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Money Supply and the Implementation of Interest Rate Targets[1]

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Abstract

In this paper, we analyze the relation between interest rate targets and money supply in a (bubble-free) rational expectations equilibrium of a standard cash-in-advance model. We examine contingent monetary injections aimed to implement interest rate sequences that satisfy interest rate target rules. An interest rate target with a positive inflation feedback in general corresponds to money growth rates rising with inflation. When prices are not completely flexible, this implies that a non-destabilizing money supply cannot implement a forward-looking and active interest rate rule. This principle also applies for an alternative model version with an interest elastic money demand. The implementation of a Taylor-rule then requires a money supply that leads to explosive or oscillatory equilibrium sequences. In contrast, an inertial interest rate target can be implemented by a non-destabilizing money supply, even if the inflation feedback exceeds one, which is often found in interest rate rule regressions.

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Non-technical summary

The stance of monetary policy is nowadays commonly summarized and announced in terms of a target for a short-run nominal interest rate. Correspondingly, people pay less attention to the behavior of monetary aggregates. The supply of reserves in open market operations, nevertheless, serves as the main instrument of most central banks. The supply and the demand for (narrow) money can thus be viewed to be "*relevant only to the question of how the central bank must adjust the instruments under its direct control so as to implement its interest-rate operating targets*" (Woodford, 2003, p. 106). Accordingly, the behavior of monetary aggregates seems to be irrelevant for the analysis of optimal monetary policy regimes, which are specified in terms of interest rate targets, as for example in form of Taylor-rules.

This paper provides a theoretical analysis, which is aimed to disclose whether money supply is really negligible for macroeconomic effects of monetary policy. The analysis, which is conducted within the current "consensus" macroeconomic framework with frictionless financial markets, i.e., the New Keynesian model, takes a closer look at the supply of money that is adjusted by the central bank in a state contingent way in order to implement a particular interest rate target. There are several possible outcomes to which this analysis might lead. One result, which would be consistent with the conventional view sketched above, would show that any interest rate target can be implemented by a particular money supply in an intuitive way. Thereby, a rise in the supply of money would be accompanied by a decline in the interest rate, such that both instruments would indicate a monetary expansion. This result is, however, not found in this paper.

Instead, the analysis in this paper leads to results which seem to be inconsistent with the conventional view. Firstly, it is found that the relation between money supply and interest rate targets is less intuitive. In particular, an increase in money supply is in general associated with higher nominal interest rates. This result, which is due to the lack of a liquidity effect, implies that an expansionary money supply is accompanied by a change in the interest rate, which looks like a contractionary monetary stance. However, the empirical evidence on liquidity effects is not unambiguous. Hence, the relation between money and interest rate as revealed in the analysis is not necessarily counterfactual, though it is different than expected or described in most undergraduate macroeconomic textbooks. Secondly, it turns out that highly stylized interest rate targets, in particular forward-looking Taylor-rules that are designed to stabilize inflation, cannot be implemented by non-destabilizing adjustments of the outstanding stock of money, i.e., by money supply procedures that for example avoid hyperinflations.

At first view, these results seem to challenge the conventional approach to monetary policy analysis, given that these types of rules are widely applied in the literature. Yet, the third main result in the paper qualifies this view: An active interest rate target can be implemented by a stabilizing (and intuitive) money supply procedure only if the interest rate target is sufficiently inertial. Notably, the latter property, which is commonly found in interest rate rule regressions, can hardly be rationalized by an optimizing central bank in the consensus macroeconomic framework. Thus, interest rate inertia might rather be an implication of a stabilizing adjustments of the underlying instrument than an indicator for an independent central bank preference in favor of smooth interest rate path. However, the analysis shows that a particular money supply procedure is consistent with infinitely many inertial interest rate targets. Hence, standard macroeconomic theory predicts that stabilizing money supply adjustments are consistent with interest rate targets as found in the data. Yet, it cannot explain how one specific target is implemented.

Overall, the findings in this paper show that the theoretical relation between money supply and interest rate targets is less intuitive than expected. On the one hand, the lack of a strong liquidity effect implies that an immediate rise in the interest rate in response to higher inflation can only be brought about by an accommodating money supply. Further, it is responsible for the central bank not to be able to implement an aggressive interest rate target by a stabilizing money supply procedure, if the interest rate target is purely forward looking. On the other hand, standard macroeconomic theory predicts that inertial interest rate targets, which are consistent with empirical evidence, can be implemented by a non-accommodating money supply, though in a non-unique way. These findings indicate that the common assumption of frictionless financial (money) markets hinders a full understanding of monetary policy implementation. Nevertheless, this assumption, which facilitates the macroeconomic analysis of monetary policy, can be viewed as a reasonable simplification, provided that it matters only for the impact of changes in money supply on interest rates. Otherwise, money market frictions might be non-negligible even for macroeconomic effects of monetary policy.

1 Introduction

It is now standard practice to describe the stance of monetary policy in terms of a short-run nominal interest rate. Correspondingly, it has become increasingly popular in recent macroeconomic theory to characterize central bank behavior by targets for the (risk-free) one-period nominal interest rate (see Woodford, 2003a). In many contributions to this literature it is, in particular, assumed that monetary policy can be summarized by simple interest rate feedback rules, such as the well-known Taylor (1993) rule. Notwithstanding, short-run nominal interest rates serve as an operating target for most real world central banks, while the supply of reserves, e.g., via transfers or open market operations, acts as the policy instrument that implements certain interest rate targets: "*In fact, of course, any particular interest rate policy must be implemented by a specific money supply policy, and this monetary policy must be implemented by a policy of fiscal transfers, open market operations, or both.*" (Lucas, 2000, p. 258). In accordance with this view, monetary policy might as well be described by a state contingent money supply that implements a sequence of nominal interest rates satisfying a particular target (rule).

The purpose of this paper is twofold. Firstly, we examine the structural relation between money supply and interest rates in a dynamic general equilibrium model with infinitely lived households and frictionless financial markets. Secondly, we derive characteristics of interest rate targets that are implemented by money supply regimes in a rational expectations equilibrium. Thereby, we focus on fundamental (or bubble-free) equilibrium solutions, which satisfy common equilibrium selection criteria (see Blanchard and Kahn, 1980, or McCallum, 1983, 1999). In contrast to related studies, we allow money supply to be contingent on endogenous variables and, thus, depart from the common practice to restrict money growth rates to be constant or exogenous. The latter approach is, for example, applied in Alvarez et al. (2002) or Monnet and Weber (2001), where the equilibrium relation between money growth and interest rates in a framework with segmented financial markets and flexible prices is examined. This paper is further related to Minford et al. (2002) and Auray and Feve (2003), who assess the observational equivalence between interest rate rules and exogenous money growth rules, and to Vegh (2001), who identifies basic equivalences between these rules in a continuous time framework. Finally, our paper relates to the research on interest rate responses to money growth shocks, which has revealed that unanticipated monetary injections tend rather to raise than to lower the nominal interest rate, i.e., the so-called "liquidity puzzle" (see, e.g., Christiano et al., 1997).[3]

The main results derived in our benchmark model with a Lucas (1982)-type cash-in-advance specification can be summarized as follows. With frictionless financial markets, the consumption Euler equation essentially governs the equilibrium relation between the money growth rate and the risk-free nominal interest rate. A higher nominal (real) interest rate

[3] This paper can further be viewed as complementary to comparisons of welfare effects under interest rate and exogenous money growth rules (see Honkapohja and Evans, 2003, Collard and Dellas, 2005, Gavin et al., 2004, or Schmidt-Grohe and Uribe, 2004).

induces consumption and, thus, nominal (real) balances to grow, since money demand is characterized by a positive income elasticity. Hence, a nominal interest rate target that rises with inflation (and output), is associated with money growth rates also rising with inflation (and output). As long as prices are flexible, forward-looking interest rate rules and state contingent reaction functions for the money growth rate can further be equivalent, in the sense that the fundamental solutions to the rational expectations equilibrium under both policy descriptions are identical.

When prices are imperfectly flexible, i.e., set in a staggered way, the initial price level can be determined in equilibrium for any policy regime. This implies that a money growth policy extends the state space, as it induces the equilibrium sequences to depend on the initial stock of money, which is predetermined. Thus, an exact equivalence between the latter policy regime and a forward-looking interest rate policy is obviously not possible. The policy induced history dependence under money growth policy further implies that a non-destabilizing money supply cannot implement equilibrium interest rate sequences that satisfy a forward-looking target rule by which the nominal interest rate is raised by more than one for one with inflation (active interest rate target). The reason is that the latter causes the equilibrium condition for consumption to become unstable, which is associated with an unique solution in a forward-looking environment, whereas it leads to divergent equilibrium sequences under a money growth regime, since the history dependent evolution of money, which is linked to consumption, becomes explosive. Notably, if the central bank is assumed to be able to control end-of-period real balances without relying on cash injections, then the implemented sequence of interest rates can satisfy an active target rule in a locally stable equilibrium. Moreover, a forward-looking interest rate target rule can be equivalent to a forward-looking rule for the real value of the end-of-period money stock that, actually, serves as an alternative operating target, but not as a monetary policy instrument.[4]

To assess the robustness of the main results, we further consider an alternative money demand specification. In particular, we allow for money demand to be interest elastic and for its income elasticity to deviate from one, by applying a money-in-the-utility function specification. The analysis of the equilibrium behavior of interest rates under sticky prices reveals that state contingent adjustments of the money growth rate cannot implement a Taylor(1993)-type interest rate target rule without causing equilibrium sequences to become explosive or oscillatory. In contrast, an *inertial* interest rate target that exhibits a (short-run or long-run) coefficient on inflation that exceeds one can be implemented by a non-destabilizing money supply, if the feedback coefficient on the lagged interest rate in the target rule is sufficiently high. This finding suggests that significant coefficients on lagged interest rates in interest rate rule regressions are rather a consequence of a history dependent money supply, which implements the operating target, than an indicator for interest rate inertia

[4] Policy equivalance then requires end-of-period real balances to be independent of the real or nominal value of the stock of money held by households at the beginning of the same period.

being a fundamental component of an optimal monetary policy.[5] Put differently, while it is known from the theoretical literature on monetary policy analyses that interest rate inertia cannot be rationalized by a central bank aiming at maximizing the "true social objective" (see, e.g., Woodford, 2003b), it is a property of the equilibrium sequence of interest rates implemented by money supply adjustments.

The remainder is organized as follows. In section 2 we develop the benchmark model. Section 3 examines the equilibrium relation between money supply and interest rates under flexible and sticky prices. In section 4 we examine an alternative money demand specification, i.e., the money-in-the-utility-function approach, and particularly focus on inertial interest rate targets. Section 5 concludes.

2 The model

This section presents a model with Lucas' (1982) cash-in-advance specification. To facilitate comparisons with the voluminous literature on New Keynesian macroeconomics we allow prices to be imperfectly flexible due to a staggered price setting. The economy is further hit by cost-push shocks, which are introduced via a stochastic wage mark-up. Throughout the paper, nominal variables are denoted by upper-case letters, while real variables are denoted by lower-case letters.

There is a continuum of households indexed with $j \in [0,1]$. Households have identical asset endowments and identical preferences. Household j maximizes the expected sum of a discounted stream of instantaneous utilities u :

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{jt}, l_{jt}\right), \tag{1}$$

where E_0 is the expectation operator conditional on the time 0 information set, and $\beta \in (0,1)$ is the subjective discount factor. The instantaneous utility U is assumed to be increasing in consumption c, decreasing in working time l, strictly concave, twice continuously differentiable, and to satisfy the usual Inada conditions. Instantaneous utility U is further assumed to be separable in private consumption and working time, $U\left(c_t, l_t\right) = u(c_{jt}) - v(l_{jt})$.

At the beginning of period t household j is endowed with holdings of money M_{jt-1} and a portfolio of state contingent claims on other households yielding a (random) payment Z_{jt}. Before the goods market opens, households enter the asset market, where they can adjust their portfolio and receive government transfers. Let $q_{t,t+1}$ denote the period t price of one unit of currency in a particular state of period $t+1$ normalized by the probability of occurrence of that state, conditional on the information available in period t. Then, the price of a random payoff Z_{jt+1} in period $t+1$ is given by $E_t[q_{t,t+1} Z_{jt+1}]$. The households further receive wage payments and dividends D_{it} from monopolistically competitive firms indexed

[5] See Rudebusch (2002) for a critical assessment of monetary policy inertia.

by $i \in [0,1]$. The asset market constraint of household j can be written as

$$M_{jt} \leq M_{jt-1} + Z_{jt} - E_t[q_{t,t+1} Z_{jt+1}] + P_t w_{jt} l_{jt} + P_t \tau_t - P_t c_{jt} + \int_0^1 D_{j,it} di, \tag{2}$$

where P_t denotes the aggregate price level, w_{jt} the (individual) real wage rate, and τ_t a lump-sum transfer. We further assume that households have to fulfill a no-Ponzi game condition, $\lim_{i\to\infty} E_t q_{t,t+i}(M_{jt+i}+Z_{jt+1+i}) \geq 0$. After they leave the asset markets, households enter the goods market, where they rely on liquid funds for transactions. In particular, consumption expenditures are restricted by the following cash-in-advance constraint, which relates to the specification in Lucas (1982):

$$P_t c_{jt} \leq M_{jt-1} + Z_{jt} - E_t[q_{t,t+1} Z_{jt+1}] + P_t \tau_t. \tag{3}$$

According to (3), the payoff from state contingent claims net of investments in a new portfolio can be used for consumption purchases. Moreover, lump-sum transfers $P_t \tau_t$ which households receive in the asset market raise the amount of liquid funds and, thus, alleviate the goods market restriction. These injections serve as the central bank's *instrument* to influence the private sector.[6]

We assume that households monopolistically supply differentiated labor services. Differentiated labor services l_j are transformed into aggregate labor input l_t where $l_t^{1-1/\eta_t} = \int_0^1 l_{jt}^{1-1/\eta_t} dj$. The elasticity of substitution between differentiated labor services $\eta_t > 1$ varies exogenously over time (see below). Cost minimization then leads to the following demand for differentiated labor services l_{jt}, $l_{jt} = (w_{jt}/w_t)^{-\eta_t} l_t$, with $w_t^{1-\eta_t} = \int_0^1 w_{jt}^{1-\eta_t} dj$, where w_t denotes aggregate real wage rate. Maximizing the objective (1), subject to the budget constraint (2), the cash-in-advance constraint (3), the labor demand condition, and the no-Ponzi-game condition, for given initial values Z_{j0} and $M_{j,-1}$ leads to the following first order conditions:

$$\begin{aligned} u_c(c_{jt}) &= \lambda_{jt} + \psi_{jt}, \quad v_l(l_{jt}) = \xi_t^{-1} w_{jt} \lambda_{jt}, \\ \lambda_{jt} &= \beta E_t \frac{\lambda_{jt+1} + \psi_{jt+1}}{\pi_{t+1}}, \quad q_{t,t+1} = \frac{\beta}{\pi_{t+1}} \frac{\lambda_{jt+1} + \psi_{jt+1}}{\lambda_{jt} + \psi_{jt}}, \end{aligned} \tag{4}$$

and the goods market constraint (3), $\psi_t \left(M_{t-1} + Z_t - E_t[q_{t,t+1} Z_{t+1}] + P_t \tau_t - P_t c_t \right) = 0$ and $\psi_t \geq 0$, where π denotes the inflation rate ($\pi_t \equiv P_t/P_{t-1}$), λ the shadow price of wealth, and ψ the Lagrange multiplier on the cash constraint. Further, $\xi_t \equiv \eta_t/(\eta_t - 1)$ denotes the stochastic wage mark-up, which will below be discussed in more detail. Furthermore, the budget constraint (2) holds with equality and the transversality condition, $\lim_{i\to\infty} \beta^{t+i} E_t[\lambda_{jt+i}(M_{jt+i} + Z_{jt+1+i})/P_{t+i}] = 0$, must be satisfied. The one-period nominal interest rate on a risk-free portfolio is defined as follows

$$R_t = [E_t q_{t,t+1}]^{-1}. \tag{5}$$

[6] Note that this specification is equivalent to a specification where open market operations – instead of lump-sum monetary injections – serve as the monetary policy instrument (see below).

