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Measuring Socioeconomic Inequality in Health, Health Care and Health Financing by Means of Rank-Dependent Indices: A Recipe for Good Practice

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Measuring socioeconomic inequality in health, health care and health financing by means of rank-dependent indices: A recipe for good practice

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Abstract

The tools to be used and other choices to be made when measuring socioeconomic inequalities with rank-dependent inequality indices have recently been debated in this journal. This paper adds to this debate by stressing the importance of the measurement scale, by providing formal proofs of several issues in the debate, and by lifting the curtain on the confusing debate between adherents of absolute versus relative health differences. We end this paper with a 'matrix' that provides guidelines on the usefulness of several rank-dependent inequality indices under varying circumstances.

KEY WORDS: Health inequality, Socioeconomic inequality, Concentration Index

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1. Introduction

The Concentration Curve and the related Concentration Index (CI) have nowadays attained the status of "workhorse in most health economic studies" (Fleurbaey and Schokkaert, 2009: 73). A characteristic feature of the CI is that it measures the socioeconomic inequality of health by taking into account every individual's *level* of health and every individual's *rank* in the socioeconomic domain. In recent years the welfare economics foundations of the CI have been explored in depth (Bommier and Stecklov, 2002; Bleichrodt and van Doorslaer, 2006). In this paper we focus on an issue which has received less attention in the literature: the properties of the variables measuring health and socioeconomic position. The matter appears to be of little importance for the socioeconomic dimension, since an ordinal variable suffices to rank individuals according to their socioeconomic position (Wagstaff and Watanabe, 2003; Lindelöw, 2006; O'Donnell *et al.*, 2008: chapter 6). In the case of health, however, the issue is less innocuous (Clarke *et al.*, 2002; Wagstaff, 2005, 2009; Erreygers, 2009a, 2009b; van Doorslaer and Van Ourti, forthcoming).

Wagstaff (2005) has shown that the minimum and maximum values of the CI calculated on the basis of a binary variable – say suffering from a chronic illness or not – depend upon the mean of this variable. This holds more generally for any bounded variable (Erreygers, 2009a), and thus complicates the comparison of the values of the CI for populations with different mean health levels. The bounded character of the health variable moreover raises the question of the relation between health and ill health inequality. Erreygers (2009a) has argued that an index of socioeconomic inequality should reveal the same 'magnitude of inequality' when calculated on the basis of the *health* variable and when calculated on the basis of the associated *ill health* variable, since health and ill health are just 'mirrors' of one another. The CI does not have this mirror property. Invariance to measurement scale is another desirable requirement put forward by Erreygers (2009a). While the CI and the related Gini index have been developed in the field of income inequality measurement where ratio-scale properties can be taken for granted – multiplying all incomes by a positive number does not affect the value of the Gini index –, health indicators often do not have ratio-scale properties. Van Doorslaer and Jones (2003) and Erreygers (2009a) have shown that the CI is invariant to proportional changes; but not to positive linear transformations that would be needed for cardinal health variables such as the Health Utility Index (Feeny *et al.* 2002, Furlong *et al.* 2001). Whereas the standard CI fails the mirror and cardinal invariance tests, both the 'corrected' CI proposed by Erreygers (2009a) and the generalized version of the modified CI proposed by Wagstaff (2005) pass the two tests.

Additional value judgements, recently discussed in this journal by Wagstaff (2009) and Erreygers (2009a, 2009b), are needed to discriminate between the two.

The purpose of this paper is threefold. First, we focus on the importance of accounting for measurement scale, and develop guidelines that discriminate between various rank-dependent inequality indices based on the measurement scale and the properties of the underlying health variable (and by extension health care and expenditure indicators). Second, we discuss and prove several of the properties of rank-dependent indices for bounded variables that were presented in Erreygers (2009a, 2009b), and provide – if possible – the corresponding properties for unbounded variables. Third, we revisit the discussion of Erreygers and Wagstaff and show its relation with the long-lasting debate in epidemiology (and health economics) on measuring absolute versus relative health differences (among others Wagstaff *et al.*, 1991; Mackenbach and Kunst, 1997; Oliver *et al.*, 2002; Regidor, 2004; Avendano Pabon, 2006; Harper and Lynch, 2007; Mackenbach *et al.*, 2008; Regidor *et al.*, 2009; Harper *et al.*, 2010). We show that a lot of confusion derives from defining absolute and relative inequality on the raw indicators. Following Erreygers (2009a, 2009b), (i) we make the case for transforming health indicators into a 'standardized representation' before defining absolute and relative inequalities, and (ii) we stress the impossibility of measuring relative inequalities only while accepting at the same time that health inequality should be the mirror image of ill health inequality.

The remainder of this paper is organised as follows. In the next section, we focus on the measurement scales of health variables. The third section introduces the class of rank-dependent inequality indices, and section 4 discusses the properties and conditions needed to narrow down this class of indices to specific indices. The fifth section discusses the implications of these properties and the measurement scale of health variables for the usefulness of the specific rank-dependent inequality indices. The final section concludes.

2. Properties and measurement scale of variables

We consider a given population of n individuals and assign to each individual $i = 1, 2, \ldots, n$ a rank λ_i based upon this person's socioeconomic position, with the least well-off individual ranked first and the best well-off ranked last. The health level of individual i is represented by the health variable h_i, a real number. The vector $h = (h_1, h_2, ..., h_n)$ represents the health situation of the whole population. We assume that a higher value of h_i indicates a better health situation of individual i, and denote the average health of the population as μ_h.

Health variables can be measured on different scales (Roberts, 1979), i.e. the level of measurement can be:

- *nominal*, implying that one can classify individuals without being able to order;
- *ordinal*, allowing to order individuals, but with the differences between individuals being meaningless;
- *cardinal*, meaning that differences between individuals make sense, but ratios not, such that the zero point is fixed arbitrarily;
- *ratio-scale*, involving that ratios between individuals have meaning and the zero point corresponding to a situation of complete absence, such that the measurement scale is unique up to a proportional scaling factor;
- *absolute*, requiring that the measurement scale is unique (or fixed or absolute) with the zero point corresponding to a situation of complete absence.

In addition, the range of the variable can be either bounded or unbounded. The range of a variable is characterized by its lower bound a_h and its upper bound b_h. A bounded variable has both a finite lower and a finite upper bound. An unbounded variable, by contrast, has at least one infinite bound. In this paper, when we deal with unbounded variables we always assume that they have an infinite upper bound and a finite lower bound.[1] With regard to unbounded ratio-scale and absolute health variables, we adopt the assumption that their lower bound a_h is zero, which means that these variables take nonnegative values only.

Given a bounded health variable, we can construct a corresponding ill health variable by calculating the shortfall with regard to the maximum. Starting from the (good) health variable h_i we define the ill health variable s_i by the following transformation:

$$s_i \equiv b_h - h_i \qquad (1)$$

The vector $s = (s_1, s_2, ..., s_n)$ represents the ill health situation of the population as a whole. The ill health variable is, of course, also bounded, since it has finite lower and upper bounds $a_s = 0$ and $b_s = (b_h - a_h)$. Observe moreover that the averages of the two variables are related to one another by the formula $\mu_s = b_h - \mu_h$.

