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Null or Zero Players: The Difference between the Shapley value and the Egalitarian Solution*

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Abstract

A situation in which a finite set of players can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility*. A solution for TU-games assigns to every TU-game a distribution of the payoffs that can be earned over the individual players. Two well-known solutions for TU-games are the *Shapley value* and the *egalitarian solution*. The Shapley value is characterized in various ways. Most characterizations use some axiom related to *null players*, i.e. players who contribute nothing to any coalition. We show that in these characterizations, replacing null players by *zero players* characterizes the *egalitarian solution*, where a player is a zero player if every coalition containing this player earns zero worth. We illustrate this difference between these two solutions by applying them to *auction games*.

Keywords: Null players, zero players, Shapley value, egalitarian solution, strong monotonicity, coalitional monotonicity, auction games.
JEL code: C71, D44

1 Introduction

A situation in which a finite set of n players $N = \{1, \ldots, n\}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any *coalition* $S \subseteq N$, $v(S)$ is the *worth* of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate.

A *payoff vector* $x \in \mathbb{R}^n$ of an n-player TU-game (N, v) is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a function that assigns a payoff vector to every TU-game (N, v).

One of the most famous single valued solutions for TU-games is the *Shapley value* (Shapley (1953)) which is widely applied in economic models, see e.g. Rothschild (2001) for allocating cartel profits, Maniquet (2003) for queueing problems and Graham, Marshall and Richard (1990) for bidder ring formation in auctions. In the game theoretic literature various axiomatic characterizations of the Shapley value can be found. Usually these characterizations contain an axiom concerning *null players*, i.e. players who do not contribute anything to any coalition. For example, the *null player property* states that such null players earn zero payoff. Chun (1989) introduced *coalitional strategic equivalence* as a weakening of *strong monotonicity* (see Young (1985)) stating that if to a game we add another game in which some player is a null player, then the payoff of this player does not change.

Instead of null players we can also consider *zero players*, i.e. players whose presence in a coalition implies that the coalition generates zero worth, see Deegan and Packel (1979). It turns out that replacing null players by zero players in axiomatic characterizations of the Shapley value yield characterizations of the *egalitarian solution* which distributes the worth of the 'grand coalition' N equally among all players. The null player property then becomes the *zero player property* stating that zero players earn a zero payoff. Coalitional strategic equivalence becomes *coalitional standard equivalence* stating that if to a game we add another game in which some player is a zero player, then the payoff of this player does not change.

We illustrate this difference between the Shapley value and the egalitarian solution by applying them to *auction games* which describe situations in which we have to assign an indivisible good to one out of a set of agents, and determine how the agent who gets the good has to compensate the others. The agents individually valuate the good at possibly different values. We discuss allocation rules that are obtained by applying the Shapley value and the egalitarian solution to the corresponding auction games. Both rules assign the good to an agent that valuates it the highest, but they differ in the compensations that are used. In this context the difference between the Shapley value and the egalitarian solution boils down to different independence axioms. The Shapley rule (obtained by applying the Shapley value) satisfies *independence on higher valuations* which states that the compensation of an agent does not depend on the valuations of agents with higher valuations for the good. The *egalitarian rule* (obtained by applying the egalitarian solution) satisfies *independence on lower valuations* which implies that the compensation of an agent does not depend on the valuations of agents with lower valuations for the good. We show that these independence axioms follow from strong monotonicity and coalitional monotonicity of solutions for TU-games.

The paper is organized as follows. In Section 2 we state some preliminaries on TU-games and the Shapley value. In Section 3 we show how replacing null players by zero players yields characterizations of the egalitarian solution. In Section 4 we apply some properties concerning null- and zero players to auction games and show the difference between the allocation rules obtained by applying the Shapley value and the egalitarian solution in such games. Finally, Section 5 contains some concluding remarks.

2 Preliminaries

In this paper we take the set of players N to be fixed, and therefore denote a TU-game (N, v) just by its characteristic function v. The collection of all characteristic functions (which we will thus refer to as games) on N is denoted by $\mathcal{G}^N$. The increase in worth when player $i \in N$ joins coalition $S \subseteq N \setminus \{i\}$ is called the *marginal contribution* of player i to coalition S in game $v \in \mathcal{G}^N$ and is denoted by

$$m_i^S(v) = v(S \cup \{i\}) - v(S).$$

Suppose that the 'grand coalition' N forms in a way such that the players enter the coalition one by one. Such an order of entrance can be represented by a permutation $\pi\colon N \to N$ of the players. We denote the collection of all permutations on N by $\Pi(N)$. The *Shapley value* (Shapley (1953)) is the solution $Sh\colon \mathcal{G}^N \to \mathbb{R}^n$ that assigns to every player its expected marginal contribution to the coalition of players that enter before him, given that every order of entrance π has equal probability to occur, i.e.

