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Balanced Consistency and Balanced Cost Reduction for Sequencing Problems

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Abstract

We investigate the implications of imposing *balanced consistency* and *balanced cost reduction* in the context of sequencing problems. *Balanced consistency* requires that the effect on the payoff from the departure of one agent to another agent should be equal between any two agents. On the other hand, *balanced cost reduction* requires that if one agent leaves a problem, then the total payoffs of the remaining agents should be affected by the amount previously assigned to the leaving agent. We show that the minimal transfer rule is the only rule satisfying *efficiency* and *Pareto indifference* together with either one of our two main axioms, *balanced consistency* and *balanced cost reduction.*

JEL Classification Numbers: D63, D71.

Keywords: Sequencing problem, minimal transfer rule, balanced consistency, balanced cost reduction.

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1. Introduction

Consider a group of agents who must be served in a facility. The facility can handle only one agent at a time and agents incur waiting costs. We assume that an agent's waiting cost is constant per unit of time, but that agents differ in the unit waiting cost and the amount of service time. Efficiency requires to minimize the total costs incurred by the agents. On the other hand, fairness requires that agents served earlier should give compensations to agents served later. We are interested in finding the order in which to serve agents and the (positive or negative) monetary compensations they should receive. Each agent's utility is equal to his monetary compensations minus his total waiting cost. This sequencing problem has been studied extensively in the recent literature: from the incentive viewpoint (Dolan, 1978; Suijs, 1996; Mitra, 2001, 2002) and from the normative viewpoint (Maniquet, 2003; Chun, 2004, 2006a, b; Mishra and Rangarajan, 2007; Moulin, 2007). Two well-known subclasses of sequencing problems are: (i) a queueing problem in which all agents are assumed to need the same amount of service time but they differ in their unit waiting cost, and (ii) a scheduling problem in which all agents are assumed to have the same unit waiting cost but need (possibly) different amount of service times.

The underlying paper focuses on the normative approach. Two solutions or rules that have played important roles in the normative approach are the minimal and the maximal transfer rules. As shown in Maniquet (2003) and Chun (2006a) for queueing problems, and their generalization to sequencing problems by Chun (2004), these two rules can be obtained by applying the Shapley value (Shapley 1953) to corresponding cooperative games in which the worth of a coalition is appropriately defined from the sequencing problem. For the minimal transfer rule, the worth of a coalition is defined to be the minimum waiting cost incurred by its members under the assumption that they are served before the non-coalitional members. For the maximal transfer rule, it is defined to be the minimum waiting cost incurred by its members under the assumption that they are served after the non-coalitional members.[1]

[1]This is different from the sequencing games with an initial order of Curiel, Pederzoli and Tijs (1989), where agents (or jobs) are ordered in an initial queue. The question then is how agents should compensate one another when re-ordering into an efficient queue, taking account of the initial order. The sequencing games with an initial order are line-

In this paper, we investigate how the minimal and the maximal transfer rules respond to changes in the set of agents. Our first main axiom is *balanced consistency,* which requires that the effect on the payoff from the departure of one agent to another agent should be equal between any two agents. We show that the minimal transfer rule is the only rule satisfying *efficiency, Pareto indifference,* and *balanced consistency.* On the other hand, the maximal transfer rule can be characterized by an alternative formulation of *balanced consistency under constant completion time*: upon the departure of an agent, all of his predecessors are assumed to move back by one position to keep the completion time constant. Under this alternative formulation, the maximal transfer rule becomes the only rule satisfying *efficiency, Pareto indifference,* and *balanced consistency under constant completion time.*

Our second main axiom is *balanced cost reduction,* which requires that if one agent leaves a problem, then the total payoffs of the remaining agents should be affected by the amount previously assigned to the leaving agent. Once again, the minimal transfer rule is the only rule satisfying *efficiency, Pareto indifference,* and *balanced cost reduction.*

The paper is organized as follows. Section 2 contains some preliminaries and introduces rules. Section 3 explores the implications of *balanced consistency* and presents our first characterization of the minimal transfer rule. Section 4 explores the implications of *balanced cost reduction* and presents our second characterization of the minimal transfer rule. Concluding remarks follow in Section 5. In the appendix, we discuss the axiom of *participation consistency* which is a game theoretic property related to *balanced cost reduction.*

2. Preliminaries

Let $I \equiv \{1, 2, \ldots\}$ be an (finite or infinite) universe of "potential" agents, and $\mathcal{N}$ the family of non-empty subsets of I. Each agent $i \in I$ is characterized by his *service time,* $r_i > 0$, and his *unit waiting cost,* $\theta_i \geq 0$. A *sequencing problem* is defined as a list (N, r, θ) where $N \in \mathcal{N}$ is the set of agents,

graph games, where only coalitions of consecutive agents (in the initial order) can have non-zero dividend (see van den Brink, van der Laan and Vasil'ev (2007)), while sequencing games as referred to in this paper are 2-games (i.e. only coalitions of size 2 can have a non-zero dividend, see Harsanyi (1959)).

