TI 2005-064/1
Tinbergen Institute Discussion Paper

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Alternating Offers in Economic Environments*

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June, 2005

Abstract

The Nash bargaining solution of a modified bargaining problem in the contract space yields the pair of stationary subgame perfect equilibrium proposals in the alternating offers model, also for positive time between proposals. As time vanishes, convergence to the Nash bargaining solution is immediate by the Maximum Theorem. Numerical implementation in standard optimization packages is straightforward.

JEL Classification: C72 Noncooperative Games, C73 Stochastic and Dynamic Games, C78 Bargaining Theory

Keywords: Bargaining, Negotiation, Computation, Convex program, Nash bargaining solution, Maximum Theorem, Applied General Equilibrium

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*The author is indebted to Michiel Keyzer for addressing the computational implementation of game-theoretic solution concepts in applied economics. The usual disclaimer applies.

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1 Introduction

This note provides a powerful framework for characterizing stationary subgame perfect equilibrium (SSPE) proposals in alternating offers directly in an 'economic environment', as defined in Roemer (1988), that is attractive to both theoretical and applied economics. The main result states that for *all* discount factors, the pair of SSPE proposals in the economic environment corresponds to the asymmetric Nash bargaining solution of a *modified* bargaining problem in the economic environment, where the players' bargaining weights are related to the players' discount factors as in Binmore et al. (1986). This program is convex and allows to invoke the Maximum Theorem, see e.g. Varian (1992), which establishes existence of a convex set of maximizers. The program is even strict convex for the interior of the parameter space of time preferences and strictly quasi-concave utility functions and, then, the optimum is unique. Moreover, the Maximum Theorem also implies upper semi-continuity of the pairs of proposals in all parameters and, in particular, in 'the time between bargaining rounds', which becomes continuity in case of a unique optimum. Therefore, the Maximum Theorem immediately establishes the convergence in the *economic environment* to the asymmetric Nash bargaining solution as the time between proposals vanishes. Finally, the single convex program makes numerical implementation of SSPE proposals in the economic environment relatively easy and allows for robust computational algorithms that are widely available in many optimization packages, such as GAMS. This will enhance serious applications in the future including applied general equilibrium (AGE) modeling as surveyed in e.g. Ginsburgh and Keyzer (2002).

This note is organized as follows. In Section 2, we follow Roemer (1988) by taking the economic environment as the main primitive, which firmly roots our results within the class of AGE models. In this section we also state the equivalence between pairs of SSPE proposals and a single convex program for the utility representation, as recently shown in Houba (2005). Our main results are derived in Section 3 and numerical implementation discussed in Section 4.

2 The model

Consider the extended version of the alternating offers model with discounting in Rubinstein (1982), where the economic environment in Roemer (1988) replaces the dollar.[1] Formally, the two players are indexed $i = 1, 2$. The economy has $n \geq 2$ commodities, a vector of initial endowments $\omega^i \in \mathbb{R}^n_+$ for player i, total endowments $\omega = \omega^1 + \omega^2 > 0$ and monotonic and concave utility functions $u_i : \mathbb{R}^n_+ \to \mathbb{R}$. The subject of the negotiations is a feasible allocation $z = (z^1, z^2)$, $z^1, z^2 \in \mathbb{R}^n_+$ and $z^1 + z^2 \leq \omega$. We assume that the initial allocation is Pareto inefficient meaning the bargaining problem is essential.

The alternating offers procedure is as follows: Bargaining rounds are indexed $t \in \mathbb{N}$. At t odd, player 1 proposes the feasible allocation $z^t = (z^{1,t}, z^{2,t})$ and, then, player 2 accepts or rejects. Accept ends the negotiations. If rejected, then $e^{-r_2\Delta}$, $r_2 \geq 0$ and 'time between bargaining round' $\Delta \geq 0$, is the probability of a next (even) round. At t even, the players' roles are reversed and the probability of a next (odd) round is $e^{-r_1\Delta}$, $r_1 \geq 0$. Utilities over outcomes in the procedure are expected utilities. The equilibrium concept is subgame perfectness. Note that we allow for $\Delta = 0$ and all $(r_1, r_2) \geq [0, \infty)^2$, $(r_1, r_2) \neq (0, 0)$. Whenever it is convenient to write δ_i instead of $e^{-r_i\Delta}$ we do so.