$$Sh_i(v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} \cdot m_i^{P(\pi,i)}(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-|S|-1)!(|S|)!}{n!} \cdot m_i^S(v) \text{ for all } i \in N,$$

where $n = |N|$, and for every $\pi \in \Pi(N)$ we denote by $P(\pi, i) = \{j \in N \mid \pi(j) < \pi(i)\}$ the set of players that enter before player i in permutation π. Various axiomatizations of the Shapley value have been given in the literature. First, it is characterized by efficiency, the null player property, symmetry and additivity. A solution f satisfies *efficiency* if it always exactly distributes the worth of the 'grand' coalition, i.e. if $\sum_{i \in N} f_i(v) = v(N)$ for all $v \in \mathcal{G}^N$. We already mentioned the null player property in the introduction. Player $i \in N$ is a *null player* in $v \in \mathcal{G}^N$ if $m_i^S(v) = 0$ for all $S \subseteq N \setminus \{i\}$. A solution f satisfies the *null player property* if $f_i(v) = 0$ whenever i is a null player in v. Next, two players $i, j \in N$ are called *symmetric* in $v \in \mathcal{G}^N$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. A solution f satisfies *symmetry* if it always assigns the same payoff to symmetric players[1], i.e. if $f_i(v) = f_j(v)$ whenever i and j are symmetric players in $v \in \mathcal{G}^N$. A solution f satisfies *additivity* if the payoffs assigned to players in the sum of two games is equal to the sum of the payoffs assigned to them in the two separate games, i.e. if $f(v + w) = f(v) + f(w)$ for every pair of games $v, w \in \mathcal{G}^N$, where $(v + w) \in \mathcal{G}^N$ is given by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

Another well-known characterization of the Shapley value is given by Young (1985) who characterized the Shapley value by efficiency, symmetry and strong monotonicity. A solution f satisfies *strong monotonicity* if $f_i(v) \geq f_i(w)$ for every pair of games $v, w \in \mathcal{G}^N$ and $i \in N$ such that $m_i^S(v) \geq m_i^S(w)$ for all $S \subseteq N \setminus \{i\}$. As argued by Chun (1989), in Young's characterization strong monotonicity can be weakened to *coalitional strategic equivalence* which says that for $v, w \in \mathcal{G}^N$ it holds that $f_i(v + w) = f_i(v)$ whenever i is a null player in w. Other characterizations on a fixed player set[2] can be found in, e.g. Feltkamp (1995) and van den Brink (2001).

3 Zero players and the egalitarian solution

The *egalitarian solution* $\gamma\colon \mathcal{G}^N \to \mathbb{R}^n$ distributes the worth $v(N)$ of the 'grand coalition' equally among all players in any game, i.e.,

$$\gamma_i(v) = \frac{v(N)}{n} \text{ for all } i \in N.$$

The egalitarian solution satisfies efficiency, symmetry and addivitity. It does not satisfy the null player property nor strong monotonicity.

Example 3.1 *Let $N = \{1, 2\}$. Consider the game $v \in \mathcal{G}^N$ given by $v(S) = 1$ if $1 \in S$, and $v(S) = 0$ otherwise. Then $\gamma_2(v) = \frac{1}{2}$ although player 2 is a null player in v. Considering also the game $w \in \mathcal{G}^N$ given by $w(S) = 0$ for all $S \subseteq N$, we see that $\gamma_2(w) = 0 < \frac{1}{2} = \gamma_2(v)$ although $m_2^S(v) = m_2^S(w)$ for $S \in \{\emptyset, \{1\}\}$.* □

[1]In the literature, this symmetry property is also refered to as *equal treatment of equals*.

[2]Characterizations on variable player sets can be found in, e.g. Hart and Mas-Colell (1988, 1989).

Replacing null players by zero players in the characterizations mentioned before characterizes the egalitarian solution. Player $i \in N$ is a *zero player* in $v \in \mathcal{G}^N$ if $v(S) = 0$ for all $S \subseteq N$ with $i \in S$. A solution f satisfies the *zero player property* if $f_i(v) = 0$ whenever i is a zero player in v (see Deegan and Packel (1979) who used this property to characterize their Deegan-Packel value).

