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Abstract

We show that if an agent is uncertain about the precise form of his utility function, his actual relative risk aversion may depend on wealth even if he knows his utility function lies in the class of constant relative risk aversion (CRRA) utility functions. We illustrate the consequences of this result for asset allocation: poor agents that are uncertain about their risk aversion parameter invest less in risky assets than wealthy investors with identical risk aversion uncertainty.

JEL classification: D81, D84, G11

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1 Introduction

Invididual preferences for risk are not necessarily stable and deterministic, as assumed in standard models of economic decision making. For example, Andersen, Lau, Harrison, and Rutstrom (2008) find sizeable within-subject differences in elicited relative risk aversion. Given the usual caution in interpreting survey research, it is quite possible that agents do not know their preferences very well, or that their preferences change between the stage of thinking about a choice and actually making the choice, see also Weber and Milliman (1997).

The effect of uncertain preferences on choice behavior has only received limited attention. In particular, it is not clear what are the effects of preference uncertainty on decision making under uncertainty. In this paper, we try to fill this gap by taking the perspective of an agent who is aware of his ambivalence or may even be averse to preference uncertainty. As a simple example, one can think of an investor who is unsure whether he is moderately risk averse or very risk averse. As an application of our theory, we illustrate the impact of this type of uncertainty on asset allocation decisions.

Existing research on preference uncertainty largely focuses on preferences for alternative (risk-less) outcomes, particularly for resource valuation, see Li and Mattsson (1995), Akter, Bennett, and Akhter (2008), and Van Kooten, Krcmar, and Bulte (2001). Another line of research rationalizes the underlying mechanisms that result in preference uncertainty. For example, Fischer, Luce, and Jia (2000) argue that preference uncertainty arises because of unfamiliarity with and the need for learning about prospects that have multiple attributes. In contrast to these earlier papers, the current paper focusses on decision making under uncertainty and the impact of preference uncertainty.

For the set-up of our model, we build on the framework of decision making under ambiguity aversion by Klibanoff, Marinacci, and Mukerji (2005, 2009). These authors study the impact of uncertainty about the probabilities of risky outcomes. In our paper, we take the probabilities of risky outcomes as given and study the impact of uncertainty about the preference structure itself.

The paper is set up as follows. Section 2 formalizes our model for preference uncertainty and states the main result. Section 3 provides an illustration and presents the implications of our main result on asset allocation decisions under power utility. Section 4 concludes.

2 The model

Consider an agent with investment horizon T and uncertain final wealth W_T. The distribution of W_T is denoted as $F(W_T)$. The agent's utility function, $U(w, \gamma)$ is indexed by a parameter γ. For example, U can be a standard power utility function $U(W; \gamma) = (1-\gamma)^{-1}W^{1-\gamma}$, with γ the coefficient of relative risk aversion. We assume there is uncertainty about the precise value of γ. This uncertainty may be the result of an intrinsic uncertainty about preferences, e.g., due to the difficulty of valuing multi-attribute alternatives as in Fischer, Luce, and Jia (2000). Alternatively, the uncertainty may be the result of possible shifts in preferences between now and the horizon date T. The uncertainty about γ is summarized by the distribution function $G(\gamma)$.

We assume the agent maximizes

$$V(F,G) = \int v\left(\int U(w;\gamma)\mathrm{d}F(w)\right)\mathrm{d}G(\gamma), \tag{1}$$

where v is a strictly increasing concave function. The function v captures the agent's aversion to preference uncertainty in a similar way as the ambiguity aversion function of Klibanoff, Marinacci, and Mukerji (2005, 2009). The more curved the function v, the higher the agent's aversion to preference uncertainty.

The objective function in (1) naturally embeds the expected utility case for fixed γ. To see this, note that if δ_g is the Dirac function that jumps from 0 to 1 at $g \in \mathbb{R}$, we obtain

$$V(F,\delta_g) = v\left(\int U(w;g)\mathrm{d}F(w)\right), \tag{2}$$

such that maximizing $V(F,\delta_g)$ is the same as maximizing expected utility.