$r = (r_i)_{i\in N} \in \mathbb{R}^N_{++}$ is the vector of service times, and $\theta = (\theta_i)_{i\in N} \in \mathbb{R}^N_+$ is the vector of unit waiting costs. Let $\mathcal{S}^N$ be the class of all sequencing problems for N and $\mathcal{S} = \bigcup_{N\in\mathcal{N}} \mathcal{S}^N$. Two subclasses of sequencing problems introduced earlier are: a *queueing problem,* where for each i, $j \in I$, $r_i = r_j$, and a *scheduling problem,* where for each i, $j \in I$, $\theta_i = \theta_j$.

An *allocation* for $(N, r, \theta) \in \mathcal{S}^N$ is a pair $(\sigma, t) \in \{1, \ldots, |N|\}^N \times \mathbb{R}^N$, where for each $i \in N$, σ_i denotes agent i's position in the queue and t_i the monetary transfer to him. Let $P_i(\sigma) = \{j \in N \mid \sigma_j < \sigma_i\}$ be the set of agents preceding agent i in σ, and $F_i(\sigma) = \{j \in N \mid \sigma_j > \sigma_i\}$ the set of agents following him. The agent who is served first incurs no waiting cost. If agents $j \in N$ are served in the σ_j^{th} position, then the waiting cost of agent $i \in N$ is $\sum_{j\in P_i(\sigma)} r_j\theta_i$. We assume that each agent $i \in N$ has a quasi-linear utility function, so that his utility from consuming the bundle (σ, t) is given by $u_i(\sigma, t) = t_i - \sum_{j\in P_i(\sigma)} r_j\theta_i$. An allocation is *feasible* if no two agents are assigned the same position and the sum of all the transfers is not positive. Thus, the set of feasible allocations $Z(N, r, \theta)$ consists of all pairs $z = (\sigma, t) \in \{1, \ldots, |N|\}^N \times \mathbb{R}^N$ such that for all i, $j \in N$, $i \neq j$ implies $\sigma_i \neq \sigma_j$ and $\sum_{i\in N} t_i \leq 0$.

Given $(N, r, \theta) \in \mathcal{S}^N$, an allocation $(\sigma, t) \in Z(N, r, \theta)$ is *queue-efficient* if it minimizes the total waiting cost among the feasible allocations, that is, for all $(\sigma', t') \in Z(N, r, \theta)$, $\sum_{i\in N}\sum_{j\in P_i(\sigma)} r_j\theta_i \leq \sum_{i\in N}\sum_{j\in P_i(\sigma'_i)} r_j\theta_i$. As shown in Smith (1956), total waiting cost is minimized if the agents are served in nonincreasing order with respect to their *urgency index* θ_i/r_i. For i, $j \in N$, if $\theta_i/r_i = \theta_j/r_j$, then agents i and j have *equivalent* urgency indexes. The efficient queues do not depend on the transfers. Moreover, it is unique except for agents with equivalent urgency indexes, who will be next to each other in the queue and can be permuted. The set of efficient queues for $(N, r, \theta) \in \mathcal{S}^N$ is denoted by $Eff(N, r, \theta)$. An allocation $(\sigma, t) \in Z(N, r, \theta)$ is *budget balanced* if $\sum_{i\in N} t_i = 0$. A feasible allocation is *efficient* if it is *queue-efficient* and *budget balanced.*

A *rule* is a mapping $\varphi : \mathcal{S} \to \bigcup_{N\in\mathcal{N}} Z(N, r, \theta)$, which associates with each problem $(N, r, \theta) \in \mathcal{S}$ a non-empty subset $\varphi(N, r, \theta) \subset Z(N, r, \theta)$ of feasible allocations. A pair $(\sigma_i, t_i) \in \varphi_i(N, r, \theta)$ represents the position σ_i of i in the queue and his transfer t_i in (N, r, θ). If the monetary transfer of an agent is positive, then this agent receives a compensation from other agents. If it is negative, he has to pay that amount as compensation to other agents.

We mention two standard axioms for rules. First, a rule should choose an *efficient* (i.e. queue-efficient and budget balanced) allocation.

Efficiency: For all $N \in \mathcal{N}$, all $(N, r, \theta) \in \mathcal{S}^N$, and all $(\sigma, t) \in \varphi(N, r, \theta)$, we have $\sigma \in Eff(N, r, \theta)$ and $\sum_{i \in N} t_i = 0$.

Second, if an allocation is chosen by a rule, then all other allocations which assign the same utility to each agent should be chosen by the rule.

Pareto indifference: For all $N \in \mathcal{N}$, all $(N, r, \theta) \in \mathcal{S}^N$, all $(\sigma, t) \in \varphi(N, r, \theta)$, and $(\sigma', t') \in Z(N, r, \theta)$: if $u_i(\sigma', t') = u_i(\sigma, t)$ for all $i \in N$, then $(\sigma', t') \in \varphi(N, r, \theta)$.

Next we recall two rules studied in Maniquet (2003) and Chun (2006a) for queueing problems, and generalized to sequencing problems by Chun (2004). The minimal transfer rule selects an efficient queue and transfers from each agent a half of his waiting cost multiplied by the sum of all his predecessors' service times minus a half of the sum of the unit waiting cost over all his followers multiplied by his own service time.