Theorem 3.2 *A solution $f\colon \mathcal{G}^N \to \mathbb{R}^n$ is equal to the egalitarian solution if and only if it satisfies efficiency, the zero player property, symmetry and additivity.*

PROOF
It is easy to verify that γ satisfies the four properties. Uniqueness follows in a similar way as for the Shapley value, but using the standard basis instead of the unanimity basis[3] of a game. For any $\alpha \in \mathbb{R}$, the α-scaled *standard game* of coalition $T \subseteq N$, $T \neq \emptyset$, is the game $b_T^\alpha \in \mathcal{G}^N$ given by $b_T^\alpha(S) = \alpha$ if $S = T$, and $b_T^\alpha(S) = 0$ otherwise. Take $T \subseteq N$, $T \notin \{N, \emptyset\}$. Then the zero player property implies that $f_i(b_T^\alpha) = 0$ for all $i \in N \setminus T$. Efficiency then implies that $\sum_{i \in N} f_i(b_T^\alpha) = \sum_{i \in T} f_i(b_T^\alpha) = b_T^\alpha(N) = 0$. Thus, with symmetry it follows that $f_i(b_T^\alpha) = 0$ for all $i \in T$. For $T = N$, efficiency and symmetry imply that $f_i(b_N^\alpha) = \frac{b_N^\alpha(N)}{n} = \frac{\alpha}{n} = \frac{v(N)}{n}$ for all $i \in N$.
Uniqueness for arbitrary $v \in \mathcal{G}^N$ follows since additivity of f and the fact that $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} b_T^{v(T)}$ implies that $f_i(v) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} f_i(b_T^{v(T)}) = f_i(b_N^{v(N)}) = \frac{v(N)}{n}$ for every $v \in \mathcal{G}^N$. □

Replacing null players by zero players in coalitional strategic equivalence yields the property which states that the payoff of a player does not change if we add a game in which this player is a zero player. So, a solution f satisfies *coalitional standard equivalence* if for $v, w \in \mathcal{G}^N$ it holds that $f_i(v + w) = f_i(v)$ whenever i is a zero player in w. Replacing coalitional strategic equivalence by this new property in the characterization of the Shapley value given in Chun (1989) yields a characterization of the egalitarian solution.

Theorem 3.3 *A solution $f\colon \mathcal{G}^N \to \mathbb{R}^n$ is equal to the egalitarian solution if and only if it satisfies efficiency, symmetry and coalitional standard equivalence.*

PROOF
It is easy to verify that γ satisfies the three properties. Uniqueness follows by induction on $d(v) = |\{T \subseteq N \mid v(T) \neq 0\}|$ (in a similar way as for the Shapley value by induction on the number of coalitions with non-zero dividend).

[3]For $\alpha \in \mathbb{R}$, the α-scaled *unanimity game* of coalition $T \subseteq N$, $T \neq \emptyset$, is the game $u_T^\alpha \in \mathcal{G}^N$ given by $u_T^\alpha(S) = \alpha$ if $T \subseteq S$, and $u_T^\alpha(S) = 0$ otherwise. As is known, every game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} u_T^{\Delta_v(T)}$ with $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ being the Harsanyi dividends, see Harsanyi (1959).

If $d(v) = 0$ then efficiency and symmetry imply that $f_i(v) = 0$ for all $i \in N$.
Proceeding by induction assume that $f_i(w) = \gamma_i(w)$ if $d(w) < d(v)$. Let $H(v) = \{i \in N \mid v(S) = 0 \text{ for all } S \subseteq N \setminus \{i\}\}$. For every $i \in N \setminus H(v)$ there exists an $S \subseteq N \setminus \{i\}$ such that $v(S) \neq 0$. Coalitional standard equivalence and the induction hypothesis then imply that $f_i(v) = f_i(v - b_S^{v(S)}) = \gamma_i(v - b_S^{v(S)}) = \gamma_i(v)$ for $i \in N \setminus H(v)$ and $S \subseteq N \setminus \{i\}$. (The last equality follows since $b_S^{v(S)}(N) = 0$ if $S \subseteq N \setminus \{i\}$.)
Symmetry and efficiency then imply that $f_i(v) = \frac{v(N) - \sum_{j \in N \setminus H(v)} \gamma_j(v)}{|H(v)|} = \gamma_i(v)$ for all $i \in H(v)$. □

Replacing null players by zero players in Young's strong monotonicity boils down to replacing marginal contributions by the worths of coalitions. This yields *coalitional monotonicity* meaning that $f_i(v) \geq f_i(w)$ for every pair of games $v, w \in \mathcal{G}^N$ and player $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$.

