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Axiomatizations of Two Types of Shapley Values for Games on Union Closed Systems

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Abstract

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game. A (single-valued) solution for TU-games assigns a payoff distribution to every TU-game. A well-known solution is the Shapley value.

In the literature various models of games with restricted cooperation can be found. So, instead of allowing all subsets of the player set N to form, it is assumed that the set of feasible coalitions is a subset of the power set of N. In this paper we consider such sets of feasible coalitions that are closed under union, i.e. for any two feasible coalitions also their union is feasible. We consider and axiomatize two solutions or rules for these games that generalize the Shapley value: one is obtained as the *conjunctive permission value* using a corresponding *superior graph*, the other is defined as the Shapley value of a modified game similar as the *Myerson rule* for conference structures.

Keywords: TU-game, restricted cooperation, union closed system, Shapley value, permission value, superior graph, axiomatization.

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1 Introduction

A *cooperative game with transferable utility*, or simply a TU-game, is a finite set of players and for any subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A (single-valued) solution is a function that assigns to every game a payoff vector which components are the individual payoffs of the players. A solution is efficient if it assigns to every game a payoff vector such that the sum of the payoffs is equal to the worth of the grand coalition consisting of all players. The most well-known efficient (single-valued) solution is the *Shapley value* (Shapley, 1953).

In its classical interpretation, a TU-game describes a situation in which the players in every coalition S of N can cooperate to form a feasible coalition and earn its worth. In the literature various restrictions on coalition formation are developed. For a survey we refer to Bilbao (2000). In this paper we assume that the set of feasible coalitions is closed under union, meaning that for any pair of feasible coalitions also their union is feasible. By convention we assume that every union closed set of feasible coalitions contains the empty coalition $\emptyset$. Some examples of cooperation structures that yield union closed systems are the following.

Example 1.1 Suppose that only coalitions of a minimal size k are feasible. Then the set of coalitions $\Omega = \{S \subseteq N \mid |S| \geq k\} \cup \{\emptyset\}$ for some $k \in \{1, \ldots |N|\}$ is closed under union.

Example 1.2 To give a more general example, consider the situation where the player set N is partitioned in a coalition structure $\mathcal{P} = \{P^1, \ldots, P^m\}$ of nonempty coalitions such that for every element P^k, $k \in \{1, \ldots, m\}$ there is a quota $q_k \in \{1, \ldots, |P^k|\}$ meaning that a coaliton $S \subseteq N$ can form if for every $k = 1, \ldots, m$, S contains at least q_k players from P^k. So, given such a *majority cooperation situation* $(N, v, \{P^1, \ldots, P^m\}, \{q_1, \ldots, q_k\}\}$ with $\{P^1, \ldots, P^m\}$ being a partition of N and $q_k \in \{1, \ldots, |P^k|\}$ for all $k \in \{1, \ldots, m\}$, the set of feasible coalitions is given by

$$\Omega = \{S \subseteq N \mid |S \cap P_k| \geq q_k \text{ for all } k \in \{1, \ldots, m\}\} \cup \{\emptyset\}.$$

Obviously, if $\min\{|S \cap P_k|, |T \cap P_k|\} \geq q_k$ for all $k \in \{1, \ldots, m\}$, then $|(S \cup T) \cap P_k| \geq q_k$ for all $k \in \{1, \ldots, m\}$, and thus Ω is closed under union.

By definition antimatroids are closed under union[1]. Thus, the games considered in this paper generalize the games on antimatroids as considered in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004a), and therefore also the *games with a permission*

[1]A set of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an antimatroid if, besides being union closed and containing $\emptyset$, it satisfies *accessibility* meaning that $S \in \mathcal{A}$ implies that there is a player $i \in S$ such that $S \setminus \{i\} \in \mathcal{A}$, see Dilworth (1940) and Edelman and Jamison (1985).

structure of Gilles, Owen and van den Brink (1992), van den Brink and Gilles (1996), Gilles and Owen (1994) and van den Brink (1997), where there are players who need permission from other players before they are allowed to cooperate. Note that the two examples given above yield union closed structures that are not antimatroids[2]. Therefore to deal with such type of situations we need a more general approach.

We define and axiomatize two solutions for games on union closed systems, one is based on games with a permission structure, the other on the approach of Myerson (1977, 1980) for communication graph games and conference structures. Both solutions generalize the Shapley value in the sense that both are equal to the Shapley value when the union closed system is the power set of player set N. First, for every union closed system we define the corresponding *superior graph* being the directed graph that is obtained by putting an arc from player i to player j if every feasible coalition containing player j also contains player i. Then we consider the game with permission structure of the original game on this superior graph, and define the superior rule as the conjunctive permission value of the game with permission structure, see Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996). We also give an axiomatization of this superior rule.

Second, we apply the method of Myerson (1977, 1980) to define another solution for games on union closed systems which generalizes the Shapley value for games on antimatroids which is axiomatized in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). First, we define a modified or restricted game in which any feasible coalition earns its own worth. By union closedness, every nonfeasible coalition has a unique largest feasible subset. The restricted game assigns to any nonfeasible coalition the worth of this largest feasible subset. Then the *union rule* for games on union closed systems is defined as the Shapley value of this restricted game. We provide an axiomatization for this solution.

This paper is organized as follows. Section 2 is a preliminary section containing cooperative TU-games and games with a permission structure. Section 3 introduces games on union closed systems. In Section 4 we define and axiomatize the superior rule. In Section 5 we define and axiomatize the union rule. The axioms discussed in Sections 4 and 5 all concern a fixed union closed system. In Section 6 we discuss several issues concerning a variable union closed system, and give a comparison with Myerson (1980)'s conference structures. Section 7 contains concluding remarks.