Minimal transfer rule, φ^M: For all $N \in \mathcal{N}$, and all $(N, r, \theta) \in \mathcal{S}$,

$$\varphi^M(N, r, \theta) = \left\{ (\sigma^M, t^M) \in Z(N, r, \theta) \;\middle|\; \begin{array}{c} \sigma^M \in Eff(N, r, \theta) \text{ and} \\ t_i^M = \sum_{j \in P_i(\sigma^M)} \frac{r_j \theta_i}{2} - \sum_{j \in F_i(\sigma^M)} \frac{r_i \theta_j}{2} \end{array} \right\}.$$

On the other hand, the maximal transfer rule selects an efficient queue and transfers to each agent a half of the sum of the unit waiting cost over all his predecessors multiplied by his own service time minus a half of his waiting cost multiplied by the sum of each of his followers' service time.

Maximal transfer rule, φ^X: For all $N \in \mathcal{N}$, and all $(N, r, \theta) \in \mathcal{S}$,

$$\varphi^X(N, r, \theta) = \left\{ (\sigma^X, t^X) \in Z(N, r, \theta) \;\middle|\; \begin{array}{c} \sigma^X \in Eff(N, r, \theta) \text{ and} \\ t_i^X = \sum_{j \in P_i(\sigma^X)} \frac{r_i \theta_j}{2} - \sum_{j \in F_i(\sigma^X)} \frac{r_j \theta_i}{2} \end{array} \right\}.$$

Note that the minimal and the maximal transfer rules assign a unique allocation if and only if all agents have different urgency index θ_i / r_i. However, even when some agents have the same urgency index, agents' utilities do not depend on the choice of efficient queues if the compensation is determined according to the minimal or the maximal transfer rule. Thus, both rules are *essentially single-valued,* in the sense that for a given problem, each agent's utility is the same at all allocations that the rule chooses. As a consequence, any efficient queue can be chosen to calculate the utilities assigned by the two rules. To be specific, for all $N \in \mathcal{N}$, and all $(N, r, \theta) \in \mathcal{S}^N$, for the minimal transfer rule, the utility of agent i is given by

$$u_i(\sigma^M, t^M) = -\sum_{j \in P_i(\sigma^M)} r_j \theta_i + t_i^M = -\sum_{j \in P_i(\sigma^M)} \frac{r_j \theta_i}{2} - \sum_{j \in F_i(\sigma^M)} \frac{r_i \theta_j}{2}, \qquad (1)$$

and for the maximal transfer rule,

$$\begin{aligned} u_i(\sigma^X, t^X) &= -\sum_{j \in P_i(\sigma^X)} r_j \theta_i + t_i^X = -\sum_{j \in P_i(\sigma^X)} r_j \theta_i + \sum_{j \in P_i(\sigma^X)} \frac{r_i \theta_j}{2} - \sum_{j \in F_i(\sigma^X)} \frac{r_j \theta_i}{2} \\ &= -\sum_{j \in N \backslash \{i\}} r_j \theta_i + \sum_{j \in P_i(\sigma^X)} \frac{r_i \theta_j}{2} + \sum_{j \in F_i(\sigma^X)} \frac{r_j \theta_i}{2}. \end{aligned}$$

As mentioned in the introduction, the minimal and the maximal transfer rules can be obtained by applying the Shapley value (1953) to corresponding TU-games in which the worth of a coalition is appropriately defined. For the minimal transfer rule, the worth of a coalition is defined to be the minimum waiting cost incurred by its members under the assumption that they are served before the non-coalitional members. For the maximal transfer rule, it is defined to be the minimum waiting cost incurred by its members under the assumption that they are served after the non-coalitional members.

3. Balanced consistency in sequencing problems

If an agent leaves a sequencing problem, then it will affect the payoffs of other remaining agents. *Balanced consistency* requires that the effect of agent i leaving a sequencing problem on the payoff of another agent $j \neq i$ should be the same as the effect of agent j leaving a sequencing problem on the payoff of agent i. It is similar to 'preservation of differences' of solutions for TU-games as discussed in Hart and Mas-Colell (1989).[2] To stress the fact that our axiom concerns situations in which an agent leaves the sequencing problem similar as players leave a game in (reduced game) consistency properties, we refer to this property as balanced consistency[3].

For all $(N, r, \theta) \in \mathcal{S}^N$ and all $j \in N$, let $r^{-j} \in \mathbb{R}_{++}^{N \setminus \{j\}}$ and $\theta^{-j} \in \mathbb{R}_{+}^{N \setminus \{j\}}$ be the projections given by $r^{-j} = (r_k)_{k \in N \setminus \{j\}}$ and $\theta^{-j} = (\theta_k)_{k \in N \setminus \{j\}}$.

Balanced consistency: For all $N \in \mathcal{N}$, all $(N, r, \theta) \in \mathcal{S}$, all i, $j \in N$, all $(\sigma, t) \in \varphi(N, r, \theta)$, all $(\sigma^{-i}, t^{-i}) \in \varphi(N \setminus \{i\}, r^{-i}, \theta^{-i})$, and all $(\sigma^{-j}, t^{-j}) \in \varphi(N \setminus \{j\}, r^{-j}, \theta^{-j})$:

$$u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) = u_j(\sigma, t) - u_j(\sigma^{-i}, t^{-i}).$$

Now we investigate the implications of *balanced consistency* in the context of sequencing problems. First, we show that the minimal transfer rule satisfies this property.

