equivalent cost of ignoring tail dependence**

The benchmark process is a Gaussian-SJC diffusion with DCC. The alternative processes have either a DCC specification (left figures) or a CCC specification (right column), and the unconditional distribution is Gaussian. All parameters of the conditional correlation specification are from the Gaussian-SJC type with DCC (left column) and from Gaussian-SJC type with CCC (right column). The Certainty Equivalent Cost is given in cents per dollar. Investment horizon is 5 years.

	The cost of disregarding tail dependence					
	(Gaussian alternative, DCC)			(Gaussian alternative, CCC)		
	HARA (b = -0.2)	CRRA (b = 0)	HARA (b = 0.2)	HARA (b = -0.2)	CRRA (b = 0)	HARA (b = 0.2)
$\gamma = 2$	1.3153	1.5158	1.7162	3.2467	3.8692	4.4916
$\gamma = 4$	0.6384	0.7438	0.8492	1.1366	1.4361	1.7357
$\gamma = 6$	0.3912	0.4619	0.5326	0.4602	0.6562	0.8523
$\gamma = 8$	0.2658	0.3189	0.3719	0.1301	0.2757	0.4212
$\gamma = 10$	0.1902	0.2327	0.2751	0.0000	0.0507	0.1664

Figure 9. Certainty Equivalent Cost

Panel. A. Certainty Equivalent Cost of ignoring dynamic conditional correlation for varying mean levels of conditional correlation

The certainty equivalent cost of disregarding dynamic conditional correlation for a benchmark Gaussian-SJC diffusion with DCC vs. a Gaussian diffusion with CCC with parameter calibrated to match the mean conditional correlation of the corresponding DCC model. Varying average values of conditional correlation through the parameter γ_o. HARA investor with $b = -0.2$ and varying degrees of relative risk aversion, and investment horizon of 1 and 5 years.
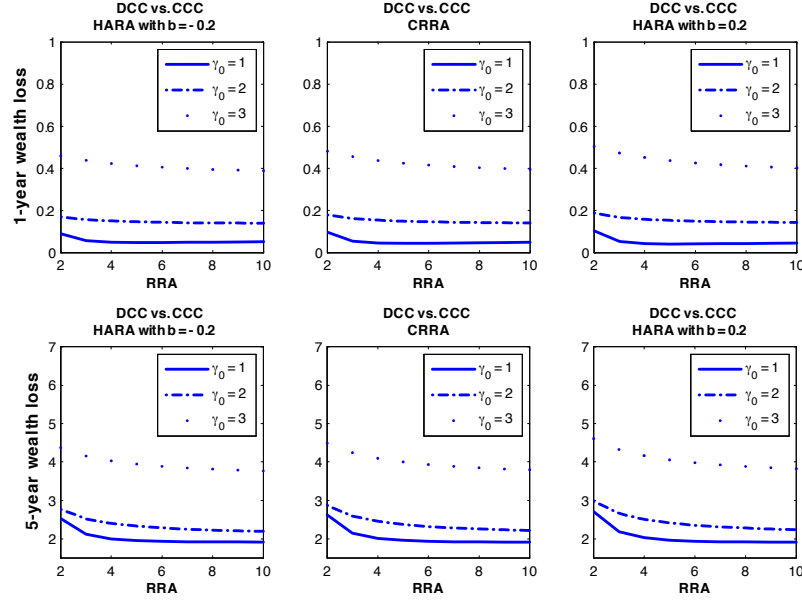
aim at reproducing the same stylized fact of increased dependence in down markets. Thus if at least one of them is taken into account when making portfolio decisions, the impact of disregarding the other in terms of wealth loss will be subdued.

As we saw in the above simulations exercise, the portfolio composition changes considerably for varying levels of the mean conditional correlation, modeled through the parameter γ_0. In order to determine the economic significance of this finding, we compute the certainty equivalent cost for disregarding correlation dynamics. Results are summarized in Panel A of Figure 9.

The certainty equivalent cost is lower for the lowest levels of correlation considered ($\gamma_0 = 1$ or average correlation of 0.45 over the estimation horizon) and increases significantly for higher correlation levels. It also increases with the investment horizon. Results are consistent over the utility specifications considered (CRRA and 2 types of HARA utility).