[Table 1 about here]

Table 1 gives an overview of the different possibilities along the axes 'measurement scale' versus bounded or unbounded, and provides example(s) of variables encountered in

[1] Variables with an infinite lower bound and a finite upper bound can be treated similarly. Variables with infinite lower and upper bounds are of a different character.

health economics studies. While most examples speak for themselves, some require more explanation. First, type of illness gets both the label bounded and unbounded. In general, one might assume there is an unlimited amount of deficiencies, but if one studies specific health conditions, these are likely to be caused by a limited amount of illnesses only. A similar argument is used for the number of illnesses in the cell (absolute, unbounded). Second, as far as we know, there are no examples of ordinal unbounded health variables in health economics, but for completeness we give the example of an ordinal utility function. Third, van Doorslaer and Jones (2003) have used the predicted linear index of an ordered probit model of categorical self-reported health responses as an indicator of an individual's health status. Due to the properties of the ordered probit model, this linear index is unbounded and its zero point is arbitrarily fixed at zero. Fourth, the examples in the rows 'ratio-scale' and 'absolute' show that $a_h = 0$ appears to be a common-sense assumption: one rarely encounters negative weight/length or health care expenditures.

We discuss the consequences of these different possibilities for inequality measurement in more detail in section 5, but it is worth pointing out here that rank-dependent inequality indices cannot be applied to nominal and ordinal health indicators since nominal and ordinal measurement scales do not allow differences between individuals to be compared. This might seem at odds with the large amount of health inequality studies using indicators of the categorical (e.g. self-assessed health) or the binary type (with 0 indicating the absence and 1 the presence of a certain condition, e.g. immunization against measles). If, however, such a variable can be transformed into or proxied by a cardinal variable, it becomes possible to compare these health differences.[2] From now on we assume cardinal, ratio-scale or absolute measurement scales; we return to nominal and ordinal scales in section 5.

3. A family of rank-dependent inequality indices

In this section, we present the rank-dependent inequality indices that have been used in the literature and define an encompassing family, separately for bounded and unbounded variables. By defining desirable properties on this family, we arrive in subsequent sections at the value judgements underlying each of these indices.

We start with the most popular rank-dependent inequality index, the health CI $C(h)$:

[2] For example, van Doorslaer and Jones (2003) have projected the ordinal self-assessed health categories upon the cardinal HUI-scale. In case of a binary 0/1 indicator, one might overcome the ordinal nature by assuming that it expresses the presence of a certain condition in percentage points, i.e. 100% or 0%. While this seems somewhat implausible at the individual level, it makes sense at the aggregate level (e.g. percentiles).

$$C(h) = \frac{2}{n^2 \mu_h} \sum_{i=1}^{n} z_i h_i \tag{2a}$$

Eq. (2a) clarifies that the health Concentration Index is a normalized sum of weighted health levels, with the weights being determined by the socioeconomic ranks, i.e. $z_i = \lambda_i - \left[(n+1)/2\right]$ (Kakwani, 1980: 173-178; Wagstaff *et al.*, 1991; Erreygers, 2009a). Following Shorrocks (1983), Wagstaff *et al.* (1991) and Clarke *et al.* (2002) defined the related Generalized health CI $V(h)$:

$$V(h) = \frac{2}{n^2} \sum_{i=1}^{n} z_i h_i \tag{3a}$$

When the health variable is bounded, we can by analogy define the ill health CI $C(s)$ and the Generalized ill health CI $V(s)$ as:

$$C(s) = \frac{2}{n^2 \mu_s} \sum_{i=1}^{n} z_i s_i \tag{2b}$$

$$V(s) = \frac{2}{n^2} \sum_{i=1}^{n} z_i s_i \tag{3b}$$

In addition, Wagstaff (2005) and Erreygers (2009a) have developed the Wagstaff Index $W(x)$ and the Erreygers Index $E(x)$, which are indices that can only be applied to bounded health variables. These can be expressed as follows:

$$W(x) = \frac{2(b_x - a_x)}{n^2 (b_x - \mu_x)(\mu_x - a_x)} \sum_{i=1}^{n} z_i x_i \tag{4a}$$

$$E(x) = \frac{8}{n^2 (b_x - a_x)} \sum_{i=1}^{n} z_i x_i \tag{5a}$$

where $x = h, s$.

Equations (2)-(5a) are all variants of a general expression which differ only with respect to the normalization applied to the weighted sum of health (c.q. ill health) levels. Following Erreygers (2009a), we define the family of rank-dependent indices by the expression:

$$I(x) = f(a_x, b_x, \mu_x, n) \sum_{i=1}^{n} z_i x_i \tag{6a}$$

where $f(.)$ is a continuous function. For unbounded variables, for which we have $b_x = +\infty$, this function can be simplified to $f(a_x, \mu_x, n)$.

4. Desirable properties of rank-dependent inequality indices

Erreygers (2009a) started on purpose with a very general form of the function $f(.)$ in (6a). As explained before, the idea is to make it more specific by looking at a number of desirable properties of the index $I(x)$. Several of these properties for *bounded* variables have been described in Erreygers (2009a, 2009b), but some remain to be proven formally. A first goal of this section therefore is to further discuss and prove the latter properties (for readability, all proofs are provided in the appendix). Secondly, we provide – if possible – the corresponding properties for *unbounded* variables. We start by describing the Sign Condition and Scale Invariance which are relevant for bounded *and* unbounded variables. Next, we cover the Mirror property, Absolute and Relative Inequality for bounded variables, the Convergence property, and Linearity.

4.1. Sign Condition

By convention, positive values of a rank-dependent index are seen as signs of a pro-rich bias in the distribution, negative values as signs of a pro-poor bias, and zero as typical for a distribution which is neither pro-rich nor pro-poor. In order to distinguish these cases, we have to identify the situations in which we want $I(x)$ to indicate that there is no systematic bias in favour of either the rich or the poor. Erreygers (2009a) did so by imposing $f(.) > 0$. It makes sense to impose a requirement on $f(.)$ only since the sign of $\sum_{i=1}^{n} z_i x_i$ is invariant to positive linear transformations, i.e. $\sum_{i=1}^{n} z_i(\alpha + \beta x_i) = \beta \sum_{i=1}^{n} z_i x_i$, where $\beta > 0$ (see also the requirement of Scale Invariance in the next section). Here we rationalize and further underpin this assumption.

Sign Condition: The sign of $I(x)$ coincides with the sign of $\sum_{i=1}^{n} z_i x_i$.

From (6a) it is obvious that $\sum_{i=1}^{n} z_i x_i = 0$ leads to $I(x) = 0$, irrespective of the value of $f(.)$. This may correspond to a situation where all persons have the same level of health (or ill health), or to a situation where differences between persons are not systematically in favour of either the rich or the poor. It is however much more doubtful whether $I(x) = 0$ should be extended to situations in which $\sum_{i=1}^{n} z_i x_i \neq 0$. Given a_x, b_x and n, the value of $f(.)$ is determined exclusively by μ_x. We can think of no good reason why for some specific value μ_x we should have $f(.) = 0$.[3] If that were the case, for any distribution x with this specific mean we would always have $I(x) = 0$. But clearly some of these distributions would be entirely pro-rich and others entirely pro-poor, so that for these we should have $I(x) \neq 0$. The Sign Condition obviously leads to the following proposition:

Proposition 1 *A rank-dependent index $I(x)$ satisfies the Sign Condition if and only if:*

(i) $f(a_x, \mu_x, n) > 0$ for $n > 0$ and $a_x < \mu_x < +\infty$ when x is an unbounded variable;

(ii) $f(a_x, b_x, \mu_x, n) > 0$ for $n > 0$ and $a_x < \mu_x < b_x$, when x is a bounded variable.