Lemma 1. *The minimal transfer rule satisfies balanced consistency.*

Proof. Without loss of generality, let $N \equiv \{1, 2, \ldots, n\}$ be such that $\theta_1/r_1 \geq \theta_2/r_2 \geq \cdots \geq \theta_n/r_n$. To simplify the notation, we do not attach the superscript M to σ and t. From the *essential single-valuedness* of φ^M, we may assume that for all $i \in N$, $\sigma_i = i$. Let i, $j \in N$ be such that $j \in P_i(\sigma)$ (and thus $i \in F_j(\sigma)$). Then, for all $(\sigma, t) \in \varphi^M(N, r, \theta)$, all $(\sigma^{-i}, t^{-i}) \in \varphi^M(N \setminus \{i\}, r^{-i}, \theta^{-i})$, and all $(\sigma^{-j}, t^{-j}) \in \varphi^M(N \setminus \{j\}, r^{-j}, \theta^{-j})$, we have

$$u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j})$$

[2]This property states that the effect of player i leaving the game on the payoff of player $j \neq i$ is equal to the effect of player j leaving the game on the payoff of player i.

[3]Note that in the balanced contributions property as introduced by Myerson (1980) for cooperative game solutions where the games have a restricted set of feasible coalitions (stating that the effect of deleting all coalitions containing player i from the set of feasible coalitions on the payoff of player $j \neq i$ is equal to the effect of deleting all coalitions containing player j from the set of feasible coalitions on the payoff of player i), the player set is fixed.

$$\begin{aligned} &= -\sum_{k\in P_i(\sigma)} \frac{r_k\theta_i}{2} - \sum_{k\in F_i(\sigma)} \frac{r_i\theta_k}{2} - \left(-\sum_{k\in P_i(\sigma)\setminus\{j\}} \frac{r_k\theta_i}{2} - \sum_{k\in F_i(\sigma)} \frac{r_i\theta_k}{2}\right) \\ &= -\frac{r_j\theta_i}{2}, \end{aligned}$$

and

$$\begin{aligned} &u_j(\sigma,t) - u_j(\sigma^{-i},t^{-i}) \\ &= -\sum_{k\in P_j(\sigma)} \frac{r_k\theta_j}{2} - \sum_{k\in F_j(\sigma)} \frac{r_j\theta_k}{2} - \left(-\sum_{k\in P_j(\sigma)} \frac{r_k\theta_j}{2} - \sum_{k\in F_j(\sigma)\setminus\{i\}} \frac{r_j\theta_k}{2}\right) \\ &= -\frac{r_j\theta_i}{2}. \end{aligned}$$

Altogether, we conclude that the minimal transfer rule satisfies *balanced consistency.* ∎

We ask whether there is any other rule satisfying *efficiency* and *Pareto indifference* together with *balanced consistency.* As it turns out, the minimal transfer rule is the only one satisfying the three axioms together. We note that if $\sigma_i \in \{1, \ldots, |N|\}$ is determined and $u_i(\sigma,t)$ is known, then also t_i is determined.

Theorem 1. *The minimal transfer rule is the only rule satisfying efficiency, Pareto indifference, and balanced consistency.*

Proof. It is well-known that the minimal transfer rule satisfies *efficiency* and *Pareto indifference*, and by Lemma 1, it satisfies *balanced consistency.* Conversely, let φ be a rule satisfying the three axioms. Let $N \in \mathcal{N}$ and $(N, r, \theta) \in \mathcal{S}^N$ be given. If $|N| = 1$, then *efficiency* implies that $\sigma_i = 1$ and $t_i = 0$ for $i \in N$.

Let N be such that $|N| = 2$. Without loss of generality, we may assume that $N \equiv \{i, j\}$ and that $\theta_i/r_i \geq \theta_j/r_j$. Let $(\sigma, t) \in \varphi(N, r, \theta)$, $(\sigma^{-i}, t^{-i}) \in \varphi(N \setminus \{i\}, r^{-i}, \theta^{-i})$, and $(\sigma^{-j}, t^{-j}) \in \varphi(N \setminus \{j\}, r^{-j}, \theta^{-j})$. By *balanced consistency,* $u_i(\sigma,t) - u_i(\sigma^{-j},t^{-j}) = u_j(\sigma,t) - u_j(\sigma^{-i},t^{-i})$. Since $u_i(\sigma^{-j},t^{-j}) = u_j(\sigma^{-i},t^{-i}) = 0$, we have $u_i(\sigma,t) = u_j(\sigma,t)$. By *efficiency,* $u_i(\sigma,t) + u_j(\sigma,t) =$

$-r_i\theta_j$. Altogether, we obtain $u_i(\sigma,t) = u_j(\sigma,t) = -\frac{r_i\theta_j}{2}$. By *efficiency* and *Pareto indifference,* we may assume that $\sigma_i = 1$ and $\sigma_j = 2$. Then, $t_i = -\frac{r_i\theta_j}{2} = -t_j$, as desired.