[2]These structures do not satisfy accessibility. In Example 1.2 this can be seen since for $S \in \Omega$ with $|S| = k$ there is no $i \in S$ such that $S \setminus \{i\} \in \Omega$. In Example 1.1 this can be seen since taking a coalition $S \in \Omega$ with $|S \cap P_k| = q_k$ for all $k \in \{1, \ldots, m\}$, there is no $i \in S$ such that $S \setminus \{i\} \in \Omega$.

2 Preliminaries

2.1 TU-games

A situation in which a finite set of players can obtain certain payoffs by cooperating can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of n players and $v\colon 2^N \to \mathbb{R}$ is a characteristic function on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e., the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. Since we take the player set N to be fixed, we denote the game (N, v) just by its characteristic function v. We denote the collection of all characteristic functions on N by $\mathcal{G}^N$ and $n = |N|$ denotes the cardinality of N. A game $v \in \mathcal{G}^N$ is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. We denote by $\mathcal{G}^N_M$ the class of all monotone TU-games on N.

A payoff vector for a game is a vector $x \in \mathbb{R}^n$ assigning a payoff x_i to every $i \in N$. In the sequel, for $S \subseteq N$ we denote $x(S) = \sum_{i\in S} x_i$.

A (single-valued) solution f is a function that assigns to any $v \in \mathcal{G}^N$ a unique payoff vector. The most well-known (single-valued) solution is the Shapley value given by

$$Sh_i(v) = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!}(v(S) - v(S \setminus \{i\})) \text{ for all } i \in N.$$

2.2 Cooperative games with a permission structure

A game with a permission structure on N describes a situation where some players in a TU-game need permission from other players before they are allowed to cooperate within a coalition. Formally, a permission structure can be described by a directed graph on N. A directed graph or *digraph* is a pair (N, D) where $N = \{1, ..., n\}$ is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. In the sequel we simply refer to D for a digraph (N, D) . We denote the collection of all digraphs on N by $\mathcal{D}^N$. For $i \in N$ the nodes in $S_D(i) := \{j \in N \mid (i, j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) := \{j \in N \mid (j, i) \in D\}$ are called the *predecessors* of i. For a set $T \subseteq N$, let $S_D(T) = \cup_{i\in T} S_D(i)$ denote the union of the sets of successors of the players in T, respectively $P_D(T) = \cup_{i\in T} P_D(i)$ the set of all predecessors of the players in T. Further, $T_D = \{i \in N \mid P_D(i) = \emptyset\}$ denotes the set of *top nodes* in D, being the set of nodes not having a predecessor. By $\widehat{S}_D(i)$ we denote the set of successors of i in the *transitive closure* of D i.e., $j \in \widehat{S}_D(i)$ if and only if there exists a sequence of players $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in S_D(h_k)$ for all $1 \leq k \leq t - 1$, and $h_t = j$.

For given $D \in \mathcal{D}^N$, a (directed) *path* between i and j in N is a sequence of distinct nodes $(i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$, and $(i_k, i_{k+1}) \in D$ for $k = 1, \ldots, m-1$. A directed path $(i_1, \ldots, i_m)$, $m \geq 1$, in D is a *cycle* in D if $(i_m, i_1) \in D$. We call digraph D *acyclic* if it does not contain any cycle. Note that acyclicity of a digraph D implies that D is irreflexive, i.e., $(i,i) \notin D$ for all $i \in N$. Also, when D is acyclic then there is at least one top node. A digraph is called *quasi-strongly connected* if there exists a node $i_0 \in N$, such that for every $j \neq i_0$ there is a directed path from i_0 to j. When D is acyclic and quasi-strongly connected then it has exactly one top node. In the sequel we denote this unique top node by i_0, i.e. $T_D = \{i_0\}$ when D is acyclic and quasi-strongly connected.

A tuple (v, D) with $v \in \mathcal{G}^N$ a TU-game and $D \in \mathcal{D}^N$ a digraph on N is called a *game with a permission structure*. In this paper we follow the *conjunctive approach* as introduced in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996) in which it is assumed that a player needs permission from all its predecessors in order to cooperate with other players[3]. Therefore a coalition is feasible if and only if for any player in the coalition all its predecessors are also in the coalition. So, for permission structure D the set of *conjunctive feasible coalitions* is given by

$$\Phi_D^c = \{E \subseteq N \,|\, P_D(i) \subseteq E \text{ for all } i \in E\}.$$

For any $E \subseteq N$, let $\overline{\sigma}_D^c(E) = E \setminus \widehat{S}_D(N \setminus E)$ be the largest conjunctive feasible subset of E in D.

Given the tuple (v, D) with $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$, under the conjunctive permission structure the induced *restricted* game $\overline{r}_{v,D}^c \colon 2^N \to \mathbb{R}$ is given by

$$\overline{r}_{v,D}^c(S) = v(\overline{\sigma}_D^c(S)) \text{ for all } S \subseteq N. \tag{2.1}$$

The *conjunctive permission value* φ^c then is the solution that assigns to every game with a permission structure the Shapley value of the restricted game, i.e.

$$\varphi^c(v, D) = Sh(\overline{r}_{v,D}^c).$$

These games with a permission structure and the conjunctive permission value are generalized to games on antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003, 2004a). In this paper we consider a further generalization to games on union closed systems.

[3]As an alternative, Gilles and Owen (1994) and van den Brink (1997) consider the disjunctive approach where it is assumed that a player needs permission to cooperate of at least one of its predecessors (if it has any).