It is easily checked that all the above mentioned indices satisfy the Sign Condition in the unbounded and/or bounded case, with the exception of $C(x)$, which runs into trouble when $a_x < \mu_x < 0$; in other words, it is not applicable to variables with a cardinal measurement scale. There is, however, an easy way around this problem. If x is a cardinal variable, the following modified version of the CI does satisfy the Sign Condition:

$$\hat{C}(x) = \frac{2}{n^2(\mu_x - a_x)} \sum_{i=1}^{n} z_i x_i \tag{7a}$$

4.2. Scale Invariance

The requirement of Scale Invariance is that we want our index to be independent of the unit of measurement of health (or ill health). We consider both cardinal and ratio-scale, and bounded and unbounded variables, but not absolute variables since these have a fixed unit of measurement. Erreygers (2009b) discussed this requirement extensively for bounded variables. Therefore, we here only define the requirement, and derive the consequences for the set of admissible indices that belong to the family of rank-dependent inequality indices defined by (6a). For unbounded variables we give some additional discussion.

[3] An exception must be made for the cases $\mu_x = a_x$ and $\mu_x = b_x$, since these can occur only if all individuals have the same level x_i, which implies that $\sum_{i=1}^{n} z_i x_i = 0$.

Scale Invariance: Consider a change in the unit of measurement which transforms the distribution x into $\tilde{x}$, the lower bound a_x into $\tilde{a}_x$, and the upper bound b_x into $\tilde{b}_x$, either by means of a positive linear transformation (i.e. $\tilde{x}_i = \alpha + \beta x_i$, $\tilde{a}_x = \alpha + \beta a_x$ and $\tilde{b}_x = \alpha + \beta b_x$, where $\beta > 0$), when x is a cardinal variable, or by means of a positive proportional transformation (i.e. $\tilde{x}_i = \beta x_i$, $\tilde{a}_x = \beta a_x$ and $\tilde{b}_x = \beta b_x$, where $\beta > 0$), when x is a ratio-scale variable. This change does not affect the value of the inequality index, i.e. $I(\tilde{x}) = I(x)$.

The requirement of Scale Invariance substantially reduces the set of allowable $f(.)$ expressions.

Proposition 2 *A rank-dependent index $I(x)$ has the Scale Invariance property if and only if:*

(i) $f(a_x, \mu_x, n) = \dfrac{1}{\mu_x - a_x} k(n)$ *when x is an unbounded variable;*

(ii) $f(a_x, b_x, \mu_x, n) = \dfrac{1}{b_x - a_x} g\left(\dfrac{\mu_x - a_x}{b_x - a_x}, n\right)$ *when x is a bounded variable.*

For unbounded variables the set of allowable $f(.)$ expressions seems rather narrow. It is so narrow, in fact, that all that is needed to arrive at a unique index is to choose the desired maximum bounds of the index. If these are fixed at −1 and +1, we are led to the expression $k(n) = 2/n^2$ and therefore to the standard CI defined in (2a) and (2b), when *x* is a ratio-scale variable, or to the modified CI defined in (7a), when *x* is a cardinal variable.[4]

We believe this is a powerful result that might help to clarify the subtleties of measuring inequalities in the health sector when the health variable is unbounded (see table 1 for some examples). We ended section 2 by stressing that rank-dependent indices cannot be applied to variables measured with a nominal or ordinal scale since differences between individuals are meaningless in those cases. Based on the Sign Condition we excluded the application of the standard CI to unbounded cardinal variables. Taking into account Scale Invariance we now find that the standard CI emerges as the natural candidate when dealing

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[4] Strictly speaking, the correct expression is $k(n) = \dfrac{2}{n(n-1)}$. The bounds of the standard and modified CI are $(1-n)/n$ and $(n-1)/n$.

with unbounded ratio-scale variables, and the modified CI when dealing with unbounded cardinal variables.

While Scale Invariance manages to sufficiently reduce the family of rank-dependent inequality indices for unbounded variables, it leaves more room for bounded variables. It turns out that of the five rank-dependent indices introduced above, only three have the property of Scale Invariance for bounded variables, irrespective of whether the variable has cardinal or ratio-scale: the modified Concentration Index, the Wagstaff Index and the Erreygers Index.[5]

In the subsequent sections, we dig deeper into the indices for bounded variables. Since we believe that the Sign Condition and Scale Invariance are indispensable for a rank-dependent inequality index, we restrict attention to the modified Concentration Index, the Wagstaff Index and the Erreygers Index. Before introducing additional conditions that reduce the set of allowable $f(.)$ expressions for bounded variables, we highlight an expositional advantage of the condition of Scale Invariance. Since Scale Invariance allows the unit of measurement to be chosen freely, we find it more convenient to represent the health variable (and likewise the ill health variable) in the form suggested by condition (ii) of Proposition 2. The *standardized representation* of a health variable h_i varying between a_h and b_h is equal to h_i^* which is defined as:

$$h_i^* \equiv \frac{h_i - a_h}{b_h - a_h} \tag{8}$$

This variable is bounded between $a_{h*} = 0$ and $b_{h*} = 1$. The corresponding standardized ill health variable is then $s_i^* \equiv (b_h - h_i)/(b_h - a_h) = 1 - h_i^*$. These standardized variables can be interpreted as the net health and ill health ratios, because they measure the distance from respectively the lower and the upper bounds, and express it as a proportion of the maximum distance. Observe furthermore that $\mu_{h*} = (\mu_h - a_h)/(b_h - a_h)$ and $\mu_{s*} = (b_h - \mu_h)/(b_h - a_h) = 1 - \mu_{h*}$.

With these standardized notations, the expressions for the modified Concentration Index, the Wagstaff Index and the Erreygers Index can be simplified. In fact we have:

$$\hat{C}(x^*) = \frac{2}{n^2 \mu_{x*}} \sum_{i=1}^{n} z_i x_i^* \tag{7b}$$

[5] The function $g(\gamma, n)$ – with $\gamma = (\mu_x - a_x)/(b_x - a_x)$ – equals $2/(n^2\gamma)$ for the modified CI, $2/\left[n^2\gamma(1-\gamma)\right]$ for the Wagstaff index and $8\gamma^0/n^2$ for the Erreygers index.

$$W\left(x^*\right) = \frac{2}{n^2 \mu_{x^*}\left(1-\mu_{x^*}\right)} \sum_{i=1}^{n} z_i x_i^* \tag{4b}$$

$$E\left(x^*\right) = \frac{8}{n^2} \sum_{i=1}^{n} z_i x_i^* \tag{5b}$$

The standardized general expression for scale-invariant rank-dependent indices for bounded variables is:

$$I\left(x^*\right) = g\left(\mu_{x^*}, n\right) \sum_{i=1}^{n} z_i x_i^* \tag{6b}$$

4.3. The Mirror property

A third requirement introduced and discussed by Erreygers (2009a, 2009b) is that we would like the health index to be the reflection of the ill health index. This Mirror property is formally defined as follows:

Mirror: Let h be a given health distribution and s its associated ill health distribution. Then the health index $I(h)$ and the ill health index are equal in absolute value but have opposite signs, i.e. $I(h) = -I(s)$.

For the class of scale-invariant indices in (6b), the Mirror property imposes further structure upon the normalization function $g\left(\mu_{x^*}, n\right)$:

Proposition 3 *A scale-invariant rank-dependent index $I\left(x^*\right)$ has the Mirror property if and only if $g\left(\mu_{x^*}, n\right) = g\left(1-\mu_{x^*}, n\right)$.*

In mathematical terms, the function $g\left(\mu_{x^*}, n\right)$ must be symmetrical around $\mu_{x^*} = 0.5$ for a given value of n. Both the Wagstaff Index and the Erreygers Index have this property, as can be seen from (4b) and (5b). The modified CI does not have this property, i.e. for $\mu_{x^*} \neq 0.5$ $g\left(\mu_{x^*}, n\right) = 2\left(n^2 \mu_{x^*}\right)^{-1} \neq 2\left[n^2\left(1-\mu_{x^*}\right)\right]^{-1} = g\left(1-\mu_{x^*}, n\right)$ (see (7b)).