For consistency with the simulations experiment, we also consider the economic loss for disregarding tail dependence, given that the dynamics of conditional correlation have been accounted for. We compute it by comparing the benchmark Gaussian-SJC diffusion with DCC with a corresponding Gaussian diffusion

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Figure 9. Panel. B. Certainty Equivalent Cost of disregarding tail dependence

The certainty equivalent cost of disregarding tail dependence by considering a Gaussian DCC diffusion instead of the benchmark data generating process of a Gaussian-SJC DCC diffusion for varying levels of the ω^{SJC} parameter determining the weight of the SJC copula in the mixture distribution. Parameters are taken from estimating the benchmark case over the whole estimation horizon, while the correlation parameter of the Gaussian copula is calibrated so that to reflect the same Kendall's tau as the one implied by the SJC copula with the estimated parameters.
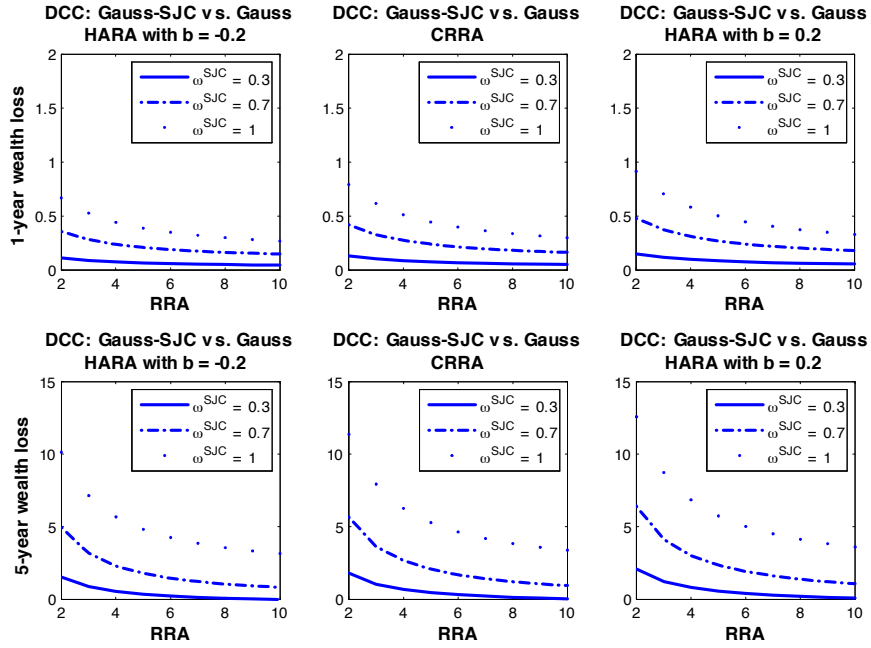
with the same conditional correlation dynamics. We do so for varying weights ω of the mixture copula $C^{Ga-SJC} = \omega C^{SJC} + (1-\omega) C^{Ga}$. Parameters are taken from the benchmark model over the whole estimation horizon, and the Gaussian correlation parameter is set so that the Kendall's tau implied by the Gaussian copula is equal to the one implied by the SJC copula, so varying the composition of the Gaussian-SJC copula will not change the Kendall's tau, but only the relative importance of tail dependence. Results are presented in Panel B in Figure 9. Even if dynamic conditional correlation has already been accounted for, there are substantial economic costs for disregarding tail dependence, reaching over ten cents per dollar for a 5-year investment horizon. They increase with increasing the weight of the SJC copula in the benchmark model (and hence the importance of tail dependence in the data generating process) and are higher for investors with lower levels of risk aversion.

3   Conclusion

In this paper we propose a model for asset dynamics where both conditional correlation and dependence between extremes are considered. Our proposed specification also permits us to link the market and economic states to the correlation dynamics of risky asset returns in a very parsimonious way. We solve for the optimal portfolio allocation in this dynamic setting and obtain in closed form the optimal portfolio components in terms of mean-variance demand as well as intertemporal hedging demands.