Our analysis builds on a recent result in Houba (2005) for essential and convex bargaining problems in utility representation. Economic environments induce such bargaining problems, denoted as (S, d) with $S \in \mathbb{R}^2$ the nonempty, compact and convex set of feasible utility pairs, disagreement point $d \in S$ and there exists an $s \in S$ such that $s > d$, see e.g. Roemer (1988). The set of individually rational utility pairs is given by $S^{IR} = \{s \in S | s \geq d\}$ and the curve $s_i = f_i(s_j)$, $i, j = 1, 2$, $i \neq j$, describes it Pareto frontier. The function f_i is decreasing and concave. Furthermore, f_2 is the inverse function of f_1.

For convex bargaining problems, the alternating offers model in utility representation yields a unique subgame perfect equilibrium in stationary strategies. Denote $x \in S$, re-

[1] Alternatively, as in Nash (1950), we could have assumed (linear) expected utility functions on a simplex representing probability distributions on a discrete outcome space or, extend the analysis to concave nonexpected utility functions over this simplex, see e.g. Houba and Bolt (2002).

spectively, $y \in S$ as arbitrary SSPE proposal for player 1 and 2, respectively. Such SSPE proposals are Pareto efficient and correspond to $x = (x_1, f_2(x_1))$ and $y = (f_1(y_2), y_2)$. Houba (2005) shows that (x_1, y_2) is the unique solution of

$$\begin{aligned} (x_1, y_2) &= \arg \max_{s_1 \geq d_1; s_2 \geq d_2} (s_1 - d_1)^{\alpha} (s_2 - d_2)^{1-\alpha}, \\ \text{s.t.} \quad s_1 &\leq f_1((1-\delta_2) d_2 + \delta_2 s_2), \\ s_2 &\leq f_2((1-\delta_1) d_1 + \delta_1 s_1), \end{aligned} \tag{1}$$

where $\alpha \equiv \alpha(r_1, r_2, \Delta) = \ln \delta_2 / (\ln \delta_1 + \ln \delta_2) = r_2 / (r_1 + r_2) \in [0, 1]$ for all $\Delta \in [0, \infty)$, provided $(r_1, r_2) \neq (0, 0)$. In the unique optimum both constraints are binding, which yields the familiar fixed point problem as in e.g. Muthoo (1999) and Houba and Bolt (2002). Moreover, by the Maximum Theorem, the unique $(x_1(r_1, r_2, \Delta), y_2(r_1, r_2, \Delta))$ of (1) is continuous in r_1, r_2 and Δ. In particular, as the time between bargaining rounds vanishes, the limit $(x_1(r_1, r_2, 0), y_2(r_1, r_2, 0))$ exists and coincides with the Nash bargaining solution with $\alpha = r_2 / (r_1 + r_2)$.

3 The convex program

In this section, we first transform program (1) directly in terms of a single convex program for the economic environment and show the latter program corresponds to the SSPE in utility representation. Furthermore, we establish limit results when the time between bargaining rounds vanishes.

Stationary strategies in the economic environment prescribe the player-dependent allocation $\hat{x} = (\hat{x}^1, \hat{x}^2)$ and $\hat{y} = (\hat{y}^1, \hat{y}^2)$ for player 1 and 2, respectively. Feasibility imposes $\hat{x}^1 + \hat{x}^2 \leq \omega$ and $\hat{y}^1 + \hat{y}^2 \leq \omega$. The link with the utility representation goes through the equalities $x_i = u_i(\hat{x}^i)$, $y_i = u_i(\hat{y}^i)$ and $d_i = u_i(\omega^i)$, $i = 1, 2$. Furthermore, in any SSPE player 1 accepts $\hat{y}$ if and only if $u_1(\hat{y}^1) \geq (1-\delta_1) d_1 + \delta_1 u_1(\hat{x}^1)$, see e.g. Houba and Bolt (2002). Similar, $u_2(\hat{x}^2) \geq (1-\delta_2) d_2 + \delta_2 u_2(\hat{y}^2)$. The (in)equality constraints pose some minor problems with respect to the convexity of the program that can be circumvented by rewriting program (1) for the economic environment as

$$\max_{s_1 \geq d_1; s_2 \geq d_2; \hat{x}^i, \hat{y}^i \in R^n_+} (s_1 - d_1)^{\alpha} (s_2 - d_2)^{1-\alpha}, \tag{2}$$