It turns out that the Mirror property is closely related to the elasticity of the normalization function with respect to the mean. Let $\varepsilon\left(\mu_{x^*}\right) \equiv \frac{\partial g\left(\mu_{x^*}, n\right)}{\partial \mu_{x^*}} \frac{\mu_{x^*}}{g\left(\mu_{x^*}, n\right)}$ denote the elasticity of the normalization function $g\left(\mu_{x^*}, n\right)$ with respect to the mean μ_{x^*}. The Mirror property restricts the set of normalization functions rather drastically.

Proposition 4 *For a scale-invariant rank-dependent index $I\left(x^*\right)$ the Mirror property holds if and only if* $\varepsilon\left(1-\mu_{x^*}\right) = -\frac{\left(1-\mu_{x^*}\right)}{\mu_{x^*}} \varepsilon\left(\mu_{x^*}\right)$.

This result enables us to confirm our earlier finding that the Wagstaff and Erreygers index satisfy the Mirror property, while the modified Concentration Index does not. For the Wagstaff Index we have $\varepsilon\left(\mu_{x^*}\right) = -\frac{\left(1-2\mu_{x^*}\right)}{\left(1-\mu_{x^*}\right)}$ and $\varepsilon\left(1-\mu_{x^*}\right) = \frac{\left(1-2\mu_{x^*}\right)}{\mu_{x^*}}$, and so the condition of Proposition 4 is satisfied. The same holds for the Erreygers Index, for which we have $\varepsilon\left(\mu_{x^*}\right) = \varepsilon\left(1-\mu_{x^*}\right) = 0$. For the modified CI, by contrast, we have $\varepsilon\left(\mu_{x^*}\right) = \varepsilon\left(1-\mu_{x^*}\right) = -1$ and so the condition cannot be satisfied. In subsequent sections, we will show that the value of the elasticity of the normalization function turns out to be crucial to further characterize the properties of the Wagstaff and Erreygers Index.

Finally, the Mirror property allows to define a parametric class of indices which all have the properties of Scale Invariance and Mirror[6]:

$$C^{\theta}\left(x^*\right) = \frac{8}{n^2\left[4\mu_{x^*}\left(1-\mu_{x^*}\right)\right]^{\theta}} \sum_{i=1}^{n} z_i x_i^* \qquad (9)$$

It encompasses the Wagstaff Index for $\theta = 1$, and the Erreygers Index for $\theta = 0$.

4.4. Absolute and relative inequality for bounded variables

Wagstaff (2009) and Erreygers (2009b) discussed the concepts of absolute and relative inequality extensively. One of the major insights emerging from their discussion is that one cannot impose the meaning of absolute and relative inequality – in the way these have been

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[6] More generally, any positive-valued continuous function $g\left(\psi, n\right)$, where $\psi = \mu_{x^*}\left(1-\mu_{x^*}\right)$, can be used to define an index which satisfies the Scale Invariance and Mirror properties.

understood for unbounded ratio-scale variables – upon bounded variables. More exactly, Erreygers (2009b) has claimed (i) that "depending on whether you look at one side (health) or the other (ill health), the same change may be seen as relative inequality preserving or as relative inequality changing" (ibidem: 522), (ii) that "it is impossible to construct an index which is sensitive to relative inequality changes only *and* satisfies the mirror condition" (ibidem), and (iii) that "in the case of a bounded variable, ... the notion of a proportional change becomes ambiguous, since what is seen as a proportional change in attainment levels is mirrored by a non-proportional change in shortfall levels" (ibidem). Our purpose here is to explore these issues in a more formal way.

It may be useful to briefly recall the notions of relative and absolute inequality indices in the case of unbounded ratio-scale variables, such as income. A relative inequality index is an index of which the value remains constant for any proportional change of all individual levels. For unbounded ratio-scale variables, the notion of a relative inequality index therefore coincides with that of a scale-invariant index. An absolute inequality index, by contrast, is characterized by the property of translation-invariance: adding the same amount to everyone's level leaves the value of the inequality index unchanged. The difficulty when trying to apply these notions to bounded variables is that some of the changes are infeasible, because the bounds of the variables act as constraints. This means that we have to adapt the definitions, and to make the distinction clear we will for bounded variables refer to 'quasi-relative' and 'quasi-absolute' indices.

Since we are interested in scale-invariant indices only, we adopt the previously introduced standardized representation of health and ill health variables. This representation has the advantage of recording the 'real' changes in health and ill health, but not the 'nominal' ones which are purely the effect of moving from one unit of measurement to another.[7] A scale-invariant rank-dependent health (c.q. ill health) inequality index is said to be quasi-relative if it is insensitive to any *feasible* proportional change of all standardized health (c.q. ill health) levels. In other words, a quasi-relative index takes into account only the relative positions of individuals, not the absolute differences among persons.[8]. Formally, we have:

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[7] A change from x_i^* to rx_i^* in standardized terms corresponds to a change from level x_i to $rx_i+(1-r)a_x$.

[8] A feasible move from x_i^* to rx_i^* ($r \neq 0, r \neq 1$) for all individuals does not change the relative positions ($x_i^*/x_j^* = rx_i^*/rx_j^*$), but changes the absolute differences ($\left|x_i^*-x_j^*\right| \neq \left|rx_i^*-rx_j^*\right|$ for $x_i^* \neq x_j^*$).

Quasi-Relativity: Consider a change from x^* to y^*, where $y_i^* = rx_i^*$ and $r \neq 0$. If for any feasible x^* and y^* we have $I\left(y^*\right) = I\left(x^*\right)$, then $I\left(x^*\right)$ is a quasi-relative index.

By analogy we define the notion of a quasi-absolute index:

Quasi-Absoluteness: Consider a change from x^* to y^*, where $y_i^* = x_i^* + \Delta$ and $\Delta \neq 0$. If for any feasible x^* and y^* we have $I\left(y^*\right) = I\left(x^*\right)$, then $I\left(x^*\right)$ is a quasi-absolute index.

Note that for bounded variables the property of Quasi-Relativity is independent of the property of scale-invariance. The property of Quasi-Absoluteness coincides with the property of Level Independence (Erreygers, 2009a). We now show a number of formal results on quasi-relative and quasi-absolute indices.

Proposition 5 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *is quasi-relative if and only if* $\varepsilon\left(\mu_{x^*}\right) = -1$.

Proposition 6 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *is quasi-absolute if and only if* $\varepsilon\left(\mu_{x^*}\right) = 0$.

Propositions 4, 5 and 6 suggest that the elasticity $\varepsilon\left(\mu_{x^*}\right)$ of the normalization function $g\left(\mu_{x^*}, n\right)$ with respect to the mean μ_{x^*} can be used to construct a measure of an index's sensitivity to (quasi-)relative and (quasi-)absolute inequality. We define the weight which an index gives to relative inequality as $-\varepsilon\left(\mu_{x^*}\right)$ and the weight which it gives to absolute inequality as $1+\varepsilon\left(\mu_{x^*}\right)$. Indices for which both weights are positive are called 'mixed inequality' indices. Indices for which the relative inequality weight is negative and the absolute inequality weight greater than 1 are called 'inverse-relative' indices. These indices have the property of increasing in magnitude when a change occurs which leaves all absolute differences the same and decreases all relative differences. Likewise, indices for which the absolute inequality weight is negative and the relative inequality weight greater than 1 are called 'inverse-absolute' indices. Such indices have the property of increasing in magnitude when a change occurs which leaves all relative differences the same and decreases all absolute differences.