The uncertainty about γ can be considered as a type of background risk, as in Guiso and Paiella (2008) and Heaton and Lucas (2000). It enters the model exogenously and cannot be hedged (completely). Usually, background risk enters the objective through F only. Here, by contrast, the background risk enters through the separate probability function G, where G does not affect the distribution of final wealth. Instead, G operates on the perception of final wealth through the utility function and through the risk-return trade-off over all possible final wealth levels.

The first and second order partial derivatives of U with respect to wealth are denoted as U' and U'', respectively. We assume $U' > 0$, and $U'' < 0$. We also introduce $\bar{U} = \bar{U}(\gamma) = \int U(w;\gamma)\mathrm{d}F(w)$ as a short-hand notation for expected utility under the fixed preference γ, with first and second order derivatives (with respect to γ) denoted as $\bar{U}'$ and $\bar{U}''$, respectively. Finally, v' and v'' denote the first and second order derivatives of v with respect to its argument $\bar{U}$, and we assume $v' > 0$ and $v'' < 0$.

A key concept to assess the effect of preference uncertainty on optimal asset allocation decisions is the Arrow-Pratt coefficient of absolute risk aversion (ARA). A textbook result is that for standard utility functions, the ARA coefficient can be obtained by computing the negative second order derivative of the certainty equivalent for a small Bernoulli gamble, i.e., $\mathrm{ARA} = -\partial^2 c(e)/\partial e^2|_{e=0}$, where

$$U(c(e)) = 0.5U(W_0 + e) + 0.5U(W_0 - e),$$

see also the appendix. Using this definition for absolute risk aversion, the following theorem gives our main result.

Theorem 2.1 *Define the risk aversion coefficients with respect to w and γ as $\mathrm{ARA}_w = -\partial^2 c(e)/\partial e^2|_{e=0}$ and $\mathrm{ARA}_\gamma = -\partial^2 g(e)/\partial e^2|_{e=0}$, where*

$$V(\delta_c, G) = V(0.5\delta_{w-e} + 0.5\delta_{w+e}, G), \tag{3}$$

$$V(F, \delta_g) = V(F, 0.5\delta_{\gamma-e} + 0.5\delta_{\gamma+e}), \tag{4}$$

for fixed w and γ. We have

$$\mathrm{ARA}_w = -\int \frac{U''(w;\tilde{\gamma})}{U'(w;\tilde{\gamma})} \,\mathrm{d}\bar{G}(\tilde{\gamma}; w), \tag{5}$$

where

$$\mathrm{d}\bar{G}(\tilde{\gamma}; w) = \frac{v'(U(w;\tilde{\gamma}))U'(w;\tilde{\gamma})\mathrm{d}G(\tilde{\gamma})}{\int v'(U(w;\tilde{\gamma}))U'(w;\tilde{\gamma})\mathrm{d}G(\tilde{\gamma})}. \tag{6}$$

Similarly,

$$\mathrm{ARA}_\gamma = -\frac{v''(\bar{U})}{v'(\bar{U})}\,\bar{U}' - \frac{\bar{U}''}{\bar{U}'}, \tag{7}$$

where $\bar{U} = \bar{U}(\gamma) = \int U(w;\gamma)\mathrm{d}F(w)$.

The proof of Theorem 2.1 can be found in the appendix.

Equation (5) shows that under preference uncertainty the risk aversion coefficient with respect to wealth, ARA_w, is the expected value of the standard risk aversion coefficient for known γ, . The expectation, however, is not taken with respect to the distribution G of preference uncertainty, but rather with respect to $\bar{G}$, as defined in Equation (6). The denominator in (6) is the integrating constant to ensure that $\bar{G}$ is a distribution function. The distribution $\bar{G}$ assigns more weight to those values of γ that have a high marginal utility U' for the current level of wealth w and/or a high marginal valuation v' of expected utility preference. In this way, the risk aversion coefficient becomes wealth dependent, even if the risk aversion coefficient of U itself does not depend on wealth.