In accordance with recent contributions to the monetary policy literature (see, e.g., Woodford, 2003a), we will consider the risk-free interest rate R_t as the central bank's *operating target.*

The final consumption good is an aggregate of differentiated goods produced by monopolistically competitive firms indexed with $i \in [0,1]$. The CES aggregator of differentiated goods is defined as $y_t^{\frac{\epsilon-1}{\epsilon}} = \int_0^1 y_{it}^{\frac{\epsilon-1}{\epsilon}} di$, with $\epsilon > 1$, where y_t is the number of units of the final good, y_{it} the amount produced by firm i, and ϵ the constant elasticity of substitution between these differentiated goods. Let P_{it} and P_t denote the price of good i set by firm i and the price index for the final good. The demand for each differentiated good is $y_{it} = (P_{it}/P_t)^{-\epsilon} y_t$, with $P_t^{1-\epsilon} = \int_0^1 P_{it}^{1-\epsilon} di$. A firm i produces good y_i employing a technology which is linear in the labor input: $y_{it} = l_{it}$, where $l_t = \int_0^1 l_{it} di$. Hence, labor demand satisfies: $mc_{it} = w_t$, where $mc_{it} = mc_t$ denotes real marginal costs.

We allow for a nominal rigidity in form of staggered price setting as developed by Calvo (1983). Each period firms may reset their prices with the probability $1-\phi$ independently of the time elapsed since the last price setting. The fraction $\phi \in [0,1)$ of firms are assumed to adjust their previous period's prices according to the simple rule $P_{it} = \overline{\pi} P_{it-1}$, where $\overline{\pi}$ denotes the average inflation rate. Firms maximize their market value, which equals the expected sum of discounted dividends $E_t \sum_{s=0}^{\infty} q_{t,t+s} D_{it+s}$, where $D_{it} \equiv (P_{it} - P_t mc_t) y_{it}$.[7] In each period a measure $1-\phi$ of randomly selected firms set new prices $\widetilde{P}_{it}$ as the solution to $\max_{\widetilde{P}_{it}} E_t \sum_{s=0}^{\infty} \phi^s q_{t,t+s} (\overline{\pi}^s \widetilde{P}_{it} y_{it+s} - P_{t+s} mc_{t+s} y_{it+s})$, s.t. $y_{it+s} = (\overline{\pi}^s \widetilde{P}_{it})^{-\epsilon} P_{t+s}^{\epsilon} y_{t+s}$. The first order condition for the price of re-optimizing producers is for $\phi > 0$ given by

$$\widetilde{P}_{it} = \frac{\epsilon}{\epsilon - 1} \frac{E_t \sum_{s=0}^{\infty} \phi^s \left[q_{t,t+s} y_{t+s} P_{t+s}^{\epsilon+1} \overline{\pi}^{-\epsilon s} mc_{t+s} \right]}{E_t \sum_{s=0}^{\infty} \phi^s \left[q_{t,t+s} y_{t+s} P_{t+s}^{\epsilon} \overline{\pi}^{(1-\epsilon)s} \right]}. \tag{6}$$

Aggregate output is given by $y_t = (P_t^*/P_t)^{\epsilon} l_t$, where $(P_t^*)^{-\epsilon} = \int_0^1 P_{it}^{-\epsilon} di$ and thus $(P_t^*)^{-\epsilon} = \phi \left(\overline{\pi} P_{t-1}^* \right)^{-\epsilon} + (1-\phi) \widetilde{P}_t^{-\epsilon}$. If prices are flexible, $\phi = 0$, then the first order condition for the optimal price of the differentiated good reads: $mc_t = \frac{\epsilon-1}{\epsilon}$.

The central bank is assumed to trade with households in the asset markets. There, it also injects money via lump sum transfers $P_t \tau_t$. Its budget is thus given by $P_t \tau_t = M_t - M_{t-1} = (\mu_t - 1) M_{t-1}$, where μ_t denotes the gross money growth rate. It should be noted that we can, alternatively, assume that money and government bonds are exclusively traded in open market operations, where their supply is characterized by "holding fiscal policy constant in the face of a government asset exchange", such that the government budget constraint can be written as $M_t - M_{t-1} = -(B_t - R_{t-1} B_{t-1})$, with B_t denoting government bonds (see Sargent and Smith, 1987). Together with a consistent goods market constraint, which then reads $P_t c_{jt} \le M_{jt-1} + Z_{jt} - E_t[q_{t,t+1} Z_{jt+1}] + R_{t-1} B_{jt-1} - B_{jt}$, this specification is equivalent, i.e.,

[7] It should be noted that the application of the households' stochastic discount factor $q_{t,t+s}$ in principle implies that dividends also deliver a liquidity value ψ, i.e., can be used for purchases of consumption goods. We neglected this property to specify the cash-constraint in a conventional way. Nonetheless, the inclusion of dividends on the right hand side of (3) would not (qualitatively) change any result in this paper, since dividends are either constant (for $\phi = 0$) or – in equilibrium – solely a function of current output (for $\phi > 0$).

leads to an identical set of equilibrium conditions.[8]

We assume that the central bank controls the money growth rate $\mu_t = m_t\pi_t/m_{t-1}$ (where m denotes real balances $m_t = M_t/P_t$) contingent on the current information set. Thus, money supply is specified as a function of the exogenous state or of endogenous variables that depend on the current state. In particular, we apply the following reaction function for money supply

$$\mu_t = \pi_t^{\mu_\pi} y_t{}^{\mu_y} \xi_t^{\mu_\xi} e^{\kappa_\mu}, \tag{7}$$

where μ_π, μ_y, and μ_ξ are arbitrary constants, and κ_μ is assumed to ensure that the average value for the money growth rate satisfies $\mu \geq 1$. The reaction function (7) encompasses money supply specifications which can typically be found in the literature, such as exogenous money growth rules, $\mu_\pi = \mu_y = 0$, or money growth rules that depend on changes in current output or inflation, $\mu_\xi = 0$, (see, e.g., McCallum, 1999). The money supply specification (7) is related to rules for the interest rate target, which are also specified in a log-linear form. To be more precise, we consider the following rule for the interest rate target[9]

$$R_t = \pi_t^{\rho_\pi} y_t^{\rho_y} \xi_t^{\rho_\xi} e^{\kappa_\rho}, \qquad \text{with} \quad R_t \geq 1. \tag{8}$$

In the last part of our analysis, we further consider lagged values of the interest rate as a determinant of the current interest rate target. It should be noted that the specification (8) is more general as for example Taylor(1993)-type rules, where a feedback from exogenous states is usually neglected, $\rho_\xi = 0$. We allow for $\rho_\xi > 0$, as it might be a reasonable policy when prices are flexible. Note that the constant κ_ρ can be chosen to ensure that the nominal interest rate exceeds its lower bound, $R_t > 1$.

In equilibrium, households' net wealth solely consists of money balances, since there is no borrowing or lending between them, $Z_t = 0$. The cash-in-advance constraint (3) therefore reduces to $P_t c_t \leq M_{t-1} + P_t\tau_t$. Using that all markets clear (including $c_t = y_t$) and the budget constraint of the central bank, gives the following relation between money growth and output $P_t y_t \leq \mu_t M_{t-1}$. The equilibrium is defined as follows.

Definition 1 *A rational expectations equilibrium is a set of sequences* $\{y_t, l_t, P_t^*, P_t, \widetilde{P}_t, mc_t, w_t, m_t, R_t, \mu_t\}_{t=0}^\infty$ *satisfying the firms' first order conditions* $mc_t = w_t$, *(6) with* $\widetilde{P}_{it} = \widetilde{P}_t$, *and* $P_t^{1-\epsilon} = \phi\left(\overline{\pi}P_{t-1}\right)^{1-\epsilon} + (1-\phi)\,\widetilde{P}_t^{1-\epsilon}$, *the households' first order conditions* $u_c(y_t)w_t = R_t v_l(l_t)\xi_t$, $u_c(y_t)/P_t = \beta R_t E_t\left[u_c(y_{t+1})/P_{t+1}\right]$, $P_t y_t = \mu_t m_{t-1}P_{t-1}$ *for* $R_t > 1$ *and* $P_t y_t \leq \mu_t m_{t-1}P_{t-1}$ *for* $R_t = 1$, $\mu_t = m_t\pi_t/m_{t-1}$, *the aggregate resource constraint* $y_t = (P_t^*/P_t)^\epsilon l_t$, *where* $(P_t^*)^{-\epsilon} = \phi\left(\overline{\pi}P_{t-1}^*\right)^{-\epsilon} + (1-\phi)\widetilde{P}_t^{-\epsilon}$, *a monetary policy satisfying (7) or (8), and the transversality condition, for a given sequence* $\{\varepsilon_t\}_{t=0}^\infty$, *and initial values* $P_{-1} > 0$, $P_{-1}^* > 0$, *and* $m_{-1}P_{-1} = M_{-1} > 0$.

[8] The households' budget constraint would then be given by $B_{jt} + M_{jt} \leq R_{t-1}B_{jt-1} + M_{jt-1} + Z_{jt} - E_t[q_{t,t+1}Z_{jt+1}] + P_t w_{jt} l_{jt} - P_t c_{jt} + \int_0^1 D_{j,it}di$, and the first order condition on bond holdings by $u_c(c_{jt}) = \beta R_t E_t \frac{u_c(c_{jt+1})}{\pi_{t+1}}$. A consistent initial value for total government liabilities would be equal to zero, $B_{j,-1} + M_{j,-1} = 0$, which ensures government solvency in any stable equilibrium.

[9] In the last part of our analysis, we further consider lagged values of the interest rate as a determinant of the current interest rate target, $R_t = R_{t-1}^{\rho_R}\pi_t^{\rho_\pi} y_t^{\rho_y} \xi_t^{\rho_\xi} e^{\kappa_\rho}$.

The wage mark-up is assumed to follow the stochastic process $\xi_t = \eta_t/(\eta_t - 1) = \xi^{1-\rho}\xi_{t-1}^{\rho}\exp(\varepsilon_t)$, where $\rho \in [0,1)$ and $\xi > 1$. Taking logs, one therefore obtains $\widetilde{\xi}_t = \rho\widetilde{\xi}_{t-1} + (1-\rho)\,\overline{\xi} + \varepsilon_t$, where $\overline{\xi} > 0$ and $\widetilde{x}_t$ denotes the log of a generic variable x_t, $\widetilde{x}_t = \log(x_t)$, and $\overline{x} = E_0\widetilde{x}_t$. The innovations ε_t are assumed to be normally distributed with mean zero and a constant variance, $\varepsilon_t \sim N(0, var_e)$. The stochastic process can also be written as $\widehat{\xi}_t = \rho\widehat{\xi}_{t-1} + \varepsilon_t$, where $\widehat{\xi}_t = \widetilde{\xi}_t - \overline{\xi}$. Throughout the paper, we further assume that the bounds on the mark-up fluctuations are sufficiently tight, such that the central bank can ensure the nominal interest rate to be larger than one, $R_t > 1$, and that the cash-in-advance constraint is binding.[10]

3 Money supply and interest rate targets

In this section, we examine the relation between money supply and interest rates in the cash-in-advance model. To be more precise, we aim at disclosing the equilibrium behavior of the risk-free nominal interest rate R_t under a state contingent money supply (7). We, further, assess conditions for policy regimes to be equivalent. Thereby, we apply the following definition of policy equivalence.

Definition 2 *Two policy regimes are equivalent if they are consistent with the same fundamental solution to the rational expectations equilibrium.*

Throughout the paper we focus on the model's fundamental solution, also known as the bubble-free solution, which satisfies commonly used equilibrium selection devices. For example, the fundamental solution is identical with the unique stable solution (see Blanchard and Kahn 1980), and it satisfies the minimum state variable criterion (see McCallum, 1983).[11] It should be noted that we do not restrict our attention to policy rules that ensure the rational expectations equilibrium to exhibit a unique solution. Nevertheless, we disregard non-fundamental solutions, i.e., solutions with extraneous states.

3.1 Efficient Interest Rate Targets

In the first part of this section we consider the case where prices are perfectly flexible ($\phi = 0$).[12] The wage mark-up leads to a macroeconomic distortion that accompanies the cash distortion brought about by the goods market restriction (3). The latter distortion is minimized by small values for the nominal interest rate, whereas a minimization of the former distortion requires the central bank to stabilize the effects from mark-up shocks. As a consequence, the Friedman rule will not be optimal if the variance of $\widetilde{\xi}_t$ is sufficiently large. Thus, an interest rate policy that stays away from the lower bound in order to undo the mark-up

[10] This can, for example, be induced by a sufficiently high value for κ_ρ in the interest rate target rule (8) and a data generating process of the stochastic variable $\widetilde{\xi}_t$ approximated by a finite state representation with a sufficiently small support (see, e.g., Tauchen, 1986).

[11] See McCallum (2004) for the relation of these equilibrium selection devices.

[12] Under certainty, the Friedman rule ($R_t = 1$) is, evidently, Pareto-optimal under flexible prices. Then, the cash-in-advance constraint is not binding, $y_t \leq \mu_t m_{t-1}\pi_t^{-1}$, such that the equilibrium allocation is consistent with any money growth rate, that satisfies the transversality condition.

shock effects can enhance welfare (see appendix 1.3). In any case, an efficient interest rate target under flexible prices satisfies $\widetilde{R}_t = \rho_\xi \widetilde{\xi}_t + \kappa_\rho$, where $\rho_\xi \leq 0$ and $\kappa_\rho \geq 0$.

When prices are imperfectly flexible, which will be assumed in the subsequent section, an efficient policy regime should predominantly aim at stabilizing inflation, as shown in several recent studies on optimal monetary policy (see, e.g., Woodford, 2003a, or Schmitt-Grohe and Uribe, 2004). Following Ravenna and Walsh's (2005) welfare analysis in a model that is (in reduced form) isomorphic to the model in this paper, it can further be shown that an interest rate target under sticky prices and discretionary optimization takes the form $\widehat{R}_t = \rho_\pi \widehat{\pi}_t$, where $\rho_\pi > 0$ (see appendix 1.3). An optimal commitment policy is in general history dependent if the private sector is forward-looking. To account for this property, we will also consider the case where the current nominal interest rate is adjusted contingent on changes in lagged output.

3.2 Policy equivalence under flexible prices

For the case where prices are flexible, we assume, for convenience, that the utility function exhibits constant elasticities of intertemporal substitution, $U^f(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\vartheta}}{1+\vartheta}$, and that $\sigma > 0$ and $\vartheta > 0$. Combining the conditions, $y_t^{-\sigma} w_t = R_t l_t^{\vartheta} \xi_t$, $w_t = \frac{\epsilon - 1}{\epsilon}$, and $y_t = l_t$, the equilibrium can be summarized as follows: A rational expectations equilibrium of the flexible price model ($\phi = 0$) with $R_t > 1$ and $U = U^f$ is a set of sequences $\{\pi_t, y_t, m_t, R_t, \mu_t\}_{t=0}^{\infty}$ satisfying

$$(\epsilon - 1)/\epsilon = \xi_t R_t y_t^{\vartheta+\sigma}, \quad y_t^{-\sigma} = \beta R_t E_t[y_{t+1}^{-\sigma}/\pi_{t+1}], \quad y_t = \mu_t m_{t-1}/\pi_t, \tag{9}$$

the transversality condition, $\lim_{i\to\infty} \beta^{t+i} \frac{\overline{\xi}\epsilon}{\epsilon-1} E_t m_{t+i}^{1+\vartheta} = 0$, and a monetary policy either in terms of R_t or $\mu_t = m_t \pi_t / m_{t-1}$, for a given sequence $\{\xi_t\}_{t=0}^{\infty}$, a given initial value $M_{-1} > 0$, and any initial price level satisfying $P_{-1} > 0$. Reducing the model and taking logs leads to the following linear equilibrium conditions in $\widetilde{m}_t = \log m_t$, $\widetilde{R}_t$, and $\widetilde{\pi}_t : \widetilde{m}_t = -(\vartheta+\sigma)^{-1} \widetilde{R}_t - (\vartheta+\sigma)^{-1} \widetilde{\xi}_t + \log(1 - 1/\epsilon)$, and $-\sigma \widetilde{m}_t = \widetilde{R}_t - \sigma E_t \widetilde{m}_{t+1} - E_t \widetilde{\pi}_{t+1} + \frac{1}{2} var_t(-\sigma \widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}) + \widetilde{\beta}$, and a policy $\widetilde{R}_t$ or $\widetilde{\mu}_t$.