Intuitively, it seems hard to make a case for inverse-relative or inverse-absolute indices, since they move in opposite directions from what we expect. Our choice is therefore confined to mixed inequality indices, with the quasi-relative and quasi-absolute indices as limit cases. It is easy to verify that of the three scale-invariant rank-dependent indices which we considered previously, only the modified CI is a quasi-relative index. We have already shown that the modified CI fails the Mirror property. This is in fact not a coincidence, but a general result:

Proposition 7 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *cannot simultaneously have the properties of Mirror and Quasi-Relativity.*

In other words, if we want both properties of Scale Invariance and Mirror, we have to leave the quasi-relative indices out of consideration. It remains to be seen whether the other indices perform any better. The following result reduces the possibilities even further.

Proposition 8 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *which has the Mirror property can be of the mixed inequality type for only half of the distributions.*

The reason is that when such an index is of the mixed inequality type for distributions with mean μ_{x^*}, it turns out to be inverse-relative for distributions with mean $1-\mu_{x^*}$. This can be illustrated by considering the class of parametric rank-dependent indices defined by (9), which satisfy both Scale Invariance and Mirror. For these indices we have $\varepsilon\left(\mu_{x^*}\right) = -\theta\dfrac{\left(1-2\mu_{x^*}\right)}{\left(1-\mu_{x^*}\right)}$. The index can be a mixed inequality index only if $\theta \neq 0$. But if $\theta > 0$, the index is inverse-relative for distributions with mean $\mu_{x^*} > ½$, and if $\theta < 0$ for distributions with mean $\mu_{x^*} < ½$.[9]

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[9] Moreover, if $\theta > 1$ the index is inverse-absolute for distributions with mean $\mu_{x^*} < \dfrac{1}{2} - \dfrac{1}{2(2\theta-1)}$, and if $\theta < 0$

Having ruled out quasi-relative and mixed inequality indices, we are left with only one option: quasi-absolute indices. Here, at last, no further difficulties arise:

Proposition 9 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *which has the Mirror property and is never inverse-relative or inverse-absolute must be a quasi-absolute index.*

Of our three rank-dependent indices, the Erreygers index is the only one which is quasi-absolute. Since it also has the properties of Scale Invariance and Mirror, this shows that Proposition 9 does not refer to an empty class.

4.5. Discussion

In the previous section we have shown that both the modified CI and the Wagstaff index must be ruled out, the first because it violates the Mirror condition, and the second because it is inverse-relative for distributions with a mean $\mu_{x^*} > \frac{1}{2}$. Whereas the violation of the Mirror condition seems rather obvious, an example may be useful to clarify when the Wagstaff index is inverse-relative. Suppose we start from the distribution $x^* = (0.5, 0.6, 0.7)$, where person 1 is the least well-off and person 3 the best well-off. Since we have $n = 3$, $\mu_{x^*} = 0.6$ and $\sum_{i=1}^{3} z_i x_i^* = 0.2$, it follows that $\hat{C}\left(x^*\right) = 2/27$, $W\left(x^*\right) = 5/27$ and $E\left(x^*\right) = 8/45$. Now let us add $\Delta = 0.2$ to everyone's level, so that the distribution becomes $y^* = (0.7, 0.8, 0.9)$. All absolute differences remain the same, and all relative differences decrease: in relative terms persons 1 and 2 come closer to one another (they move from ratio 5/6 to ratio 7/8), but so do persons 1 and 3 (from 5/7 to 7/9) and persons 2 and 3 (from 6/7 to 8/9). Since we have $n = 3$, $\mu_{y^*} = 0.8$ and $\sum_{i=1}^{3} z_i y_i^* = 0.2$, it follows that $\hat{C}\left(y^*\right) = 1/18$, $W\left(y^*\right) = 5/18$ and $E\left(y^*\right) = 8/45$. In other words, the modified CI decreases, the Wagstaff index increases, and the Erreygers index remains the same.

Although the case for the Erreygers index should be clear by now, additional arguments can be advanced in its favour. Erreygers (2009b) has argued that one should look also at what happens when a given distribution is gradually reduced to a perfectly equal distribution. More particularly, he suggests to look at the limit value of an index when all

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for distributions with mean $\mu_{x^*} > \dfrac{1}{2} + \dfrac{1}{2(1-2\theta)}$.

individual levels are reduced to zero by means of a proportional reduction. When all individuals have zero, the distribution is equal and so the value of the index should tend to zero. This is what we call Convergence:

Convergence: Let x^* be a given distribution. Consider a change which reduces everyone's position from x_i^* to rx_i^*. Then $\lim_{r \to 0} I\left(rx^*\right) = 0$.

The following result relates the property of Convergence to the normalization function.

Proposition 10 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *has the property of Convergence if and only if* $\lim_{\mu_{x^*} \to 0} g\left(\mu_{x^*}, n\right)\mu_{x^*} = 0$.

It turns out that the Wagstaff Index does not have the property of Convergence. For the parametric class of indices defined by (9) we have $\lim_{\mu_{x^*} \to 0} g\left(\mu_{x^*}, n\right)\mu_{x^*} = \lim_{\mu_{x^*} \to 0} \frac{8\left(\mu_{x^*}\right)^{1-\theta}}{n^2\left(4 - 4\mu_{x^*}\right)^{\theta}}$. If $\theta < 1$, the limit value is zero; if $\theta = 1$, however, it is equal to $2/n^2 > 0$, and if $\theta > 1$, it is equal to $+\infty$. The reason why the Wagstaff Index does not have the property of Convergence is that the index ceases to attach any weight to absolute inequality when the value of μ_{x^*} approaches 0 (in other words, for the Wagstaff index we have $\varepsilon\left(0\right) = -1$).

The Convergence property looks at the limit value of the index when there is an equiproportional reduction of all individual levels. Even if the limit value of an index tends to 0, many different types of trajectories remain possible. Erreygers (2009b) has argued that a linear trajectory is obvious and simple.

Linearity: Let x^* be a given distribution. Consider a change which reduces everyone's position from x_i^* to rx_i^*, with $0 \leq r < 1$. Then we have: $I\left(rx^*\right) = rI\left(x^*\right)$.

Proposition 11 *A scale-invariant rank-dependent index* $I\left(x^*\right)$ *has the property of Linearity if and only if* $g\left(\mu_{x^*}, n\right) = k\left(n\right)$.

With regard to the parametric class of indices defined by (9), the condition of Proposition 11 holds only for $\theta = 0$. Hence, the Erreygers index has the property of Linearity, but not the Wagstaff index.

5. Properties and measurement scale of health variables and implications for the class of rank-dependent inequality indices

We have previously discussed the measurement scales of health variables and presented an overview with examples in table 1. The purpose of this section is to link these measurement scales to the properties discussed in the previous sections in order to provide guidelines to discriminate between the rank-dependent inequality indices. These guidelines are summarized in table 2.