It is easy to verify that the egalitarian solution satisfies coalitional monotonicity. From the previous theorem it then follows that the egalitarian solution is characterized by replacing strong monotonicity by coalitional monotonicity in Young's characterization (since f satisfying coalitional monotonicity implies that f satisfies coalitional standard equivalence). In this case, symmetry even can be weakened to weak symmetry which only requires symmetry for games in which all players are symmetric, i.e., a solution f satisfies *weak symmetry* if for every $v \in \mathcal{G}^N$ such that all players in N are symmetric in v, there exists a $c^* \in \mathbb{R}$ such that $f_i(v) = c^*$ for all $i \in N$.

Theorem 3.4 *A solution $f\colon \mathcal{G}^N \to \mathbb{R}^n$ is equal to the egalitarian solution if and only if it satisfies efficiency, weak symmetry and coalitional monotonicity.*

PROOF
It is easy to verify that γ satisfies efficiency and weak symmetry. Coalitional monotonicity is satisfied since $v(S) \geq w(S)$ for all $S \subseteq N$ with $i \in S$, implies that $v(N) \geq w(N)$ and thus $\gamma_i(v) = \frac{v(N)}{n} \geq \frac{w(N)}{n} = \gamma_i(w)$.
To show uniqueness, suppose that the solution f satisfies the three axioms, and let $v \in \mathcal{G}^N$. For $i \in N$, define $w^i \in \mathcal{G}^N$ by

$$w^i(S) = \begin{cases} v(N) & \text{if } S = N \\ \min_{\substack{T \subseteq N \\ i \in T}} v(T) & \text{otherwise.} \end{cases}$$

Efficiency and weak symmetry imply that $f_j(w^i) = \frac{v(N)}{n}$ for all $j \in N$. In particular, $f_i(w^i) = \frac{v(N)}{n}$. Since $v(S) \geq w^i(S)$ for all $i \in N$ and $S \subseteq N$ with $i \in S$, coalitional monotonicity implies that $f_i(v) \geq f_i(w^i) = \frac{v(N)}{n}$ for all $i \in N$. Efficiency then implies that $f_i(v) = \frac{v(N)}{n} = \gamma_i(v)$ for all $i \in N$. □

4 An application: auction situations

4.1 Auction games

Consider the situation in which one indivisible good is to be assigned to one out of the set of n agents $N = \{1, \ldots, n\}$. The agents valuate the good at possibly different values. This situation can be described by the pair (N, V) where N is the set of agents and $V \in \mathbb{R}^n_+$ is the vector which i^{th} component $V_i \in \mathbb{R}_+$ is the value at which agent $i \in N$ valuates the indivisible good. The agent who gets the good can compensate the others by giving them an amount of some *numeraire* good. We assume that all agents value each unit of the numeraire good at the same value, normalized to be 1.

An *allocation-compensation scheme* is a pair $(i, c) \in N \times \mathbb{R}^n_+$ where $i \in N$ denotes the agent who gets the good and $c \in \mathbb{R}^n$ satisfying $\sum_{j \in N} c_j = 0$ is the vector of compensations. So, c_j is the amount of the numeraire good that agent i gives to agent j as compensation if $j \neq i$, and c_i is the total compensation that has to be paid by i to the other agents. The *value* of an allocation-compensation scheme (i, c) is the vector $\phi(i, c) \in \mathbb{R}^n_+$ with $\phi_j(i, c) = c_j$ if $j \in N \setminus \{i\}$, and $\phi_i(i, c) = V_i + c_i$.

The main question in this situation is who gets the indivisible good and what is a 'fair' way to compensate the others. A popular way to handle such an allocation problem is using auctions. Therefore, we refer to a pair (N, V) as described above as an *auction situation*. Given valuations $V \in \mathbb{R}^n_+$ we define the corresponding *auction game* as the TU-game $v \in \mathcal{G}^N$ on a set of players N that coincides with the set of agents in the auction situation, and which characteristic function assigns to every coalition $S \subseteq N$ the maximal valuation over the agents in S, i.e.,

$$v(S) = \max_{i \in S} V_i \text{ for all } S \subseteq N. \tag{4.1}$$

So, according to the game v, in every coalition we assign the good to the agent who valuates it the most[4,5].

Graham, Marshall and Richard (1990) use a slightly different game to represent auction situations. Given valuations $V \in \mathbb{R}^n_+$, for every $S \subseteq N$ they define a two-player strategic game[6] between the two 'players' S and $N \setminus S$. They show that a dominant strategy for S in this game is to remain active until the bidding reaches $v(S)$ as given in (4.1), and

[4] To be precise we should define the auction mapping $v\colon \mathbb{R}^n_+ \to \mathcal{G}^N$ such that $v(V)$ is the auction game corresponding to the auction situation with valuations $V \in \mathbb{R}^n_+$. With slight abuse of notation we write this game just as v if there is no confusion with respect to the valuations.