3 Restricted games on union closed systems

We consider tuples (v, Ω), where v is a TU-game on player set N and $\Omega \subseteq 2^N$ is a collection of subsets of N. We call such a tuple a *game with limited cooperation.* In such a game the collection of subsets Ω restricts the cooperation possibilities of the players in N. A set $S \subseteq N$ of players can only attain its value $v(S)$ if $S \in \Omega$. When $S \notin \Omega$ then not all players are able to cooperate within S, so that $v(S)$ can not be realised. We say that a coalition $S \in 2^N$ is *feasible* if $S \in \Omega$. In this paper we only consider sets of feasible coalitions that are closed under union.

Definition 3.1 *A collection $\Omega \subseteq 2^N$ is a union closed system of coalitions if*
1. $\emptyset, N \in \Omega$,
2. If $S,\ T \in \Omega$, then $S \cup T \in \Omega$.

We assume that the 'grand coalition' N is feasible for notational convenience. The results in this paper can be modified to hold without this assumption if in the axioms we distinguish between players that belong to at least one feasible coalition and those that do not belong to any feasible coalition. Note that by condition 2 the 'grand coalition' is feasible if every player belongs to at least one feasible coalition. So, instead of assuming that $N \in \Omega$ we could do with the weaker *normality* assumption stating that every player belongs to at least one feasible coalition. In the sequel we denote the collection of all union closed systems in 2^N by $\mathcal{C}^N$.

Example 3.2
1. $\Omega = \{\emptyset, N\}$ is union closed.

2. $\Omega = 2^N$ is union closed.

3. Every antimatroid is union closed by definition, and thus also the sets of conjunctive and disjunctive feasible coalitions of an acyclic permission structure are union closed systems.

4. Note that the set of connected coalitions in an (undirected) communication graph is not union closed, because in a communication graph the union of two connected coalitions might not be connected.

For a system $\Omega \in \mathcal{C}^N$, we define the function $\sigma_\Omega \colon 2^N \to \Omega$ by

$$\sigma_\Omega(S) = \bigcup_{\{U \in \Omega | U \subseteq S\}} U,$$

so $\sigma_\Omega(S)$ is the largest feasible subset of S. By union closedness this largest feasible subset is unique. For the tuple (v,Ω), the *restricted* game $r_{v,\Omega} \in \mathcal{G}^N$ is defined by

$$r_{v,\Omega}(S) = v(\sigma_\Omega(S)).$$

The restricted game assigns to each coalition $S \subseteq N$ the worth of its largest feasible subset. Notice that when v is monotone, it holds that for every $\Omega \in \mathcal{C}^N$ also the restricted game $r_{v,\Omega}$ is monotone since $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$.

Example 3.3
1. If $\Omega = \{\emptyset, N\}$ then $\sigma_\Omega(N) = N$ and $\sigma_\Omega(S) = \emptyset$ for all $S \neq N$. So, $r_{v,\Omega}(N) = v(N)$ and $r_{v,\Omega}(S) = 0$ for every $S \neq N$. Thus the restricted game $r_{v,\Omega}$ is a multiple of the unanimity game of N.

2. If $\Omega = 2^N$ then $\sigma_\Omega(S) = S$ and $r_{v,\Omega}(S) = v(S)$ for every $S \subseteq N$. The restricted game $r_{v,\Omega}$ coincides with v.

3. If Ω is the set of conjunctive feasible coalitions of some permission structure then $\sigma_\Omega(E) = E \setminus \widehat{S}_D(N \setminus E) = \cup\{T \in \Phi^c_D \mid T \subseteq E\}$, and $r_{v,\Omega}$ is the conjunctive restriction. Similar for the disjunctive case.

4 The superior rule

A solution for games on union closed systems is a function f that assigns a payoff distribution $f(v,\Omega) \in \mathbb{R}^N$ to every $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. In this section we introduce and axiomatize a solution for games on union closed systems that is based on the conjunctive permission value of a digraph associated to the union closed system.

4.1 The superior graph

The *superior graph* of a union closed system $\Omega \in \mathcal{C}^N$ assigns an arc from player i to player j if every feasible coalition containing player j also contains player i. So, the arcs can be seen as some kind of dominance relation in the sense that a player is a subordinate of another player if it 'needs' the other player to be in a feasible coalition.

Definition 4.1 *For two players $i,j \in N$, $i \neq j$, player i is a* superior *of player j in $\Omega \in \mathcal{C}^N$, if $i \in S$ for every $S \in \Omega$ with $j \in S$. In that case we call player j a* subordinate *of player i. For $\Omega \in \mathcal{C}^N$, the superior graph of Ω is the directed graph $D^\Omega \in \mathcal{D}^N$ with*

$$D^\Omega = \{(i,j) \in N \times N \mid i \text{ is superior of } j \text{ in } \Omega\}.$$

Notice that i is a subordinate (superior) of j in $\Omega \in \mathcal{C}^N$ if and only if i is a successor (predecessor) of j in D^Ω. The next corollary is straightforward for $\Omega \in \mathcal{C}^N$.

Corollary 4.2 *If i is a superior of j in Ω and k is a superior of i in Ω then k is a superior of j in Ω.*

Example 4.3
1. If $\Omega = \{\emptyset, N\}$ then for every $S \in \Omega$ and for every $i, j \in N$ it holds that $i \in S$ when $j \in S$. So every $i \in N$ is a superior of every $j \in N \setminus \{i\}$, and thus $D^\Omega = \{(i,j) \in N \times N \mid i, j \in N,\ i \neq j\}$.

2. If $\Omega = 2^N$ then $\{i\} \in \Omega$ for every $i \in N$, and thus $D^\Omega = \emptyset$.

3. Let $D \in \mathcal{D}^N$ be a directed graph representing a permission structure and let $\Omega = \Phi_D^c$ be the union closed system of feasible coalitions under the conjunctive approach. Then D^Ω is the transitive closure of D.[4]

Having constructed the superior graph D^Ω of a union closed system Ω, we consider now the set of feasible coalitions of the permission structure D^Ω according to the conjunctive approach, and we denote this collection of coalitions by $\Sigma^\Omega = \Phi_{D^\Omega}^c$. Notice that this set is again a union closed system.