$$\begin{aligned}
\text{s.t.} \qquad s_1 &\leq u_1\left(\hat{x}^1\right), \\
s_2 &\leq u_2\left(\hat{y}^2\right), \\
(1-\delta_1) d_1 + \delta_1 s_1 &\leq u_1\left(\hat{y}^1\right), \\
(1-\delta_2) d_2 + \delta_2 s_2 &\leq u_2\left(\hat{x}^2\right), \\
\hat{x}^1 + \hat{x}^2 &\leq \omega, & (p^x) \\
\hat{y}^1 + \hat{y}^2 &\leq \omega, & (p^y)
\end{aligned}$$

where p^x and p^y denote the vector of shadow prices associated with the player-dependent proposals. This program states the formula for the Nash bargaining solution in the modified economic environment for *all* parameter values. The assumptions made are sufficient to ensure that program (2) is a convex program that meets Slater's constraint qualification. Excluding the trivial dictatorship weights corresponding to $\alpha(r_1, r_2, \Delta)$ equal to 0 or 1, we establish that in any optimum of program (2) all six constraints are binding and, therefore such optimum coincides with the optimum of program (1).

Proposition 1 *Let $r_1, r_2 > 0$. All constraints in program (2) are binding.*

Proof.

In any optimum, Pareto efficiency of $\hat{x}$ and $\hat{y}$ and monotonic utility functions imply binding feasibility constraints in (2). Next, the other constraints in (2) impose

$$\begin{aligned}
s_1 &\leq \min\left\{u_1\left(\hat{x}^1\right), \delta_1^{-1} u_1\left(\hat{y}^1\right) - \delta_1^{-1}(1-\delta_1) d_1\right\}, \\
s_2 &\leq \min\left\{u_2\left(\hat{y}^2\right), \delta_2^{-1} u_2\left(\hat{x}^2\right) - \delta_2^{-1}(1-\delta_2) d_2\right\},
\end{aligned}$$

which are necessarily binding when $\alpha(r_1, r_2, \Delta) \in (0, 1)$. By symmetry of arguments, it suffices to investigate two cases.

1. $u_1(\hat{x}^1) < \delta_1^{-1} u_1(\hat{y}^1) - \delta_1^{-1}(1-\delta_1) d_1$ and $u_2(\hat{y}^2) < \delta_2^{-1} u_2(\hat{x}^2) - \delta_2^{-1}(1-\delta_2) d_2$. Then,

slightly increasing $\hat{x}^1$ and $\hat{y}^2$ at the expense of $\hat{x}^2$ and $\hat{y}^1$, respectively, relaxes both constraints and improves the objective. Hence, a contradiction. The argument can be accommodated if one of the inequalities is an equality.

2. $u_1(\hat{x}^1) < \delta_1^{-1} u_1(\hat{y}^1) - \delta_1^{-1}(1-\delta_1) d_1$ and $u_2(\hat{y}^2) > \delta_2^{-1} u_2(\hat{x}^2) - \delta_2^{-1}(1-\delta_2) d_2$. Then, substitution yields

$$\begin{aligned} (x_1 - d_1)^\alpha (y_2 - d_2)^{1-\alpha} &= \left(u_1(\hat{x}^1) - d_1\right)^\alpha \left(\delta_2^{-1}\left[u_1(\hat{x}^2) - d_2\right]\right)^{1-\alpha} \\ &< \left(\delta_1^{-1}\left[u_1(\hat{y}^1) - d_1\right]\right)^\alpha \left(u_2(\hat{y}^2) - d_2\right)^{1-\alpha}, \end{aligned}$$

because $\delta_1^\alpha = \delta_2^{1-\alpha}$ in e.g. Muthoo (1999) and Houba and Bolt (2002). So, the Pareto efficient allocation $\hat{x}$ maximizes the Nash product over $\hat{x}^1 + \hat{x}^2 \leq \omega$, which implies that $\hat{y}$ is infeasible. Hence, a contradiction. The argument can be accommodated if one of the inequalities is an equality, provided $\alpha(r_1, r_2, \Delta) \in (0, 1)$ to preserve the inequality in the Nash product. ■

Uniqueness of SSPE allocations can be easily extended, because positive bargaining weights and strictly quasi-concave utility functions make program (2) strictly convex. This extension generalizes the strict log-concavity required in Hoel (1986) for the 'divide the dollar'. Houba and Bolt (2002) provide nongeneric examples of multiplicity in the contract space, but they conclude that the generic case is that each pair of SSPE utilities corresponds to a unique pair of SSPE contracts in case of quasi-concave utility functions. These arguments combined with the Maximum Theorem yield the following convergence result.[2]