We will establish the claim for an arbitrary number of agents by an induction argument. As induction hypothesis, suppose that $\varphi(N', r', \theta') = \varphi^M(N', r', \theta')$ whenever $|N'| \leq |N| - 1$. Let $N \equiv \{1, 2, \ldots, n\}$, and let $(\sigma, t) \in \varphi(N, r, \theta)$. By *efficiency* and *Pareto indifference,* we may assume without loss of generality that $\theta_1/r_1 \geq \theta_2/r_2 \geq \cdots \geq \theta_n/r_n$, and that $\sigma_i = i$ for all $i \in N$. For any $i \in N$, let $(\sigma^{-i}, t^{-i}) \in \varphi(N \setminus \{i\}, r^{-i}, \theta^{-i})$. By *balanced consistency,* $u_i(\sigma,t) - u_i(\sigma^{-j}, t^{-j}) = u_j(\sigma,t) - u_j(\sigma^{-i}, t^{-i})$ for any i, $j \in N$. Now fix i, change $j \neq i$ from 1 to n, and add up the $(n-1)$ equations obtained in this way. We have:

$$(n-1)u_i(\sigma,t) - \sum_{j \in N \setminus \{i\}} u_i(\sigma^{-j}, t^{-j}) = \sum_{j \in N \setminus \{i\}} (u_j(\sigma,t) - u_j(\sigma^{-i}, t^{-i})).$$

Adding $u_i(\sigma,t) + \sum_{j \in N \setminus \{i\}} u_i(\sigma^{-j}, t^{-j})$ to both sides gives

$$nu_i(\sigma,t) \;=\; \sum_{j \in N} u_j(\sigma,t) - \sum_{j \in N \setminus \{i\}} u_j(\sigma^{-i}, t^{-i}) + \sum_{j \in N \setminus \{i\}} u_i(\sigma^{-j}, t^{-j}). \quad (2)$$

By *efficiency,*

$$\sum_{j \in N} u_j(\sigma,t) = -\sum_{j \in N} \sum_{k \in P_j(\sigma)} r_k\theta_j$$

and

$$\sum_{j \in N \setminus \{i\}} u_j(\sigma^{-i}, t^{-i}) = -\sum_{j \in P_i(\sigma)} \sum_{k \in P_j(\sigma)} r_k\theta_j - \sum_{j \in F_i(\sigma)} \sum_{k \in P_j(\sigma) \setminus \{i\}} r_k\theta_j.$$

Subtracting these two expressions from each other yields

$$\sum_{j \in N} u_j(\sigma,t) - \sum_{j \in N \setminus \{i\}} u_j(\sigma^{-i}, t^{-i}) = -\sum_{k \in P_i(\sigma)} r_k\theta_i - \sum_{k \in F_i(\sigma)} r_i\theta_k. \quad (3)$$

From the induction hypothesis, it follows that

$$\begin{aligned}
\sum_{j\in N\setminus\{i\}} u_i(\sigma^{-j}, t^{-j}) &= -\sum_{j\in P_i(\sigma)}\left(\sum_{k\in P_i(\sigma)\setminus\{j\}}\frac{r_k\theta_i}{2} - \sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2}\right)\\
&\quad + \sum_{j\in F_i(\sigma)}\left(\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - \sum_{k\in F_i(\sigma)\setminus\{j\}}\frac{r_i\theta_k}{2}\right)\\
&= -(i-2)\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - (i-1)\sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2}\\
&\quad -(n-i)\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - (n-i-1)\sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2}\\
&= -(n-2)\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - (n-2)\sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2}. \qquad (4)
\end{aligned}$$

Substituting (3) and (4) in (2) yields

$$\begin{aligned}
nu_i(\sigma, t) &= -2\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - 2\sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2}\\
&\quad -(n-2)\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - (n-2)\sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2}\\
&= -n\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - n\sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2},
\end{aligned}$$

or equivalently,

$$u_i(\sigma, t) = -\sum_{k\in P_i(\sigma)}\frac{r_k\theta_i}{2} - \sum_{k\in F_i(\sigma)}\frac{r_i\theta_k}{2} = u_i(\sigma^M, t^M). \qquad (5)$$

By *efficiency*, $\sigma \in Eff(N, r, \theta)$, and thus (5) fixes the transfers

$$t_i = \sum_{j\in P_i(\sigma)}\frac{r_j\theta_i}{2} - \sum_{j\in F_i(\sigma)}\frac{r_i\theta_j}{2} = t_i^M,$$

as desired. ∎

Remark 1: Although 'preservation of differences' is typical for the Shapley value, it is not obvious for sequencing problems that it characterizes the minimal transfer rule since the maximal transfer rule is also obtained as the Shapley value of an associated TU-game.