We find significant distinction in the portfolio composition for the risky funds between both alternative specifications of the dependence structure. In the presence of dynamic conditional correlation the spread between the hedging demands for the two funds increases, while in the presence of tail dependence it decreases. Translating our findings into the certainty equivalent costs, we show that indeed there are substantial economic costs for investors in ignoring either the dynamics of conditional correlation or the clustering of extreme events. This welfare loss increases dramatically with the investment horizon, during both recessionary and volatile periods, and for low levels of the agent's relative risk aversion.

In addition, we document that both correlation hedging demands and intertemporal hedges due to increased tail dependence have distinct portfolio implications and they cannot act as substitutes to each other. There is a substantial utility loss for disregarding dependence between extreme realizations, even when dynamic conditional correlation has already been accounted for, and vice versa. Our findings suggest that it is economically crucial for investors to consider both asymmetric tail dependence and dynamic conditional correlation in their investment decisions.

We also shed light in this paper on an important conclusion concerning the use of macroeconomic and market factors. By isolating the correlation hedging demands required to hedge against fluctuations in these observed variables, we show that the total correlation demands due to those variables are generally negative throughout the period we consider. The impact of the macroeconomic variable turns out to be more significant than the one due to the market influence and directs the behavior of the hedging

demands. This finding can be investigated further by asking if there is any welfare gain from using observed factors in general instead of relying more on latent factors in the modeling of asset dynamics. We leave this simple, yet important question to later research.

Finally, there are also a number of ways in which the present study could be extended. For simplicity, we have assumed that the bond and stock dynamics are independent from each other. There is compelling evidence of co-movement between bond and stock returns that are, in turn, linked to the same macroeconomic states (e.g. Li 2002). It would be of interest to incorporate this stylized fact in the present portfolio solution setup. One could also test the sensitivity of the results to an increased number of assets in the portfolio, as we would expect that hedging demands should increase as a result of the higher level of uncertainty linked to both the conditional correlation structure and the dependence through the stationary distribution. At the modeling level, our paper can be extended in a different way as well. We have documented that the dependence structure changes dramatically from a Gaussian regime during calm periods to a dependence that exhibits asymmetries and high tail dependence coefficients during volatile periods. This could motivate researchers to consider further a specification where the copula composition changes from a normal to an extreme dependent one. One way this could be done is through varying weights of the copula in the spirit of Patton (2004).

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APPENDIX

A   The affine setup for the bond price

The interest rate is given by $r\left(t, Y_t^r\right) = \delta_0 + \delta_1 Y_t^r$, where the state variable Y_t^r follows an affine diffusion under the risk neutral measure Q:

$$dY_t^r = \overline{\kappa_r}\left(\overline{\theta^r} - Y_t^r\right)dt + \Sigma_r\sqrt{S_t^r}dW_t^{*r}, \tag{A.1}$$

where W^{*r} is a $Q-$Brownian motion, and $S_t^r = \alpha^r + \beta^r Y_t^r$. We adopt a completely affine model in that we specify the market price of risk as $\lambda\sqrt{S_t^r}$ for a constant λ (e.g. Dai and Singleton 2000). The state variable process under the physical measure P is then given by:

$$dY_t^r = \kappa_r\left(\theta^r - Y_t^r\right)dt + \Sigma_r\sqrt{S_t^r}dW_t^r, \tag{A.2}$$

for $\kappa_r = \overline{\kappa_r} - \Sigma_r\lambda\beta^r$ and $\theta^r = \kappa_r^{-1}\left(\overline{\kappa_r}\overline{\theta^r} + \Sigma_r\lambda\alpha^r\right)$. Given the above setup and provided an admissible parameterization, we know from Duffie and Kan (1996) and Dai and Singleton (2000) that the price of a bond for a maturity τ is given by:

$$B\left(t, \tau\right) = \exp\left\{a\left(\tau\right) - b\left(\tau\right)Y_t^r\right\}, \tag{A.3}$$

where $a\left(\tau\right)$ and $b\left(\tau\right)$ solve the following system of ODEs:

$$
\begin{aligned}
\frac{\partial a\left(\tau\right)}{\partial\tau} &= -\overline{\theta^r}\overline{\kappa_r}b\left(\tau\right) + \frac{1}{2}\left[\Sigma_r b\left(\tau\right)\right]^2\alpha^r - \delta_0 \\
\frac{\partial b\left(\tau\right)}{\partial\tau} &= -\overline{\kappa_r}b\left(\tau\right) - \frac{1}{2}\left(\Sigma_r b\left(\tau\right)\right)^2\beta^r + \delta_1.
\end{aligned}
$$

with initial conditions $a\left(\tau\right) = 0$ and $b\left(\tau\right) = 0$. An application of Ito's lemma implies that the discount bond price dynamics can be expressed as follows:

$$
\begin{aligned}
\frac{dB(t, \tau)}{B(t, \tau)} &= \mu^B\left(t, \tau, Y_t^r\right)dt + \sigma^B\left(t, \tau, Y_t^r\right)dW_t^r \\
\text{where } \mu^B\left(t, \tau, Y_t^r\right) &= r\left(t, Y_t^r\right) - b\left(\tau\right)\Sigma_r S_t^r\lambda \\
\text{and } \sigma^B\left(t, \tau, Y_t^r\right) &= -b\left(\tau\right)\Sigma_r\sqrt{S_t^r}.
\end{aligned} \tag{A.4}
$$

B   Copula Functions

The following d-dimensional copula functions are used in the paper.

- **Gaussian copula**

$$C^{Ga}\left(u_1, u_2, ..., u_d \mid R_{Ga}\right)$$

$$= \int_{-\infty}^{\Phi^{-1}(u_1)} ... \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{2\pi \left|R_{Ga}\right|^{1/2}} \exp\left\{-\frac{1}{2} x^{\intercal} R_{Ga}^{-1/2} x\right\} dx_1 ... dx_d$$

where R_{Ga} denotes the correlation matrix, and $\Phi^{-1}\left(u_i\right)$ is the inverse of the univariate standard normal CDF.

- **Symmetrized Joe-Clayton copula**

This copula function was introduced by Patton (2004) and is based on the bivariate Joe-Clayton copula, that is a two-parameter copula function with parameters $\tau_L \in (0,1)$ and $\tau_U \in (0,1)$ that are a measure of the lower and upper tail dependence. The Joe-Clayton copula has the following form:

$$C^{JC}\left(u_1, u_2 \mid \tau_L, \tau_U\right)$$

$$= 1 - \left\{1 - \left[\left(1 - (1 - u_1)^{\kappa}\right)^{-\gamma} + \left(1 - (1 - u_2)^{\kappa}\right)^{-\gamma} - 1\right]^{-\frac{1}{\gamma}}\right\}^{\frac{1}{\kappa}}$$

$$\text{where } \kappa = \frac{1}{\log_2\left(2 - \tau_U\right)}$$

$$\gamma = -\frac{1}{\log_2\left(2 - \tau_L\right)}$$

The symmetrized version of the copula, designed to render it completely symmetric for equal values of the lower and upper tail dependence parameters has the following form:

$$C^{SJC}\left(u_1, u_2 \mid \tau_L, \tau_U\right)$$

$$= \frac{1}{2}\left[C^{JC}\left(u_1, u_2 \mid \tau_L, \tau_U\right) + C^{JC}\left(1 - u_1, 1 - u_2 \mid \tau_U, \tau_L\right) + u_1 + u_2 - 1\right]$$

The coefficients of upper (τ_U) and lower (τ_L) tail dependence are defined as:

$$\tau_U = \lim_{u \to 1} \Pr\left[X_1 > F_{X_1}^{-1}(u) \mid X_2 > F_{X_2}^{-1}(u)\right]$$

$$\tau_L = \lim_{u \to 0} \Pr\left[X_1 \le F_{X_1}^{-1}(u) \mid X_2 \le F_{X_2}^{-1}(u)\right].$$

for random variables X_1 and X_2. Both coefficients can be represented in terms of the copula function: $\tau_U = \lim_{u \to 1} \frac{(1 - 2u + C(u,u))}{1 - u}$ and $\tau_L = \lim_{u \to 0} \frac{C(u,u)}{u}$. The SJC copula function is parameterised directly in terms of these two coefficients.