[5] We remark that in an auction situation the good initially is not owned by one of the players. On the other hand, in market games the initial owner is one of the players in the game.

[6] For details about this strategic game we refer to Graham, Marshall and Richard (1990).

for $N \setminus S$ to remain active until the bidding reaches $v(N \setminus S)$. From this they derive the worth of a coalition $S \subseteq N$ to be equal to

$$w(S) = \max\left\{\max_{i \in S} V_i - \max_{j \in N \setminus S} V_j, 0\right\}.$$

It is straightforward to verify that the game w is equal to the *dual game* v^* of the auction game v, and thus is also equal to the auction game that is studied in Brânzei, Fragnelli and Tijs (2002) and is given by[7]

$$v^*(S) := v(N) - v(N \setminus S) = \max_{j \in N} V_j - \max_{j \in N \setminus S} V_j \text{ for all } S \subseteq N.$$

In the next subsections we apply the Shapley value and the egalitarian solution to auction games. Since, the Shapley value and egalitarian solution of a game are equal to the Shapley value, respectively, egalitarian solution of its dual game, it does not matter whether we use the game v or its dual game $w = v^*$ corresponding to an auction situation.

4.2 Allocation rules for auction situations

An allocation rule for auction situations is a function $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ which assigns to every vector of valuations $V \in \mathbb{R}^n_+$ the value $\varphi(V) \in \mathbb{R}^n_+$ of some allocation-compensation scheme[8]. In this section we discuss two of such allocation rules. Without loss of generality we assume that $V_1 \leq \ldots \leq V_n$.

Applying the Shapley value to the corresponding auction game as defined in (4.1) yields the *Shapley rule* $\varphi^{Sh}\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ given by

$$\varphi_i^{Sh}(V) = \sum_{j=1}^{i} \frac{V_j - V_{j-1}}{n-j+1} \text{ for } i \in N, \tag{4.2}$$

where $V_0 = 0$.[9] The use of the Shapley value in auction games is motivated by Graham, Marshall and Richard (1990). On the other hand, applying the egalitarian solution to this game yields the egalitarian rule $\varphi^{\gamma}\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ which is given by

$$\varphi_i^{\gamma}(V) = \frac{\max_{i \in N} V_i}{n} = \frac{V_n}{n} \text{ for all } i \in N.$$

Both solutions assign the indivisible good to the agent with the highest valuation but differ in the way this agent compensates the others.

[7]This can be seen as follows. If $\max_{i \in S} V_i \geq \max_{j \in N \setminus S} V_j$ then $v^*(S) = \max_{i \in S} V_i - \max_{j \in N \setminus S} V_j \geq 0$, and thus $v^*(S) = w(S)$. If $\max_{i \in S} V_i \leq \max_{j \in N \setminus S} V_j$ then $v^*(S) = \max_{j \in N \setminus S} V_j - \max_{j \in N \setminus S} V_j = 0 = w(S)$.

[8]We again take the set of agents N fixed, and thus represent an auction situation (N, V) just by its vector of valuations V.

[9]Alternatively, $V_0 \in [0, V_1]$ can be taken as the reservation value of the seller of the good.

4.3 Axiomatizations of the Shapley- and egalitarian rule

Next we illustrate the use of axioms that characterize the Shapley value and the egalitarian solution for TU-games, to characterize the corresponding allocation rules for auction situations. We do this by applying Young's (1985) characterization of the Shapley value and Theorem 3.4 for the egalitarian solution. First, efficiency and symmetry for TU-games applied to auction situations yield the following two properties[10]. Efficiency states that the sum of all values over all agents equals the highest valuation that is attached to the indivisible good. If we do not allow for negative compensations this implies that the good is assigned to an agent who valuates it the most.

Axiom 1 (Efficiency) *For every* $V \in \mathbb{R}^n_+$ *it holds that* $\sum_{i \in N} \varphi_i(V) = \max_{i \in N} V_i$.

Symmetry or equal treatment of equals states that two agents who valuate the good equally should earn the same. This implies that if the good is not assigned to one of them then they should be compensated by the same amount. If the good is assigned to one of them, then this agent should compensate the other such that their values in the resulting allocation-compensation scheme are equal.

Axiom 2 (Symmetry) *For every* $V \in \mathbb{R}^n_+$ *and* $i, j \in N$ *with* $V_i = V_j$ *it holds that* $\varphi_i(V) = \varphi_j(V)$.

Applying strong monotonicity to auction situations yields a property that states that the value obtained by an agent does not change if we increase the valuation of agents who valuate the good at least as high as i. In the context of auction situations we refer to this property as independence on higher valuations.