[table 2 about here]

Recall that we have differentiated health variables (and by extension health care and expenditure indicators) along two axes, i.e. (i) health variables can be bounded or unbounded, and (ii) the measurement scale can be nominal, ordinal, cardinal, ratio-scale, or absolute. We explained in section 2 that rank-dependent inequality indices cannot be applied to nominal and ordinal health indicators since differences between individuals are meaningless with those measurement scales. More generally, we do not see how one could meaningfully measure inequalities in nominal health indicators (using rank-dependent indices, other classes of indices, dominance relations or other measurement frameworks) as these indicators do not rank health states. By contrast, ordinal health indicators might be subjected to inequality measurement. While rank-dependent inequality indices cannot be applied directly to the raw ordinal health indicators, applied health inequality researchers have projected cardinal scales upon these ordinal health indicators, circumventing the meaninglessness of ordinal health differences (for example, van Doorslaer and Jones, 2003; see also section 2). Although in the health inequality literature the projections have until now always generated cardinally scaled variables, they can in principle also lead to ratio-scaled variables. This method solves the incompatibility between rank-dependent inequality indices and ordinal health differences, but that does not mean that any rank-dependent inequality index can be used. In view of our findings on Scale Invariance in section 4.2, one should apply the modified CI when the projection gives rise to an unbounded cardinal variable, the CI when the projection leads to an unbounded ratio-scale variable, and any member of the class of indices defined in (6b) when the projection generates a bounded cardinal or ratio-scale variable.[10]

The property of Scale Invariance is in our view indispensable as it makes the value of an index independent of the specific unit in which a variable is expressed. Scale-invariant indices allow to compare inequality levels of variables with different cardinalizations (for example inequality in health care expenditures versus inequality in the health utility index). For unbounded variables, Scale Invariance has important implications for the kind of inequality that can be measured. If the variable is of the cardinal or ratio-scale type, one can only meaningfully measure relative inequalities. Measuring absolute inequalities using the Generalized CI is only feasible for unbounded indicators with an absolute scaling.

For bounded variables, whether they be of the cardinal or of the ratio-scale type, the choice narrows down to the class of indices defined by (6b). This class includes the modified CI, which is not mentioned in table 2 since it does not satisfy the Mirror property. The Wagstaff and Erreygers Indices belong to the subset of this class containing the indices satisfying the Mirror property. What index to choose from this subset depends on one's judgements on how the rank-dependent inequality index should react to changes in average *standardized* health (or ill health).[11] We have presented two equivalent ways to characterize the properties of this subset of indices.

The first starts from the concepts of absolute and relative inequality. These concepts have a well-known meaning for unbounded variables such as income. Relative inequality indices are insensitive to equiproportional changes, whereas absolute inequality indices are insensitive to equal additions. However, these concepts are difficult to apply to bounded variables since for many distributions equiproportional changes or equal additions are infeasible due to the bounds of the variables acting as constraints. As a result, we have introduced the notions of quasi-relative and quasi-absolute inequality which state that a rank-dependent inequality index should be insensitive to any *feasible* equiproportional change (Quasi-Relativity) or any *feasible* equal addition (Quasi-Absoluteness) of the *standardized* variables. It is essential that these notions are stated in terms of the standardized

[10] Alternatively, one could desert the class of rank-dependent inequality indices and resort to other measurement approaches that take the ordinal nature of the health indicators explicitly into account. Allison and Foster (2004) have developed a median-based partial inequality ordering for ordinal health variables for the measurement of pure health inequalities. Abul Naga and Yalcin (2008) have extended this approach to inequality indices (see Madden (2009) for an application). Apouey (2007) has proposed a class of health polarization indices that is also median-based. Erreygers (2009c) has explored an Atkinson approach to the measurement of socioeconomic health inequalities, and Zheng (2006) has developed dominance conditions to evaluate bivariate health-income distributions that are free from any cardinal valuation of health status.

[11] Note that this standardized representation is also an easy way out if individuals have different minimum and maximum values of the health indicator (for example, it is common to use different threshold levels for the BMI of males and females).

representation in order to rule out the impact of nominal changes which are purely the effect of moving from one unit of measurement to another. We have shown in section 4.4 that the indices belonging to the class defined by (6b) cannot simultaneously have the properties of Mirror and Quasi-Relativity. Since Mirror is in our view an essential property, this rules out quasi-relative rank-dependent indices for bounded variables. We have also shown that the Erreygers Index is the only rank-dependent inequality measure that has the properties of Mirror and Quasi-Absoluteness. All other indices belonging to the class defined by (6b) which have the Mirror property, including the Wagstaff index, turn out to be 'inverse-relative' for at least half of the distributions. We believe that these indices are hard to justify since they increase in magnitude when there is a change that decreases relative differences while keeping absolute differences the same. To sum up, in our view 'Quasi-Absoluteness' – and therefore the Erreygers Index – is the only reasonable inequality concept when measuring inequality in bounded variables.

The alternative (and equivalent) characterization of the properties of the subset of indices defined by (6b) was based on the idea that proportional reductions of the *standardized* health (cf. ill health) levels should in the limit reduce the value of the index to zero. The Wagstaff index does not have this property. The Erreygers Index differs from others indices which do have this property in that it moves to this situation of perfect equality in a linear way.

6. Discussion and conclusion

The title of this paper reflects our goal of providing guidelines to applied health inequality researchers on the suitability of various rank-dependent inequality indices when dealing with health, health care, and health expenditure indicators with different properties and measurement scales. In order to come up with these guidelines, we have formally proven some of the properties discussed in Erreygers (2009a-b) and added several new properties.

We differentiated these indicators according to two criteria: (i) whether they are bounded or unbounded and (ii) whether they have a nominal, ordinal, cardinal, ratio-scale or absolute measurement scale. We summarize our guidelines in table 2. Our main conclusions are:

- Inequality measurement is meaningless for nominal variables.
- Rank-dependent inequality indices are in principle meaningless for ordinal variables, but this can be circumvented by projection into a cardinal or ratio- scale variable.
- The most popular rank-dependent inequality index – i.e. the CI that measures relative

inequalities – can be applied only to unbounded variables with ratio- or absolute scale, while another popular variant – the Generalized CI that measures absolute inequalities – can be applied only to unbounded variables with absolute scale. We have also introduced a Modified CI that can be applied to unbounded variables with cardinal scale.

- For bounded variables with cardinal, ratio-, or absolute scale, we have advocated the Erreygers Index which simultaneously satisfies the properties of Mirror, Scale Invariance and Linearity. We have also shown that it is the only index measuring quasi-absolute inequality, i.e. it is insensitive to any *feasible* equal addition to the *standardized* variable. If one does not want to impose this assumption, one could use the other members of the class of scale-invariant rank-dependent indices (including the Wagstaff index) that satisfy the Mirror property. In this case, one implicitly agrees that for some distributions inequality increases in magnitude when there is a ceteris paribus decrease of relative differences, which we think is unacceptable.

In addition to providing guidelines, we highlighted the implications of the new properties derived in this paper (and in the work of Wagstaff (2009) and Erreygers (2009a-b)) for the long-lasting debate in health economics and epidemiology on measuring absolute versus relative health differences. In this literature, it has become common practice to report both absolute and relative health differences as these might rank distributions differently. For example, Mackenbach *et al.* (2008) report that absolute inequalities in overall mortality rates between lower and higher educated are the highest in Estonia, while relative inequalities are the highest in the Czech Republic and Estonia only ranks fifth in relative terms. More generally, it is traditionally conceived that analyzing relative inequalities is conceptually different from analyzing absolute inequalities and that one should analyze each separately. For example, Harper and Lynch (2007) give the example of using the CI for measuring relative inequalities, and the Generalized CI for measuring absolute inequalities in smoking prevalence, which indeed is a bounded variable. If one accepts the Mirror property – and we believe one should – the findings in this paper show however that the only case in which the CI and the Generalized CI should be combined to measure both absolute and relative inequalities is when health is unbounded and measured on an absolute scale. When the (raw) indicator is bounded, there remains no ground for sticking to the absolute/relative dichotomy for two reasons. First, notions of absolute and relative inequality should only be defined for the *standardized* representation of health variables to make sure that one is not 'confused' by mere changes in the measurement scale of the health indicator, and to account for the fact that the bounded nature places restrictions on the feasible relative and absolute changes (for example: individuals with maximum health cannot improve their health level). Second and taking the standardized representation for granted, we have shown that quasi-relative indices are incompatible with the Mirror property, but that quasi-absolute are not. As stated by Erreygers (2009b; 3): "Apparently, many people find it hard to accept that the simple change from an unbounded to a bounded variable can make much of a difference. But it does."