Remark 2: Upon the departure of an agent, if we assume that all of his predecessors are moving back by one position to keep the same completion time, an alternative *balanced consistency under constant completion time* property can be formulated. The maximal transfer rule is the only rule satisfying *efficiency, Pareto indifference,* and *balanced consistency under constant completion time.*

4. Balanced cost reduction in sequencing problems

Suppose that an agent leaves a sequencing problem. Since the agent is not in the queue anymore, the total waiting cost of all the remaining agents will be decreased. In other words, the presence of an agent generates a negative externality to any other agent. *Balanced cost reduction* requires that the total (over all remaining agents) decrease in this negative externality as a result of the departure of an agent is equal to the negative of the payoff of the departing agent when he is still present.

Balanced cost reduction: For all $N \in \mathcal{N}$, all $(N, r, \theta) \in \mathcal{S}$, all $j \in N$, all $(\sigma, t) \in \varphi(N, r, \theta)$, and all $(\sigma^{-j}, t^{-j}) \in \varphi(N \setminus \{j\}, r^{-j}, \theta^{-j})$,

$$\sum_{i \in N \setminus \{j\}} \Big(u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) \Big) = u_j(\sigma, t).$$

We explore the implications of *balanced cost reduction* in the context of sequencing problems. First, we show that the minimal transfer rule satisfies this property.

Lemma 2. *The minimal transfer rule satisfies balanced cost reduction.*

Proof. Without loss of generality, let $N \equiv \{1, 2, \ldots, n\}$ be such that $\theta_1/r_1 \geq \theta_2/r_2 \geq \cdots \geq \theta_n/r_n$. To simplify the notation, we do not attach the superscript M to σ and t. From the *essential single-valuedness* of φ^M, we may assume that for all $i \in N$, $\sigma_i = i$. Let $j \in N$. Then, for all $(\sigma, t) \in \varphi^M(N, r, \theta)$, and all $(\sigma^{-j}, t^{-j}) \in \varphi^M(N \setminus \{j\}, r^{-j}, \theta^{-j})$, if $i \in P_j(\sigma)$, then

$$\begin{aligned} & u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) \\ = & -\sum_{k \in P_i(\sigma)} \frac{r_k \theta_i}{2} - \sum_{k \in F_i(\sigma)} \frac{r_i \theta_k}{2} - \left(-\sum_{k \in P_i(\sigma)} \frac{r_k \theta_i}{2} - \sum_{k \in F_i(\sigma) \setminus \{j\}} \frac{r_i \theta_k}{2} \right) \\ = & -\frac{r_i \theta_j}{2}, \end{aligned}$$

and if $i \in F_j(\sigma)$, then

$$\begin{aligned} & u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) \\ = & -\sum_{k \in P_i(\sigma)} \frac{r_k \theta_i}{2} - \sum_{k \in F_i(\sigma)} \frac{r_i \theta_k}{2} - \left(-\sum_{k \in P_i(\sigma) \setminus \{j\}} \frac{r_k \theta_i}{2} - \sum_{k \in F_i(\sigma)} \frac{r_i \theta_k}{2} \right) \\ = & -\frac{r_j \theta_i}{2}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \sum_{i \in N \setminus \{j\}} \left(u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) \right) &= -\sum_{i \in P_j(\sigma)} \frac{r_i \theta_j}{2} - \sum_{i \in F_j(\sigma)} \frac{r_j \theta_i}{2} \\ &= u_j(\sigma, t), \end{aligned}$$

showing that the minimal transfer rule satisfies *balanced cost reduction.* ∎

Now we investigate whether there exists any other *efficient* and *Pareto indifferent* rule satisfying this property. As it turns out, once again, the minimal transfer rule is the only rule satisfying *balanced cost reduction* in addition to *efficiency* and *Pareto indifference.*

Theorem 2. *The minimal transfer rule is the only rule satisfying efficiency, Pareto indifference, and balanced cost reduction.*

Proof. It is well-known that the minimal transfer rule satisfies *efficiency* and *Pareto indifference*, and by Lemma 2, it satisfies *balanced cost reduction.* Conversely, let φ be a rule satisfying the three axioms. If $|N| = 1$, then *efficiency* implies that for any $i \in N$, $\sigma_i = 1$ and $t_i = 0$.

Let N be such that $|N| = 2$. Without loss of generality, we may assume that $N \equiv \{i, j\}$ and that $\theta_i/r_i \geq \theta_j/r_j$. Let $(\sigma, t) \in \varphi(N, r, \theta)$ and $(\sigma^{-j}, t^{-j}) \in \varphi(N \setminus \{j\}, r^{-j}, \theta^{-j})$. By *balanced cost reduction,* $u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) = u_j(\sigma, t)$. Since $u_i(\sigma^{-j}, t^{-j}) = 0$, we have $u_i(\sigma, t) = u_j(\sigma, t)$. By *efficiency,* $u_i(\sigma, t) + u_j(\sigma, t) = -r_i\theta_j$. Altogether, we obtain $u_i(\sigma, t) = u_j(\sigma, t) = -\frac{r_i\theta_j}{2}$. By *efficiency* and *Pareto indifference,* we may assume that $\sigma_i = 1$ and $\sigma_j = 2$, and thus the transfer $t_i = -\frac{r_i\theta_j}{2} = -t_j$ is determined.