Axiom 3 (Independence on higher valuations) *Let* $V, W \in \mathbb{R}^n_+$ *and* $i, j \in N$, $i \neq j$, *be such that* $\min\{W_j, V_j\} \geq V_i$ *and* $W_h = V_h$ *for all* $h \in N \setminus \{j\}$. *Then* $\varphi_i(V) = \varphi_i(W)$.

Proposition 4.1 *Let* $f\colon \mathcal{G}^N \to \mathbb{R}^n$ *be a solution for TU-games, and let the allocation rule* $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *for auction situations be given by* $\varphi(V) = f(v)$, *where* v *is the auction game corresponding to* $V \in \mathbb{R}^n_+$ *given by (4.1). Then* φ *satisfies independence on higher valuations if* f *satisfies strong monotonicity.*

Proof

Let $V, W \in \mathbb{R}^n_+$ and $i, j \in N$, $i \neq j$, be such that $\min\{W_j, V_j\} \geq V_i$ and $W_h = V_h$ for all $h \in N \setminus \{j\}$. Take $S \subseteq N \setminus \{i\}$. If $j \notin S$ then $m_i^S(v) = m_i^S(w)$ since $W_h = V_h$ for

[10] For these two properties we use the same name in auction situations and TU-games. It will be clear from the context when we speak about auction situations or about TU-games.

all $h \in N \setminus \{j\}$. If $j \in S$ then $m_i^S(v) = m_i^S(w) = 0$ since $\min\{W_j, V_j\} \geq V_i = W_i$, and thus $\max_{h \in S \cup \{i\}} V_h = \max_{h \in S} V_h$ and $\max_{h \in S \cup \{i\}} W_h = \max_{h \in S} W_h$. Solution f satisfying strong monotonicity then implies that $\varphi_i(V) = \varphi_i(W)$, and thus φ satisfies independence on higher valuations. □

The three axioms described above characterize the Shapley rule[11].

Theorem 4.2 *An allocation rule $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is equal to the Shapley rule if and only if it satisfies efficiency, symmetry and independence on higher valuations.*

Proof

The Shapley rule satisfying efficiency and symmetry follows easily from its definition, while independence on higher valuations follows from Proposition 4.1.

Now, suppose that $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies the three axioms, and let $V \in \mathbb{R}^n_+$. Suppose without loss of generality that $V_i \leq V_j$ if $i < j$. We prove that $\varphi_i(V) = \varphi_i^{Sh}(V)$ by induction on the label i.

Let $V^1 = (V_1^1, \ldots, V_n^1)$ be such that $V_i^1 = V_1$ for all $i \in N$. Symmetry and efficiency imply that $\varphi_i(V^1) = \frac{V_1}{n}$ for all $i \in N$. Independence on higher valuations then implies that $\varphi_1(V) = \varphi_1(V^1) = \frac{V_1}{n}$.

Proceeding by induction, assume that we have determined $\varphi_i(V)$ for all $i \leq j-1$.

Define $V^j = (V_1^j, \ldots, V_n^j)$ such that $V_i^j = V_i$ for all $i < j$, and $V_i^j = V_j$ for all $i \geq j$. The induction hypothesis implies that we have determined the values $\varphi_i(V)$ for all $i < j$. Independence on higher valuations then implies that the values $\varphi_i(V^j) = \varphi_i(V)$ are uniquely determined for all $i < j$. Symmetry implies that all $\varphi_i(V^j)$ should be equal for all $i \geq j$. With efficiency it then follows that $\varphi_j(V^j) = \frac{V_j - \sum_{h=1}^{j-1} \varphi_h(V)}{n-j+1}$ which is uniquely determined since all $\varphi_h(V),\ h \in \{1, \ldots, j-1\}$, are uniquely determined. According to independece on higher valuations it then follows that $\varphi_j(V) = \varphi_j(V^j)$ is uniquely determined. □

Clearly, the egalitarian rule φ^γ does not satisfy independence on higher valuations. Instead it satisfies independence on lower valuations which states that the value obtained by an agent does not change if we lower the valuation of agents who value the good at most as high as i.

[11] Alternatively, we can prove this (and the next) theorem by characterizing the class of auction games and characterizing the Shapley value and egalitarian solution restricted to this class. As already noted by Dubey (1975), axiomatizations of the Shapley value on $\mathcal{G}^N$ not necessarily characterize the Shapley value on subclasses of games. For our purpose it is more easy and transparent to give direct proofs without refering to auction games.

Axiom 4 (Independence on lower valuations) *Let $V, W \in \mathbb{R}^n_+$ and $i, j \in N$, $i \neq j$, be such that $\max\{W_j, V_j\} \leq V_i$ and $W_h = V_h$ for all $h \in N \setminus \{j\}$. Then $\varphi_i(V) = \varphi_i(W)$.*

This independence axiom is obtained by applying coalitional monotonicity to the class of auction games.