The question we want to answer is how money supply has to look like in order to lead to identical equilibrium sequences as a feedback rule for the nominal interest rate. Given that the model is sufficiently simple, we do not restrict our attention to stationary equilibria.[13] After eliminating real balances, the model can be reduced to the following equilibrium condition

$$\begin{aligned} -\vartheta \widetilde{R}_t = {} & \sigma E_t \widetilde{R}_{t+1} - \sigma(1-\rho)\widetilde{\xi}_t - (\vartheta+\sigma) E_t \widetilde{\pi}_{t+1} + (\vartheta+\sigma)\widetilde{\beta} + \sigma(1-\rho)\overline{\xi} \\ & + \frac{\vartheta+\sigma}{2} var_t \left[\frac{\sigma}{\vartheta+\sigma} \left(\widetilde{R}_{t+1} + \widetilde{\xi}_{t+1} \right) - \widetilde{\pi}_{t+1} \right]. \end{aligned} \tag{10}$$

Given that output does not appear in the equilibrium condition (10), we focus – for simplicity – on the case where the interest rate feedback rule (8) is only a function of the exogenous

[13] This approach is applied in the next section for a local approximation to the model when prices are assumed to be imperfectly flexible.

state and of current inflation, $\rho_y = 0$. Taking logs, the feedback rule (8) reads:

$$\widetilde{R}_t = \widetilde{R}_t\left(\widetilde{\pi}_t, \widetilde{\xi}_t\right) = \rho_\pi \widetilde{\pi}_t + \rho_\xi \widetilde{\xi}_t + \kappa_\rho, \tag{11}$$

where κ_ρ is a constant that ensures $\widetilde{R}_t > 0$. Given that there is no backward-looking element, the fundamental solution does not feature an endogenous state variable. Eliminating the nominal interest rate with (11) in (10), gives a stochastic difference equation in inflation. The fundamental solution for inflation can then be derived in a straightforward way, by using the generic form $\widetilde{\pi}_t = \gamma_{\pi e}\widetilde{\xi}_t + \psi_\pi$, where ψ_π collects terms that are independent of $\widetilde{\xi}_t$. In particular, the coefficient $\gamma_{\pi e}$ is given by $\gamma_{\pi e} = [(\rho - 1)\,\sigma + (\vartheta + \rho\sigma)\,\rho_\xi]/[\rho\,(\sigma + \vartheta) - \rho_\pi\,(\vartheta + \rho\sigma)]$. In contrast to ψ_π, which is given in appendix 1.1, the coefficient $\gamma_{\pi e}$ is independent of the moments of the exogenous state. It should further be noted that the existence of the fundamental solution for inflation requires $\rho > 0$ or $\rho_\pi > 0$.[14]

Once the fundamental solution is derived, one can easily identify a money growth policy which implements identical equilibrium sequences. For a money supply policy to be equivalent to (11), the money growth rate has to be set contingent on the current state. In fact, we can always find a money growth reaction function of the form $\widetilde{\mu}_t = \mu_\pi \widetilde{\pi}_t + \kappa_\mu$ or $\widetilde{\mu}_t = \mu_\xi \widetilde{\xi}_t + \kappa_\mu^*$ that is equivalent to (11).[15]

Lemma 1 *Consider the interest rate feedback rule (11). If $\rho > 0$ or $\rho_\pi > 0$, there exists an equilibrium solution without endogenous states and infinitely many money supply reaction functions of the form $\widetilde{\mu}_t = \mu_\pi \widetilde{\pi}_t + \mu_\xi \widetilde{\xi}_t + \kappa_\mu$ that are equivalent. An equivalent money supply reaction function satisfying $\widetilde{\mu}_t = \mu_\pi \widetilde{\pi}_t + \kappa_\mu$ is uniquely determined and is characterized by*

$$\mu_\pi = \frac{\rho_\pi + (\sigma - 1)\,\rho}{\sigma\rho} + \frac{\rho_\xi}{\sigma\rho}\frac{\rho\,(\sigma + \vartheta) - \rho_\pi\,(\vartheta + \rho\sigma)}{(\rho - 1)\,\sigma + (\vartheta + \rho\sigma)\,\rho_\xi}, \tag{12}$$

while κ_μ collects terms that are independent of the current state.

Proof. See appendix 1.2.

Further details on the term κ_μ are presented in appendix 1.2.[16] Some notable implications of the equivalence condition (12) should now be discussed. Consider the case where the central bank aims to undo all mark-up shocks, which is a reasonable policy if the variance var_e is sufficiently large (see appendix 1.3). In order to eliminate uncertainty, the interest rate target (11) has to satisfy $\rho_\xi = -1$ and $\rho_\pi = 0$. Thus, (12) implies the response of an equivalent money growth policy to inflation to be equal to one, $\mu_\pi = 1$, $\forall t \geq 1$. The supply of nominal balances then rises one for one with inflation, such that real balances grow

[14] The solution for real balances (and output) can, nevertheless, for $\rho_\pi = 0$ be obtained from $\widetilde{m}_t = \widetilde{y}_t = -(1 + \rho_\xi)\,(\vartheta + \sigma)^{-1}\,\widetilde{\xi}_t + \log(1 - 1/\epsilon)$.

[15] Evidently, a money growth policy $\widetilde{\mu}_t = \mu_\pi \widetilde{\pi}_t + \kappa_\mu$ can be transformed to $\widetilde{\mu}_t = \mu_\xi \widetilde{\xi}_t + \kappa_\mu^*$, by using the fundamental inflation solution, i.e., both are equivalent for $\mu_\xi = \mu_\pi \gamma_{\pi e}$ and $\kappa_\mu^* = \mu_\pi \psi_\pi + \kappa_\mu$.

[16] For $\rho = 0$ and $\rho_\pi > 0$, it can for example be written as $\kappa_\mu = \psi_\pi \frac{\rho_\xi}{\gamma_{e\pi}} - \kappa_\rho + (\sigma\mu_\pi + 1 - \sigma)\,(\psi_\pi + \gamma_{e\pi}) - (\frac{\sigma - \sigma\rho_\pi - \vartheta\rho_\xi}{\rho\sigma + \rho\vartheta - \vartheta\rho_\pi - \rho\sigma\rho_\pi})^2 \frac{1}{2} var_e$

with a constant rate κ_μ : $\widetilde{\mu}_{t+1} = \widetilde{m}_{t+1} + \widetilde{\pi}_{t+1} - \widetilde{m}_t = \widetilde{\pi}_{t+1} + \kappa_\mu \Leftrightarrow m_{t+1}/m_t = e^{\kappa_\mu}$. Next, consider an endogenous interest rate rule, $\rho_\xi = 0$ and $\rho_\pi \geq 0$ that has often been applied in theoretical studies (see, e.g., Benhabib et al., 2001, Carlstrom and Fuerst, 2001, or Dupor, 2001). Equivalence then requires the following condition to be satisfied $\sigma\mu_\pi = \rho_\pi\rho^{-1} + \sigma - 1$. According to this condition, an interest rate peg, $\rho_\xi = \rho_\pi = 0$, is equivalent to a money supply satisfying $\mu_\pi = \frac{\sigma-1}{\sigma}$ and therefore implies money supply to be accommodating, i.e., to be positively related to inflation, if $\sigma > 1$. The inflation feedback of equivalent money supply and interest rate reaction functions are further positively related. Some main implications are summarized in the following proposition.

Proposition 1 *Consider an interest rate rule (11) and a money supply policy satisfying* $\widetilde{\mu}_t = \mu_\pi\widetilde{\pi}_t + \kappa_\mu$. *Then, i.) an interest rate policy that eliminates uncertainty,* $\rho_\xi = -1$ *and* $\rho_\pi = 0$*, is equivalent to a money growth policy satisfying* $\mu_\pi = 1$*, and ii.) an active interest rate policy,* $\rho_\pi > 1$ *and* $\rho_\xi = 0$*, is equivalent to an accommodating money growth policy* $\mu_\pi > 0$*. If households are risk averse or risk neutral,* $\sigma \geq 1$*, then iii.) any interest rate policy satisfying* $\rho_\pi \geq 0$ *and* $\rho_\xi = 0$ *implies money supply to be accommodating, while iv.) a constant money growth policy is equivalent to an interest rate policy with a non-positive inflation feedback,* $\rho_\pi \leq 0$ *and* $\rho_\xi = 0$*.*

One main implication of proposition 1 is that when the central bank aims to raise its interest rate target with inflation, then it has to increase the money growth rate with inflation, if $\sigma \geq 1$ (see *iii.*) Put differently, an interest rate target that might be meant to stabilize inflation, $\rho_\pi > 0$, is associated with an accommodating money supply, which is at odds with the view that a central bank should reduce the supply of money when inflation is high (see, e.g., McCallum, 1999). This relation between money growth rates and interest rates is implied by the consumption Euler equation and the money demand specification. It will be shown in the subsequent sections that this main property will also emerge under sticky prices or alternative money demand specifications with a positive output elasticity.

Regarding the issue of equilibrium solution multiplicity, it can immediately be shown that if a money growth policy $\widetilde{\mu}_t = \mu_\pi\widetilde{\pi}_t + \kappa_\mu$ satisfies $\frac{\sigma-1}{\sigma} < \mu_\pi < 1$ or $1 + \frac{2}{\vartheta-\sigma} < \mu_\pi$ and $\sigma < \vartheta$, then the fundamental solution is the unique solution (see appendix 1.4). On the other hand, an interest rate rule, $\widetilde{R}_t = \rho_\pi\widetilde{\pi}_t + \kappa_\rho$, ensures equilibrium determinacy if $\rho_\pi \in (1, 1 + \frac{\vartheta}{\sigma})$, or if $\rho_\pi > 1 + \frac{\vartheta}{\sigma}$ for $\vartheta > \sigma$, and $\rho_\pi \in \left(1 + \frac{\vartheta}{\sigma}, 1 + \frac{2\vartheta}{\sigma-\vartheta}\right)$ for $\vartheta < \sigma$.[17] Suppose, for example, that the intertemporal elasticities of substitution for consumption and labor are equal to one, $\sigma = \vartheta = 1$ and that $\rho = 0.75$. Then, an interest rate rule $\rho_\pi = 1.5$ is associated with a uniquely determined equilibrium, i.e., the minimum state solution, which can equivalently be implemented by a money supply satisfying $\widetilde{\mu}_t = 2\widetilde{\pi}_t + \kappa_\mu$. The latter, however, allows for multiple solutions to the rational expectations equilibrium. Hence, a policy equivalence does, evidently, not imply these policies to be associated with the same determinacy properties.

[17] Note that without a cash-credit friction the determinacy condition would be just $\rho_\pi > 1$ (see, e.g., Woodford, 2003a).

3.3 Money supply and interest rates under sticky prices

Turning to the sticky price case ($\phi > 0$), we focus on stationary equilibrium sequences and apply a first-order Taylor-expansion of the model at the steady state with a target inflation rate $\overline{\pi} : \overline{\pi} \geq \beta$. Hence, the model's second moments are now irrelevant for the solution. Monetary policy rules, which will only be applied in a log-linearized form, are assumed to be consistent with the same steady state. The rational expectations equilibrium of the log-linear model is defined as follows: A rational expectations equilibrium of the log-linear approximation to the model at the steady state with sticky prices ($\phi > 0$) and $R_t > 1$ is a set of sequences $\{\widehat{\pi}_t, \widehat{m}_t, \widehat{y}_t, \widehat{R}_t\}_{t=0}^{\infty}$ satisfying

$$\sigma\widehat{y}_t = \sigma E_t\widehat{y}_{t+1} - \widehat{R}_t + E_t\widehat{\pi}_{t+1}, \tag{13}$$

$$\widehat{\pi}_t = \beta E_t\widehat{\pi}_{t+1} + \omega\widehat{y}_t + \chi\widehat{R}_t + \chi\widehat{\xi}_t, \tag{14}$$

$$\widehat{y}_t = \widehat{m}_t, \tag{15}$$

where $\omega = \chi(\vartheta + \sigma)$, $\sigma = -u_{cc}(\overline{c})\overline{c}/u_c(\overline{c}) > 0$ and $\vartheta = v_{ll}(\overline{l})\overline{l}/v_l(\overline{l}) > 0$, and $\chi = (1-\phi)(1-\beta\phi)/\phi$ (see Yun, 1996), and the transversality condition, for a monetary policy in terms of $\widehat{R}_t$ or $\widehat{\mu}_t = \widehat{m}_t + \widehat{\pi}_t - \widehat{m}_{t-1}$ given a sequence $\{\widehat{\xi}_t\}_{t=0}^{\infty}$, and initial values for nominal balances M_{-1} and the price level P_{-1}. The equilibrium conditions (13) and (14) differ from the standard New Keynesian model (see Clarida et al., 1999) just with regard to the gross nominal interest rate entering the aggregate supply constraint (14). It should be noted that the main results are qualitatively unchanged if one applies the more conventional specification of the aggregate supply constraint $\widehat{\pi}_t = \beta\widehat{\pi}_{t+1} + \omega\widehat{y}_t$.

3.3.1 Implementing interest rate targets

When prices are imperfectly flexible, policy equivalence between a money supply reaction function and an interest rate target rule are less likely, as will be shown below. We, therefore, focus on the equilibrium relation between a state contingent money supply rule (7) and the implemented sequence for the nominal interest rate. Thus, we examine how the central bank has to supply money to implement a particular endogenous interest rate target.[18] Thereby, we assume that the latter takes the form $\widehat{R}_t = \rho_\pi\widehat{\pi}_t + \rho_y\widehat{y}_t$, which relates for example to the interest rate rule suggested by Taylor (1993). The model (13)-(15) is closed by the following log-linearized version of the money supply reaction function (7)

$$\widehat{m}_t + \widehat{\pi}_t - \widehat{m}_{t-1} = \widehat{\mu}_t = \mu_\pi\widehat{\pi}_t + \mu_y\widehat{y}_t + \mu_\xi\widehat{\xi}_t. \tag{16}$$

The fundamental (minimum state) solution of the sticky price model under (16) turns out to feature beginning-of-period real money $\widehat{m}_{t-1}$ as a relevant predetermined state variable. The fundamental solution thus exhibits the generic form $\widehat{m}_t = \widehat{y}_t = \delta_m\widehat{m}_{t-1} + \delta_{me}\widehat{\xi}_t$, $\widehat{\pi}_t = \delta_{\pi m}\widehat{m}_{t-1} + \delta_{\pi e}\widehat{\xi}_t$, and $\widehat{R}_t = \delta_{Rm}\widehat{m}_{t-1} + \delta_{Re}\widehat{\xi}_t$, where δ_m is the single eigenvalue. The following

[18] A discussion of the vantages of endogenous interest rate rules ($\rho_\xi = 0$) can be found in Woodford (2003a).

lemma summarizes the main properties of the fundamental solution under a money supply policy (16) and an interest rate policy.