Proposition 4.4 *For $\Omega \in \mathcal{C}^N$ it holds that $\Omega \subseteq \Sigma^\Omega$.*

Proof. Let $S \in \Omega$. By definition of superior it holds that S includes all superiors of i for every $i \in S$. On the other hand it holds that $(j, i) \in D^\Omega$ if and only if j is superior of i, $i \in S$. It follows that S is feasible for the permission structure D^Ω according to the conjunctive approach. Hence $\Omega \subseteq \Sigma^\Omega$. $\square$

4.2 The superior rule

Now we define the superior rule SUP as the solution for games on union closed systems which assigns to every (v, Ω) the conjunctive permission value of the game v with permission structure D^Ω, i.e.

$$SUP_i(v, \Omega) = \varphi_i^c(v, D^\Omega) = Sh_i(\overline{r}_{v,D^\Omega}^c) \text{ for all } i \in N.$$

[4] Let $D \in \mathcal{D}^N$ be an acyclic, quasi-strongly connected directed graph. Let Ω be the union closed system of feasible coalitions under the disjunctive permission structure. Then $(i, j) \in D^\Omega$ if and only if every path from i_0 to j contains player i.

Next we give an axiomatization of the superior rule as a solution for games on union closed systems. The axioms are generalizations of axioms used to axiomatize the conjunctive permission value in van den Brink and Gilles (1996) and the Shapley value for games on poset antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). First, efficiency states that the total sum of payoffs equals the worth of the 'grand' coalition.

Axiom 4.5 (Efficiency) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $\sum_{i \in N} f_i(v, \Omega) = v(N)$.*

Additivity is a straightforward generalization of the well-known additivity axiom for TU-games.

Axiom 4.6 (Additivity) *For every pair of cooperative TU-games $v, w \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f(v + w, \Omega) = f(v, \Omega) + f(w, \Omega)$.*

Next we introduce a generalization of the inessential player property stating that a null player in v whose subordinates in Ω are all null players in v, earns a zero payoff. We say that player $i \in N$ is *inessential* in (v, Ω) if $v(E \cup \{j\}) = v(E)$ for all $j \in \{i\} \cup S_{D^\Omega}(i)$ and $E \subseteq N \setminus \{j\}$. For $v \in \mathcal{G}^N$, $\Omega \in \mathcal{C}^N$, we denote by $I(v, \Omega)$ the set of all inessential players in (v, Ω).

Axiom 4.7 (Inessential player property) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have that $f_i(v, \Omega) = 0$ for all $i \in I(v, \Omega)$.*

The next axiom generalizes the necessary player property (which holds for monotone TU-games) in a straightforward way, stating that a necessary player in a monotone game earns at least as much as any other player, irrespective of the coalitions in the union closed system. A player $i \in N$ is *necessary* in game v if $v(E) = 0$ for all $E \subseteq N \setminus \{i\}$.

Axiom 4.8 (Necessary player property) *For every monotone game $v \in \mathcal{G}_M^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f_i(v, \Omega) \geq f_j(v, \Omega)$ for all $j \in N$, when $i \in N$ is a necessary player in v.*

Finally, structural monotonicity is generalized using the superior graph, stating that whenever player i is a superior of player j in the union closed system and the game is monotone, then player i earns at least as much as player j.

Axiom 4.9 (Structural monotonicity) *For every monotone game $v \in \mathcal{G}_M^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f_i(v, \Omega) \geq f_j(v, \Omega)$ if $i \in N$ and $j \in S_{D^\Omega}(i)$.*

The five axioms above characterize the superior rule for games on union closed systems.

Theorem 4.10 *A solution f for cooperative games on union closed systems is equal to the superior rule SUP if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity.*

PROOF

By efficiency of the conjunctive permission value (i.e. $\sum_{i\in N}\varphi_i^c(v,D) = v(N)$ for every $v \in \mathcal{G}^N$ and $D \in \mathcal{D}^N$) we have that $\sum_{i\in N} SUP_i(v,\Omega) = \sum_{i\in N}\varphi_i^c(v,D^\Omega) = v(N)$, showing that the superior rule satisfies efficiency. Additivity, the inessential player property, the necessary player property and structural monotonicity follow from the corresponding axioms of the conjunctive permission value for games with a permission structure, see van den Brink and Gilles (1996).

To prove uniqueness, suppose that solution f satisfies the five axioms. Let v_0 be the *null game* given by $v_0(E) = 0$ for all $E \subseteq N$. The inessential player property then implies that $f_i(v_0,\Omega) = 0$ for all $i \in N$.

Next, consider union closed system Ω and the game $w_T = c_T u_T$, $c_T > 0$, $T \subseteq N$, $T \neq \emptyset$. We distinguish the following three cases with respect to $i \in N$:

1. If $i \in T$ then the necessary player property implies that there exists a $c^* \in \mathbb{R}$ such that $f_i(w_T,\Omega) = c^*$ for all $i \in T$, and $f_i(w_T,\Omega) \leq c^*$ for all $i \in N \setminus T$.

2. If $i \in N \setminus T$ and $T \cap (\{i\} \cup S_{D^\Omega}(i)) \neq \emptyset$ then structural monotonicity implies that $f_i(w_T,\Omega) \geq f_j(w_T,\Omega)$ for every $j \in T \cap (\{i\} \cup S_{D^\Omega}(i))$, and thus with case 1 that $f_i(w_T,\Omega) = c^*$.