Proposition 4.3 *Let $f\colon \mathcal{G}^N \to \mathbb{R}^n$ be a solution for TU-games, and let the allocation rule $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ for auction situations be given by $\varphi(V) = f(v)$, where v is the auction game corresponding to $V \in \mathbb{R}^n_+$ given by (4.1). Then φ satisfies independence on lower valuations if f satisfies coalitional monotonicity.*

Proof

Let $V, W \in \mathbb{R}^n_+$ and $i, j \in N$, $i \neq j$, be such that $\max\{W_j, V_j\} \leq V_i$ and $W_h = V_h$ for all $h \in N \setminus \{j\}$. Take $S \subseteq N$ with $i \in S$. If $j \notin S$ then $v(S) = w(S)$ since $W_h = V_h$ for all $h \in N \setminus \{j\}$. If $j \in S$ then $v(S) = w(S) \geq V_j$ since $i \in S$ and $W_i = V_i \geq \max\{W_j, V_j\}$, and thus $\max_{h \in S} V_h = \max_{h \in S} W_h$. Solution f satisfying coalitional monotonicity then implies that $\varphi_i(V) = \varphi_i(W)$, and thus φ satisfies independence on lower valuations. □

Replacing independence on higher valuations in Theorem 4.2 by independence on lower valuations yields a characterization of the egalitarian rule.

Theorem 4.4 *An allocation rule $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is equal to the egalitarian rule φ^γ if and only if it satisfies efficiency, symmetry and independence on lower valuations.*

Proof

The egalitarian rule satisfying satisfying efficiency and symmetry follows easily from its definition, while independence on lower valuations follows from Proposition 4.3.
Now, suppose that $\varphi\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies the three axioms. Again suppose without loss of generality that $V_i \leq V_j$ if $i < j$. We again prove that $\varphi_i(V) = \varphi_i^\gamma(V)$ by induction on the label i. (The proof goes along the same lines as the proof of Theorem 4.2, but now we start with the agent with the highest valuation instead of the one with the lowest valuation.)
Let $V^n = (V_1^n, \ldots, V_n^n) \in \mathbb{R}^n_+$ be such that $V_i^n = V_n$ for all $i \in N$. Symmetry and efficiency imply that $\varphi_i(V^n) = \frac{V_n}{n}$ for all $i \in N$. Independence on lower valuations then implies that $\varphi_n(V) = \varphi_n(V^n) = \frac{V_n}{n}$.
Proceeding by induction, assume that $\varphi_i(V) = \frac{V_n}{n}$ for all $i \geq j + 1$.
Define $V^j = (V_1^j, \ldots, V_n^j) \in \mathbb{R}^n_+$ such that $V_i^j = V_i$ for all $i > j$ and $V_i^j = V_j$ for all $i \leq j$. The induction hypothesis implies that $\varphi_i(V) = \frac{V_n}{n}$ for all $i > j$. Independence on lower valuations then implies that $\varphi_i(V^j) = \varphi_i(V) = \frac{V_n}{n}$ for all $i > j$. Symmetry implies that all $\varphi_i(V^j)$ should be equal for all $i \leq j$. With efficiency it then follows that

$\varphi_j(V^j) = \frac{V_n - \sum_{h=j+1}^{n} \varphi_h(V)}{j} = \frac{V_n - (n-j)\frac{V_n}{n}}{j} = \frac{V_n}{n}$. Independence on lower valuations finally implies that $\varphi_j(V) = \varphi_j(V^j) = \frac{V_n}{n}$. □

Above we stated results in case the valuations $V = (V_1, \ldots V_n) \in \mathbb{R}_+^n$ are non-negative. The results can be re-stated for arbitrary valuations $V \in \mathbb{R}^n$. If $V \in \mathbb{R}_-^n$ then the good can be seen as an indivisible 'bad' that has to be assigned to one of the agents, who all prefer not to own the good. According to the rules described above, the 'bad' will be assigned to one of the agents who dislikes it the least, who will be compensated by the others. According to the egalitarian rule all others pay to the agent who obtains the 'bad' in such a way that every agent has the same negative value. According to the Shapley rule the agent who obtains the 'bad' might be compensated in such a way that he even obtains a positive value[12]. If there are both negative and positive valuations, then some agents like to have the good while at the same time others do not want the good. Again, according to the egalitarian rule and the Shapley rule the good will be assigned to the agent with the highest positive valuation[13]. According to the egalitarian rule this agent will compensate the others. According to the Shapley rule not all other agents get compensated by a positive amount. If the negative valuations are small enough it can even be the case that the agent with highest positive valuation besides obtaining the good also gets a positive compensation[14].