Theorem 2 *Let $(r_1, r_2) \neq (0, 0)$ and u_i quasi-concave. The set of maximizing allocations $\{\hat{x}(r_1, r_2, \Delta), \hat{y}(r_1, r_2, \Delta)\}$ in program (2) form a nonempty, compact, convex-valued and upper semi-continuous correspondence in r_1, r_2 and Δ (including $\Delta = 0$). Moreover, the set $\{\hat{x}(r_1, r_2, 0), \hat{y}(r_1, r_2, 0)\}$ exists and coincides with the set of Nash bargaining solutions with weight α. If additionally, $r_1, r_2 > 0$ and u_i strict quasi-concave, then the unique pair $\hat{x}(r_1, r_2, \Delta)$ and $\hat{y}(r_1, r_2, \Delta)$ are continuous in r_1, r_2 and Δ.*

[2] For completeness, upper semi-continuity in the parameters ω^1 and ω^2 also holds.

4 Numerical implementation

This section contains a brief discussion on the numerical implementation of program (2).

The reformulation of program (1) into the convex program (2) enhances numerical implementation of the alternating offers model in economic environments, because many optimization packages, such as GAMS, are nowadays available that allow for robust computational algorithms for convex programs. In AGE models, the Nash bargaining solution at $\Delta = 0$ is already popular and it features $\hat{x} = \hat{y} = z$. It can be implemented by less variables and constraints by solving

$$\max_{z^1, z^2 \in R_+^n} \left(u_1\left(z^1\right) - d_1\right)^{\alpha} \left(u_2\left(z^2\right) - d_2\right)^{1-\alpha}, \quad \text{s.t.} \quad z^1 + z^2 \leq \omega. \tag{3}$$

So, the additional costs in terms of additional variables and constraints of solving SSPE proposals in (2) instead of the Nash bargaining solution in (3) amounts to $2n + 2$ variables and $n + 4$ constraints of which 4 nonlinear.

The optimum of program (2) can be decentralized through markets and price-taking behavior on behalf of the two 'consumers'. In that case, the shadow prices p^x and p^y are the market clearing prices associated with allocations $\hat{x}$ and $\hat{y}$, respectively. By the Maximum Theorem, the shadow prices p^x and p^y are nonnegative and also form a nonempty, compact convex-valued and upper semi-continuous correspondence in the parameters r_1, r_2, $\Delta \geq 0$. By the Second Welfare Theorem, decentralization through markets requires (possibly negative) lump-sum taxation of player 1 equal to $T^x = p^x \left(\omega^1 - \hat{x}^1\right)$ in case the players agree to implement $\hat{x}$ and $T^y = p^y \left(\omega^1 - \hat{y}^1\right)$ for $\hat{y}$. Hence, program (2) facilitates studies of the first-mover advantage in terms of market clearing prices and lump-sum taxation. Decentralization is important in AGE modeling involving convex production technologies, because the extended program (2) then admits smaller subprograms solving each producer's profit maximization problem.

5 Concluding Remark

Quasi-concave utility functions are sufficient for the generic uniqueness of the optimum of (2). Such utility functions impose the generalized class of strongly comprehensive bargaining problems in utility representation (UR). Application of the method in Shaked and Sutton (1984) for two-player strictly comprehensive bargaining problems in UR implies that uniqueness in subgame perfect equilibrium strategies is equivalent to uniqueness in SSPE strategies. For such bargaining problems, the results in Houba (2005) can be extended such that each pair of SSPE proposals corresponds to one intersection point of the two constraints in (1) and in each intersection point the Nash product is tangent to the Pareto frontier spanned by these two constraints. Then, the maximum of program (1) corresponds to a pair of SSPE strategies in UR that is axiomatized in Kaneko (1980), whereas the set of all intersection point on the Pareto frontier of (1) is axiomatized as in Herrero (1989) and multiple intersection points may exists. These results for UR directly translate into local maxima and minima in (2). For economic environments, generic uniqueness of the optimum of (2) is therefore not sufficient to conclude overall uniqueness in SSPE strategies. What is needed is the stronger uniqueness in tangency points. A simple numerical test consists of minimizing the Nash product in (2) and reversing the inequality signs. If the minimization program yields the same value for the objective as (2), then overall uniqueness is obtained.

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