Lemma 2 *Consider the fundamental solution to the sequences* $\{\widehat{\pi}_t, \widehat{m}_t, \widehat{y}_t, \widehat{R}_t\}_{t=0}^{\infty}$ *satisfying (13)-(15), and a monetary policy.*

1. *Suppose that the money growth rate satisfies (16). Then, the equilibrium sequences of inflation and output depend on beginning-of-period real money balances* $\partial\widehat{\pi}_t \backslash \partial\widehat{m}_{t-1} = \delta_{\pi m} \neq 0$ *if* $\mu_\pi \neq 1$ *and* $\partial\widehat{y}_t \backslash \partial\widehat{m}_{t-1} = \delta_m \neq 0$. *The equilibrium sequences are locally stable and non-oscillatory,* $\delta_m \in (0,1)$, *only if* $\mu_\pi < 1 + \varkappa_1 \mu_y$, *where* $\varkappa_1 = \frac{\beta+\chi-1}{\omega}$.

2. *Suppose that the risk-free nominal interest rate satisfies* $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$. *Then, the equilibrium sequences are locally stable and unique if and only if* $\rho_\pi > 1 + \varkappa_1 \rho_y$ *for* $\vartheta \geq \sigma$, *or* $\rho_\pi \in \left(1 + \varkappa_1 \rho_y,\ \varkappa_2 + \varkappa_3 \rho_y\right)$ *for* $\vartheta < \sigma$, *where* $\varkappa_2 = \frac{2\sigma+\omega+2\sigma\beta}{\chi(\sigma-\vartheta)}$ *and* $\varkappa_3 = \frac{1+\beta+\chi}{\chi(\sigma-\vartheta)}$.

Proof. See appendix 2.1.

It should be noted that the condition in the first part of lemma 2 does not ensure uniqueness of the equilibrium sequences. It can be shown that there is only one positive and stable eigenvalue, $\delta_m \in (0,1)$, if and only if $\mu_\pi \in (\widetilde{\mu}_{1\pi}, \widetilde{\mu}_{2\pi})$ or $\mu_\pi < \min\{\widetilde{\mu}_{1\pi}, \widetilde{\mu}_{2\pi}, \widetilde{\mu}_{3\pi}\}$ for $\sigma < \vartheta$ or $\widetilde{\mu}_{3\pi} < \mu_\pi < \min\{\widetilde{\mu}_{1\pi}, \widetilde{\mu}_{2\pi}\}$ for $\sigma > \vartheta$, where $\widetilde{\mu}_{1\pi} = 1 + (\beta + \chi)\left(\mu_y - 1\right)(\sigma\chi)^{-1}$, $\widetilde{\mu}_{2\pi} = 1 + \mu_y \varkappa_1$, and $\widetilde{\mu}_{3\pi} = 1 + \left(\mu_y - 2\right)\varkappa_3$ (see appendix 2.1).

If the central bank controls the nominal interest rate – rather than the money growth rate – contingent on $\widehat{\pi}_t$ and $\widehat{y}_t$, then all endogenous variables are (under the fundamental solution) solely a function of the exogenous state $\delta_m = \delta_{\pi m} = 0$. It should be noted that equilibrium uniqueness, requires interest rates to be raised by more than one for one with inflation as in a model without a cash-credit-distortion (see Woodford, 2001). The latter is however responsible for the exact determinacy condition to be more severe than in the standard case. As can be seen from the second part of lemma 2, a positive response to changes in output $\rho_y > 0$, tends to raise the lower bound on the inflation coefficient ρ_π, given that $\varkappa_1$ is likely to be positive. In any case, an interest rate policy can, obviously, not lead to the identical fundamental solution as the money growth policy (16). Next, we characterize the properties of an endogenous interest rate target under a money supply (16).

Lemma 3 *Consider a central bank that controls the supply of money according to (16). Under the fundamental solution, there exists a unique value* μ_ξ^*, *such that the central bank implements an equilibrium sequence of interest rates satisfying* $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$ *if* $\mu_\xi = \mu_\xi^*$. *This target rule for the nominal interest rate is characterized by*

$$\rho_\pi = [\sigma(\mu_\pi - 1) + 1]\,\delta_m \quad \text{and} \quad \rho_y = \mu_y \delta_m. \tag{17}$$

Further, there exists no equivalent interest rate rule satisfying (8).

Proof. See appendix 2.2

The solution for μ_ξ^* can be found in appendix 2.2.[19] Note that the condition for the response of money supply to the exogenous state, $\mu_\xi = \mu_\xi^*$, ensures that the nominal interest rate target only depends on inflation and output. The feedback from the latter variables, which is governed by the coefficients ρ_π and ρ_y, then relies on the coefficients μ_π and μ_y in (16). The equations in (17) reveal that the responses of money supply and the interest rate to changes in inflation and output are positively related, if the equilibrium sequences are non-oscillatory, $\delta_m > 0$. The following proposition summarizes some main implications.

Proposition 2 *Suppose that the central bank supplies money in a way that ensures the equilibrium sequences to be locally stable. Then, i.) the central bank has to increase money supply with output to implement an equilibrium sequence of interest rates satisfying* $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$ *with* $\rho_y \geq 0$*, ii.) an inflation feedback satisfying* $\rho_\pi > 1$ *requires real balances to grow with inflation,* $\mu_\pi > 1$*, and iii.) an interest rate target satisfying* $\rho_\pi > 1 + \varkappa_1 \rho_y$ *for* $\vartheta \geq \sigma$*, or* $\rho_\pi \in \left(1 + \varkappa_1 \rho_y,\ \varkappa_2 + \varkappa_3 \rho_y\right)$ *for* $\vartheta < \sigma$ *cannot be implemented.*

Proof. The claims made in *i.)* and *ii.)* immediately follow from (17) in lemma 3, $\delta_m \in (0,1)$ and $\sigma > 0$. Part *iii.)* follows from part 2 of lemma 2, which implies that if the equilibrium interest rate sequence satisfies $\rho_\pi > 1 + \varkappa_1 \rho_y$ for $\vartheta \geq \sigma$ and $\rho_\pi \in \left(1 + \varkappa_1 \rho_y,\ \varkappa_2 + \varkappa_3 \rho_y\right)$ for $\vartheta \leq \sigma$, then there cannot be a stable eigenvalue, which is required for stable equilibrium sequences under a money growth policy. ■

The results on the relation between money supply and the interest rate target in part *i.)* and *ii.)* of proposition 2 accord to the properties of equivalent policies under flexible prices (see proposition 1). Part *iii.)* of proposition 2 presents a novel result. A simple active interest rate target satisfying $\rho_\pi > 1$ and $\rho_y = 0$ (for $\vartheta \geq \sigma$), cannot be implemented by the central bank if it supplies money according to a state contingent money growth rate (16). The reason is that such an interest rate target requires the existence of two unstable eigenvalues, which necessarily leads to explosive equilibrium sequences if there exists a relevant predetermined state variable. Under a money growth policy, the supply of money and, therefore, consumption depend on the beginning-of-period stock of money, such that the central bank induces the economy to evolve in a history dependent way.[20] Stability then requires the existence of at least one stable eigenvalue which is inconsistent with the active interest rate target.

More generally, a forward-looking interest rate target that ensures the absence of stable eigenvalues (see part 2 of lemma 2) leads to unstable equilibrium sequences if the central bank aims to implement this target by a history dependent money supply. This result is evidently not particular to interest rate targets that depend on current realizations of endogenous variables. Given that the fundamental solution implies $\partial E_t \widehat{\pi}_{t+1} / \partial \widehat{\pi}_t = \delta_m$, the results summarized in proposition 2 also apply for an interest rate rule featuring expected future inflation rates, $\widehat{R}_t = \rho_\pi E_t \widehat{\pi}_{t+1} + \rho_y \widehat{y}_t$.

[19] For $\rho = 0$ it is for example given by $\mu_\xi^* = (\mu_\pi - 1)\,\chi[\delta_m(\beta + \chi + \vartheta\chi - \mu_y - \vartheta\mu_\pi + 1) + \delta_m^2(\sigma\chi - \chi - \beta + \beta\mu_y + \chi\mu_y - \sigma\chi\mu_\pi) - 1]^{-1}$.

[20] It should be noted that this property relies on the fact that the equilibrium sequence of the price level can be determined for all periods (nominal determinacy) if prices are imperfectly flexible.

It should be noted that the results in this section are derived for interest rate rules, by which the central bank adjusts the particular interest rate target contingent on changes in current inflation and current output. Under this restriction it has been shown that some target rules cannot be implemented by a contingent money growth policy leading to stable equilibrium sequences (see proposition 2). If one relaxes this restriction and considers interest rate reaction functions, which feature the exogenous state variable, $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t + \rho_\xi \widehat{\xi}_t$, there is a multiplicity of interest rate rules that are consistent with a money supply satisfying (16).[21] Thus, for this class of interest rate rules there is no simple relation between the money supply behavior and the interest rate feedback coefficients. Yet, this type of interest rate reaction function is usually not used as a description of interest rate targets (see Woodford, 2003a). Further, the stochastic term can, evidently, not be interpreted as a policy shock.

3.3.2 Equivalence under alternative policy specifications

While the benchmark policy rule specifications cannot be equivalent under sticky prices, we now briefly examine alternative specifications of monetary policy that facilitate equivalences. To assess the possibility of interest rate policy to be equivalent to a money supply reaction function satisfying (16), we consider an alternative rule for the interest rate target, by which the current interest rate is related to lagged output, $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_{ly} \widehat{y}_{t-1}$.[22] Given that $\widehat{m}_t = \widehat{y}_t$, we can easily derive a money growth reaction function, $\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t$, which leads to a fundamental solution that is identical with the one under the backward-looking interest rate rule. The following proposition presents the relation between equivalent policies for the case where the autocorrelation of the mark-up shocks is assumed to be zero, for convenience.

Proposition 3 *Suppose that $\rho = 0$. Then, a backward-looking interest rate policy $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_{ly} \widehat{y}_{t-1}$ is equivalent to a money supply satisfying $\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t$, where $\mu_\pi = 1 + \left(\rho_\pi / \rho_{ly}\right)$ and $\mu_y = 1 + \frac{(1-\delta_m)(\mu_\pi - 1)\sigma + 1}{\rho_\pi - \delta_m}$.*

Proof. See appendix 2.3.

Hence, the main principle regarding the structural behavior of interest rates and money growth rates also applies in this case: When interest rates rise with inflation, $\rho_\pi > 0$, an equivalent money supply has to be accommodating, $\mu_\pi > 1$. According to proposition 3, policy equivalence relies on $\rho_{ly} \neq 0$. Regarding the issue of local stability it can be shown that the backward-looking interest rate rule has to satisfy $\rho_\pi < 1 + \frac{\beta + \chi - 1}{\omega} \rho_{ly}$ for $\rho_{ly} \geq 0$ and $\beta + \chi > 1$, in order to be consistent with non-explosive and non-oscillatory equilibrium

[21] Using the fundamental solutions for output and inflation under (16), which take the form $\widehat{m}_t = \delta_m \widehat{m}_{t-1} + \delta_{me} \widehat{\xi}_t$ and $\widehat{\pi}_t = \delta_{\pi m} \widehat{m}_{t-1} + \delta_{\pi e} \widehat{\xi}_t$, we get the relation $\widehat{y}_t = \frac{\delta_m}{\delta_{\pi m}} \widehat{\pi}_t + (\delta_{me} - \frac{\delta_m}{\delta_{\pi m}} \delta_{\pi e}) \widehat{\xi}_t$. Thus, starting with an interest rate target rule described in part two of lemma 2, there are infinitely many equivalent equilibrium relations of the form $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t + \rho_\xi \widehat{\xi}_t$.

[22] Such a history dependent interest rate policy, can for example serve as a characterization of a monetary policy regime that aims to minimize a quadratic loss function, featuring inflation and output(gap) variances as arguments, under commitment (see Woodford, 2003a).

sequences (see appendix 2.4). Thus, a backward-looking interest rate policy should not be too aggressive (active) to implement stable equilibrium sequences.

We now demonstrate that policy equivalence can further be established if money supply is specified in a non-backward-looking way. Suppose that the central bank aims to control the real value of the end-of-period stock of money $\widehat{m}_t$. Given that monetary injections rather than (end-of-period) real balances serve as the central bank instrument, this can be viewed as an alternative operating target for the central bank. If it applies a forward-looking rule for this target, such as $\widehat{m}_t = \eta_\pi \widehat{\pi}_t + \eta_y \widehat{y}_t$, then one can easily derive conditions for the equivalence between such a real balance rule and a forward-looking interest rate rule.[23]

Proposition 4 *Consider the target rule $\widehat{m}_t = \eta_\pi \widehat{\pi}_t + \eta_y \widehat{y}_t$ for end-of-period real balances. Then, there exists a continuum of equivalent interest rate rules of the form $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$. If $\rho_\pi = 0$ or $\rho_y = 0$, then there exists a unique equivalent interest rate rule. For $\rho_y = 0$, it satisfies $\rho_\pi = \rho - (1-\rho)\,\sigma\eta_\pi/(1-\eta_y)$.*

Proof. The consumption Euler equation $\sigma E_t \widehat{y}_{t+1} - \sigma \widehat{y}_t = \widehat{R}_t - E_t \widehat{\pi}_{t+1}$ and $\widehat{m}_t = \widehat{y}_t$, lead to the equilibrium relation $\left(1-\eta_y\right) \widehat{R}_t = \left(\left(1-\eta_y\right) + \sigma\eta_\pi\right) E_t \widehat{\pi}_{t+1} - \sigma\eta_\pi \widehat{\pi}_t$. The fundamental solution under the interest rate rule (and the real balance rule) reads $\widehat{y}_t = \delta_{ye} \widehat{\xi}_t$ and $\widehat{\pi}_t = \delta_{\pi e} \widehat{\xi}_t$ (and thus $E_t \widehat{\pi}_{t+1} = \rho \widehat{\pi}_t$), such that $\widehat{R}_t = [\left(\left(1-\eta_y\right) + \sigma\eta_\pi\right)\rho - \sigma\eta_\pi](1-\eta_y)^{-1}\widehat{\pi}_t$. Thus, one can construct multiple equivalent interest rate rules of the form $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$. ■

Rearranging the condition given in proposition 4, $\eta_\pi = \frac{1-\eta_y}{1-\rho}\left(\rho_\pi - \rho\right)/\sigma$, shows that an active interest rate target rule $\rho_\pi > 1$, is equivalent to a real balance target rule with a negative feedback from inflation, $\eta_\pi < 0$. The relation between these targets seems to be more intuitive than the relation between interest rate targets and money growth rates described in proposition 2. Evidently, there is a corresponding relation between end-of-period real balance targets and state contingent money growth reaction functions.

It should further be noted that interest rate smoothing, where lagged values of the interest rate are included in the policy rule ($\partial \widehat{R}_t / \partial \widehat{R}_{t-1} = \rho_r$) does not facilitate equivalence. With a non-zero coefficient on past interest rates, interest rate policy can certainly induce any degree of history dependence. However, the policy rule parameters can, in general, not be set in a way that reproduces the structural relations between inflation, real balances and the interest rate implied by a money growth regime.[24] The implementation of inertial interest rate targets will subsequently be discussed in detail when money demand is interest elastic.

4 Interest elastic money demand

In this section we want to assess the robustness of the main results derived for the sticky price case. For this we introduce an alternative money demand specification that is widely used in

[23] Evidently, such a real balance rule is equivalent to a particular money growth rule satisfying $\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_{ly} \widehat{y}_{t-1}$, where $\mu_{ly} = -1$, $\mu_\pi = \eta_\pi + 1$, and $\mu_y = \eta_y$, given that $\widehat{y}_t = \widehat{m}_t$.