5 Concluding remarks

We showed that in characterizations of the Shapley value, replacing null players by zero players characterizes the egalitarian solution. We illustrated this difference between these two solutions by applying them to auction games which describe a situation in which the

[12]With the Shapley- and egalitarian rule we mean the rule that is obtained by applying the Shapley-, respectively, eqalitarian solution to the corresponding auction game (4.1). Note that expression (4.2) of the Shapley rule needs some adaptation (e.g. by labeling the agents such that $V_1 \geq V_2 \geq \ldots \geq V_n$). Consider, for example, the set of agents $N = \{1, 2, 3\}$ with valuations $V = (-1, -2, -3)$. The egalitarian rule and Shapley rule assign to this situation the value vectors $(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $(\frac{1}{2}, -\frac{1}{2}, -1)$, respectively. So, agent 1 gets the 'bad' but according to the Shapley rule gets compensated in a way such that his value is positive.

[13]Again, we mean the rules obtained by applying the Shapley- respectively eqalitarian solution to the corresponding auction game (4.1). Expression (4.2) of the Shapley rule must adapted to distinguish between the agents with positive and negative valuation.

[14]Consider, for example, the set of agents $N = \{1, 2, 3\}$ with valuations $V = (1, -1, -2)$. The egalitarian rule and Shapley rule assign to this situation the value vectors $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(1\frac{5}{6}, -\frac{1}{6}, -1\frac{2}{3})$, respectively. So, agent 1 gets the good which gives him a positive value of 1, but according to the Shapley rule he even gets a positive compensation of $\frac{5}{6}$ because the other agents do not want the good.

allocation of an indivisible good is done by an auction such that the agent (bidder) who obtains the good compensates the other agents (bidders). Graham and Marshall (1989) argue that cooperation appears in English open and Second price sealed bid auctions by the bidders forming bidder rings. Graham, Marshall and Richard (1990) showed that the Shapley rule might be the outcome in such auctions when the agents (bidders) form such bidder rings. On the other hand, if bidder rings are not formed we usually see that the bidders who do not 'win' the auction earn the same payoff. If there is no compensation then this payoff is zero, so the winner might be better off than the others.

In this application we concentrated on the characterization of the Shapley value given by Young (1985) and the similar characterization of the egalitarian solution. We saw that applying these characterizations to the restricted class of auction games characterizes these solutions. This is not the case for all characterizations of the Shapley value. For example, taking the characterization that uses efficiency, the null player property, symmetry and additivity, we saw that efficiency and symmetry can be applied straightforward to the class of auction games. The same holds for the null player property, which then states that a player with zero valuation earns a zero payoff[15]. However, we can apply additivity only to two auction games if the sum of these two games is also an auction game. Therefore, applying additivity to the class of auction games yields the property of *additivity over order preserving valuations*[16], meaning that $\varphi(V + W) = \varphi(V) + \varphi(W)$ whenever $V, W \in \mathbb{R}^n_+$ satisfy $V_i \geq V_j$ if and only if $W_i \geq W_j$. Although the Shapley rule satisfies these four properties, it is not the unique allocation rule satisfying these properties. For example, the allocation rule that equally distributes the highest valuation over the agents with the highest valuation (i.e. the allocation rule $\tilde{\varphi}\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ given by $\tilde{\varphi}_i(V) = \frac{\max_{j \in N} V_j}{|M(V)|}$, and $\tilde{\varphi}_i(V) = 0$ otherwise, where $M(V) = \{j \in N \mid V_j = \max_{h \in N} V_h\}$), also satisfies these four properties[17].

The idea of replacing null players by zero players to obtain characterizations of some egalitarian solution also can be applied to games with restricted cooperation, such as games in coalition structure, games with limited communication (graph) structure, games with a hierarchical permission structure, or games on more general combinatorial structures. A next step after that can be to look at the consequences for network formation as done by Jackson and Wolinsky (1996) for the games with limited communication (graph) structure as introduced in Myerson (1977).

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[15]Note that zero players appear in auction games only when all players have zero valuation.

[16]It can be shown that adding two auction games that arise from valuation vectors that are not order preserving yields a game that cannot be the auction game corresponding to some valuation vector.

[17]A characterization of the Shapley value using additivity over order preserving valuations for (dual) auction games is given in Algaba, Bilbao, van den Brink and Jiménez-Losada (2003) who add a property called structural monotonicity.

We end by mentioning that all axioms in the theorems in this paper are logically independent.

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