We will establish the claim for an arbitrary number of agents by an induction argument. As induction hypothesis, suppose that $\varphi(N', r', \theta') = \varphi^M(N', r', \theta')$ whenever $|N'| \leq |N| - 1$. Let $N \equiv \{1, 2, \ldots, n\}$, and let $(\sigma, t) \in \varphi(N, r, \theta)$. By *efficiency* and *Pareto indifference,* we may assume without loss of generality that $\theta_1/r_1 \geq \theta_2/r_2 \geq \cdots \geq \theta_n/r_n$, and that $\sigma_i = i$ for all $i \in N$. Let $j \in N$ be a leaving agent and $(\sigma^{-j}, t^{-j}) \in \varphi(N \setminus \{j\}, r^{-j}, \theta^{-j})$. By *balanced cost reduction,*

$$\sum_{i \in N \setminus \{j\}} \Big(u_i(\sigma, t) - u_i(\sigma^{-j}, t^{-j}) \Big) = u_j(\sigma, t).$$

Adding $u_j(\sigma, t)$ to both sides gives

$$\sum_{i \in N} u_i(\sigma, t) - \sum_{i \in N \setminus \{j\}} u_i(\sigma^{-j}, t^{-j}) = 2u_j(\sigma, t). \tag{6}$$

By *efficiency,*

$$\sum_{i \in N} u_i(\sigma, t) = -\sum_{i \in N} \sum_{k \in P_i(\sigma)} r_k\theta_i$$

and

$$\sum_{i \in N \setminus \{j\}} u_i(\sigma^{-j}, t^{-j}) = -\sum_{i \in P_j(\sigma)} \sum_{k \in P_i(\sigma)} r_k\theta_i - \sum_{i \in F_j(\sigma)} \sum_{k \in P_i(\sigma) \setminus \{j\}} r_k\theta_i.$$

Therefore, by subtracting these two equations from each other, we obtain

$$\sum_{i\in N} u_i(\sigma,t) - \sum_{i\in N\setminus\{j\}} u_i(\sigma^{-j},t^{-j}) = -\sum_{i\in P_j(\sigma)} r_i\theta_j - \sum_{i\in F_j(\sigma)} r_j\theta_i. \tag{7}$$

Substituting (7) in (6) yields

$$2u_j(\sigma,t) = -\sum_{i\in P_j(\sigma)} r_i\theta_j - \sum_{i\in F_j(\sigma)} r_j\theta_i,$$

which implies that

$$u_j(\sigma,t) = -\sum_{i\in P_j(\sigma)} \frac{r_i\theta_j}{2} - \sum_{i\in F_j(\sigma)} \frac{r_j\theta_i}{2} = u_j(\sigma^M,t^M). \tag{8}$$

By efficiency, $\sigma \in \mathit{Eff}(N,r,\theta)$, and thus (8) fixes the transfers

$$t_j = \sum_{i\in P_j(\sigma)} \frac{r_i\theta_j}{2} - \sum_{i\in F_j(\sigma)} \frac{r_j\theta_i}{2} = t_j^M,$$

the desired expression. ∎

6. Concluding remarks

In this paper, we presented two axiomatic characterizations of the minimal transfer rule in the context of sequencing problems on the basis of *balanced consistency* or *balanced cost reduction* in addition to *efficiency* and *Pareto indifference.* We note that all our results carry over to two subclasses of sequencing problems: queueing problems and scheduling problems.

Another axiom widely discussed in the literature specifying how a rule should respond to changes in the population is *population solidarity* (Thomson 1983, Chun 1986, and others[4]): it requires that upon the departure of an agent, all the remaining agents should be affected in the same direction,

[4]See Thomson (1995) for a survey.

all gain or all lose. As discussed in Chun (2006a) for queueing problems, the minimal transfer rule for sequencing problems also satisfies *population solidarity,* but the maximal transfer rule does not satisfy it. On the other hand, as in Remark 1, upon the departure of an agent, if we assume that all of his predecessors are moving back by one position to keep the completion time constant, then both the minimal and the maximal transfer rules satisfy the alternative *population solidarity under constant completion time* property. It remains an open question whether the minimal or the maximal transfer rules can be characterized on the basis of *population solidarity.*

Another question for future research is to investigate axioms concerning changes in the parameters of the sequencing problem without changing the set of agents, such as the before mentioned *balanced contributions* property (Myerson, 1980), *fairness* (Myerson, 1977; van den Brink, 2001) or *monotonicity* (Young, 1985; van den Brink, 2007).

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Appendix: Participation consistency of the Shapley value for 2-games