3. If $i \in N \setminus T$ and $T \cap (\{i\} \cup S_{D^\Omega}(i)) = \emptyset$ then the inessential player property implies that $f_i(w_T,\Omega) = 0$.

From 1 and 2 follows that $f_i(w_T,\Omega) = c^*$ for $i \in T \cup P_{D^\Omega}(T)$. Efficiency and 3 then imply that $\sum_{i\in N} f_i(w_T,\Omega) = |T \cup P_{D^\Omega}(T)|c^* = c_T$, implying that c^*, and thus $f(w_T,\Omega)$, is uniquely determined.

Next, consider (w_T,Ω) with $w_T = c_T u_T$ for some $c_T < 0$ (and thus we cannot apply the necessary player property and structural monotonicity since w_T is not monotone). Since $-w_T = -c_T u_T$ with $-c_T > 0$, and $v_0 = w_T + (-w_T)$, it follows from additivity of f that $f(w_T,\Omega) = f(v_0,\Omega) - f(-w_T,\Omega) = -f(-w_T,\Omega)$ is uniquely determined because $-w_T$ is monotone.

Finally, since every characteristic function $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{T \subseteq N} \Delta_v(T) u_T$ (with $\Delta_v(T)$ the *Harsanyi dividend* of coalition T, see Harsanyi (1959)), additivity uniquely determines $f(v, \Omega) = \sum_{T \subseteq N} f(\Delta_v(T) u_T, \Omega)$ for any $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$. □

We end this section by showing logical independence of the five axioms stated in Theorem 4.10.

1. The solution that assigns to every game on union closed system simply the Shapley value of game v, i.e. $f(v, \Omega) = Sh(v)$, satisfies efficiency, additivity, the inessential player property and the necessary player property. It does not satisfy structural monotonicity.

2. For $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$, let $\overline{v} \in \mathcal{G}^N$ be given by $\overline{v}(E) = v(\bigcup_{i \in E} \{i\} \cup S_{D^\Omega}(i))$ for all $S \subseteq N$. The solution $f(v, \Omega) = Sh(\overline{v})$ satisfies efficiency, additivity, the inessential player property and structural monotonicity. It does not satisfy the necessary player property.

3. The equal division solution given by $f_i(v, \Omega) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property and structural monotonicity. It does not satisfy the inessential player property.

4. The equal division over essential players, given by

$$f_i(v, \Omega) = \begin{cases} \frac{v(N)}{|N \setminus I(v,\Omega)|} & \text{if } i \in N \setminus I(v, \Omega) \\ 0 & \text{if } i \in I(v, \Omega), \end{cases}$$

   satisfies efficiency, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy additivity.

5. The zero solution given by $f_i(v, \Omega) = 0$ for all $i \in N$ satisfies additivity, the inessential player property, the necessary player property and structural monotonicity. It does not satisfy efficiency.

5 The union rule

In this section we introduce and axiomatize the *union rule* as a solution for games on union closed systems. This rule is defined similar as the Myerson rule for conference structures in Myerson (1980) and the Shapley value for games on antimatroids in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003). The union rule U assigns to every (v,Ω) the Shapley value of the restriced game $r_{v,\Omega}$, i.e.

$$U_i(v,\Omega) = \mathit{Sh}_i(r_{v,\Omega}) \text{ for all } i \in N.$$

This solution is different from the superior rule as illustrated in the following example.

Example 5.1 Consider the unanimity game $v = u_{\{1\}}$ and union closed system $\Omega = \{\emptyset, \{1,2\}, \{1,3\}, \{1,2,3\}\}$ on $N = \{1,2,3\}$. The superior graph of Ω is $D^\Omega = \{(1,2),(1,3)\}$. Therefore, the superior rule equals $\mathit{SUP}(v,\Omega) = (1,0,0)$.

On the other hand, the restricted game is given by $r_{v,\Omega}(\{1\}) = r_{v,\Omega}(\{2\}) = r_{v,\Omega}(\{3\}) = r_{v,\Omega}(\{2,3\}) = 0$, $r_{v,\Omega}(\{1,2\}) = r_{v,\Omega}(\{1,3\}) = r_{v,\Omega}(\{1,2,3\}) = 1$, and thus the union rule equals $U(v,\Omega) = \mathit{Sh}(r_{v,\Omega}) = (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

From the axioms that are used to characterize the superior rule in Theorem 4.10, the union rule satisfies all the axioms except the inessential player property. The union rule not satisfying the inessential player property is illustrated by the following example.

Example 5.2 Consider the union closed system $\Omega = \{\emptyset, \{1,2\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,3,4\}\}$ and game $v = u_{\{3\}}$ on $N = \{1,2,3,4\}$. The superior graph is given by $D^\Omega = \{(1,2),(3,4)\}$, and $I(v,\Omega) = \{1,2,4\}$. However, the restricted game is $r_{v,\Omega} = u_{\{1,3\}} + u_{\{3,4\}} - u_{\{1,3,4\}}$, and thus $U(v,\Omega) = (\frac{1}{6}, 0, \frac{2}{3}, \frac{1}{6})$.

The union rule satisfies a weaker axiom requiring zero payoffs for inessential players only in games where the worth of any coalition equals the worth of its largest feasible subset[5].

Axiom 5.3 (Inessential player property for union closed games) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$ such that $v(E) = v(\sigma_\Omega(E))$ for all $E \subseteq N$, we have that $f_i(v,\Omega) = 0$ for every $i \in I(v,\Omega)$.*

To characterize the union rule we add one more axiom which states that the payoffs only depend on the worths of feasible coalitions.