[24] The reason is that the fundamental solution now features the lagged nominal interest rate as the single endogenous state variable, which implies that the remaining coefficients of the fundamental solution are only indirectly affected via their impact on the eigenvalue $\delta_r \equiv \widehat{R}_t / \widehat{R}_{t-1}$ (see Schabert, 2003).

the literature, namely, the money-in-the-utility function approach. Consider that households have preferences which are characterized by real balances entering the utility function. This specification can be rationalized as a short-cut for a more explicit specification of money demand, such as a shopping time specification (see Brock, 1974). In particular, we apply a specification where end-of-period money holdings enter the utility function in a separable way (see, e.g., Woodford, 2003a):

$$U^m(c_t, l_t, M_t/P_t) = U(c_t, l_t) + \nu(M_t/P_t), \tag{18}$$

where $\nu(m_t)$ is strictly increasing, concave, and twice continuously differentiable. Log-linearization of the households' first order conditions at a steady state with $\overline{R} > 1$ and a positive elasticity of intertemporal substitution of money $\sigma_m \equiv -\frac{\overline{m}\overline{\nu}_{mm}}{\overline{\nu}_m} > 0$ leads to the following money demand condition

$$\sigma_m \widehat{m}_t = \sigma \widehat{c}_t - \left(\overline{R} - 1\right)^{-1} \widehat{R}_t. \tag{19}$$

Compared to the benchmark specification, money demand, evidently, takes a more general form under a money-in-the-utility function specification, since it is now also interest rate elastic and exhibits a consumption (income) elasticity that might be different from one. Due to the absence of the cash-constraint (3), the aggregate supply relation (14) now reads $\widehat{\pi}_t = \beta E_t \widehat{\pi}_{t+1} + \omega \widehat{y}_t + \chi \widehat{\xi}_t$. The latter, the aggregate demand condition (13), money demand (19), and a monetary policy then characterize the rational expectations equilibrium.

As in the former version, the fundamental solution of the model exhibits no endogenous state variable for a non-backward-looking interest rate rule, while a money growth policy induces the economy to evolve in a history dependent way. Thus, these policy regimes can again not be equivalent. In this section we will explicitly consider interest rate targets to be inertial, i.e., to satisfy

$$\widehat{R}_t = \rho_R \widehat{R}_{t-1} + \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t. \tag{20}$$

Introducing lagged interest rates as a relevant state variable does in general not facilitate an equivalence. Yet, it will turn out that the previous results on the implementability of interest rate targets are altered when the central bank's interest rate target is sufficiently inertial. The novel results thereby rely on the property of money demand to be interest rate elastic. The following lemma presents the stability and determinacy conditions for the equilibrium sequences under both policy regimes. The results for interest rate policy correspond to those in Woodford (2001, 2003a).

Lemma 4 *Suppose that households' preferences are given by (18) and that prices are imperfectly flexible ($\phi > 0$).*

1. *When the money growth rate satisfies (16), there exists a stable and non-oscillatory solution to the rational expectations equilibrium if and only if*

$$\mu_\pi + \varkappa_4 \mu_y < 1. \tag{21}$$

Further, there exists no other stable and non-oscillatory solution if (21) is satisfied.

2. *When the risk-free interest rate satisfies (20), there exists a stable and uniquely determined solution to the rational expectations equilibrium if and only if*

$$\rho_\pi + \varkappa_4 \rho_y > 1 - \rho_R. \tag{22}$$

where $\varkappa_4 = (1-\beta)/\omega > 0$.

Proof. The proof of the first part is given in appendix 3.1. A proof of the second part can be found in Woodford (2003a).

It should be noted that there are also stable but oscillatory solutions if (21) is violated.[25] Evidently, stability demands the monetary policy instrument not to rise too strongly with inflation and output under a money growth regime, whereas local equilibrium uniqueness for the latter interest rate rule requires exactly the opposite. Turning to the properties of the implied interest rate equilibrium sequences under a money supply (16), we find that the relation derived in the previous section is qualitatively unaffected for a forward-looking interest rate target rule. Yet, the fact that money demand is interest elastic changes the relation for an inertial interest rate target rule (20). To implement this type of interest rate targets, money supply has to be adjusted contingent on lagged states in the following way

$$\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_\xi \widehat{\xi}_t + \mu_{l\xi} \widehat{\xi}_{t-2}. \tag{23}$$

The history dependence of an (inertial) interest rate targets, thus, requires current money supply to depend on past realizations of the exogenous state ξ. The following lemma summarizes this result.

Lemma 5 *Suppose that households' preferences are given by (18), that prices are imperfectly flexible ($\phi > 0$), and that money supply satisfies (23). Under the fundamental solution, there exist unique values for μ_ξ^* and for $\mu_{l\xi}^*$, such that the central bank can implement an interest rate target (20) if $\mu_\xi = \mu_\xi^*$ and $\mu_{l\xi} = \mu_{l\xi}^*$. The equilibrium sequence of interest rates then satisfies*

$$\rho_\pi = \Gamma \left[\sigma_m (\mu_\pi - 1) + 1\right](\delta_m - \rho_R) \quad \text{and} \quad \rho_y = \Gamma \mu_y \sigma_m (\delta_m - \rho_R), \tag{24}$$

where $\Gamma = \left(\overline{R} - 1\right) / \left(\overline{R} - \delta_m\right)$ *and* $\mu_{l\xi}^* = 0$ *if* $\rho_R = 0$.

Proof. See appendix 3.2.

Note that a (non-accommodating) money supply satisfying $\mu_\pi + \varkappa_4 \mu_y < 1$ implies $\delta_m \in (0,1)$ and $\Gamma < 1$. Consider the case, where the interest rate target is not inertial ($\rho_R = 0$), as in the former section. According to lemma 4 and 5, a central bank can then not implement a sequence of nominal interest rates satisfying an active interest rate target rule with $\rho_\pi \geq 1$

[25] It can be shown that this possibility is ruled out if money supply satisfies $\mu_\pi + \mu_y \frac{1+\beta}{\omega} < 1 + \frac{2(\omega + \sigma(R+1)(1+\beta))}{\omega \sigma_m (R-1)}$, which is hardly restrictive, given that the second term on the right hand side is very large for reasonable interest rate values.

and $\rho_y \geq 0$, by supplying money such that $\mu_\xi = \mu_\xi^*$, $\mu_{l\xi} = \mu_{l\xi}^*$, and in particular $\mu_\pi + \varkappa_4 \mu_y < 1$, which is necessary and sufficient for unique, non-oscillatory, and stable equilibrium sequences. If, however, we allow for interest rate inertia $\rho_R > 0$, then the sign and the magnitude of the coefficients ρ_π and ρ_y crucially depend on the exact feedback from lagged interest rates. If the feedback is sufficiently large such that $\rho_R > \delta_m$, then the interest rate sequence can be consistent with $\rho_\pi + \varkappa_4 \rho_y > 1 - \rho_R$ under $\mu_\pi + \varkappa_4 \mu_y < 1$.

Proposition 5 *Suppose that households' preferences are given by (18), that prices are imperfectly flexible ($\phi > 0$), and that money supply satisfies (23), $\mu_\xi = \mu_\xi^*$ and $\mu_{l\xi} = \mu_{l\xi}^*$. Then, i.) the equilibrium interest rate sequence is consistent with infinitely many interest rate targets (20), ii.) an interest rate target satisfying (22), cannot be implemented by a non-destabilizing money supply regime (21) if $\rho_R = 0$, and iii.) for $\rho_R > \delta_m$, the central bank can implement any interest rate target characterized by $\rho_\pi \geq 0$ and $\rho_y \geq 0$ in a stable and non-oscillatory way.*

Proof. Part *i.*) immediately follows from (24) indicating that there are just two conditions for three unknowns. Part *ii.*) follows from the fact that there exists no stable eigenvalue if $\rho_\pi + \varkappa_4 \rho_y > 1$ and $\rho_R = 0$. Hence, a money supply policy that aims to implement such a target will necessarily be associated with unstable equilibrium sequences. Finally (*iii.*), the conditions in (24) that can be rewritten as $\mu_\pi = 1 - \frac{1}{\sigma_m}(\frac{\rho_\pi}{\rho_R - \delta_m} \frac{\overline{R} - \delta_m}{\overline{R} - 1} + 1)$ and $\mu_y = -\frac{1}{\rho_R - \delta_m} \frac{\rho_y}{\sigma_m} \frac{\overline{R} - \delta_m}{\overline{R} - 1}$, indicating that if $\rho_R > \delta_m$ money supply satisfies $\mu_\pi < 1$ and $\mu_y < 0$ for $\rho_\pi \geq 0$ and $\rho_y \geq 0$, while $\delta_m \in (0, 1)$. Thus, (21) is satisfied. ■

According to part *ii.*) of proposition 5, an active forward-looking interest rate rule cannot be implemented by a stabilizing money supply policy. This corresponds to the result for the cash-in-advance model (see 2). Yet, considering inertial interest rate targets changes the picture (see *iii.*). If the degree of interest rate inertia is sufficiently high, $\rho_R > \delta_m$, which is for example ensured by values for ρ_R close to one, then a negative feedback of inflation (output) on money supply is associated with a positive feedback of inflation (output) on the interest rate target (see 24). Hence, the central bank can for $\rho_R > \delta_m$ implement an equilibrium sequence of interest rates consistent with positive elasticities ρ_π and ρ_y by an appropriate choice of the money supply response coefficient μ_π, without inducing the equilibrium sequences to become explosive. For $\rho_R > \delta_m$, a stabilizing money supply regime is therefore consistent with (long-run) active interest rate targets $\left(\rho_\pi + \varkappa_4 \rho_y\right) / (1 - \rho_R) > 1$, which have been found in interest rate rule regressions (see, e.g., Clarida et al., 2000, for the pre-1979 period).[26]

As implied by part *i.*) of proposition 5, the exact values for the coefficients of the interest rate target rule ρ_R, ρ_π, and ρ_y are not uniquely determined under a particular money growth reaction function (23). Thus, for any given contingent money growth policy, there are infinitely many representations of an inertial interest rate target rule. If, for example, one restricts the interest rate target to satisfy $\rho_R = 1$, such that the target rule is specified

[26] If however $\rho_R < \delta_m$, an implementation of an interest rate target (20) with $\rho_\pi + \varkappa_4 \rho_y > 1$ requires money supply to violate (21), which leads to unstable or oscillatory equilibrium sequences.

in differences, then the central bank can implement exactly one interest rate target by state contingent money supply adjustments. Moreover, such an interest rate target with feedback coefficients $\rho_\pi \geq 0$ and $\rho_y \geq 0$ is associated with a stable equilibrium, and is consistent with the empirical evidence by Ireland (2004), who estimated an interest rate rule in terms of growth rates and found positive feedback coefficients on current inflation and the output gap.

5 Conclusion

When a short-run nominal interest rate serves as the operating target, the central bank has to adjust the stock of money in a state contingent way, in order to implement a particular feedback rule for its interest rate target. In this paper an analysis of the relation between money supply and interest rate targets is conducted in a general equilibrium model with standard money demand specifications and frictionless financial markets. The implementation of simple interest rate targets that rise with inflation (and output), is shown to require the accompanying money growth rates also to rise with inflation (and output). When prices are imperfectly flexible, such an accommodating money supply tends to destabilize equilibrium sequences if the interest rate target rule is forward-looking and active, such as the Taylor (1993) rule. In contrast, a sufficiently *inertial* interest rate target can be implemented by a stabilizing money supply policy, even if the (long-run) inflation feedback exceeds one. Thus, when econometricians find significant coefficients on lagged interest rates in interest rate regressions, this might rather originate in a non-destabilizing adjustment of the outstanding stock of money, than being an indication for inertia as a fundamental component of the monetary strategy. Yet, the coefficients of an inertial interest rate target rule are not uniquely identified under a particular money supply regime. We view these findings as an indication that the assumption of frictionless financial (money) market is less useful for a full understanding of monetary policy implementation. This issue is aimed to be examined in future research.

Appendix

1 Appendix to the flexible price model

1.1 Fundamental solution

The model under flexible prices can be reduced to (10). Eliminating the nominal interest rate with the policy rule (11) gives

$$0 = [\sigma\rho_\pi - (\vartheta+\sigma)]\, E_t\widetilde{\pi}_{t+1} + \sigma\rho_\xi E_t\xi_{t+1} - \sigma(1-\rho)\widetilde{\xi}_t + (\vartheta+\sigma)\left(\widetilde{\beta} + \kappa_\rho\right) + \sigma(1-\rho)\overline{\xi}$$
$$+ (\vartheta+\sigma)\frac{1}{2}var_t\left(\frac{\sigma}{\vartheta+\sigma}\left(\rho_\pi\widetilde{\pi}_{t+1} + \left(1+\rho_\xi\right)\xi_{t+1} + \kappa_\rho\right) - \widetilde{\pi}_{t+1}\right) + \vartheta\rho_\pi\widetilde{\pi}_t + \vartheta\rho_\xi\xi_t.$$

Evidently, there is no endogenous state variable, such that the fundamental solution for inflation takes the form $\widetilde{\pi}_t = \gamma_{\pi e}\widetilde{\xi}_t + \psi_\pi$, implying $E_t\widetilde{\pi}_{t+1} = \rho\widetilde{\pi}_t + (1-\rho)\overline{\pi}$. Thus, we get

$$\widetilde{\pi}_t = \frac{(\rho-1)\sigma + (\vartheta+\rho\sigma)\rho_\xi}{\rho(\sigma+\vartheta) - \rho_\pi(\vartheta+\rho\sigma)}\widetilde{\xi}_t$$
$$+\frac{(\sigma(\rho_\pi-1)-\vartheta)(1-\rho)\overline{\pi} + \sigma\left(\rho_\xi+1\right)(1-\rho)\overline{\xi} + (\sigma+\vartheta)\kappa_\rho + (\vartheta+\sigma)\widetilde{\beta}}{\rho\sigma+\rho\vartheta-\vartheta\rho_\pi-\rho\sigma\rho_\pi}$$
$$+\frac{\vartheta+\sigma}{\rho\sigma+\rho\vartheta-\vartheta\rho_\pi-\rho\sigma\rho_\pi}\frac{1}{2}var_t\left(\left(\frac{\sigma}{\vartheta+\sigma}\rho_\pi - 1\right)\widetilde{\pi}_{t+1} + \frac{\sigma}{\vartheta+\sigma}\left(\rho_\xi+1\right)\xi_{t+1} + \frac{\sigma}{\vartheta+\sigma}\kappa_\rho\right).$$