We give a clear illustration of this case in the next section for the case of constant relative risk aversion (CRRA). Interestingly, the transform from G to $\bar{G}$ resembles the transform from actual to risk neutral probabilities via a pricing kernel, see for example Cochrane (2001). In this case, however, the transform is not applied to wealth uncertainty, but to preference uncertainty.

The risk aversion coefficient for preference uncertainty (ARA_γ) is composed of two terms. The first term of (7) reflects the curvature of v, which operates on expected utility. Clearly, the more curved v, the higher ARA_γ. The effect is multiplied by the derivative of expected utility with respect to γ. If expected utility hardly moves if γ is changed, the curvature of v matters less. The second component of ARA_γ is the curvature of expected utility $\bar{U}$ with respect to γ (rather than w). Though the notation is similar to the standard notation for risk aversion, the expression for familiar utility specifications U is substantially different. For example, even for the CRRA case, no closed form expressions for $\bar{U}$ are readily available.

3 Asset allocation with CRRA utility

To illustrate Theorem 2.1, we consider an expected utility maximizing agent ($v(\bar{U}) = \bar{U}$), endowed with a power utility function

$$U(W_T, \gamma) = (1-\gamma)^{-1} W_T^{1-\gamma}, \tag{8}$$

where $\gamma = -W_T U''/U'$ denotes the relative risk aversion of the agent. The agent is unsure about his precise value of γ, which can take either a high value γ^H or a low value γ^L with equal probability.

In our context of optimal asset allocation, the agent can invest in a risky and a risk-free asset with returns $r^f + r$ and r^f, respectively. The risky asset's excess return above the risk-free rate r has probability distribution $F(r)$. If α denotes the fraction invested in the risky asset, end-of-period wealth W_T equals $W_T = W_0 \cdot (1 + r^f + \alpha \cdot r)$. Using Theorem 2.1, we obtain the relative risk aversion coefficient

$$\mathrm{RRA}_w = W \cdot \mathrm{ARA}_w = \frac{\gamma_L W^{\Delta\gamma} + \gamma_H}{W^{\Delta\gamma} + 1}, \tag{9}$$

where $\Delta\gamma = \gamma_H - \gamma_L > 0$. The RRA_w coefficient clearly depends on wealth, even though the RRA_w for fixed γ does not. Looking more closely at (9), we see that risk aversion monotonically decreases in W with an upper limit γ_H for small values of W, and a lower limit γ_L for high values of W. Put differently, the uncertainty in γ induces decreasing relative risk aversion. Figure 1 illustrates the results.

The baseline case in Figure 1 is the setting without preference uncertainty: $\gamma^H = \gamma^L = 5$. We obtain the familiar result that the fraction invested in the risky asset is constant in the initial wealth level. If the uncertainty in γ is increased by a mean preserving spread, the pattern changes substantially. For high wealth levels, the relative risk aversion coefficient in (9) is substantially lower than 5. This results in higher allocations to the risky asset. Ultimately, the allocation tends to that for γ^L. For low wealth levels, a similar result emerges. At low wealth levels, the agent becomes more prudent, ultimately converging to the allocation for γ^H.