Axiom 5.4 (Independence of irrelevant coalitions) *For every pair of cooperative TU-games $v, w \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f(v,\Omega) = f(w,\Omega)$ whenever $v(S) = w(S)$ for all $S \in \Omega$.*

[5]Note that the union rule satisfies the stronger property requiring zero payoffs for all null players in games v such that $v(E) = v(\sigma_\Omega(E))$ for all $E \subseteq N$.

For $\Omega \in \mathcal{C}^N$ and $T \subseteq N$, we define $\Omega_T = \{H \in \Omega \mid T \subseteq H\}$ as the set of feasible coalitions containing coalition T. In the proof of uniqueness in Theorem 5.6 we use the following lemma.

Lemma 5.5 *For every $\Omega \in \mathcal{C}^N$, $T \subseteq N$ and $c \in \mathbb{R}$, there exist numbers $\delta_H \in \mathbb{R}$, $H \in \Omega_T$, such that $r_{cu_T,\Omega} = \sum_{H \in \Omega_T} \delta_H u_H$.*

PROOF

Consider $\Omega \in \mathcal{C}^N$, $T \subseteq N$ and $c \in \mathbb{R}$. If $T \in \Omega$ then $T \in \Omega_T$ and we have $\delta_T = c$ and $\delta_H = 0$ for all $H \in \Omega_T \setminus \{T\}$. If $T \notin \Omega$, then define

$$\mathcal{T}^1 = \{H \in \Omega \mid T \subset H \text{ and there is no } Z \in \Omega \text{ such that } T \subset Z \subset H\}$$

and, recursively, for $k = 2, \ldots$

$$\mathcal{T}^k = \left\{ H \in \Omega \mid T \subset H \text{ and for every } Z \in \Omega \text{ such that } T \subset Z \subset H \text{ it holds that } Z \in \bigcup_{p=1}^{k-1} \mathcal{T}^p \right\}.$$

Since N is finite there exists an $M < \infty$ such that $\mathcal{T}^k \neq \emptyset$ for all $k \in \{1, \ldots M\}$, $\mathcal{T}^{M+1} = \emptyset$ and $\bigcup_{k=1}^{M} \mathcal{T}^k = \Omega_T$. Since by definition $\mathcal{T}^k \cap \mathcal{T}^l = \emptyset$ for all $k, l \in \mathbb{N}$, we have that $\mathcal{T}^1, \ldots, \mathcal{T}^M$ is a partition of the set $\{H \in \Omega \mid T \subset H\}$ of feasible coalitions containing non-feasible coalition T. (Note that this set equals Ω_T since $T \notin \Omega$.) Then $\delta_H = c$ for all $H \in \mathcal{T}^1$ and, recursively for $k = 2, \ldots, M$, the numbers δ_H, $H \in \mathcal{T}^k$, are determined by

$$\delta_H + \sum_{\left\{Z \subset H \mid Z \in \bigcup_{l=1}^{k-1} \mathcal{T}^l\right\}} \delta_Z = c.$$

□

Weakening, in Theorem 4.10, the inessential player property by requiring only the inessential player property for union closed games, and adding independence of irrelevant coalitions, characterizes the union rule.

Theorem 5.6 *A solution f for cooperative games on union closed systems is equal to the union rule U if and only if it satisfies efficiency, additivity, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions.*

PROOF

We first prove that U satisfies the five axioms. Let $v \in \mathcal{G}^N$ and $\Omega \in \mathcal{C}^N$.

1. By efficiency of the Shapley value and $\sigma_\Omega(N) = N$, we have that $\sum_{i \in N} U_i(v, \Omega) = \sum_{i \in N} Sh_i(r_{v,\Omega}) = v(N)$, showing that U satisfies efficiency.

2. Additivity of the Shapley value and the fact that $r_{v,\Omega}(S) + r_{w,\Omega}(S) = v(\sigma_\Omega(S)) + w(\sigma_\Omega(S)) = (v+w)(\sigma_\Omega(S)) = r_{v+w,\Omega}(S)$ for all $S \subseteq N$, imply for $i \in N$ that $U_i(v,\Omega) + U_i(w,\Omega) = Sh_i(r_{v,\Omega}) + Sh_i(r_{w,D^\Omega}) = Sh_i(r_{v+w,\Omega}) = U_i(v+w,\Omega)$, showing that U satisfies additivity.

3. U satisfying the inessential player property for union closed games follows directly from the null player property of the Shapley value.

4. Let v be a monotone game on N. Since $S \subseteq T$ implies that $\sigma_\Omega(S) \subseteq \sigma_\Omega(T)$, by monotonicity of v we have that $r_{v,\Omega}$ is a monotone game on N. The necessary player property then follows from the necessary player property of the Shapley value.

5. If $v(S) = w(S)$ for all $S \in \Omega$, then $r_{v,\Omega} = r_{w,\Omega}$, showing that the union rule U satisfies independence of irrelevant coalitions.