The solution for $\widetilde{\pi}_t$ can be summarized by the following term which governs the impact of the current state $\widetilde{\xi}_t$ on inflation

$$\gamma_{\pi e} = \frac{(\rho-1)\sigma + (\vartheta+\rho\sigma)\rho_\xi}{\rho(\sigma+\vartheta) - \rho_\pi(\vartheta+\rho\sigma)},$$

and by ψ_π which collects the remaining terms that are independent of the current state. Using that $\overline{\pi} = \gamma_{\pi e}\overline{\xi} + \psi_\pi$ and the solution for $\gamma_{\pi e}$ we get

$$\psi_\pi = \frac{1}{2}\frac{\left(\sigma\rho_\pi - \sigma + \vartheta\rho_\xi\right)^2}{(1-\rho_\pi)(\rho\sigma+\rho\vartheta-\vartheta\rho_\pi-\rho\sigma\rho_\pi)^2}var_e + \Xi +$$
$$\frac{\rho_\pi\left(\rho\sigma\left(\overline{\xi}-\widetilde{\beta}-\kappa_\rho\right) - \sigma\overline{\xi} - \vartheta\kappa_\rho - \widetilde{\beta}\vartheta\right) + \rho_\xi\overline{\xi}\vartheta(\rho-1) + \rho\left(\widetilde{\beta}(\sigma+\vartheta) + \sigma\left(\kappa_\rho - \overline{\xi}\right) + \vartheta\kappa_\rho\right) + \sigma\overline{\xi}}{(\rho_\pi-1)(\vartheta\rho_\pi - \rho\vartheta - \rho\sigma + \rho\sigma\rho_\pi)},$$

where Ξ summarizes terms of higher order than two. It should be noted that these terms vanish if shocks are not autocorrelated, $\rho = 0$. For example, when $\rho = 0$ and $\rho_\pi > 0$, the solution for inflation is given by $\widetilde{\pi}_t = \gamma_{\pi e}\widetilde{\xi}_t + \psi_\pi$, where

$$\gamma_{\pi e} = \frac{\sigma - \vartheta\rho_\xi}{\rho_\pi(\vartheta+\rho\sigma)} \text{ and } \psi_\pi = \frac{\rho_\pi\left(\sigma\overline{\xi} + \vartheta\kappa_\rho + \widetilde{\beta}\vartheta\right) + \rho_\xi\overline{\xi}\vartheta - \sigma\overline{\xi}}{(1-\rho_\pi)\vartheta\rho_\pi} + \frac{1}{2}\frac{\left(\sigma\rho_\pi - \sigma + \vartheta\rho_\xi\right)^2}{(1-\rho_\pi)\rho_\pi^2\vartheta^2}var_e. \tag{25}$$

1.2 Proof of lemma 1

We want to derive a money growth policy that is equivalent to (11). For this we combine the equilibrium condition $-\sigma\widetilde{m}_t = \widetilde{R}_t - \sigma E_t\widetilde{m}_{t+1} - E_t\widetilde{\pi}_{t+1} + \frac{1}{2}var_t\left(-\sigma\widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}\right) + \widetilde{\beta}$ and the interest rate feedback rule (11) to give

$$\sigma E_t\widetilde{m}_{t+1} - \sigma\widetilde{m}_t = \rho_\xi\widetilde{\xi}_t + \rho_\pi\widetilde{\pi}_t + \kappa_\rho - E_t\widetilde{\pi}_{t+1} + \frac{1}{2}var_t\left(-\sigma\widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}\right) + \widetilde{\beta}.$$

Further using $E_t\widetilde{\pi}_{t+1} = \rho\widetilde{\pi}_t + (1-\rho)\overline{\pi}$, the money growth rate $\widetilde{\mu}_t = \widetilde{m}_t + \widetilde{\pi}_t - \widetilde{m}_{t-1}$, and $E_t\widetilde{\mu}_{t+1} = \rho\widetilde{\mu}_t + (1-\rho)\overline{\mu}$, we get

$$\sigma\rho\widetilde{\mu}_t + \sigma(1-\rho)\overline{\mu} = \rho_\xi\widetilde{\xi}_t + (\rho_\pi + (\sigma-1)\rho)\widetilde{\pi}_t + \frac{1}{2}var_t\left(-\sigma\widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}\right) + \widetilde{\beta} + \kappa_\rho + (\sigma-1)(1-\rho)\overline{\pi}.$$

Together with the inflation solution, we can therefore derive an equivalent money growth policy either as a function of the exogenous state $\widetilde{\mu}_t = \widetilde{\mu}_t(\widetilde{\xi}_t)$ or as a function of an endogenous variable, e.g. current inflation $\widetilde{\mu}_t = \widetilde{\mu}_t(\widetilde{\pi}_t)$. For this, we apply $\widetilde{\xi}_t = \frac{1}{\gamma_{\pi e}}\widetilde{\pi}_t - \frac{\psi_\pi}{\gamma_{\pi e}}$, leading to

$$\begin{aligned}\widetilde{\mu}_t =& \left(\frac{\rho_\pi + (\sigma-1)\rho}{\sigma\rho} + \frac{\rho_\xi}{\sigma\rho}\left(\gamma_{\pi e}\right)^{-1}\right)\widetilde{\pi}_t \\ &-\frac{\rho_\xi}{\sigma\rho}\frac{\psi_\pi}{\gamma_{\pi e}} + \frac{\widetilde{\beta} + \kappa_\rho + (\sigma-1)(1-\rho)\overline{\pi} - \sigma(1-\rho)\overline{\mu}}{\sigma\rho} + \frac{1}{\sigma\rho}\frac{1}{2}var_t\left(-\sigma\widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}\right).\end{aligned}$$

Further using that $\overline{\pi} = \gamma_{\pi e}\overline{\xi} + \psi_\pi$ and $\overline{\mu} = \mu_\pi\left(\gamma_{\pi e}\overline{\xi} + \psi_\pi\right) + \kappa_\mu$, we can write the term κ_μ, which collects terms in the money supply reaction function $\widetilde{\mu}_t = \mu_\pi\widetilde{\pi}_t + \kappa_\mu$ that are independent of inflation, as

$$\begin{aligned}\kappa_\mu =& -(\sigma\rho-1)\rho_\xi\psi_\pi\left(\gamma_{\pi e}\right)^{-1} + (\sigma\rho-1)\left(\widetilde{\beta} + \kappa_\rho + (1-\rho)(\sigma - 1 - \sigma\mu_\pi)\left(\gamma_{\pi e}\overline{\xi} + \psi_\pi\right)\right) \\ &+ (\sigma\rho-1)\frac{1}{2}var_t\left(-\sigma\widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}\right).\end{aligned}$$

The last term can be decomposed into a variance term and Ξ collecting terms of higher order $(\sigma\rho-1)\frac{1}{2}var_t\left(-\sigma\widetilde{m}_{t+1} - \widetilde{\pi}_{t+1}\right) = (\sigma\rho-1)\left(\frac{\sigma-\sigma\rho_\pi-\vartheta\rho_\xi}{\rho\sigma+\rho\vartheta-\vartheta\rho_\pi-\rho\sigma\rho_\pi}\right)^2\frac{1}{2}var_e + \Xi$. The last term vanishes for $\rho = 0$. To give an example, for $\rho = 0$ and $\rho_\pi > 0$ the policy parameter κ_μ is given by $\kappa_\mu = \psi_\pi\frac{\rho_\xi}{\gamma_{e\pi}} - \kappa_\rho + (\sigma - \sigma\mu_\pi - 1)(-\psi_\pi - \gamma_{e\pi}) - \left(\frac{\sigma-\sigma\rho_\pi-\vartheta\rho_\xi}{\rho\sigma+\rho\vartheta-\vartheta\rho_\pi-\rho\sigma\rho_\pi}\right)^2\frac{1}{2}var_e$, where the coefficients of the inflation solution ψ_π and $\gamma_{e\pi}$ can be found in (25). ■

1.3 Efficiency of interest rate targets

Flexible prices The following discussion is aimed to demonstrate the welfare implications of a state contingent interest rate rule (11) for $\rho_\pi = 0$, which encompasses the Friedman rule for $\kappa_\rho = \rho_\xi = 0$. Suppose, for simplicity, that shocks are not autocorrelated, $\rho = 0$, such that $\widetilde{\xi}_t = \overline{\xi} + \varepsilon_t$. Inserting the interest rate rule, $\widetilde{R}_t = \rho_\xi\widetilde{\xi}_t + \kappa_\rho$, into the static equilibrium condition in (9), gives $\widetilde{c}_t = -(\vartheta+\sigma)^{-1}\left(\kappa_\rho + \rho_\xi\widetilde{\xi}_t\right) - (\vartheta+\sigma)^{-1}\widetilde{\xi}_t + \log(1-1/\epsilon)$. Hence, the solution for consumption is given by $\widetilde{c}_t = -\frac{1+\rho_\xi}{\vartheta+\sigma}\widetilde{\xi}_t + \log(1-1/\epsilon) - (\vartheta+\sigma)^{-1}\kappa_\rho$. In what follows we compare the implications of interest rate policy on households' welfare. In particular, we consider the Friedman rule $\widetilde{R}_t = 0$ ($\kappa_\rho = \rho_\xi = 0$) and an interest rate policy that eliminates consumption fluctuations induced by aggregate shocks, $\rho_\xi = -1$. The former policy is associated with an equilibrium sequence of consumption satisfying

$$\widetilde{c}_t^F = -\frac{1}{\vartheta + \sigma}\widetilde{\xi}_t + \log(1 - 1/\epsilon),$$

and, therefore, $\overline{c}^F = -\frac{1}{\vartheta+\sigma}\overline{\xi} + \log(1 - 1/\epsilon)$ and $var(\widetilde{c}_t^F) = (\vartheta + \sigma)^2 var_e$, while the state contingent policy is associated with a constant consumption value given by

$$\widetilde{c}_t^S = \overline{c} = \log(1 - 1/\epsilon) - (\vartheta + \sigma)^{-1}\kappa_\rho,$$

and, therefore, $var(\widetilde{c}_t^S) = 0$. Correspondingly, the equilibrium sequences for working time satisfy $\widetilde{l}_t^F = -\frac{1}{\vartheta+\sigma}\widetilde{\xi}_t + \log(1 - 1/\epsilon)$ and $\widetilde{l}_t^S = \log(1 - 1/\epsilon) - (\vartheta + \sigma)^{-1}\kappa_\rho$, respectively. Given that the model exhibits no endogenous state variable, the maximization problem of the households is entirely static. Hence, the households' objective (1) reduces to $\max E_0 u_t = (1-\sigma)^{-1}E_0 c_t^{1-\sigma} - (1+\vartheta)^{-1}E_0 l_t^{1+\vartheta}$. Using that all variables are log-normally distributed, the problem can be written as

$$\max E_0 u_t = (1-\sigma)^{-1} e^{(1-\sigma)\overline{c}+(1/2)(1-\sigma)^2 var_c} - (1+\vartheta)^{-1} e^{(1+\vartheta)\overline{l}+(1/2)(1+\vartheta)^2 var_l}. \tag{26}$$

Now, consider the Friedman rule. Inserting the solution for consumption and working time in the expression for expected utility (26), gives the following value

$$\begin{aligned}(E_0 u_t)^F &= (1-\sigma)^{-1}\exp\left[(1-\sigma)\left(-(\vartheta+\sigma)^{-1}\overline{\xi} + \log(1-1/\epsilon)\right) + (1/2)(1-\sigma)^2(\vartheta+\sigma)^2 var_e\right] \\ &- (1+\vartheta)^{-1}\exp\left[(1+\vartheta)\left(-(\vartheta+\sigma)^{-1}\overline{\xi} + \log(1-1/\epsilon)\right) + (1/2)(1+\vartheta)^2(\vartheta+\sigma)^2 var_e\right].\end{aligned}$$

It can immediately be seen that expected utility $(E_0 u_t)^F$ declines monotonically with the variance of mark-up shocks, var_ε, if (but not only if) $\sigma \geq 1$. Under the state contingent policy $\rho_\xi = -1$, expected utility is given by

$$\begin{aligned}(E_0 u_t)^S = &(1-\sigma)^{-1}\exp\left[(1-\sigma)\left(\log(1-1/\epsilon) - (\vartheta+\sigma)^{-1}\kappa_\rho\right)\right] \\ &- (1+\vartheta)^{-1}\exp\left[(1+\vartheta)\left(\log(1-1/\epsilon) - (\vartheta+\sigma)^{-1}\kappa_\rho\right)\right].\end{aligned}$$

When $\sigma \geq \vartheta + 2$, expected utility $(E_0 u_t)^S$ declines with the average value for the nominal interest rate $\kappa_\rho : \partial (E_0 u_t)^S / \partial \kappa_\rho < 0$. Nevertheless, the central bank can raise expected utility by switching from the Friedman rule to a state contingent rule satisfying $\rho_\xi = -1$, if the variance var_ε is sufficiently large.

Sticky prices As shown by Ravenna and Walsh (2005), household welfare of an isomorphic model can be approximated by applying a second-order Taylor expansion at the (undistorted) steady state in the following form

$$E_0 \sum_{t=0}^{\infty} \beta^t U_t \approx \overline{U} - \Omega E_0 \sum_{t=0}^{\infty} \beta^t \left[\widehat{\pi}_t^2 + \left(\chi \frac{\sigma + \vartheta}{\epsilon} \right) \widehat{y}_t^2 \right].$$

Applying this objective and the equilibrium conditions, which are log-linearized at the steady state, as constraints, the policy problem under discretionary optimization leads to the following first order condition $\widehat{\pi}_t = -\frac{\sigma+\vartheta}{\epsilon\vartheta}\widehat{y}_t$. This condition deviates from the condition of the corresponding cashless model, which reads $\widehat{\pi}_t = -\frac{1}{\epsilon}\widehat{y}_t$ (see Woodford, 2003a). Applying the log-linearized consumption Euler equation $\sigma\widehat{c}_t = \sigma E_t\widehat{c}_{t+1} - \widehat{R}_t + E_t\widehat{\pi}_{t+1}$, where $\sigma = -u_{cc}(\overline{c})\overline{c}/u_c(\overline{c}) > 0$ and using $\widehat{y}_t = \widehat{c}_t$, we get the following equilibrium relation for the interest rate under the fundamental solution for the optimal discretionary plan

$$\widehat{R}_t = \left(\rho + \frac{\sigma (1-\rho) \epsilon\vartheta}{\sigma + \vartheta} \right) \widehat{\pi}_t,$$

where $\vartheta = v_{ll}(\bar{l})\bar{l}/v_l(\bar{l}) > 0$ we used that $E_t\widehat{\pi}_{t+1} = \rho\widehat{\pi}_t$ and $E_t\widehat{y}_{t+1} = \rho\widehat{y}_t$. Thus, the term in the round bracket is strictly positive and can be smaller or larger than one.

1.4 Equilibrium solution uniqueness of the flexible price model

To assess the conditions for equilibrium solution uniqueness we apply the deterministic version, for convenience. We start with the case of a money growth policy $\widetilde{\mu}_t = \mu_\pi\widetilde{\pi}_t + \kappa_\mu$. Replacing the nominal interest rate and consumption, the equilibrium conditions can be reduced to $\sigma\widetilde{m}_{t+1} = -\vartheta\widetilde{m}_t - \widetilde{\pi}_{t+1} + \widetilde{\beta} + \widetilde{\frac{\epsilon-1}{\epsilon}}$. Combining the latter with the logged money growth reaction function for the period $t+1$, leads to the following difference equation in real balances

$$\widetilde{m}_{t+1} = \{(1-\mu_\pi)(\sigma+\vartheta)/[1+\sigma(\mu_\pi - 1)]\}\widetilde{m}_t + \varrho, \tag{27}$$

and where ϱ collects constant terms. Given that the difference equation is forward looking, uniqueness requires the eigenvalue of (27) to be unstable, i.e., to lie outside the unit circle. Using that the term in the curly brackets is strictly larger than one if $\frac{\sigma-1}{\sigma} < \mu_\pi < 1$, and smaller than -1 if $\sigma < \vartheta$ and $1 + \frac{2}{\vartheta-\sigma} < \mu_\pi$, the equilibrium is uniquely determined if and only if $\frac{\sigma-1}{\sigma} < \mu_\pi < 1$ or $1 + \frac{2}{\vartheta-\sigma} < \mu_\pi$ for $\sigma < \vartheta$.