C Form and properties of the univariate Normal Inverse Gaussian (NIG) distribution

The Normal Inverse Gaussian (NIG) distribution is a member of the family of Generalized Hyperbolic distributions, constructed as normal mean-variance mixtures with the Generalized Inverse Gaussian (GIG)

as the mixing distribution. Its density is given by:

$$
\begin{aligned}
f_{NIG}\left(x;\alpha,\beta,\delta,\mu\right) &= c\left(\alpha,\delta\right)\left(\delta^2+\left(x-\mu\right)^2\right)^{\frac{1}{2}}\times \\
&\quad K_1\left(\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}\right)e^{\delta\sqrt{\alpha^2-\beta^2}+\beta\left(x-\mu\right)} \\
\text{where } c\left(\alpha,\delta\right) &= \frac{\alpha\delta}{\pi} \\
x &\in \mathbb{R}
\end{aligned}
$$

where $\delta > 0$, $\alpha \geq |\beta| \geq 0$, $\mu \in \mathbb{R}$, and K_λ is the modified Bessel function of the third kind with index λ, defined as :

$$
K_\lambda\left(x\right)=\frac{1}{2}\int_0^\infty y^{\lambda-1}e^{-\frac{x}{2}\left(y+y^{-1}\right)}dy, \quad x > 0
$$

Its tail behavior is given by

$$
\lim_{x\to\pm\infty} f_{NIG}\left(x;\alpha,\beta,\delta,\mu\right) \sim |x|^{-3/2}\, e^{\left(\mp\alpha+\beta\right)x}
$$

and it has the interesting property of being closed under convolution, so that the sum of two independent random variables that have a NIG distribution $X_i \sim NIG\left(x;\alpha,\beta,\delta_i,\mu_i\right), i = 1,2$ is also NIG-distributed: $X_1 + X_2 \sim NIG\left(x;\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2\right)$.

In the portfolio application we use several properties of the modified Bessel function that we summarize bellow (following Bibby and Sorensen (2003)):

$$
\begin{aligned}
K_{-\lambda}\left(x\right) &= K_\lambda\left(x\right) \\
K_\lambda'\left(x\right) &= -\frac{\lambda}{x}K_\lambda\left(x\right)-K_{\lambda-1}\left(x\right) \\
K_{n+\frac{1}{2}}\left(x\right) &= \sqrt{\frac{\pi}{2x}}\exp\left(-x\right)\left[1+\sum_{i=1}^{n}\frac{\left(n+i\right)!}{\left(n-i\right)!i!}\left(2x\right)^{-i}\right], \quad n = 1,2,3,...
\end{aligned}
$$

D Malliavin Derivatives of the State Variables

Recall that the Malliavin derivatives of the state variables $Y \equiv \left(X_1, X_2, F^V, F^M, Y^r\right)$ can be represented as the solutions to a linear stochastic differential equation[17]:

$$
D_t Y_s = \sigma^Y\left(t,Y_t\right)\exp\left\{\int_t^s dL_v\right\}
$$

where $\sigma^Y\left(t,Y_t\right)$ is the 5×5 matrix of diffusion terms of the state variables, and dL_t is defined by:

$$
dL_t \equiv \left(\partial_2\mu^Y\left(t,Y_t\right)-\frac{1}{2}\sum_{j=1}^{5}\partial_2\sigma_{\cdot j}^Y\left(t,Y_t\right)\partial\sigma_{\cdot j}^Y\left(t,Y_t\right)^{\mathsf{T}}\right)dt + \sum_{j=1}^{5}\partial_2\sigma_{\cdot j}^Y\left(t,Y_t\right)dW_{jt}
$$