In this appendix, we discuss a game theoretic property for a special class of TU-games that is related to balanced cost reduction. As mentioned in footnote 1, it is known that the sequencing problems are a special class of 2-games, being games (N,v) with $N \subset \mathcal{N}$ and $v\colon 2^N \to \mathbb{R}$ such that $v(S) = 0$ if $|S| \leq 1$, and $v(S) = \sum_{\substack{T \subseteq N \\ |T|=2}} v(T)$ if $|S| \geq 2$. Equivalently, a game (N,v) is a 2-game if and only if only coalitions of size two can have a nonzero dividend, i.e. $\Delta_v(S) \neq 0$ implies that $|S| = 2$, where $\Delta_v(S)$ is the Harsanyi dividend of coalition S in game (N,v), see Harsanyi (1959). It is known that for 2-games the Shapley value (Shapley, 1953) coincides with several other TU-game solutions such as the nucleolus, the τ-value and the *CIS* (Center of the Imputation Set-value), and for $|N| \geq 2$ is given by $Sh_i(N,v) = \frac{1}{2}(v(N) - v(N \setminus \{i\}))$ for all $i \in N$. [5] Note that this gives a simple game theoretic characterization of the minimal tranfer rule for sequencing problems as $\varphi_i^M(N,r,\theta) = \frac{1}{2}(v_\theta^M(N) - v_\theta^M(N \setminus \{i\}))$ for all $i \in N$, where $v_\theta^M(N)$ is the total waiting cost in the efficient queue, and $v_\theta^M(N \setminus \{i\})$ is the total waiting cost in the efficient queue on agents $N \setminus \{i\}$ (i.e. after i has left). Note that these are the only two worths we need to know to determine $\varphi_i^M(N,r,\theta)$.

A solution f for TU-games assigns a payoff vector $f(N,v) \in \mathbb{R}^{|N|}$ to every game (N,v). Our purpose now is to see what property for TU-game solutions is related to *balanced cost reduction*. We state the following property for 2-games.

Participation consistency: Let (N,v) be a 2-game with $|N| \geq 2$, and $j \in N$. Then

$$\sum_{i \in N \setminus \{j\}} \Big(f_i(N,v) - f_i(N \setminus \{j\}, v^{-j}) \Big) = f_j(N,v),$$

where $(N \setminus \{j\}, v^{-j})$ is the restricted game on $N \setminus \{j\}$, i.e. $v^{-j}(S) = v(S)$ for all $S \subseteq N \setminus \{j\}$.

[5] See A. van den Nouweland, P. Borm, W. van Golstein Brouwers, R. Groot Bruinderink, and S. Tijs, A game theoretic approach to problems in telecommunication, *Management Science* **42** (1996), 294-303.

Next we define a weak efficiency axiom which requires efficiency only for games with at most two players.

2-Efficiency: For every game (N, v) with $|N| \leq 2$, $\sum_{i \in N} f_i(N, v) = v(N)$.

Together with 2-efficiency for 2-games, participation consistency characterizes the Shapley value on the class of 2-games.

Theorem A. *The solution f for 2-games satisfies 2-efficiency and participation consistency if and only if it is the Shapley value.*

Proof. Since the Shapley value is efficient, it satisfies 2-efficiency. To show that the Shapley value satisfies participation consistency, consider 2-game (N, v), $|N| \geq 2$, and $j \in N$. Then

$$\begin{aligned}
\sum_{i \in N \setminus \{j\}} \Big(Sh_i(N, v) - Sh_i(N \setminus \{j\}, v^{-j}) \Big) &= \sum_{i \in N \setminus \{j\}} \left(\sum_{\substack{S \subseteq N, |S|=2 \\ i \in S}} \frac{\Delta_v(S)}{2} - \sum_{\substack{S \subseteq N, |S|=2 \\ i \in S, j \notin S}} \frac{\Delta_v(S)}{2} \right) \\
&= \sum_{i \in N \setminus \{j\}} \sum_{\substack{S \subseteq N, |S|=2 \\ i, j \in S}} \frac{\Delta_v(S)}{2} \\
&= \sum_{\substack{S \subseteq N, |S|=2 \\ j \in S}} \frac{\Delta_v(S)}{2} \\
&= Sh_j(N, v),
\end{aligned}$$

showing that the Shapley value satisfies participation consistency.[6]

We show uniqueness by induction on $|N|$ (similar as the uniqueness part of the proof of Theorem 2, but in terms of TU-games). If $|N| = 1$ then $f_i(N, v) = v(\{i\}) = 0 = Sh_i(N, v)$ by 2-efficiency. If $|N| = 2$ then participation consistency implies that $f_i(N, v) - f_i(\{i\}, v^{-j}) = f_j(N, v)$, and thus $f_i(N, v) = f_j(N, v)$ if $N = \{i, j\}$. With 2-efficiency it then follows that $f_i(N, v) = f_j(N, v) = \frac{v(N)}{2}$.

[6] Note that the third equality follows from only two-player coalitions having a nonzero dividend, and does not hold for arbitrary games.

We will establish the claim for an arbitrary number of players by an induction argument. As induction hypothesis, suppose that uniqueness holds for all $N' \in \mathcal{N}$ such that $2 \leq |N'| \leq |N| - 1$. Let $N \equiv \{1, 2, \ldots, n\}$. Consider 2-game (N, v). For any $j \in N$, participation consistency yields

$$\sum_{i \in N \setminus \{j\}} \Big(f_i(N, v) - f_i(N \setminus \{j\}, v^{-j}) \Big) = f_j(N, v). \tag{9}$$

By the induction hypothesis the $f_i(N \setminus \{j\}, v^{-j}),\ i, j \in N,\ i \neq j$, are uniquely determined. Since $|N| \geq 3$ this yields a system of n linearly independent equations in the n unkowns $f_j(N, v),\ j \in N$, which thus are uniquely determined. ∎