Finally, we have assumed that the minimum and maximum values of the indicator are known. This is a plausible assumption for several indicators (e.g. the number of nights an individual spends in hospital), but might be implausible for others. For example, while it is generally agreed that life expectancy is bounded, it is not exactly clear what the maximum bound might be; and the measured degree of inequality will depend on the value of this maximum bound. A possible way out might be to develop indices that make distributional assumptions on the length of life, much along the lines of Kakwani (1995) who developed a class of poverty measures that takes account of the uncertainty involved in the specification of the poverty line.

Appendix: Proofs of the Propositions

Proposition 2

(i) Suppose that x is a distribution of an unbounded cardinal variable, and consider the positive linear transformation which transforms x into $\tilde{x}$, with $\tilde{x}_i = \alpha + \beta x_i$. Scale invariance means that $I(x) = I(\tilde{x})$ must hold for any x, any α, and any $\beta > 0$. Since $\sum_{i=1}^{n} z_i \tilde{x}_i = \beta \sum_{i=1}^{n} z_i x_i = \beta Z$, the equality $I(x) = I(\tilde{x})$ holds if and only if $f(a_x, \mu_x, n) Z = \beta f(\alpha + \beta a_x, \alpha + \beta \mu_x, n) Z$. Since Z can be positive, negative or zero, we must always have $f(a_x, \mu_x, n) = \beta f(\alpha + \beta a_x, \alpha + \beta \mu_x, n)$. If we take $\beta > 0$ and $\alpha = -\beta a_x$, it follows that we must have $f(a_x, \mu_x, n) = \beta f(0, \beta(\mu_x - a_x), n) = \beta g(\beta(\mu_x - a_x), n)$. This implies that $\beta^{-1} g(\mu_x - a_x, n) = g(\beta(\mu_x - a_x), n)$, i.e. $g(.)$ is homogeneous of degree -1 in $(\mu_x - a_x)$, for n kept fixed. Using Euler's homogeneous function theorem, it follows that $g(\mu_x - a_x, n) = (\mu_x - a_x)^{-1} k(n)$. This establishes the necessity part. The sufficiency part is shown by noting that a positive linear transformation changes $(\mu_x - a_x)^{-1} k(n)$ into $[\beta(\mu_x - a_x)]^{-1} k(n)$, from we which it follows that $I(\tilde{x}) = I(x)$.

If x is a distribution of an unbounded ratio-scale variable, we have $a_x = 0$ and therefore

$f(a_x,\mu_x,n)=g(\mu_x,n)$. Since only positive proportional transformations are allowed, $I(x)=I(\tilde{x})$ now implies $g(\mu_x,n)=\beta g(\beta\mu_x,n)$, i.e. $g(.)$ is homogeneous of degree -1 in μ_x, for n kept fixed. Again applying Euler's homogeneous function theorem, it follows that $g(\mu_x,n)=\mu_x{}^{-1}k(n)$ and necessity is proved. Sufficiency is obvious.

(ii) Suppose that x is a distribution of a bounded cardinal variable. Following the same reasoning as before, we derive that $I(x)=I(\tilde{x})$ implies $f(a_x,b_x,\mu_x,n)=\beta f(\alpha+\beta a_x,\alpha+\beta b_x,\alpha+\beta\mu_x,n)$. For $\alpha=-a_x/(b_x-a_x)$ and $\beta=1/(b_x-a_x)$ we obtain $f(a_x,b_x,\mu_x,n)=(b_x-a_x)^{-1}f\left(0,1,\frac{\mu_x-a_x}{b_x-a_x},n\right)=(b_x-a_x)^{-1}g\left(\frac{\mu_x-a_x}{b_x-a_x},n\right)$. This establishes the necessity part. The sufficiency part can be checked by inspection: a positive linear transformation has no effect on (μ_x-a_x/b_x-a_x), and multiplies the value of b_x-a_x by β. Hence $f(\alpha+\beta a_x,\alpha+\beta b_x,\alpha+\beta\mu_x,n)=f(a_x,b_x,\mu_x,n)/\beta$, and therefore $I(\tilde{x})=I(x)$.

If x is a distribution of a ratio-scale variable, necessity and sufficiency are proved in a similar way.

Proposition 3

For scale-invariant indices the Mirror property holds if and only if $I(x^*)=-I(1-x^*)$ for all x^*. Since $\sum_{i=1}^{n}z_i(1-x_i^*)=-\sum_{i=1}^{n}z_ix_i^*=-Z$ and $\mu_{x^*}=1-\mu_{x^*}$, this holds if and only if $g(\mu_{x^*},n)Z=g(1-\mu_{x^*},n)Z$. Since Z can be positive, negative or zero, we must always have $g(\mu_{x^*},n)=g(1-\mu_{x^*},n)$.

Proposition 4

By definition we have $\varepsilon(1-\mu_{x^*})=\frac{\partial g(1-\mu_{x^*},n)}{\partial(1-\mu_{x^*})}\frac{(1-\mu_{x^*})}{g(1-\mu_{x^*},n)}$. By the rules of derivation we have $\frac{\partial g(1-\mu_{x^*},n)}{\partial\mu_{x^*}}=\frac{\partial g(1-\mu_{x^*},n)}{\partial(1-\mu_{x^*})}\frac{\partial(1-\mu_{x^*})}{\partial\mu_{x^*}}=-\frac{\partial g(1-\mu_{x^*},n)}{\partial(1-\mu_{x^*})}$. Because of the Mirror property we have $g(\mu_{x^*},n)=g(1-\mu_{x^*},n)$, and we derive that $\frac{\partial g(1-\mu_{x^*},n)}{\partial\mu_{x^*}}=\frac{\partial g(\mu_{x^*},n)}{\partial\mu_{x^*}}$.

This means that $\varepsilon\left(1-\mu_{x^*}\right)=-\frac{\partial g\left(\mu_{x^*},n\right)}{\partial\mu_{x^*}}\frac{\left(1-\mu_{x^*}\right)}{g\left(1-\mu_{x^*},n\right)}=-\frac{\partial g\left(\mu_{x^*},n\right)}{\partial\mu_{x^*}}\frac{\mu_{x^*}}{g\left(\mu_{x^*},n\right)}\frac{\left(1-\mu_{x^*}\right)}{\mu_{x^*}}$, and we obtain $\varepsilon\left(1-\mu_{x^*}\right)=-\frac{\left(1-\mu_{x^*}\right)}{\mu_{x^*}}\varepsilon\left(\mu_{x^*}\right)$.

Proposition 5

Since we have $\sum_{i=1}^{n} z_i y_i^* = \sum_{i=1}^{n} z_i (r x_i^*) = r\sum_{i=1}^{n} z_i x_i^* = rZ$ and $\mu_{y^*} = r\mu_{x^*}$, it follows that $I\left(x^*\right)-I\left(y^*\right)=\left[g\left(\mu_{x^*},n\right)-rg\left(r\mu_{x^*},n\right)\right]Z$. The Quasi-Relativity property holds if and only if $I\left(x^*\right)-I\left(y^*\right)=0$ for any feasible x^* and y^*. Since Z can be positive, negative or zero, and $r\mu_{x^*}$ can in principle take any value between 0 and 1, this means that we must have $r^{-1}g\left(\mu_{x^*},n\right)=g\left(r\mu_{x^*},n\right)$. In other words, $g\left(\mu_{x^*},n\right)$ is homogeneous of degree -1 in μ_{x^*}, which means we have $\varepsilon\left(\mu_{x^*}\right)=-1$.