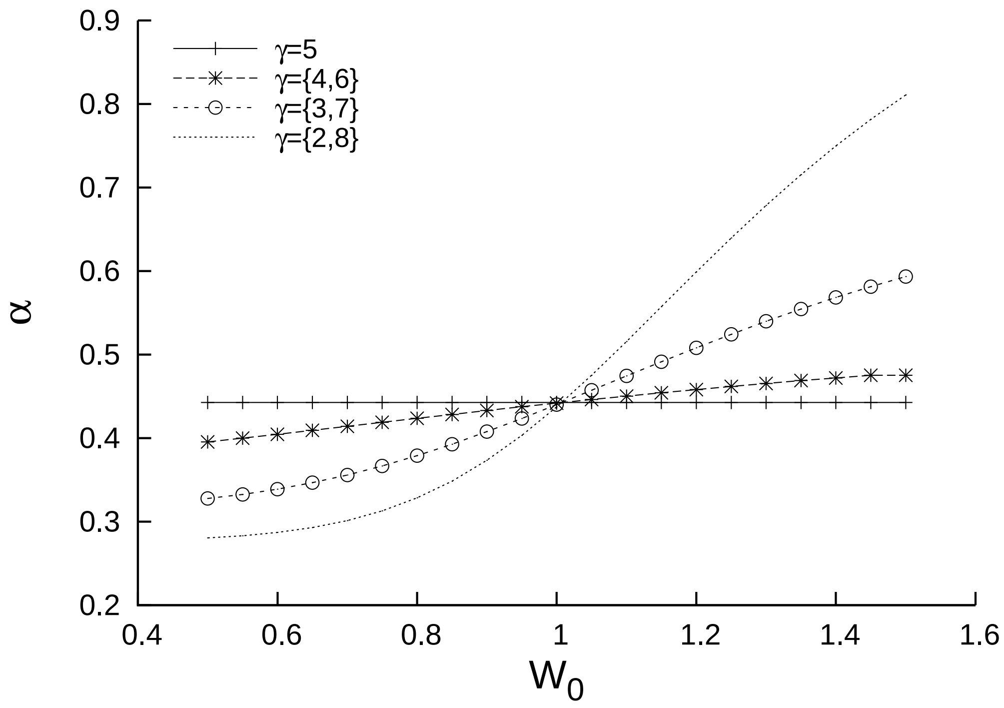

Figure 1: Optimal fraction for different noise levels.
The figure shows the optimal fraction in stock for uncertain γ, where γ takes a low or high value with equal probability as indicated in the legend. The horizontal line in the figure corresponds to $\gamma = 5$, the baseline case.

All curves in Figure 1 cross the point $(1, \alpha_{\bar{\gamma}})$, where $\alpha_{\bar{\gamma}}$ is the optimal asset allocation for the expected level of risk aversion $\bar{\gamma} = (\gamma^H + \gamma^L)/2$. Note that at $W = 1$, $\bar{\gamma}$ is the risk aversion for $0.5U(1, \gamma^L) + 0.5U(1, \gamma^H)$. This result indicates that under preference uncertainty, scaling of wealth starts to matter. This feature is shared with other utility functions without preference uncertainty, such as the exponential or constant absolute risk aversion (CARA) utility function.

The effect of preference uncertainty appears negligible if wealth at the horizon W_T is scaled by current wealth W_0, i.e., around the point $W = 1$ in the graph. However, this only holds in the static one-period model presented above. For the general multi-period context, wealth drifts from its initial starting value W_0 as time progresses. This re-introduces the changing asset allocations over wealth levels at later stages. As a result, our current set-up produces succinctly different results from the familiar Merton-Samuelson multi-period result for CRRA utility functions without preference uncertainty, see Merton and Samuelson (1990).

Further intuition for the pattern in Figure 1 can be obtained from the first order conditions,

$$\mathrm{E}\left[\left(x(W_0)\cdot(1+r^f+\alpha\cdot r)^{-\gamma^L}+(1-x(W_0))\cdot(1+r^f+\alpha\cdot r)^{-\gamma^H}\right)\cdot r\right]=0, \tag{10}$$

where $x(W_0) = W_0^{\Delta\gamma}/(1+W_0^{\Delta\gamma})$ is a weight function that increases from zero for $W_0 = 0$ to 1 for large values of W_0. Equal weights are implied by $W_0 = 1$. The use of weights in (10) has an obvious effect. For large initial wealth levels, only the first order condition for a standard CRRA optimization problem for known $\gamma = \gamma^L$ plays a role. The opposite holds for low wealth levels. The phenomenon is linked to the use of the transformed probabilities $\bar{G}$ in Theorem 2.1 and can be understood from the different curvatures of the utilities for the two different levels of risk aversion. For a large wealth level, the trade-off between a risky and a safe prospect is dominated by the lowest risk aversion utility function. The curvature of the high-risk aversion (γ^H) utility at high wealth levels is negligible compared to the curvature of its γ^L counterpart. The converse holds for low wealth levels, where the curvature of $U(\cdot,\gamma^H)$ dominates that of $U(\cdot,\gamma^L)$. This causes $x(W)$ to go to 0 and the asset allocation (and the first order condition) in this area to be dominated by γ^H.