To prove uniqueness, let $\Omega \in \mathcal{C}^N$. We first consider $v = cu_T$ for some $c \in \mathbb{R}$ and $\emptyset \neq T \subseteq N$. We distinguish two cases.
1. Let $T \in \Omega$, i.e. T is feasible. Then $r_{cu_T,\Omega} = cu_T$. From the necessary player property it follows that there exists a $c^* \in \mathbb{R}$ such that $f_i(cu_T,\Omega) = c^*$ for all $i \in T$. Since $i \in N \setminus T$ is a null player in cu_T, and $cu_T(E) = cu_T(\sigma_\Omega(E))$ for all $E \subseteq N$ if $T \in \Omega$, the inessential player property for union closed games implies that $f_i(cu_T,\Omega) = 0$ for all $i \in N \setminus T$. Then efficiency implies that $c^* = f_i(cu_T,\Omega) = \frac{c}{|T|}$ for all $i \in T$, and thus $f(cu_T,\Omega)$ is determined.
2. Suppose that $T \notin \Omega$, i.e. T is not feasible. Let $\Omega_T = \{H \in \Omega \mid T \subseteq H\}$ be the collection of feasible subsets of N that contain T. (Note that $T \notin \Omega_T$ since $T \notin \Omega$.) By Lemma 5.5 there exist numbers δ_H, $H \in \Omega_T$, such that $r_{cu_T,\Omega} = \sum_{H\in\Omega_T} \delta_H u_H$. Since $cu_T(E) = r_{cu_T,\Omega}(E)$ for all $E \in \Omega$, by independence of irrelevant coalitions it then follows that $f(cu_T,\Omega) = f(r_{cu_T,\Omega},\Omega) = f(\sum_{H\in\Omega_T} \delta_H u_H,\Omega)$. By additivity we then have that

$$f(cu_T,\Omega) = f\left(\sum_{H\in\Omega_T} \delta_H u_H,\Omega\right) = \sum_{H\in\Omega_T} f(\delta_H u_H,\Omega). \tag{5.2}$$

Since all $H \in \Omega_T$ are feasible in Ω, we know from case 1 that $f(\delta_H u_H,\Omega)$ is uniquely determined for every $H \in \Omega_T$. Thus, with (5.2) also $f(cu_T,\Omega)$ is uniquely determined.
It further follows that additivity uniquely determines $f(v,\Omega) = \sum_{T\subseteq N} f(\Delta_v(T)u_T,\Omega)$ for any $v \in \mathcal{G}^N$. □

The following example illustrates that the superior rule does not satisfy independence of irrelevant coalitions.

Example 5.7 Consider the tuple (v, Ω) of Example 5.2, and let game w be the restriction of v on Ω. Obviously, $r_{v,\Omega} = r_{w,\Omega}$. However, since the superior graph is given by $D^\Omega = \{(1, 2), (3, 4)\}$, we have that $\overline{r}^c_{v,D^\Omega} = u_{\{3\}} = v$ and $\overline{r}^c_{w,D^\Omega} = u_{\{1,3\}} + u_{\{3,4\}} - u_{\{1,3,4\}} = w$, and thus $SUP(v, \Omega) = (0, 0, 1, 0)$ and $SUP(w, \Omega) = (\frac{1}{6}, 0, \frac{2}{3}, \frac{1}{6})$.

We end this section by showing logical independence of the five axioms stated in Theorem 5.6.

1. The superior rule satisfies efficiency, additivity, the inessential player property for union closed games and the necessary player property. It does not satisfy independence of irrelevant coalitions.

2. The solution that assigns to every game on union closed system the weighted Shapley of the restricted game $r_{v,\Omega}$ for some exogenous weight system $\omega \in \mathbb{R}^N$ with $\omega_i \neq \omega_j$ for some $i, j \in N$, satisfies efficiency, additivity, the inessential player property for union closed games and independence of irrelevant coalitions. It does not satisfy the necessary player property.

3. The equal division solution given by $f_i(v, \Omega) = \frac{v(N)}{|N|}$ for all $i \in N$, satisfies efficiency, additivity, the necessary player property and independence of irrelevant coalitions. It does not satisfy the inessential player property for union closed games.

4. The equal division over non-null players, given by

$$f_i(v, \Omega) = \begin{cases} \frac{v(N)}{|N \setminus Null(v,\Omega)|} & \text{if } i \in N \setminus \text{Null}(v, \Omega) \\ 0 & \text{if } i \in \text{Null}(v, \Omega), \end{cases}$$

   where $\text{Null}(v, \Omega)$ denotes the set of null players in the restricted game $r_{v,\Omega}$, satisfies efficiency, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions. It does not satisfy additivity.

5. The zero solution given by $f_i(v, \Omega) = 0$ for all $i \in N$ satisfies additivity, the inessential player property for union closed games, the necessary player property and independence of irrelevant coalitions. It does not satisfy efficiency.

6 Irrelevant players and fairness

As mentioned before in Section 3, we can define and axiomatize the superior- and union rule also if we do not assume that the 'grand coalition' N is feasible in Definition 3.1. By condition 2 in that definition, the players that do not belong to the largest feasible subset of N do not belong to any feasible coalition. We refer to these players as *irrelevant* players.

For such a union closed system Ω we denote by $R(\Omega) = \{i \in N \mid \text{there is an } S \in \Omega \text{ with } i \in S\}$ the set of *relevant* players, i.e. the players that belong to at least one feasible coalition. Then we can define the superior rule and the union rule by applying these two rules to the game and union closed system restricted to $R(\Omega)$, and assign the payoff zero to all irrelevant players. The corresponding two rules can be axiomatized similar as done before in this paper, by adapting the axioms in a similar way (i.e. distinguishing between relevant and irrelevant players), and adding the axiom which states that irrelevant players get zero payoff.

Axiom 6.1 (Irrelevant player property) *For every game $v \in \mathcal{G}^N$ and union closed system $\Omega \in \mathcal{C}^N$, we have $f_i(v, \Omega) = 0$ for all $i \in N \setminus R(\Omega)$.*

As mentioned in the preliminaries, Myerson (1980) characterized the Myerson rule for conference structures by component efficiency[6] and fairness. Although a conference structure is any set of feasible coalitions on N, i.e. any subset of 2^N, by Myerson (1980)'s definition of connectedness all singletons are connected and thus earn their own worth in the restricted game. So, even singletons that are not feasible, in the sense that they do not belong to the conference structure, earn their worth in the restricted game. Note that in our approach we took $r_{v,\Omega}(\{i\}) = v(\{i\})$ only if $\{i\}$ is feasible, and $r_{v,\Omega}(\{i\}) = 0$ otherwise. Alternatively, in line with Myerson (1980) we could always take $r_{v,\Omega}(\{i\}) = v(\{i\})$ irrespective of whether $\{i\}$ is feasible or not.