Proposition 6

Since we have $\sum_{i=1}^{n} z_i y_i^* = \sum_{i=1}^{n} z_i (x_i^* + \Delta) = \sum_{i=1}^{n} z_i x_i^* + \Delta\sum_{i=1}^{n} z_i = \sum_{i=1}^{n} z_i x_i^* = Z$ and $\mu_{y^*} = \mu_{x^*} + \Delta$, it follows that $I\left(x^*\right)-I\left(y^*\right)=\left[g\left(\mu_{x^*},n\right)-g\left(\mu_{x^*}+\Delta,n\right)\right]Z$. The Quasi-Absoluteness property holds if and only if $I\left(x^*\right)-I\left(y^*\right)=0$ for any feasible x^* and y^*. Since Z can be positive, negative or zero, and $\mu_{x^*}+\Delta$ can in principle take any value between 0 and 1, this means that $g\left(\mu_{x^*},n\right)$ must be independent of μ_{x^*}. In other words, we have $\varepsilon\left(\mu_{x^*}\right)=0$.

Proposition 7

According to Proposition 5 the Quasi-Relativity property holds if and only if $\varepsilon\left(\mu_{x^*}\right)=\varepsilon\left(1-\mu_{x^*}\right)=-1$. According to Proposition 4 the Mirror property holds if and only if $\varepsilon\left(1-\mu_{x^*}\right)=-\frac{\left(1-\mu_{x^*}\right)}{\mu_{x^*}}\varepsilon\left(\mu_{x^*}\right)$. Hence both conditions can hold simultaneously only if $\left(1-\mu_{x^*}\right)/\mu_{x^*}=-1$, which is impossible.

Proposition 8

Let an index be of the mixed inequality type for some μ_{x^*}, which means that $-1 < \varepsilon\left(\mu_{x^*}\right) < 0$.

If it has the Mirror property, we know from Proposition 4 that $\varepsilon\left(1-\mu_{x^*}\right) = -\frac{\left(1-\mu_{x^*}\right)}{\mu_{x^*}}\varepsilon\left(\mu_{x^*}\right)$.

Hence $\varepsilon\left(1-\mu_{x^*}\right) > 0$, which means that the index is inverse-relative for $1-\mu_{x^*}$.

Proposition 9

An index is never inverse-relative or inverse-absolute if and only if $-1 \leq \varepsilon\left(\mu_{x^*}\right) \leq 0$ for all $0 \leq \mu_{x^*} \leq 1$. From Proposition 8 we know that if $-1 < \varepsilon\left(\mu_{x^*}\right) < 0$ for some μ_{x^*}, then for $1-\mu_{x^*}$ the index will be inverse-relative if it has the Mirror property. If $\varepsilon\left(\mu_{x^*}\right) = -1$ for all $0 \leq \mu_{x^*} \leq 1$, then the index does not have the Mirror property. If $\varepsilon\left(\mu_{x^*}\right) = 0$ for all $0 \leq \mu_{x^*} \leq 1$, then the index has the Mirror property and is never inverse-relative or inverse-absolute.

Proposition 10

Let $y^* = rx^*$, and assume that $\sum_{i=1}^{n} z_i x_i^* = Z \neq 0$. Since $\sum_{i=1}^{n} z_i y_i^* = r\sum_{i=1}^{n} z_i x_i^*$ and $\mu_{y^*} = r\mu_{x^*}$, we have $I\left(rx^*\right) = g\left(r\mu_{x^*}, n\right) rZ = g\left(\mu_{y^*}, n\right)\mu_{y^*}\left(Z/\mu_{x^*}\right)$. Since Z/μ_{x^*} is a constant different from zero, the condition $\lim_{r\to 0} I\left(rx^*\right) = 0$ is equivalent to the condition $\lim_{\mu_{y^*}\to 0} g\left(\mu_{y^*}, n\right)\mu_{y^*} = 0$. If $Z = 0$, we have $I\left(rx^*\right) = 0$ whatever may be the value of r.

Proposition 11

Let $y^* = rx^*$, and assume that $\sum_{i=1}^{n} z_i x_i^* = Z \neq 0$. Linearity means we have $I(rx^*) = rI(x^*)$ for any x^* and any $0 \leq r \leq 1$. Since $\sum_{i=1}^{n} z_i y_i^* = r\sum_{i=1}^{n} z_i x_i^*$, we have $I\left(rx^*\right) = g\left(r\mu_{x^*}, n\right) rZ$. Hence, the condition $I(rx^*) = rI(x^*)$ is equivalent to the condition $g\left(r\mu_{x^*}, n\right) = g\left(\mu_{x^*}, n\right)$ Because this must hold for any $0 \leq \mu_{x^*} \leq 1$ and any $0 \leq r \leq 1$, this means that the value of $g\left(\mu_{x^*}, n\right)$ must remain constant for any $0 \leq \mu_{x^*} \leq 1$. If $Z = 0$, we have $I\left(rx^*\right) = 0$ whatever may be the value of r.

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Table 1: Nature of health variables: some examples

		Unbounded	**Bounded**
measurement scale	*nominal*	type of illness	type of illness
	ordinal	ordinal utility function	self-assessed health \ binary 0/1 indicator
	cardinal	latent health variable from ordered probit	health utility index \ body temperature
	ratio-scale	health care expenditures	body length \ life expectancy
	absolute	number of illnesses	visits to the medical sector in a time period

Table 2: Nature of health variables and rank-dependent inequality indices

<table>
<tr><th></th><th></th><th>Unbounded</th><th>Bounded</th></tr>
<tr><td rowspan="5">measurement scale</td><td>nominal</td><td>inequality measurement is meaningless</td><td>inequality measurement is meaningless</td></tr>
<tr><td>ordinal</td><td>inequality measurement is meaningless, unless the variable is transformed into a cardinal variable (see row ‘cardinal’) or into a ratio-scale variable (see row ‘ratio-scale’)</td><td>inequality measurement is meaningless, unless the variable is transformed into a cardinal or ratio-scale variable (see below)</td></tr>
<tr><td>cardinal</td><td>Modified Concentration Index*</td><td rowspan="3">Subset of the class of indices defined by (6b) satisfying the Mirror property
<u>Option a</u>: Never inverse-relative, never inverse-absolute, and convergent: Erreygers Index*
<u>Option b</u>: Partly inverse-relative, never inverse-absolute, and non-convergent: Wagstaff Index
<u>Option c</u>: Partly inverse-relative, possibly partly inverse-absolute, and convergent: Indices defined by (9), with $\theta < 1, \theta \neq 0$</td></tr>
<tr><td>ratio-scale</td><td>Concentration Index*</td></tr>
<tr><td>absolute</td><td>Concentration Index*
Generalized Concentration Index**</td></tr>
</table>

Note: This table assumes that unbounded and bounded indices should satisfy the Sign Condition. All indices, except the ones applied to health variables measured with an absolute scale, also satisfy Scale Invariance.

*: These indices are unique if one assumes that the maximum bounds of the indices are –1 and +1.

**: The Generalized Concentration Index is unique assuming the maximum bounds are $-\mu_h$ and $+\mu_h$.