4 Conclusion

We have shown that uncertainty about risk aversion impacts the relation between wealth and risk aversion, so that the asset allocation implications of traditional utility functions are altered. The relation between wealth and risk taking depends on the specification of uncertainty. Our example for a power utility maximizer shows that some uncertainty on risk aversion leads to a positive relation between wealth and risk taking. This has implications for analyzing actual risk taking behavior: preference uncertainty helps to reconcile power utility implied decision making with observed decreasing absolute risk aversion (DRRA) behavior.

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Appendix

Derivation of ARA

Consider the gamble $W = W_0 + e$ or $W = W_0 - e$ with equal probability 0.5. The certainty equivalent $c(e)$ is the dollar amount for which $U(c(e)) = \mathrm{E}[U(W)]$, such that $U(c(0)) = U(W_0)$. We assume $U'(W_0) > 0$. Define $c = c(e)$, $\dot{c} = \dot{c}(e) = \partial c(e)/\partial e$, and $\ddot{c} = \ddot{c}(e) = \partial \dot{c}(e)/\partial e$. Taking first and second order derivatives of $U(c) = \mathrm{E}[U(W)]$ with respect to e and evaluating in $e = 0$, we obtain

$$U'(c)\dot{c} = 0.5U'(W_0) - 0.5U'(W_0) = 0 \qquad \Rightarrow \qquad \dot{c}(0) = 0,$$

and

$$U''(c)\dot{c}^2 + U'(c)\ddot{c} = U''(W_0) \qquad \Rightarrow \qquad \ddot{c}(0) = U''(W_0)/U'(W_0),$$

such that $-\ddot{c}$ is the standard absolute risk aversion (ARA) coefficient.

Proof of Theorem 2.1

First note that $\partial\bar{U}(\gamma)/\partial e|_{e=0} = 0$ and $\partial^2\bar{U}(\gamma)/\partial e^2|_{e=0} = U''(w)$ for $F = 0.5\delta_{w-e}+0.5\delta_{w+e}$. Using the definition equation for c, $V(\delta_c, G) = V(0.5\delta_{w-e} + 0.5\delta_{w+e}, G)$, and taking first and second order derivatives with respect to e on both sides and evaluating in $e = 0$, we obtain

$$\int v'(U(c))\ U'(c)\ \dot{c}\ \mathrm{d}G = \int v'(\bar{U}(\gamma))\ \frac{\partial\bar{U}(\gamma)}{\partial e}\ \mathrm{d}G \quad \stackrel{e=0}{\Longrightarrow} \quad \dot{c}(0, G) = 0,$$

and

$$\int v''(U(c))\ U'(c)^2\ \dot{c}^2\ \mathrm{d}G + \int v'(U(c))\ U''(c)\ \dot{c}^2\ \mathrm{d}G + \int v'(U(c))\ U'(c)\ \ddot{c}\ \mathrm{d}G =$$

$$\int v''(\bar{U}(\gamma))\ \left(\frac{\partial\bar{U}(\gamma)}{\partial e}\right)^2\ \mathrm{d}G + \int v'(\bar{U}(\gamma))\ \frac{\partial^2\bar{U}(\gamma)}{\partial e^2}\ \mathrm{d}G \quad \stackrel{e=0}{\Longrightarrow}$$

$$\begin{aligned} \ddot{c}(0, G) &= \frac{\int v'(U(w))\ U''(w)\mathrm{d}G}{\int v'(U(c))\ U'(c)\ \mathrm{d}G} = \int \frac{U''(w)}{U'(w)}\ \frac{v'(U(w))\ U'(w)\ \mathrm{d}G}{\int v'(U(w))\ U'(w)\ \mathrm{d}G} \\ &= \int \frac{U''(w)}{U'(w)}\ \mathrm{d}\bar{G}, \end{aligned}$$

where we used $c(0) = w$. The result for ARA_γ follows similarly.