Because of the definition of connectedness, and thus the restricted game, in Myerson (1980), it does not matter whether a conference structure does or does not contain $\{i\}$ for any $i \in N$. Consequently, a conference structure $\mathcal{F}$ yields the same Myerson rule payoffs as conference structure $\mathcal{F} \cup \{\{i\} \mid i \in N\}$. Considering the subclass of conference structures where all singletons are feasible (i.e. $\{i\} \in \mathcal{F}$ for all $i \in N$), the proof that there is a unique solution satisfying component efficiency and fairness is similar to that in Myerson (1980)[7]. However, for union closed systems typically we do not have $\{i\} \in \Omega$ for all $i \in N$, since the unique union closed system satisfying this property is $\Omega = 2^N$. Therefore, we only require the conditions in Definition 3.1.

Continuing our comparison with conference structures, we now discuss a fairness axiom for union closed systems similar as the one for conference structures. However,

[6]For any conference structure, two players are called connected if there is a feasible coalition that contains both players. Moreover, also all players are defined to be connected with themselves. A component in the conference structure then is a maximally connected set of players. Component efficiency states that the sum of payoffs over all players in one component equals the worth of that component in the game.

[7]Allowing $\{i\} \notin \mathcal{F}$ for some $i \in N$ the axiomatization can be stated adding the irrelevant player property.

while applying fairness to conference structures any coalition can be deleted from the set of feasible coalitions, for union closed systems we can only delete coalitions such that the remaining set of feasible coalitions is still union closed. (Similar restrictions on deleting feasible coalitions hold for other types of structures satisfying specific properties.) In other words, we can only delete coalitions that are not the union of other feasible coalitions.

Definition 6.2 *Let $\Omega \in \mathcal{C}^N$. A coalition $T \in \Omega$ is a* basis *coalition in Ω if there do not exist $U, V \in \Omega$ with $T = U \cup V$.*

Alternatively we can say that a coalition $T \in \Omega \in \mathcal{C}^N$ is a basis coalition in Ω if $\Omega \setminus \{T\} \in \mathcal{C}^N$.

Axiom 6.3 (Fairness) *For every game $v \in \mathcal{G}^N$, union closed system $\Omega \in \mathcal{C}^N$ and basis coalition $S \in \Omega$, we have $f_i(v, \Omega) - f_i(v, \Omega \setminus \{S\}) = f_j(v, \Omega) - f_j(v, \Omega \setminus \{S\})$ for all $i, j \in S$.*

The superior rule does not satisfy fairness, as can be seen by comparing the game $v = u_{\{2\}}$ on union closed systems $\Omega = \{\emptyset, \{1\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ and $\Omega \setminus \{\{2,3\}\}$ (on $N = \{1,2,3\}$), where $D^{\Omega} = \{(2,3)\}$ and $D^{\Omega \setminus \{\{2,3\}\}} = \{(1,2)\}$, and thus $SUP_2(v, \Omega) - SUP_2(v, \Omega \setminus \{\{2,3\}\}) = 1 - \frac{1}{2} = \frac{1}{2} \neq 0 = SUP_3(v, \Omega) - SUP_3(v, \Omega \setminus \{\{2,3\}\})$.

The union rule satisfies fairness. However, not allowing all coalitions to be deleted from the set of feasible coalitions restricts the applicability of the fairness axiom to characterize solutions.

Besides fairness being a weaker axiom on union closed systems than on arbitrary systems, also component efficiency is weak since, by union closedness, it boils down to just efficiency and the irrelevant player property. Efficiency, fairness and the irrelevant player property do not characterize the union rule for games on union closed systems. Another solution that satisfies these axioms on the class of games on union closed systems is the *modified union rule* where we take two disjoint coalitions of equal cardinality and in case both are feasible we subtract a fixed amount, say 1, from all players in one coalition and give it to all players in the other coalition. Formally, take two disjoint coalitions $S, T \subseteq N$ with $|S| = |T|$. Then the (S,T)-union rule is the rule $\overline{U}^{(S,T)}$ given by

$$\overline{U}^{(S,T)}(v, \Omega) = \begin{cases} U(v, \Omega) & \text{if } \{S, T\} \not\subset \Omega \\ \tilde{U}^{(S,T)}(v, \Omega) & \text{otherwise,} \end{cases}$$

where

$$\tilde{U}_i^{(S,T)}(v, \Omega) = \begin{cases} U_i(v, \Omega) + 1 & \text{if } i \in S \\ U_i(v, \Omega) - 1 & \text{if } i \in T \\ U_i(v, \Omega) & \text{otherwise.} \end{cases}$$

Note that the axioms discussed in the previous sections (see Theorems 4.10 and 5.6) all are applied to a fixed union closed system Ω. Applying axioms like fairness requires that we allow to change the set of feasible coalitions. This type of axiomatizations will be studied in future research.

7 Concluding remarks

In this paper we introduced two generalizations of the Shapley value to games on union closed systems. The superior rule is based on the conjunctive permission value of an associated game with permission structure, while the union rule is based on the Shapley value of the restricted game. We axiomatized both rules such that they differ in only one axiom. Both rules satisfy efficiency, additivity, the inessential player property for union closed games and the necessary player property. We obtain an axiomatization of the superior rule by strengthening the inessential player property for union closed games to the stronger inessential player property. We obtain an axiomatization of the union rule by adding independence of irrelevant coalitions.

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