---

[17]See Theorem 1 in Detemple et al. (2003)

where $\partial_2 \mu^Y (t, Y_t)$ and $\partial_2 \sigma_{\cdot j}^Y (t, Y_t)$ denote the derivatives of $\mu^Y (t, Y_t)$ and $\sigma_{\cdot j}^Y (t, Y_t)$ with respect to Y_t, and $\sigma_{\cdot j}^Y (t, Y_t)$ denotes the jth column of the matrix $\sigma^Y (t, Y_t)$. The particular forms of the drift $\mu^Y (t, Y_t)$ and the diffusion term $\sigma^Y (t, Y_t)$ of the state variables are given by:

$$\mu^Y (t, Y) = \begin{pmatrix} \mu_1^X (t, X_t, F^V, F^M) \\ \mu_2^X (t, X_t, F^V, F^M) \\ \mu^{F^V} (t, F^V) \\ \mu^{F^M} (t, F^M) \\ \mu^{Y^r} (t, Y^r) \end{pmatrix}$$

where $\mu_i^X (t, X_t, F^V, F^M), i = 1, 2$ are given by (1.9), $\mu^{F^V} (t, F^V) = \kappa^V (\theta^V - F^V)$, $\mu^{F^M} (t, F^M) = \kappa^M (\theta^M - F^M)$, $\mu^{Y^r} (t, Y^r) = \kappa_r (\theta^r - Y_t^r)$.

$$\sigma^Y (t, Y) = \begin{pmatrix} \sigma_{11}^X (t, X) & \sigma_{12}^X (t, X) & 0 \\ \sigma_{21}^X (t, X) & \sigma_{22}^X (t, X) & 0 \\ \sigma^{F^V} (t, F^V) & \sigma^{F^V} (t, F^V) & 0 \\ \sigma^{F^M} (t, F^M) & \sigma^{F^M} (t, F^M) & 0 \\ 0 & 0 & \sigma^{Y^r} (t, Y^r) \end{pmatrix}$$

where $\sigma^X (t, X)$ is given by (1.5), $\sigma^{F^V} (t, F^V) = \sigma^V \sqrt{F^V}$, $\sigma^{F^M} (t, F^M) = \sigma^M$, and $\sigma^{Y^r} (t, Y^r) = \sigma_r \sqrt{Y_t^r}$.

Given the chosen specifications for the state variables, we can solve separately for the Malliavin derivatives of state variable driving the short rate, as well as for the Malliavin derivatives of the two factors. The processes that we have assumed for the observable factors (F^V for the VIX and F^M for CFNAI), as well as for the state variable Y^r, allow for either closed form solutions for the Malliavin derivatives (in the case of a Vasicek process) or for significant variance reduction in their simulation following the Doss transformation[18] that eliminates the stochastic term in the Malliavin derivative (for a CIR process).

In the Vasicek case, the Malliavin derivative of F^M simplifies significantly to:

$$D_{i,t} F_s^M = \sigma^M \exp \{-\kappa^M (s - t)\}, i = 1, 2$$

For the other two state variables, Y^r and F^V, we have assumed a CIR process, that can be reduced to have constant diffusion term through a suitable change of variable technique, which then eliminates the stochastic terms for the simulation of the corresponding Malliavin derivatives. For a univariate diffusion, this variance stabilizing transformation is described in detail in Proposition 2 of Detemple et al. (2003) and we reproduce it here for completeness.

Consider a state variable Y satisfying a stochastic differential equation

$$dY_t = \mu (t, Y_t) dt + \sigma (t, Y_t) dW_t$$

We can replace it with a new state variable $Z_t = F (t, Y_t)$ where the function $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that $\partial_2 F = \frac{1}{\sigma^Y}$. Then for a continuously differentiable drift μ, twice continuously differentiable diffusion term σ, that also satisfy the growth conditions that $\mu (t, 0)$ and $\sigma (t, 0)$ are bounded for all $t \in [0, T]$, then we have for $t \leq s$:

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[18]See Detemple et al. (2003)

$$
\begin{aligned}
D_t Y_s &= \sigma\left(t, Y_t\right) D_t Z_s \\
\text{where } D_t Z_s &= \exp\left\{\int_t^s \partial_2 m\left(v, Z_v\right) dv\right\} \\
m\left(t, Z\right) &\equiv \left[\frac{\mu}{\sigma} - \frac{1}{2}\partial_2\sigma + \partial_1 F\right]\left(t, Y\right)
\end{aligned}
$$