Next, consider the case where monetary policy follows an interest rate rule $\widetilde{R}_t = \rho_\pi\widetilde{\pi}_t + \kappa_\rho$. Then the structural part of (10) reads

$$\widetilde{\pi}_{t+1} = -\frac{\vartheta\rho_\pi}{(\sigma\rho_\pi - (\vartheta+\sigma))}\widetilde{\pi}_t + \varrho^*.$$

and where ϱ^* collects constant terms. Uniqueness thus requires $\left|\frac{\vartheta\rho_\pi}{\sigma\rho_\pi - (\vartheta+\sigma)}\right| > 1$. Suppose that $\vartheta + \sigma > \sigma\rho_\pi$, then $-\frac{\vartheta\rho_\pi}{\sigma\rho_\pi-(\vartheta+\sigma)} > 1$ if $\rho_\pi > 1$. For $\vartheta + \sigma < \sigma\rho_\pi$, we have to check in this case if $-\frac{\vartheta\rho_\pi}{\sigma\rho_\pi-(\vartheta+\sigma)} < -1$. For this, we have to distinguish the cases where $\vartheta \geq \sigma$ and $\vartheta < \sigma$. If $\vartheta \geq \sigma$ it follows immediately that $\frac{\vartheta\rho_\pi}{-\sigma\rho_\pi+\vartheta+\sigma} < -1$. Hence, determinacy also arises if $\rho_\pi > 1 + \frac{\vartheta}{\sigma}$ for $\vartheta > \sigma$. If $\vartheta < \sigma$, determinacy further requires $\rho_\pi < 1 + \frac{2\vartheta}{\sigma-\vartheta}$. Thus, equilibrium uniqueness is ensured by $\rho_\pi \in (1, 1+\frac{\vartheta}{\sigma})$, or $\rho_\pi > 1+\frac{\vartheta}{\sigma}$ for $\vartheta \geq \sigma$, or $\rho_\pi \in \left(1+\frac{\vartheta}{\sigma}, 1+\frac{2\vartheta}{\sigma-\vartheta}\right)$ for $\vartheta < \sigma$.

2 Appendix to the sticky price model

2.1 Proof of lemma 2

To establish the claim made in the **first** part of the lemma, we apply the equilibrium conditions (13)-(14) together with an interest rate "rule " $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$, where $\rho_\pi \geq 0$ and $\rho_y \geq 0$, leading to the system

$$\begin{pmatrix} 1 & \sigma \\ \beta & 0 \end{pmatrix} \begin{pmatrix} E_t \widehat{\pi}_{t+1} \\ E_t \widehat{y}_{t+1} \end{pmatrix} = \begin{pmatrix} \rho_\pi & \sigma + \rho_y \\ 1 - \chi\rho_\pi & -\left(\omega + \chi\rho_y\right) \end{pmatrix} \begin{pmatrix} \widehat{\pi}_t \\ \widehat{y}_t \end{pmatrix} + \begin{pmatrix} 0 \\ -\chi \end{pmatrix} \widehat{\xi}_t,$$

The characteristic polynomial of this system is given by

$$M(X) = X^2 + X \frac{\sigma\chi\rho_\pi - \omega - \sigma\beta - \beta\rho_y - \chi\rho_y - \sigma}{\beta\sigma} + \frac{\sigma + \rho_y + \omega\rho_\pi - \sigma\chi\rho_\pi}{\beta\sigma}.$$

Equilibrium uniqueness requires both eigenvalues to be unstable. To derive the conditions for this we firstly consider $M(0) = \frac{\sigma+\rho_y+\vartheta\chi\rho_\pi}{\sigma\beta} > 1$, where use that $\omega = \sigma\chi + \vartheta\chi$. Thus, there exists at least one unstable eigenvalue. Next, we examine $M(X)$ at $X = 1$, which is given by

$$M(1) = \left[\omega\left(1-\rho_\pi\right) + \left(\beta + \chi - 1\right)\rho_y\right] / (\beta\sigma),$$

implying that $M(1) < 0 \Leftrightarrow \rho_\pi > 1 + \frac{\beta+\chi-1}{\omega}\rho_y$. Then, the existence of a stable eigenvalue can be ruled out, if $M(X)$ at $X = -1$, which is given by

$$M(-1) = \left[2\sigma + \omega + 2\sigma\beta + \left(1 + \beta + \chi\right)\rho_y - \chi\left(\sigma - \vartheta\right)\rho_\pi\right] / (\beta\sigma),$$

is positive. This is ensured for $\vartheta \geq \sigma$, while it requires $\rho_\pi < \frac{2\sigma+\omega+2\sigma\beta}{\chi(\sigma-\vartheta)} + \frac{(1+\beta+\chi)}{\chi(\sigma-\vartheta)}\rho_y$ for $\vartheta < \sigma$. Hence, equilibrium sequences satisfying $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$ are uniquely determined if and only if $\rho_\pi > 1 + \frac{\beta+\chi-1}{\omega}\rho_y$ for $\vartheta \geq \sigma$ and $\rho_\pi \in \left(1 + \frac{\beta+\chi-1}{\omega}\rho_y, \frac{2\sigma+\omega+2\sigma\beta}{\chi(\sigma-\vartheta)} + \frac{(1+\beta+\chi)}{\chi(\sigma-\vartheta)}\rho_y\right)$ for $\vartheta < \sigma$.

Turning to the **second** part of the lemma, we eliminate the nominal interest rate in the equilibrium conditions (13) and (15), and use $\widehat{y}_t = \widehat{m}_t$, leading – together with the money supply rule – to the following set of equilibrium conditions

$$\widehat{\pi}_t = \left(\beta + \chi\right) E_t \widehat{\pi}_{t+1} + \left(\omega - \chi\sigma\right) \widehat{m}_t + \chi\sigma E_t \widehat{m}_{t+1} + \chi\widehat{\xi}_t, \quad (28)$$

$$\left(1 - \mu_y\right) \widehat{m}_t - \widehat{m}_{t-1} = \left(\mu_\pi - 1\right) \widehat{\pi}_t + \mu_\xi \widehat{\xi}_t. \quad (29)$$

Given that the predetermined value of beginning-of-period real balances enters the set of equilibrium conditions, the generic form for the fundamental solution reads

$$\widehat{m}_t = \delta_m \widehat{m}_{t-1} + \delta_{me} \widehat{\xi}_t, \quad \widehat{\pi}_t = \delta_{\pi m} \widehat{m}_{t-1} + \delta_{\pi e} \widehat{\xi}_t. \quad (30)$$

Inserting the latter solution in (28)-(29), leads to the following conditions for the undetermined coefficients

$$0 = \delta_m (\beta+\chi)\delta_{\pi m} - \delta_{\pi m} + \sigma\chi\delta_m^2 + \delta_m(\omega - \sigma\chi), \tag{31}$$

$$0 = \left(1-\mu_y\right)\delta_m - 1 - (\mu_\pi - 1)\delta_{\pi m}, \tag{32}$$

$$0 = \chi - \delta_{\pi e} + \sigma\chi(\rho\delta_{me} + \delta_m\delta_{me}) + (\beta+\chi)(\rho\delta_{\pi e} + \delta_{me}\delta_{\pi m}) + \delta_{me}(\omega - \sigma\chi), \tag{33}$$

$$0 = -\left(1-\mu_y\right)\delta_{me} + (\mu_\pi - 1)\delta_{\pi e} + \mu_\xi. \tag{34}$$

We now use the conditions (31)-(32) to examine the eigenvalues of the model. Suppose that $\mu_\pi = 1$. Then, the eigenvalue δ_m is given by $\delta_m = \frac{1}{1-\mu_y}$. Otherwise, we have to check the roots of the following quadratic (characteristic) polynomial

$$G(X) = X^2 + X\frac{(\mu_\pi - 1)\omega - \chi - \beta - (\mu_\pi - 1)\sigma\chi + \mu_y - 1}{\sigma\chi(\mu_\pi - 1) + (\beta+\chi)\left(1-\mu_y\right)} + \frac{1}{\sigma\chi(\mu_\pi - 1) + (\beta+\chi)\left(1-\mu_y\right)}.$$

Given that the model exhibits one predetermined variable, we want to establish the existence of a stable eigenvalue. For this, we examine the value of G at $X = 0$, which is given by

$$G(0) = \left[\sigma\chi(\mu_\pi - 1) + (\beta+\chi)\left(1-\mu_y\right)\right]^{-1}.$$

Consider first the case where $G(0) < 0 \Leftrightarrow \mu_\pi < 1 + \frac{\beta+\chi}{\sigma\chi}\left(\mu_y - 1\right)$. Then the model exhibits a stable root if $G(1)$, which is given by

$$G(1) = \frac{\mu_y(1-\beta-\chi) + \omega(\mu_\pi - 1)}{\sigma\chi(\mu_\pi - 1) + (\beta+\chi)\left(1-\mu_y\right)},$$

is positive, $G(1) > 0$. This, obviously requires $\mu_\pi < 1 + \frac{\mu_y(\beta+\chi-1)}{\omega}$, which is assumed to be satisfied in what follows. To ensure that there is a single stable root, $G(-1)$, which is given by

$$G(-1) = -\frac{(\vartheta - \sigma)\chi(\mu_\pi - 1) + (\beta+\chi+1)\left(\mu_y - 2\right)}{\sigma\chi(\mu_\pi - 1) + (\beta+\chi)\left(1-\mu_y\right)},$$

has to be negative, $G(-1) < 0$. To disclose the conditions for this, we have to distinguish the cases $\sigma > \vartheta$ and $\sigma < \vartheta$. Suppose that $\sigma > \vartheta \Leftrightarrow 2\sigma\chi - \omega > 0$. Then, we can conclude that $G(-1) < 0$ if $\mu_\pi > 1 + \frac{(\beta+\chi+1)}{2\sigma\chi-\omega}\left(\mu_y - 2\right)$. Now suppose that $\sigma < \vartheta \Leftrightarrow 2\sigma\chi - \omega < 0$. Then, $G(-1) < 0$ if $\mu_\pi < 1 + \frac{(\beta+\chi+1)}{2\sigma\chi-\omega}\left(\mu_y - 2\right)$. Finally, consider the case where $G(0) > 0 \Leftrightarrow \mu_\pi > 1 + \frac{\beta+\chi}{\sigma\chi}\left(\mu_y - 1\right)$. Then, the existence of a stable root again requires $\mu_\pi < 1 + \frac{\mu_y(\beta+\chi-1)}{\omega}$. In this case there cannot be another stable root. Hence, the model exhibits a unique stable and positive eigenvalue if and only if

$$\begin{aligned}
&i) && \mu_\pi \in (\widetilde{\mu}_{1\pi}, \widetilde{\mu}_{2\pi}), \text{ or} \\
&ii) && \mu_\pi < \min\{\widetilde{\mu}_{1\pi}, \widetilde{\mu}_{2\pi}, \widetilde{\mu}_{3\pi}\} \text{ for } \sigma < \vartheta \\
& && \widetilde{\mu}_{3\pi} < \mu_\pi < \min\{\widetilde{\mu}_{1\pi}, \widetilde{\mu}_{2\pi}\} \text{ for } \sigma > \vartheta,
\end{aligned}$$

where $\widetilde{\mu}_{1\pi} = 1 + \frac{(\beta+\chi)\left(\mu_y - 1\right)}{\sigma\chi}$, $\widetilde{\mu}_{2\pi} = 1 + \frac{\mu_y(\beta+\chi-1)}{\omega}$, and $\widetilde{\mu}_{3\pi} = 1 + \frac{(\beta+\chi+1)\left(\mu_y - 2\right)}{2\sigma\chi-\omega}$, while the existence of a non-oscillatory and stable equilibrium, $\delta_m \in (0,1)$, requires $\mu_\pi < 1 + \frac{\mu_y(\beta+\chi-1)}{\omega}$. ■

2.2 Proof of lemma 3

To assess the equilibrium behavior of the nominal interest rate under the money growth regime (16),[27] we use the equilibrium conditions (13) and $\widehat{y}_t = \widehat{m}_t$ to get $\sigma E_t \widehat{m}_{t+1} - \sigma \widehat{m}_t = \widehat{R}_t - E_t \widehat{\pi}_{t+1}$, which can be rewritten in terms of the expected money growth rate

$$\sigma E_t \widehat{\mu}_{t+1} = \widehat{R}_t + (\sigma - 1) E_t \widehat{\pi}_{t+1}.$$

Using the state contingent money growth reaction function $\widehat{m}_t + \widehat{\pi}_t - \widehat{m}_{t-1} = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_\xi \widehat{\xi}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{m}_t + \mu_\xi \widehat{\xi}_t$, we obtain the following relation between, the current nominal interest rate and the expected future values for inflation and real balances and the cost push shock

$$\widehat{R}_t = (-(\sigma - 1) + \sigma \mu_\pi) E_t \widehat{\pi}_{t+1} + \mu_y E_t \widehat{m}_{t+1} + \mu_\xi \rho \widehat{\xi}_t,$$

where we used that $E_t \widehat{\xi}_{t+1} = \rho \widehat{\xi}_t$. In order to rewrite this equilibrium relation in terms of current realizations of the endogenous variables we use the solution for real balances $\widehat{m}_t = \delta_m \widehat{m}_{t-1} + \delta_{me} \widehat{\xi}_t$ and that $E_t \widehat{\pi}_{t+1} = E_t \left(\delta_{\pi m} \widehat{m}_t + \delta_{\pi e} \widehat{\xi}_{t+1} \right) = \delta_m \widehat{\pi}_t + ((\rho - \delta_m) \delta_{\pi e} + \delta_{\pi m} \delta_{me}) \widehat{\xi}_t$. Collecting terms, we end up with the following expression

$$\begin{aligned} &\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t + \rho_\xi \widehat{\xi}_t, \\ \text{where} \quad &\rho_\pi = (-(\sigma - 1) + \sigma \mu_\pi) \delta_m \quad \text{and} \quad \rho_y = \mu_y \delta_m, \\ &\rho_\xi = (-(\sigma - 1) + \sigma \mu_\pi) ((\rho - \delta_m) \delta_{\pi e} + \delta_{\pi m} \delta_{me}) + \mu_y \delta_{me} \rho + \mu_\xi \rho, \end{aligned}$$

where we used the money demand condition $\widehat{m}_t = \widehat{y}_t$. Hence, a money growth policy $\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_\xi \widehat{\xi}_t$, can implement a sequence of interest rates satisfying an endogenous target rule $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$, if its responds to current shocks by choosing μ_ξ such that $\rho_\xi = 0$. To examine the existence and the uniqueness of such a value for μ_ξ we have to take into account that the solution coefficients are functions of the policy rule parameter. We know from (31)-(32) that δ_m and $\delta_{\pi m} = \frac{(1-\mu_y)\delta_m - 1}{\mu_\pi - 1}$ are independent of μ_ξ. In contrast, $\delta_{\pi e}$ and δ_{me} depend on the latter. Combining (33)-(34) reveals that $\delta_{\pi e}$ and, therefore, $\delta_{me} = \frac{(\mu_\pi - 1)\delta_{\pi e} + \mu_\xi}{1 - \mu_y}$ are linear in μ_ξ :

$$\delta_{\pi e} = -\frac{\mu_\xi (\omega - \sigma\chi + \sigma\chi(\rho + \delta_m) + (\beta + \chi)\delta_{\pi m}) + \chi(1 - \mu_y)}{(\mu_\pi - 1)(\omega - \sigma\chi + \sigma\chi(\rho + \delta_m) + (\beta + \chi)\delta_{\pi m}) + (\rho(\beta + \chi) - 1)(1 - \mu_y)}.$$

[27] Alternatively, one can derive all coefficients of the fundamental solution (30), determine the equilibrium sequences of real balances and of inflation, and derive with (13)-(15) the equilibrium solution of the nominal interest rate, which takes the form $\widehat{R}_t = \delta_{Rm} \widehat{m}_{t-1} + \delta_{Re} \widehat{\xi}_t$. With the latter and the solutions for inflation and real balances one can then obtain an equilibrium relation which takes the form of an interest rate target feedback rule.

As a consequence, ρ_ξ is also linear in μ_ξ. Hence, for any particular solution under a money growth policy there is a unique value μ_ξ^*, such that ρ_ξ is equal to zero if the central bank sets $\mu_\xi = \mu_\xi^*$. Then, the central bank implements an equilibrium sequence of interest rates satisfying $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t$. ■

2.3 Proof of proposition 3

Consider that the central bank sets the nominal interest rate in a backward-looking way $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_{ly} \widehat{y}_{t-1}$. Using that $\widehat{m}_t = \widehat{y}_t$, we can write the sticky price model (13)-(15) in terms of inflation, real balances, and the nominal interest rate. Given that the model now features a predetermined endogenous state variable, $\widehat{m}_{t-1}$, the fundamental solutions under the money growth policy and the backward-looking interest rate rule take the same form. To assess the equivalence between both types of rules, we use the equilibrium condition $\widehat{R}_t = \sigma E_t \widehat{m}_{t+1} - \sigma \widehat{m}_t + E_t \widehat{\pi}_{t+1}$, and rewrite it as $\widehat{R}_t = \sigma E_t \widehat{\mu}_{t+1} - E_t \sigma \widehat{\pi}_{t+1} + E_t \widehat{\pi}_{t+1}$. A money supply policy (16) then implies

$$R_t = \sigma E_t \left(\mu_\pi \widehat{\pi}_{t+1} + \mu_y \widehat{m}_{t+1} + \mu_\xi \widehat{\xi}_{t+1} \right) - (\sigma - 1) E_t \widehat{\pi}_{t+1}.$$

Now suppose that shocks are not autocorrelated, $\rho = 0$. Using the fundamental solution (30), we then get $\widehat{R}_t = \left(\sigma \mu_\pi \delta_{\pi m} + \sigma \mu_y \delta_m - (\sigma - 1) \delta_{\pi m} \right) \widehat{m}_t$, or

$$\begin{aligned} \widehat{R}_t = {} & \left(\sigma \mu_\pi \delta_{\pi m} + \sigma \mu_y \delta_m - (\sigma - 1) \delta_{\pi m} \right) \left(\delta_m - \frac{\delta_{me}}{\delta_{\pi e}} \delta_{\pi m} \right) \widehat{m}_{t-1} \\ & + \left(\sigma \mu_\pi \delta_{\pi m} + \sigma \mu_y \delta_m - (\sigma - 1) \delta_{\pi m} \right) \frac{\delta_{me}}{\delta_{\pi e}} \widehat{\pi}_t. \end{aligned}$$

This equilibrium relation under a money supply policy already takes the form of the backward-looking interest rate rule. In order get an expression that (mainly) consists of structural parameters, we further use that $\delta_{\pi m} = \frac{(1-\mu_y)\delta_m - 1}{\mu_\pi - 1}$ and $\delta_{me} = \frac{(\mu_\pi - 1)\delta_{\pi e}}{1 - \mu_y}$. Then, the terms $\rho_{ly} = (\sigma \mu_\pi \delta_{\pi m} + \sigma \mu_y \delta_m - (\sigma - 1) \delta_{\pi m})(\delta_m - \frac{\delta_{me}}{\delta_{\pi e}} \delta_{\pi m})$ and $\rho_\pi = (\sigma \mu_\pi \delta_{\pi m} + \sigma \mu_y \delta_m - (\sigma - 1) \delta_{\pi m}) \frac{\delta_{me}}{\delta_{\pi e}}$ can be written as

$$\rho_{ly} = \frac{(\sigma - 1 - \sigma \mu_\pi)(1 - \delta_m) - \delta_m \mu_y}{(1 - \mu_\pi)(\mu_y - 1)} \quad \text{and} \quad \rho_\pi = \frac{(\sigma - 1 - \sigma \mu_\pi)(1 - \delta_m) - \delta_m \mu_y}{(1 - \mu_y)},$$

which relate the feedback parameter of both policy rules. It is more convenient to rewrite these relations as

$$\mu_\pi = 1 + \frac{\rho_\pi}{\rho_{ly}} \quad \text{and} \quad \mu_y = 1 + \frac{(1 - \delta_m)(\mu_\pi - 1)\sigma + 1}{\rho_\pi - \delta_m},$$

which give the conditions for a money supply $\widehat{\mu}_t = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t$ and the backward-looking interest rate rule to implement the same fundamental solution. ■

2.4 Stability under a backward-looking interest rate rule

In order to assess the conditions for stability under the backward-looking interest rate rule $\widehat{R}_t = \rho_\pi \widehat{\pi}_t + \rho_{ly} \widehat{y}_{t-1}$, we examine the characteristic polynomial which is given by

$$K(X) = X^3 - X^2 \frac{\sigma + \omega + \sigma\beta - \sigma\chi\rho_\pi}{\sigma\beta} + X \frac{\sigma + \omega\rho_\pi - \sigma\chi\rho_\pi - \beta\rho_{ly} - \chi\rho_{ly}}{\sigma\beta} + \frac{\rho_{ly}}{\sigma\beta}.$$

Evidently, when the central bank does not react to lagged output, $\rho_{ly} = 0$, then one root is given by $X = 0$, which is assigned to the predetermined variable. Hence, the latter is irrelevant for the equilibrium sequences. If $\rho_{ly} > 0$, we know that $K(0) = \frac{1}{\sigma\beta}\rho_{ly} > 0$, which implies that there are either three or one negative roots. In order to exhibit a stable and positive root, K has to be negative at $X = 1$. Given that $K(1) = -[(1 - \rho_\pi)\,\omega + (\beta + \chi - 1)\left(\rho_y + \rho_{ly}\right)]\sigma^{-1}\beta^{-1}$, we can conclude that there is a stable root if $\rho_\pi < 1 + \frac{(\beta+\chi-1)}{\omega}\rho_{ly}$. Since there is at least one negative root, we further know that there cannot be another positive and stable root.

3 Appendix to the money-in-the-utility-function specification

3.1 Proof of lemma 4

To derive the conditions for stable and non-oscillatory equilibrium sequences in the money-in-the-utility function model under a state contingent money growth policy (16), we replace the nominal interest rate with the money demand condition, $\frac{1}{R-1}\widehat{R}_t = \sigma\widehat{y}_t - \sigma_m \widehat{m}_t$, in the consumption Euler equation to give $R\sigma\widehat{y}_t = \sigma E_t \widehat{y}_{t+1} + (R-1)\,\sigma_m \widehat{m}_t + E_t \widehat{\pi}_{t+1}$. Together with the aggregate supply constraint and the state contingent money growth rule we get the following three-dimensional system

$$\begin{pmatrix} 0 & \beta & 0 \\ (R-1)\,\sigma_m & 1 & \sigma \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{m}_t \\ E_t \widehat{\pi}_{t+1} \\ E_t \widehat{y}_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -\omega \\ 0 & 0 & R\sigma \\ 1 & \mu_\pi - 1 & \mu_y \end{pmatrix} \begin{pmatrix} \widehat{m}_{t-1} \\ \widehat{\pi}_t \\ \widehat{y}_t \end{pmatrix} + \begin{pmatrix} -\chi \\ 0 \\ 0 \end{pmatrix} \widehat{\xi}_t,$$

which exhibits the following characteristic polynomial

$$\begin{aligned} Q(X) = {} & X^3 + X^2 \frac{(R-1)\beta\sigma_m\mu_y - \omega - \sigma\beta(1+R) - \sigma}{\beta\sigma} + \\ & X \frac{\sigma + \omega + R\sigma(1+\beta) - \omega\sigma_m(1-R) + \sigma_m(1-R)\left(\mu_y + \omega\mu_\pi\right)}{\beta\sigma} - \frac{R}{\beta}. \end{aligned}$$

Given that $Q(0) = -R/\beta < -1$, we know that the product of the eigenvalues is strictly positive and larger than one. Thus, there exists at least one unstable root, and either three or one positive root. Further, the value of $Q(1)$ is given by $Q(1) = \frac{(R-1)\sigma_m}{\beta\sigma}(\omega\,(1 - \mu_\pi) - (1 - \beta)\,\mu_y)$. The existence of a stable root lying between zero and one, necessarily requires $Q(1) > 0$, and therefore $\mu_\pi + \frac{1-\beta}{\omega}\mu_y < 1$. Then, there is no other stable and positive root of $Q(X)$. Yet, $Q(X)$ might exhibit a negative stable eigenvalue, which requires $Q(-1) > 0$. ■

3.2 Proof of lemma 5

To characterize the equilibrium behavior of the nominal interest rates under a state contingent money growth policy $\widehat{\mu}_t = \widehat{m}_t + \widehat{\pi}_t - \widehat{m}_{t-1} = \mu_\pi \widehat{\pi}_t + \mu_y \widehat{y}_t + \mu_\xi \widehat{\xi}_t + \mu_{l\xi} \widehat{\xi}_{t-2}$ in a money-in-the-utility function framework, we proceed as in the benchmark model (see appendix 2.2). Using the consumption Euler equation, money demand $\sigma_m \widehat{m}_t + \frac{1}{\overline{R}-1}\widehat{R}_t = \sigma \widehat{y}_t$, and $\widehat{y}_t = \widehat{c}_t$, we get $\sigma_m E_t \widehat{\mu}_{t+1} = (\sigma_m - 1) E_t \widehat{\pi}_{t+1} - \frac{1}{\overline{R}-1} E_t \widehat{R}_{t+1} + \frac{\overline{R}}{\overline{R}-1}\widehat{R}_t$ and with the money growth reaction function

$$\frac{\overline{R}}{\overline{R}-1}\widehat{R}_t - \frac{1}{\overline{R}-1}E_t\widehat{R}_{t+1} = (\sigma_m\mu_\pi - (\sigma_m - 1)) E_t\widehat{\pi}_{t+1} + \mu_y\sigma_m E_t\widehat{y}_{t+1} + \mu_\xi\sigma_m\rho\widehat{\xi}_t + \mu_{l\xi}\sigma_m\widehat{\xi}_{t-1}.$$

Now use that the fundamental solution under a money growth policy implies $E_t\widehat{R}_{t+1} = \delta_m\widehat{R}_t + ((\rho - \delta_m)\delta_{Re} + \delta_{Rm}\delta_{me})\widehat{\xi}_t$. Thus, the current nominal interest rate is characterized by the following equilibrium relation

$$\begin{aligned}\widehat{R}_t =& \frac{\overline{R}-1}{\overline{R}-\delta_m}\left[(\sigma_m\mu_\pi - (\sigma_m - 1)) E_t\widehat{\pi}_{t+1} + \mu_y\sigma_m E_t\widehat{y}_{t+1}\right] \\ &+ \frac{\overline{R}-1}{\overline{R}-\delta_m}\left(\mu_\xi\sigma_m\rho + \frac{1}{\overline{R}-1}((\rho-\delta_m)\delta_{Re} + \delta_{Rm}\delta_{me})\right)\widehat{\xi}_t + \frac{\overline{R}-1}{\overline{R}-\delta_m}\mu_{l\xi}\sigma_m\widehat{\xi}_{t-1}.\end{aligned}$$

By combining these two expressions, we can write

$$\begin{aligned}E_t\widehat{R}_{t+1} =& \rho_R\widehat{R}_t - \left(\overline{R}-1\right)\left[(\sigma_m\mu_\pi - (\sigma_m-1)) E_t\widehat{\pi}_{t+1} + \mu_y\sigma_m E_t\widehat{y}_{t+1} + \mu_\xi\sigma_m\rho\widehat{\xi}_t\right] \\ &-(\rho_R - \overline{R})\frac{\overline{R}-1}{\overline{R}-\delta_m}\left[(\sigma_m\mu_\pi - (\sigma_m-1)) E_t\widehat{\pi}_{t+1} + \mu_y\sigma_m E_t\widehat{y}_{t+1}\right] \\ &-(\rho_R - \overline{R})\frac{\overline{R}-1}{\overline{R}-\delta_m}\left[\left(\mu_\xi\sigma_m\rho + \frac{1}{\overline{R}-1}((\rho-\delta_m)\delta_{Re} + \delta_{Rm}\delta_{me})\right)\widehat{\xi}_t - \mu_{l\xi}\sigma_m\widehat{\xi}_{t-1}\right].\end{aligned} \tag{35}$$

where $\Gamma = \frac{\overline{R}-1}{\overline{R}-\delta_m}$. Further using that $E_t\widehat{\pi}_{t+1} = \delta_m\widehat{\pi}_t + ((\rho-\delta_m)\delta_{\pi e} + \delta_{\pi m}\delta_{me})\widehat{\xi}_t$, $E_t\widehat{y}_{t+1} = \delta_m\widehat{y}_t + ((\rho-\delta_m)\delta_{ye} + \delta_{ym}\delta_{me})\widehat{\xi}_t$, and $\widehat{R}_t = \delta_m\widehat{R}_{t-1} + \delta_{Re}\widehat{\xi}_t + (\delta_{Rm}\delta_{me} - \delta_m\delta_{Re})\widehat{\xi}_{t-1}$, we can rewrite (35) as in form of a backward-looking interest rate relation

$$\begin{aligned}\widehat{R}_t =& \rho_R\widehat{R}_{t-1} - [\sigma_m(\mu_\pi - 1) + 1]\left(\overline{R}-1\right)\frac{\rho_R-\delta_m}{\overline{R}-\delta_m}\widehat{\pi}_t - \mu_y\sigma_m\left(\overline{R}-1\right)\frac{\rho_R-\delta_m}{\overline{R}-\delta_m}\widehat{y}_t \\ &-\frac{\overline{R}-1}{\delta_m}\left\{\begin{array}{c}\mu_\xi\sigma_m\rho + \frac{\rho_R-\overline{R}}{\overline{R}-\delta_m}\left(\mu_\xi\sigma_m\rho + \frac{(\rho-\delta_m)\delta_{Re}+\delta_{Rm}\delta_{me}}{\overline{R}-1}\right) \\ \frac{\rho_R-\delta_m}{\overline{R}-\delta_m}(\sigma_m\mu_\pi - (\sigma_m-1))((\rho-\delta_m)\delta_{\pi e} + \delta_{\pi m}\delta_{me}) \\ +\frac{\rho_R-\delta_m}{\overline{R}-\delta_m}\mu_y\sigma_m((\rho-\delta_m)\delta_{ye} + \delta_{ym}\delta_{me}) + \frac{((\rho-\delta_m)\delta_{Re}+\delta_{Rm}\delta_{me})-\rho_R\delta_{Re}}{\overline{R}-1}\end{array}\right\}\widehat{\xi}_t. \\ &+\left\{\frac{\rho_R}{\delta_m}(\delta_{Rm}\delta_{me} - \delta_m\delta_{Re}) - (\rho_R - \overline{R})\frac{\overline{R}-1}{\overline{R}-\delta_m}\sigma_m\cdot\mu_{l\xi}\right\}\widehat{\xi}_{t-1}\end{aligned} \tag{36}$$

Now, suppose that there exists a value μ_ξ^* and a value $\mu_{l\xi}^*$ such that the terms in the curly brackets equal zero if the central bank sets $\mu_\xi = \mu_\xi^*$ and $\mu_{l\xi} = \mu_{l\xi}^*$. Then, we end up with an expression which can be interpreted as an inertial interest rate target

$$\widehat{R}_t = \rho_R \widehat{R}_{t-1} + \rho_\pi \widehat{\pi}_t + \rho_y \widehat{y}_t,$$
$$\text{where } \rho_\pi = -\left[\sigma_m \left(\mu_\pi - 1\right) + 1\right] \Gamma \left(\rho_R - \delta_m\right) \quad \text{and} \quad \rho_y = -\mu_y \sigma_m \Gamma \left(\rho_R - \delta_m\right),$$

Following the line of arguments for the corresponding problem in appendix 2.2, we further know that there exist unique values for μ_ξ^* and for $\mu_{l\xi}^*$, respectively, since all solution coefficients in the curly brackets in (36) are either independent of μ_ξ^* and $\mu_{l\xi}^*$, or are linear in μ_ξ^* and $\mu_{l\xi}^*$. ■

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